ACCELERATION OF BOUNCING BALLS IN EXTERNAL FIELDS

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We introduce two models, the Fermi-Ulam model in an external field and a one dimensional system of bouncing balls in an external field above a periodically oscillating plate. For both models we investigate the possibility of unbounded motion. In a special case the two models are equivalent.

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1. Introduction

In this article we investigate the possibility of unbounded growth of energy in periodically forced Hamiltonian systems. The first models of this type were proposed by Poincaré [P] in relation to the problem of the growth of entropy. In [Pu8] these questions were studied in Newtonian as well as relativistic mechanics. One of the most popular simple models for which this question has been investigate is the Fermi-Ulam model [F][U]: a point particle moves vertically between two periodically oscillating plates and reflects elastically from them. For rationally related frequencies of the oscillating plates it was shown [Pu4][Pu5][Pu9] that the velocity of the ball is bounded for each trajectory. Another popular model is that of an elastic ball bouncing vertically on a periodically oscillating plate under the influence of an external gravitational field. In contrast to the Fermi-Ulam model it was shown in [Pu1]-[Pu3] that for a sufficiently fast oscillating plate for an open set of initial condition with infinite measure the velocity of the ball is unbounded.

In this paper we propose two new models for the investigation of Fermi acceleration which are natural generalizations of the above described models. The first is the Fermi-Ulam model in an external gravitational field which will be described precisely in section 2. The second is a one dimensional system of bouncing balls in an external gravitational field above a periodically oscillating plate which will be described precisely in section 3. For each of these models we investigate the question, does there exist a trajectory with unbounded growth of the absolute value of the velocity and the energy. Furthermore we show that in a certain special case the Fermi-Ulam model in an external field is equivalent

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to the one dimensional system of two balls. In contrast to the Fermi-Ulam model, for the bouncing ball model we study only special cases.

2. The Fermi-Ulam model in an external field with and without singularities

At first we describe the (classical) Fermi-Ulam model [U]. We consider two infinitely heavy parallel plates, the lower plate oscillates according to the law $z = f_1(t)$ and the upper plate according to the law $z = f_2(t)$. Here t is time and we assume the both functions are smooth, periodic with period $T > 0$ and $f_2(t) > f_1(t)$. Note that the period of both functions is assumed the same! A ball moves freely in the direction z between the plates and collides elastically with them. Here the laws of motion and collision are Newtonian. The Newtonian mechanics play an important role, in [Pu6][Pu7] a relativistic analogue of the Fermi-Ulam model was introduced and it was shown that the energy of the ball has unbounded growth. Below we will describe the behavior of the classical Fermi-Ulam model which is in stark contrast to the relativistic case.

Ulam [U] posed the following problem in connection with mechanism of Fermi acceleration [F]: do there exist trajectories for which the energy or the velocity of the ball grow unboundedly. This problem for the Fermi-Ulam model was first solved affirmatively in [Pu4][Pu5] for analytic functions $f_1(t)$ and $f_2(t)$. Ulam himself only considered the case when $f_1(t)$ and $f_2(t)$ are sinusoidal and moreover the special case when one of them is constant. The main result of [Pu4][Pu5] is: for any initial condition the energy and the velocity of the ball are bounded by a certain constant which depends on the initial conditions. The proofs in [Pu4][Pu5] actually hold for smooth enough differentiable functions $f_1(t)$ and $f_2(t)$.

In this article we study the Fermi-Ulam model in an external gravitational field, the ball falls toward the lower plate with constant acceleration $g \geq 0$ (fig. 1). The main result for this model is that if $f_1(t)$ and $f_2(t)$ are smooth enough and do not touch (i.e. $f_1(t) < f_2(t)$ for all t) then the energy and the velocity of the ball are always bounded, i.e. are smaller than a constant which does not depend on t.

We say that the orbit of the ball which has speed v_0 at time t_0 has *bounded velocity* if there exists $c := c(v_0) > 0$ such that $\sup_\tau |v(\tau)| \leq c$. If the ball does not have bounded velocity we say that it has *unbounded velocity*. For simplicity we state the next theorem for analytic functions even though the proof holds for finitely differentiable functions with enough derivatives.

Theorem 2,1. *Suppose $f_1(t), f_2(t)$ are analytic, periodic with the same period and $f_1(t) < f_2(t)$ for all t. Then the ball has bounded velocity.*

Since the ball has zero radius we consider an important special case when the two plates are allowed to touch which we call a singularity, i.e. there exists a time t^* when $f_1(t^*) = f_2(t^*)$ (fig. 2). In this case the ball can make an infinite number of collisions with both plates in a finite time interval. We say that the ball which has speed v at time t has *unbounded velocity at time* $t^* > t$ if $\lim_{\tau \nearrow t^*} |v(\tau)| = \infty$.

Theorem 2,2. *Suppose f_1, f_2 are continuous function and there is an $\epsilon_1 > 0$ such that for $k = 1$ or $k = 2$ the f_i are C^k functions on $(t^* - \epsilon_1, t^*)$. Furthermore assume that $f_1(t) < f_2(t)$ for $t \in (t^* - \epsilon_1, t^*)$, and $f_i, f_i^{(1)}, \ldots, f_i^{(k)}$ extend by continuity from the left to the points t^*. If $f_1^{(j)}(t^*) = f_2^{(j)}(t^*)$ for $j = 0, 1 \ldots k-1$ and $f_1^{(k)}(t^*) \neq f_2^{(k)}(t^*)$ then there exist positive constants K, ϵ such that if the initial condition (v_0, t_0) satisfies $|v_0| > K$ and $t_0 \in (t^* - \epsilon, t^*)$ then the particle will have unbounded velocity at time t^*.*

Theorems 2,1 and 2,2 will be proven in section 4.

3. A one dimensional system of bouncing balls in an external gravitational field with oscillating plates

In this article we will only consider the case of two balls. Call the balls P_1 and P_2 and the masses m_1 and m_2. The balls are elastic point particles and move on a vertical line in a gravitational field with constant acceleration $g > 0$. They collide elastically with each other and with an infinitely heavy horizontal plate which oscillated periodically in time according to the law $z = f(t)$. Here $f(t)$ is a smooth function having period $T > 0$. In the case of triple collisions, standard mechanical laws do not apply in most cases. In the special case when instead of two ball we have one ball then this model reduces to the model studied in [Pu1]-[Pu3]. We remark that several balls under the influence of an external field bouncing vertically above a fixed plate was studied in [W1][W2] and [Ch]. Our first result is

Proposition 3,1. *If the masses of both balls are positive but finite then either both balls have unbounded velocity or both balls have bounded velocity.*

To produce acceleration we consider a special class of functions which was introduced in [Pu1]-[Pu3]. Let $\mathcal{C} := \{f(t) : f$ periodic with period T and $\exists t_0$ with $\dot{f}(t_0) = KTg/2$ for some $K \in \mathbf{N}^+\}$. Note that if f is of class C^1 then there exists a positive constant $c_0(f)$ such that $f(ct) \in \mathcal{C}$ for all $c \leq c_0$ and $f(ct) \notin \mathcal{C}$ for all $c > c_0$, that is if we scale f to oscillate quickly then it is in the class $\mathcal{C}$ and if we scale f to oscillate slowly then it is not in $\mathcal{C}$. The constant $c_0(f)$ depends only of the variation of f.

We next consider the special case when the balls are of equal masses. We investigate which different kinds of possible acceleration can happen. Namely suppose that P_2 is the lower ball and P_1 is the upper ball. Let $v_i^{(2)}$ be the velocity of ball P_2 at the moment just after the ith collision with the plate and $v_i^{(1)}$ be the velocity of the ball P_1 at the moment just after the ith collision with ball P_2.

Theorem 3,2. *If the masses of both balls are equal and if $f \in \mathcal{C}$ then there exists unbounded motions of both balls such that*
(a) $\lim_{i\to\infty} |v_i^{(1)}| = \lim_{i\to\infty} |v_i^{(2)}| = \infty$,
(b) $\limsup_{i\to\infty} |v_i^{(1)}| = \limsup_{i\to\infty} |v_i^{(2)}| = \infty$,
$\max(\liminf_{i\to\infty} |v_i^{(1)}|, \liminf_{i\to\infty} |v_i^{(2)}|) < \infty$,

Next we consider the special cases when the mass of ball P_1 is zero while P_2 has non-zero mass. In analogy to celestial mechanics this case is called the restricted system. There are two possible positions of the balls, in the first case the ball P_2 with non-zero mass is situated between the plate and the ball P_1 (fig. 3) and in the second case the ball P_1 is situated between the plate and the ball P_2 (fig. 4). In both case the two balls interact between each other according to the law of elastic collisions. The ball P_1 does not influence the ball P_2 since it has zero mass.

Theorem 3,3. *If f is a periodic function of class C^1 then in case (1) it is impossible that P_2 has unbounded velocity while P_1 has bounded velocity. If $f \in \mathcal{C}$ then there are initial conditions for which both balls have unbounded velocity.*

If $f \notin \mathcal{C}$ does the motion (unbounded, unbounded) occur? Arnold has shown for a class $\mathcal{D} \subset \mathcal{C}$ of slowly oscillating functions the ball P_2 always has bounded velocity [A].

Theorem 3,4. *In case (2), if f is a periodic function of class C^1 then there exists bounded motions of P_2 and if $f \in \mathcal{C}$ then there exists unbounded motions of P_2, for each of* *which an open set with infinite measure of initial conditions for ball P_1 lead to P_1 having infinite velocity in finite time.*

Note that the motion of ball P_2 can, in this case, be regularized through a triple collision since P_1 has zero mass and does not affect P_2. The motion of P_1 can not be regularized through this collision. In case (2) if P_2 has periodic motion then the model is equivalent to a special case of the Fermi-Ulam model in an external gravitational field.

4. Proofs of Fermi-Ulam theorems

Proof of theorem 2,1: We prove theorem 2,1 by applying the KAM theorem of Moser [M]. Namely let (r,θ) be polar coordinates of the annulus $0 < a \le r \le b$ and difine a mapping of the annulus by

$$\begin{aligned} \theta_1 &:= \theta + \alpha(r) + F(r,\theta) \\ r_1 &:= r + G(r,\theta). \end{aligned} \tag{4.1}$$

If $h(r,\theta)$ is a function with continuous derivatives up to order s we define the sth derivative norm by

$$|h|_s := \sup \left| \left(\frac{\partial}{\partial r}\right)^{\sigma_1} \left(\frac{\partial}{\partial \theta}\right)^{\sigma_2} h(r,\theta)\right|, \quad \sigma_1 + \sigma_2 \le s \tag{4.2}$$

where (r,θ) range over the domain in which h is defined.

Theorem [M]. *Fix $\epsilon > 0$ and an integer $s \ge 1$. Assume for the mapping (4.1) that every closed curve near a circle and its image curve intersect. Assume further $b - a \ge 1$ and*

$$b_0^{-1} \le \frac{d\alpha(r)}{dr} \le b_0 \tag{4.3}$$

with some constant $b_0 > 1$. Finally assume F, G have continuous derivatives up to order l and satisfy the inequalities

$$\begin{aligned} |F|_0 + |G|_0 &< \delta_0 \\ |\alpha|_l + |F|_l + |G|_l &< b_0 \end{aligned} \tag{4.4}$$

where $\delta_0 = \delta_0(\epsilon, s, c_0)$ is a sufficiently small positive real number and $l = l(s)$ is a large enough integer. Then the mapping (4.1) has a closed invariant curve

$$\begin{aligned} \theta &= \theta' + p(\theta') \\ r &= r_0 + q(\theta') \end{aligned} \tag{4.5}$$

where the functions p, q are functions of period 2π with s continuous derivatives satisfying

$$|p|_s + |q|_s < \epsilon. \tag{4.6}$$

The application of Moser's theorem follows the same path as the proofs in [Pu4][Pu5] and [Pu9]. Let (t, v) be polar coordinates for the plane $\mathbf{R}^2$, $t \in [0, T)$ the angular coordinate and $v \geq 0$ the radial coordinate. Let $D_r \subset \mathbf{R}^2$ be the disk of radius r and center $v = 0$. For $r > 0$ sufficiently large let $A : \mathbf{R}^2 \backslash D_r \to \mathbf{R}^2$ having the form $A(t, v) = (t', v')$ be given implicitly by the following formulas:

$$\begin{aligned} f_2(\tilde{t}) - f_1(t) &= v \cdot (\tilde{t} - t) - \frac{g \cdot (\tilde{t} - t)^2}{2} \\ \tilde{v} &= -v + g \cdot (\tilde{t} - t) + 2\dot{f}_2(\tilde{t}) \\ f_2(\tilde{t}) - f_1(t') &= -\tilde{v} \cdot (t' - \tilde{t}) + \frac{g \cdot (t' - \tilde{t})^2}{2} \\ v' &= v + g \cdot (t' - 2\tilde{t} + t) + 2\dot{f}_1(t') - 2\dot{f}_2(\tilde{t}). \end{aligned} \tag{4.7}$$

The formulas always have an implicit solution if r is sufficiently large. Here we take the minimal $\tilde{t} > t$ which is a solution of the first equation and then the minimal $t' > \tilde{t}$ which is a solution of the third equation. Then the dynamics of the ball for large enough v is implicitly given by the mapping A. Namely, if at the moment of collision with the bottom plate the ball has large enough velocity then it will collide exactly once with the top plate before hitting the bottom plate again. If these three events occur at times which we call $t < \tilde{t} < t'$ and the velocities at these times (at the moment after the collision) are called $v > 0$, $\tilde{v} < 0$ and $v' > 0$ respectively then formula (4.7) holds.

We first verify that the intersection hypothesis of Moser's theorem holds.

Lemma 4,1. *Let γ be a simple closed smooth curve on $\mathbf{R}^2$ (parametrized by t) enclosing the disk D_r for large enough radius r. Then*

$$\oint_{\gamma} \left(\frac{v^2}{2} + f_1(t) g - v \dot{f}_1(t) \right) dt = \oint_{A\gamma} \left(\frac{v^2}{2} + f_1(t) g - v \dot{f}_1(t) \right) dt. \tag{4.8}$$

Proof of Lemma 4,1: We represent A as the composition of four mappings $A = A_4 \circ A_3 \circ A_2 \circ A_1$ where

$$\begin{aligned} A_1(t, v) :=& (t_1, v_1) = (\tilde{t}, v - g(\tilde{t} - t)), \\ A_2(t_1, v_1) :=& (t_2, v_2) = (\tilde{t}, \tilde{v}), \\ A_3(t_2, v_2) :=& (t_3, v_3) = (t', \tilde{v} - g(t' - \tilde{t})), \\ A_4(t_3, v_3) :=& (t_4, v_4) = (t', 2\dot{f}_1(t') - \tilde{v} + g(t' - \tilde{t})). \end{aligned} \tag{4.9}$$

The physical meaning of these mappings is as follows. A_1 accounts for the movement from the lower plate to the upper plate, A_2 accounts for the reflection due to the collision with the upper plate, A_3 accounts for the drop from the upper plate to the lower plate and A_4 for the reflection due to the collision with the lower plate.

The integral invariant of Poincaré-Cartan [C] gives

$$\begin{aligned}&\oint_{\gamma} v dz - H(z,v)dt = -\oint_{A_1(\gamma)} v_1 dz_1 - H(z_1,v_1)dt_1\\ &-\oint_{A_2\circ A_1(\gamma)} v_2 dz_2 - H(z_2,v_2)dt_2 = -\oint_{A_3\circ A_2\circ A_1(\gamma)} v_3 dz_3 - H(z_3,v_3)dt_3\end{aligned} \tag{4.10}$$

where $z = f_1(t), H(z,v) = v^2/2 + gz, z_1 = z_2 = f_2(t_1) = f_2(t_2), z_3 = f_1(t_3)$.

Set $dz = \dot{f}_1(t),\ dz_1 = dz_2 = \dot{f}_2(t_1)dt_1 = \dot{f}_2(t_2)dt_2,\ dz_3 = \dot{f}_1(t_3)dt_3$. This transforms equations (4.10) to

$$\begin{aligned}&\oint_{\gamma} \Big(v\dot{f}_1(t) - (v^2/2 + f_1(t))g\Big)\,dt = -\oint_{A(\gamma)} \Big(v_1\dot{f}_2(t_1) - (v_1^2/2 + f_2(t_1)g)\Big)\,dt_1\\ &\oint_{A_2\circ A_1(\gamma)} \Big(v_2\dot{f}_2(t_2) - (v_2^2/2 + f_2(t_2)g)\Big)\,dt_2 =\\ &\qquad = -\oint_{A_3\circ A_2\circ A_1(\gamma)} \Big(v_3\dot{f}_1(t_3) - (v_3^2/2 + f_1(t_3)g)\Big)\,dt_3.\end{aligned} \tag{4.11}$$

Equation (4.11) and $v_2 = 2f_2(t_1) - v_1,\ t_1 = t_2,\ v_4 = 2\dot{f}_1(t_3) - v_3,\ t_4 = t_3$ yields

$$\oint_{\gamma} \left(\frac{v^2}{2} f_1(t)g - v\dot{f}_1(t)\right) dt = \oint_{A(\gamma)} \left(\frac{(v')^2}{2} + f_1(t')g - v\dot{f}_1(t')\right) dt'. \tag{4.12}$$

For large r the the curves $A(\gamma)$ and γ have the same orientation. Thus the equality (4.12) is equivalent to the statement of Lemma 4,1. ■

Lemma 4,2. *Let γ be a parametrized simple closed smooth curve on $\mathbf{R}^2$ enclosing D_r for large enough radius r. Assume additionally that the mapping A takes γ to the curve $A(\gamma)$ of the same orientation. Then γ intersects $A(\gamma)$.*

Lemma 4,2 follows directly from Lemma 4,1. This completes the verification of the intersection hypothesis in Moser's theorem. Next we must verify the small parameter condition (4.4) in Moser's theorem. To do this we will make several changes in coordinate as in [Pu4] and [Pu5]. Namely let l be a positive number. We introduce new polar coordinates $(t,y) := U(t,v) := (t, 2l/v)$ (t is the angle variable, y the radius variable). We consider the mapping A in these coordinates. Equation (4.7) transforms to $A' = U \circ A \circ U^{-1} : (t,y) \to (t',y')$

$$y' = \frac{2l}{v} = y + \phi(t,y) := y - \frac{y^2(2\dot{f}_1(t') - 2\dot{f}_2(\tilde{t}) + (t' - 2\tilde{t} + t)g)}{2l(t) + y(2\dot{f}_1(t') - 2\dot{f}_2(\tilde{t}) + (t' - 2\tilde{t} + t)g)} \tag{4.13}$$

and there exists a $r > 0$ such that in the region $|y| \le r, 0 \le t \le T$

$$|\phi(t,y)| < c_1 y^2. \tag{4.14}$$

Here c_1 is a positive constant not depending on t or y. Furthermore from (4.7) follows

$$\begin{aligned} \tilde{t} - t =& \frac{2(f_2(\tilde{t}) - f_1(t))}{v + \sqrt{v^2 + 2g(f_2(\tilde{t}) - f_1(t))}} \\ t' - \tilde{t} =& \frac{2(f_2(\tilde{t}) - f_1(t'))}{-\tilde{v} + \sqrt{\tilde{v}^2 + 2g(f_2(\tilde{t}) - f_1(t'))}}; \quad (\tilde{v} < 0). \end{aligned} \tag{4.15}$$

Combining these two equations gives

$$t' = t + \frac{f_2(\tilde{t}) - f_1(t)}{v} + \frac{f_1(t') - f_2(\tilde{t})}{\tilde{v}} + K \tag{4.16}$$

where

$$K = \frac{-\alpha_1(f_2(\tilde{t}) - f_1(t))}{v(2v + \alpha_1)} + \frac{\alpha_2(f_2(\tilde{t}) - f_1(t'))}{\tilde{v}(2\tilde{v} - \alpha_2)} \tag{4.17}$$

$$\alpha_1 = \frac{K_1}{\sqrt{v^2 + K_1} + v}, \; \alpha_2 = \frac{K_2}{\sqrt{\tilde{v}^2 + K_2} - \tilde{v}} \tag{4.18}$$

$$K_1 := -2g(f_2(\tilde{t}) - f_1(t)), \; K_2 := 2g(f_2(\tilde{t}) - f_1(t')). \tag{4.19}$$

Applying the change of variables $y = 2l/v$ to (4.7) and (4.16) yields

$$\tilde{v} = \frac{-2l + y(g(\tilde{t} - t) + 2\dot{f}_2(\tilde{t}))}{y}, \tag{4.20}$$

$$t' = t + y + \psi(t,y) := t + y + \Big\{ \frac{y^2 l(g(\tilde{t}-t) + 2\dot{f}_2(\tilde{t}))}{2l - y(g(\tilde{t}-t) + \dot{f}_2(\tilde{t}))} + \frac{y(f_2(\tilde{t}) - l - f_1(t))}{2l} + $$
$$+ \frac{y(f_2(\tilde{t}) - l - f_1(t'))}{2l - y(g(\tilde{t}-t) + 2\dot{f}_2(\tilde{t}))} + K \Big\}. \tag{4.21}$$

Next choose l large enough so that

$$l > \sup_{t_1,t_2} |f_2(t_2) - f_1(t_1)|. \tag{4.22}$$

Then for sufficiently small $r > 0$

$$|\psi(t,y)| < c_0|y|, \quad \left|\frac{\partial \psi}{\partial y}\right| < c_0, \tag{4.23}$$

for $|y| < r,\ 0 \le t < T$ where $0 < c_0 < 1$ is a constant not depending on t, y.

If the constants c_0 and c_1 in equations (4.23) and (4.14) where sufficiently small then Moser's theorem would imply that in any neighborhood of the point $y = 0$ there exists an A'-invariant curve which surrounds the point $y = 0$. Theorem 2,1 follows from this fact.

In our case the constants c_0 and c_1 from (4.23) and (4.14) are not necessarily small. The only inequality which c_0 satisfies is $0 < c_0 < 1$. Thus we apply the proof of theorem 2 in [Pu5] which is a generalization of Moser's theorem in the case that the mapping has the form given by equations (4.13) and (4.21) which satisfy Moser's intersection condition and satisfy the inequalities (4.14) and (4.23) with an arbitrary constant c_1 and $0 < c_0 < 1$. The main part of the proof of theorem 2 in [Pu5] is the construction of change of variables which transforms the mapping A' into a mapping which satisfies Moser's theorem, that is the newly constructed mapping satisfies Moser's intersection condition of has constants analogous to c_0 and c_1 which are small. This fact finishes the proof of theorem 2,1. ∎

Proof of Theorem 2,2: Equation (4.7) implies

$$\begin{aligned} \tilde{t} &= t + \frac{f_2(\tilde{t}) - f_1(t)}{v(1 - \frac{g}{2v}(\tilde{t} - t))} \\ t' &= \tilde{t} + \frac{f_2(\tilde{t}) - f_1(t')}{-\tilde{v}(1 - \frac{g}{2\tilde{v}}(t' - \tilde{t}))} \end{aligned} \tag{4.24}$$

We assume without loss of generality that $T = 1,\ t^* = 0$ and $f_1(0) = f_2(0) = 0$. The assumption of the theorem imply that $f_1^{(k-1)}(0) = f_2^{(k-1)}(0)$ and $f_1^{(k)}(0) < f_2^{(k)}(0)$ for $k = 1$ or 2.

Choose $\epsilon > 0$ so small that

$$\left|\sum_{j=0}^{k}(f_i^{(j)}(0)/j!)t^j - f_i(t)\right| < 2\max(1, |f_1^{(k+1)}(0)|, |f_2^{(k+1)}(0)|)t^{k+1} \tag{4.25}$$

for $t \in (-\epsilon, 0]$, $i = 1, 2$. Fix a positive constant C_1. We choose the constant K so large that if (v_0, t_0) satisfies $|v_0| > K$ and $t_0 \in (-\epsilon, 0)$ then the ball will in its three next hits alternate hitting the top and bottom plates. We denote the next hit of the bottom plate by (v, t) (the velocity is taken to be the velocity just after the hit) and the next hit after time t of the top plate by $(\tilde{v}, \tilde{t})$ (again the velocity is that just after the hit). We further assume K is so large that $v, -\tilde{v} > 2C_1$.

If $-\epsilon < t < 0$ then

$$\begin{aligned} -\tilde{v}\left(1 - \frac{g}{2\tilde{v}}(t' - \tilde{t})\right) &> C_1 \\ v\left(1 - \frac{g}{2v}(\tilde{t} - t)\right) &> C_1. \end{aligned} \tag{4.26}$$

Putting together (4.24) and (4.26) yields

$$t' \leq t + \frac{|f_2(\tilde{t}) - f_1(t)|}{C_1} + \frac{|f_2(\tilde{t}) - f_1(t')|}{C_1}. \tag{4.27}$$

Now equations (4.24) and (4.26) imply $\max(|\tilde{t}^k - t^k|, |\tilde{t}^k - t'^k|) = O(t^{k+1})$ and thus

$$\max(f_2(\tilde{t}) - f_1(t), f_2(\tilde{t}) - f_1(t')) = \frac{f_2^{(k)}(0) - f_1^{(k)}(0)}{k!}t^k + O(t^{k+1}). \tag{4.28}$$

Putting (4.27) and (4.28) together yields

$$t' < t + \frac{f_2^{(k)}(0) - f_1^{(k)}(0)}{k!C_1}t^k + O(t^{k+1}). \tag{4.29}$$

We inductively assume that $v_i > K$ for $i = 1, 2, \ldots n-1$. Let $(t_n, v_n) = A^n(t, v)$, that is the time and velocity of the nth hit on the bottom plate. Furthermore let $\tilde{t}_n$ play the role of $\tilde{t}$ when $t = t_n$ in equation (4.7). Equations (4.24) and (4.26) imply that $\max(|t_n - \tilde{t}_n|, |t_{n+1} - \tilde{t}_n|) = O(t_n^2)$ and (4.7) implies

$$\begin{aligned} v_{n+1} &= v_n + 2\dot{f}_1(t_{n+1}) - 2\dot{f}_2(\tilde{t}_n) + O(t_n^2) \\ &= v_n + 2(f_1^{(k)}(0)t_{n+1}^{k-1} - f_2^{(k)}(0)\tilde{t}_n^{k-1}) + O(t_n^k) \\ &= v_n + 2(f_1^{(k)}(0) - f_2^{(k)}(0))t_n^{k-1} + O(t_n^k) \\ &= v_0 + 2(f_1^{(k)}(0) - f_2^{(k)}(0))\sum_{i=1}^{n}(t_i^{k-1} + O(t_i^k)) \\ &\geq v_0 + 2(f_1^{(k)}(0) - f_2^{(k)}(0))\sum_{i=1}^{n}(t_i^{k-1} + (1/2)|t_i^{k-1}|). \end{aligned} \tag{4.30}$$

Equation (4.30) implies that $v_n \geq v_{n-1} > K$, verifying the inductive assumption. If $k = 1$ then it is clear that $v_n \to \infty$. If $k = 2$ then let $s_0 := t_0$ and $s_{n+1} := s_n + \frac{\ddot{f}_2(0) - \ddot{f}_1(0)}{3C_1} s_n^2$. Then the following lemma implies that $\sum s_n = -\infty$. If $-t_0 > 0$ is sufficiently small then equation (4.29) implies that $t_{n+1} < t_n + \frac{\ddot{f}_2(0) - \ddot{f}_1(0)}{3C_1} t_n^2$ and then simple algebra yields that $t_n \leq s_n$ for all n. Thus it follows that $\sum t_n = -\infty$ and $v_n \to \infty$. ■

Lemma 4,3. *Supposed $\tau(s) \not\equiv 0$ is a positive C^3 function in the domain $(-\delta, 0]$ where δ is a positive constant. Furthermore we assume $\tau(0) = \dot{\tau}(0) = 0$. Consider the sequence $\{s_n : n \geq 0\}$ given by $s_{n+1} = s_n + \tau(s_n)$. Then there exists an $\epsilon_0 > 0$ so that if $-s_0 \in (0, \epsilon_0)$ then $\sum_{n=0}^{\infty}(-s_n) = \infty$.*

Proof of Lemma 4,3: Suppose that

$$\sum_{n=0}^{\infty}(-s_n) < \infty. \tag{4.31}$$

Then

$$\sum_{k=0}^{\infty} \frac{-\tau(s_k)}{s_k} < \infty \tag{4.32}$$

since $\tau(0) = \dot{\tau}(0) = 0$. For any n we have the identity

$$s_{n+1} = s_0 \prod_{k=0}^{n} \left(1 + \frac{\tau(s_k)}{s_k}\right). \tag{4.33}$$

Thus by taking logarithms of (4.32) we see that the limit

$$\lim_{n\to\infty} \prod_{k=0}^{n} \left(1 + \frac{\tau(s_k)}{s_k}\right) = s^* \tag{4.34}$$

exists and $s^* \neq 0$. Therefore, from (4.33) $\lim_{n\to\infty} s_n = s_0 s^* \neq 0$ which contradicts equation (4.31). ■

5. Proofs of bouncing ball theorems

Proof of proposition 3,1: Suppose the masses of the balls are m_1 and m_2. Let

$$\alpha := \frac{m_1 - m_2}{m_1 + m_2}. \tag{5.1}$$

Let the velocities of the balls just before a collision be given by $v_-^{(1)}, v_-^{(2)}$ and just after the collision by $v_+^{(1)}, v_+^{(2)}$. Then the following equations hold

$$\begin{aligned} v_+^{(1)} =& \alpha v_-^{(1)} + (1-\alpha) v_-^{(2)} \\ v_+^{(2)} =& (1+\alpha) v_-^{(1)} + -\alpha v_-^{(2)}. \end{aligned} \tag{5.2}$$

Then proposion follows from equation (5.2). ■

Proof of theorem 3,2: We first assume there is only one ball above the plate and will build two trajectories γ_1, γ_2 for it, where γ_1 is periodic and γ_2 has unbounded velocity. Suppose that after the collision with the plate at time t the ball has velocity $v > 0$, after the next collision with the plate at time t' the ball has velocity v' and in the time interval $[t, t']$ the ball does not collide with the plate. The dynamics of the ball is implicitly given by the mapping $A : (t, v) \to (t', v')$:

$$\begin{aligned} f(t) + v \cdot (t'-t) - \frac{g}{2} \cdot (t'-t)^2 =& f(t') \\ g \cdot (t'-t) - v + 2\dot{f}(t') =& v'. \end{aligned} \tag{5.3}$$

Here we take the minimal $t' > t$ which is a solution of the first equation. If $f \in C^1$ then there exists $\bar{t}_0$ so that $\dot{f}(\bar{t}_0) = 0$. If $f \in \mathcal{C}$ then there exists $\hat{t}_0$ so that $\dot{f}(\hat{t}_0) = Tg/2$. Set $v_0 := Tgm_2/2$. Here we have assumed that $K = 1$ in the definition of the class $\mathcal{C}$.

Using the points $\bar{t}_0, \hat{t}_0$ and elementary algebra we produce two trajectories,

$$\begin{aligned} \gamma_1 &:= \{(\bar{t}_0, v_0), (\bar{t}_1, \bar{v}_1), \cdots, (\bar{t}_n, \bar{v}_n), \cdots\}, \\ \gamma_2 &:= \{(\hat{t}_0, v_0), (\hat{t}_1, \hat{v}_1), \cdots, (\hat{t}_n, \hat{v}_n), \cdots\} \end{aligned} \tag{5.4}$$

such that $v_0 = Tgm_2/2 > 0$ where m_2 is a sufficiently large positive integer, $\bar{t}_0, \hat{t}_0$ satisfy the equations $\dot{f}(\bar{t}_0) = 0,\ \dot{f}(\hat{t}_0) = Tg/2$ and for all $n \geq 0$ the following hold:

$$\begin{aligned} \bar{v}_{n+1} = v_0,\ \hat{v}_{n+1} = \hat{v}_n + Tg, \\ t_0 = \bar{t}_{n+1} = \hat{t}_{n+1} \bmod(T), \\ \bar{t}_{n+1} - \bar{t}_n = Tm_2,\ \hat{t}_{n+1} - \hat{t}_n = T(m_2 + 2n). \end{aligned} \tag{5.5}$$

Now we return to the case of two balls. The case when $\lim_{i\to\infty} |v_i^{(1)}| = \lim_{i\to\infty} |v_i^{(2)}| = \infty$, occurs when the initial condition of ball P_1 is $(\hat{t}_0, \hat{v}_0)$ and the initial condition of ball P_2 is $(\hat{t}_0 + nT, \hat{v}_0)$ for any positive integer n. The case $\limsup_{i\to\infty} |v_i^{(1)}| = \limsup_{i\to\infty} |v_i^{(2)}| = \infty$, but $\max(\liminf_{i\to\infty} |v_i^{(1)}|, \liminf_{i\to\infty} |v_i^{(2)}|) < \infty$, occurs when the initial condition of the balls is $(\hat{t}_0, \hat{v}_0)$ and $(\bar{t}_0, \bar{v}_0)$. In both cases the behavior described occurs because when two ball of equal mass collide they exchange velocities. ■

Proof of theorem 3,3: Let $v^{(1)}(t), v^{(2)}(t)$ be the velocities of balls P_1, P_2 at time t. Suppose that there is a constant $C_1 > 0$ such that $\sup_{t\geq 0} |v^{(1)}(t)| < C_1$ and $\sup_{t\geq 0} |v^{(2)}(t)| = \infty$. Let $0 \leq t_1 < t_2 < \ldots$ be the collision times between the particles and $\Delta t_i = t_{i+1} - t_i$.

If $\sup_{i\geq 0} \Delta t_i = \infty$ then there is an $i \geq 0$ for which $\Delta t_i > 4C_1/g$. Then for $t_* = t_i + (1/2)\Delta t_i$ we have

$$v^{(1)}(t_*) = v^{(1)}(t_i) - g\frac{\Delta t_i}{2} \leq C_1 - 2C_1 < -C_1 \tag{5.6}$$

a contradiction.

It remains to study the case when $\exists C_2 > 0$ such that $\sup_{i\geq 0} \Delta t_i < C_2$. Since the second ball has unbounded velocity there are arbitrarily large time intervals in which it has arbitrarily large speed. In particular there exists $\hat{t}$ such that for all $t \in [\hat{t}, \hat{t} + C_2]$ we have

$$|v^{(2)}(t)| > C_1. \tag{5.7}$$

But since $\Delta t_i < C_2$ there is always a t_i in the interval $[\hat{t}, \hat{t}+C_2]$. Now $v^{(1)}(t_i) = -v^{(1)}(t_i^-) + 2v^{(2)}(t_1)$. Thus using (5.6) and (5.7) yields

$$|v^{(1)}(t_i)| > -C_1 + 2C_1 = C_1 \tag{5.8}$$

a contradiction.

We have already shown in theorem 3,2 that for a single ball there is an unbounded trajectory γ_2 if $f \in \mathcal{C}$. However the zero mass particle does not effect the dynamics of the ball P_2 so this trajectory occurs in the case (1) as well. The above discussion shows that any initial condition for P_1 above the trajectory γ_2 leads to both balls having unbounded velocity. ■

Proof of theorem 3,4 We apply Theorem 2,2. The role of the function $f_2(t)$ in theorem 2,2 will be played by the orbit of P_2 and the role of $f_1(t)$ will be played by $f(t)$. ■

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Captions for figures.

Figure 1 The Fermi-Ulam model in an external gravitational field.

Figure 2 Touching plates.

Figure 3 Zero mass ball on top.

Figure 4 Zero mass ball in the middle.