Hénon Mappings in the Complex Domain
II: projective and inductive limits of polynomials

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Abstract. Let $H : \mathbb{C}^2 \to \mathbb{C}^2$ be the Hénon mapping given by

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} p(x) - ay \\ x \end{bmatrix}.$$

The key invariant subsets are $K_\pm$, the sets of points with bounded forward images, $J_\pm = \partial K_\pm$ their boundaries, $J = J_+ \cap J_-$, and $K = K_+ \cap K_-$. In this paper we identify the topological structure of these sets when p is hyperbolic and $|a|$ is sufficiently small, *i.e.*, when H is a small perturbation of the polynomial p. The description involves projective and inductive limits of objects defined in terms of p alone.

Contents

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1. Introduction

This paper continues the study, begun with [**HO**], of the Hénon family of mappings as a family of mappings of two complex variables. Let $p(z)$ be a polynomial in one variable and $a \neq 0$ a complex number. A Hénon mapping is one which can be written

$$H = H_{p,a} : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} p(x) - ay \\ x \end{bmatrix}.$$

Such a mapping has Jacobian a, and if $a \neq 0$, it is invertible:

$$H_{p,a}^{-1} : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} y \\ (p(y) - x)/a \end{bmatrix}.$$

The key invariant subsets under such a mapping are

$$K_{\pm} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{C}^2 \;\middle|\; \left\| H^{\circ n} \begin{bmatrix} x \\ y \end{bmatrix} \right\| \text{ bounded as } n \to \pm\infty \right\}$$

as well as

$$J_{\pm} = \partial K_{\pm}, \quad K = K_+ \cap K_-, \quad \text{and} \quad J = J_+ \cap J_-.$$

When $a = 0$, the degenerate Hénon mapping

$$H_{p,0} : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} p(x) \\ x \end{bmatrix}$$

is not invertible, but maps all of $\mathbb{C}^2$ to the curve C_p of equation $x = p(y)$, and reduces to $x \mapsto p(x)$ in the first coordinate.

According to the theory that hyperbolic dynamics is stable under perturbations, you would expect that $H_{p,a}$ could be understood as a perturbation of p for a sufficiently small when p is hyperbolic. Many people (*e.g.*, Holmes, Whitley, and Williams, *cf.*, [**Ho**], [**HWh**], and [**HWi**] for further references) have done this in the real domain, at least for $|a|$ small. Benedicks and Carleson have gone further in this direction [**BC**]. In this article we will do the same in the complex domain. By the techniques used here we can only deal with perturbations of hyperbolic polynomials, and not the much more difficult ones studied by Carleson and Benedicks.

There is a fundamental conflict between the Hénon mapping and polynomials: polynomials are not injective and Hénon mappings are. We will describe two ways of creating from a polynomial p objects which do carry bijective dynamics; both appear as invariant subsets of $\mathbb{C}^2$ for Hénon mappings which are sufficiently small perturbations of hyperbolic polynomials.

The projective limit construction.

Let

$$\hat{\mathbb{C}}_p = \varprojlim(\mathbb{C}, p).$$

A point of this projective limit is a point $z_0 \in \mathbb{C}$ and a *history of the point* z_0 under the iteration of p. More precisely,

$$\hat{\mathbb{C}}_p = \left\{ (\ldots, z_{-2}, z_{-1}, z_0) \;\middle|\; p(z_{-i-1}) = z_{-i} \text{ for all } i = \ldots, -2, -1, 0 \right\}.$$

The mapping p induces a mapping $\hat{p} : \hat{\mathbb{C}}_p \to \hat{\mathbb{C}}_p$ by

$$\hat{p}(\ldots, z_{-2}, z_{-1}, z_0) = (\ldots, p(z_{-2}), p(z_{-1}), p(z_0)) = (\ldots, z_{-1}, z_0, p(z_0))$$

which is of course bijective:

$$\hat{p}^{-1}(\ldots, z_{-2}, z_{-1}, z_0) = (\ldots, z_{-2}, z_{-1}).$$

In section 7, we will give a description of this space which makes it reasonably understandable when p is hyperbolic.

The inductive limit construction.

Recall that if $f : X \to X$ is a mapping from a space to itself, then the inductive limit

$$\check{X}_f = \varinjlim(X, f)$$

is the quotient $(X \times \mathbb{N})/\sim$, where $\sim$ is generated by setting $(x, n) \sim (f(x), n + 1)$.

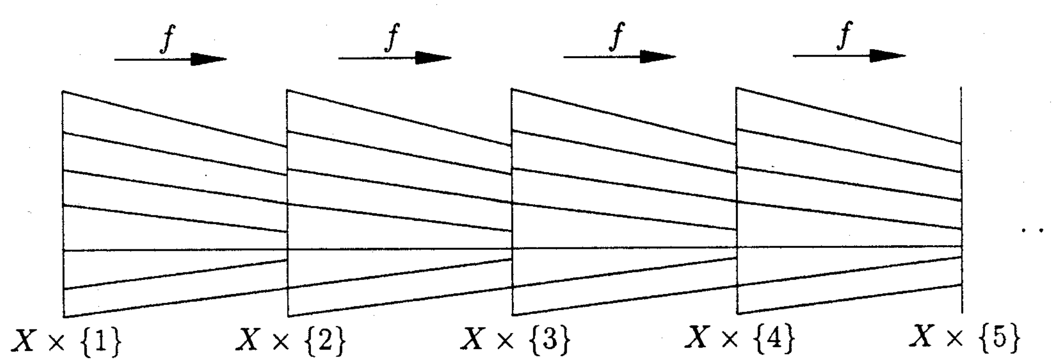

Figure 1.1: Inductive limit as an increasing union

Inductive limits are pathological objects in general, and will be Hausdorff only when f has some nice properties. We will consider them only when f is open and injective, in which case the inductive limit is an increasing union of subsets homeomorphic to X, hence locally as nice as X.

The inductive limit comes with a map to itself: $\check{f} : \check{X}_f \to \check{X}_f$ induced by

$$\check{f} : (x, n) \mapsto (f(x), n) \sim (x, n-1).$$

This mapping is obviously bijective, as an inverse is induced by $(x, n) \mapsto (x, n+1)$.

We will now apply this construction to polynomials. Our construction only makes sense for polynomials p with no critical points in the Julia set; however, we will only apply it to hyperbolic polynomials, which all have this property. Let $D \subset \mathbb{C}$ be a disk of radius R sufficiently large so that $J_p \subset D$, where J_p is the Julia set of p.

Consider the mapping $f_{p,\alpha,R} : J_p \times D \to J_p \times \mathbb{C}$ given by

$$f_{p,\alpha,R}(\zeta, z) = \left(p(\zeta), \zeta + \alpha \frac{z}{p'(\zeta)} \right),$$

which is well defined since $p'(\zeta) \neq 0$.

Lemma 1.2. *If p is hyperbolic, and if $|\alpha| \neq 0$ is sufficiently small, then the image $f_{p,\alpha,R}(J_p \times D)$ is contained in $J_p \times D$ and $f_{p,\alpha,R}$ is open and injective.*

Proof. Recall that if p is hyperbolic, there are no critical points of p in J_p (in fact, this is the only property of hyperbolic polynomials this lemma requires). Thus the formula is well defined, and clearly if $|\alpha|$ is sufficiently small, the image lies in $J_p \times D$. Moreover, if there are no critical points in J_p, then there exists $\varepsilon > 0$ such that when $\zeta_1 \neq \zeta_2 \in J_p$ and $p(\zeta_1) = p(\zeta_2)$, then $|\zeta_1 - \zeta_2| > \varepsilon$. If we choose

$$0 < |\alpha| < \frac{\varepsilon R}{\inf\limits_{\zeta \in J_p} |p'(\zeta)|},$$

then $f_{p,\alpha,R}$ is clearly injective. The mapping is open because it is a local homeomorphism. $\square$

Thus when p is hyperbolic and $|\alpha|$ is sufficiently small and R is sufficiently large, we may set

$$\check{\mathbb{C}}_p = \check{\mathbb{C}}_{p,\alpha,R} = \varinjlim (J_p \times D, f_{p,\alpha,R}),$$

and we will denote by

$$\check{p} = \check{f}_{p,\alpha,R} : \check{\mathbb{C}}_p \to \check{\mathbb{C}}_p$$

the bijective mapping above.

If $\psi : X \to Y$ is a homeomorphism conjugating $f : X \to X$ and $g : Y \to Y$, then ψ induces a homeomorphism $\check{\psi} : \check{X}_f \to \check{Y}_g$ conjugating $\check{f} : \check{X}_f \to \check{X}_f$ to $\check{g} : \check{Y}_g \to \check{Y}_g$. Thus the following proposition, which is proved in Section 6, shows that we can drop the indices α and R, and speak simply of $\check{p} : \check{\mathbb{C}}_p \to \check{\mathbb{C}}_p$.

Proposition 6.13. *For all α_1, α_2 sufficiently small and all R_1 and R_2 sufficiently large, there is a homeomorphism*

$$\psi : J_p \times D_{R_1} \to J_p \times D_{R_2}$$

conjugating f_{p,α_1,R_1} to f_{p,α_2,R_2}.

This justifies writing simply f_p and $\check{\mathbb{C}}_p$. The space $\check{\mathbb{C}}_p$ is quite difficult to understand. The only case where it is anything familiar is when J_p is a Jordan curve; in that case $\check{\mathbb{C}}_p$ is homeomorphic to the complement of a solenoid in a 3-sphere. Proposition 6.1 gives some important information, and much more is shown in Section 7. In Section 8, we show that when p is a real hyperbolic polynomial, the real part $\check{\mathbb{R}}_p$ is often the common separator of Lakes of Wada. This illustrates some of the unavoidable complexity.

An embedding of $\hat{J}_p$ into $\check{\mathbb{C}}_p$.

The inductive and projective limits above are related: the projective limit $\hat{J}_p$ is naturally an invariant subset of both. This is obvious for $\hat{\mathbb{C}}_p$; let us see why it is true for $\check{\mathbb{C}}_p$.

Let $\underline{\zeta} = (\dots, \zeta_{-2}, \zeta_{-1}, \zeta_0) \in \hat{J}_p$, and consider the intersection

$$(\{\zeta_0\} \times D) \cap \check{p}(\{\zeta_{-1}\} \times D) \cap \cdots \cap \check{p}^{\circ n}(\{\zeta_{-n}\} \times D) \cap \cdots$$

Lemma 1.3. *This is a nested sequence of embedded disks, and the intersection is a single point.*

Proof. The nesting is obvious. As we have defined it, there exists a disk D_1 relatively compact in D such that

$$f_{p,\alpha,R}(J_p \times D) \subset J_p \times D_1.$$

There are infinitely many disjoint conformal copies of the annulus $D - \bar{D}_1$ surrounding the intersection above. This shows that the intersection is a point. □

Let us call $\psi : \hat{J}_p \to \check{\mathbb{C}}_p$ the mapping which associates to $\underline{\zeta}$ the unique point in the above intersection. Clearly the diagram

$$\begin{array}{ccc} \hat{J}_p & \xrightarrow{\psi} & \check{\mathbb{C}}_p \\ {\scriptstyle \hat{p}}\downarrow & & \downarrow{\scriptstyle \check{p}} \\ \hat{J}_p & \xrightarrow[\psi]{} & \check{\mathbb{C}}_p \end{array}$$

commutes.

We will see in section 7 some examples of the objects above. In particular, we will see that the construction above corresponds to seeing the solenoid as a projective limit of circles or a decreasing intersection of solid tori.

Riemann surface laminations.

It is rather difficult to find any category to which $\hat{\mathbb{C}}_p$ and $\check{\mathbb{C}}_p$ belong. A first attempt is to say that they are are (or have large subsets which are) *Riemann surface laminations.* For future reference, we define this category to have:

Objects: Hausdorff spaces which are locally products of Riemann surfaces by topological spaces, glued together by local isomorphisms;

Morphisms: Continuous mappings, analytic on each Riemann surface.

You should imagine the topological factor to be like a Julia set, either a Cantor Julia set (for $\hat{\mathbb{C}}_p$) or a connected Julia set (for $\check{\mathbb{C}}_p$). This category has recently turned up in several fields, and Sullivan's paper [**S**] contains some basic material about this category. Pictures of the Hénon attractor [**Hé2**] or of basin boundaries will show that such structures should be relevant to dynamical systems.

The main result.

Both of the constructions above give objects which arise in the dynamical plane $\mathbb{C}^2$ of Hénon mappings.

Theorem 1.4. *Let p be a hyperbolic polynomial. There exists A such that if $0 < |a| < A$, then there exist homeomorphisms*

$$\Phi_- : \hat{\mathbb{C}}_p \to J_- \qquad \text{and} \qquad \Phi_+ : \check{\mathbb{C}}_p \to J_+$$

such that the diagrams

$$\begin{array}{ccc} \hat{\mathbb{C}}_p & \xrightarrow{\Phi_-} & J_- \\ {\scriptstyle \hat{p}}\downarrow & & \downarrow{\scriptstyle H} \\ \hat{\mathbb{C}}_p & \xrightarrow[\Phi_-]{} & J_- \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \check{\mathbb{C}}_p & \xrightarrow{\Phi_+} & J_+ \\ {\scriptstyle \check{p}}\downarrow & & \downarrow{\scriptstyle H} \\ \check{\mathbb{C}}_p & \xrightarrow[\Phi_+]{} & J_+ \end{array}$$

commute. On $\hat{J}_p$, the mappings Φ_+ and Φ_- coincide, i.e., we have

$$\Phi_- \,|_{\hat{J}_p} = \Phi_+ \circ \psi.$$

Outline of the paper.

The proof we will give of Theorem 1.4 is an adaptation of the technique of *telescopes*, which we learned from Sullivan many years ago.

In section 2, we will review Sullivan's construction. This will serve several purposes: it will motivate our construction, it will provide us with some constructions which we need, and it will provide a written account of Sullivan's proof, which was never published.

In section 3, we will define our 2-dimensional analogs of expanding maps, which we call *crossed mappings*. It seems clear that these are going to be of interest in many other settings, and we have proved the basic results concerning them with considerable care.

In section 4, we show that for Hénon mappings which are small perturbations of hyperbolic polynomials, the mappings analogous to the telescope mappings are crossed mappings. This will give us a homeomorphism $\Phi : \hat{J}_p \to J$ conjugating the Hénon mapping to $\hat{p}$, and locally the stable and unstable manifolds will also drop out of the construction.

In section 5, we identify the unstable manifold of J with $\hat{\mathbb{C}}_p$, and in section 6 we identify the stable manifold with $\check{\mathbb{C}}_p$. This last step is quite delicate, and is surely the hardest proof in the paper.

Finally, in sections 7 and 8, we show in some examples exactly what these results give us for the topology of Hénon mappings, including Lakes of Wada.

2. Telescopes and Hyperbolic Polynomials

Many years ago, we learned from Sullivan that hyperbolic polynomials (and rational functions) are structurally stable on their Julia sets. Sullivan used *telescopes* in his proof, and we are planning to adapt this construction to Hénon mappings.

Let $p(z)$ be a hyperbolic polynomial. In fact, everything we will say goes over to rational functions without modification. We will take as our definition of hyperbolic that all critical points are attracted to attractive periodic cycles. As we will see below, this is equivalent to saying that p is strongly expanding on the Julia set J_p.

Call Ω the Fatou set of p, C the set of attracting periodic points of p (including ∞), and

$$X_0 = \{\, z \in \Omega \mid d_\Omega(z, C) \le 1 \,\},$$

where d_Ω is the Poincaré metric on Ω. (The number 1 in the definition is arbitrary; everything would go through for any positive constant.)

Further set $X_n = p^{-n}(X_0)$. The X_n form an increasing collection of compact subsets of Ω which exhaust Ω, and they are strictly increasing in the sense that X_{n-1} is contained in the interior of X_n. Similarly, the sets $U_n = \overline{\mathbb{C}} - X_n$ form a basis of nested open neighborhoods of J_p, each relatively compact in the previous.

Let N be the smallest index such that all the critical points of p are in X_N. Such an N exists since there are only finitely many critical points, all in Ω.

Proposition 2.1. *The mapping $p : U_n \to U_{n-1}$ is a covering map for $n \ge N$. In particular, it is strongly infinitesimally expanding for the Poincaré metric of U_{n-1}.*

Proof. Clearly $p : U_n \to U_{n-1}$ is proper and a local homeomorphism, hence a covering map and a local isometry from the Poincaré metric of U_n to the Poincaré metric of U_{n-1}. Since

U_n is relatively compact in U_{n-1}, the inclusion is strongly contracting for the Poincaré metric of U_{n-1}. □

We will call $U = U_{n-1}$ and $U' = U_n$, so that $p : U' \to U$ is a covering map. Choose $\varepsilon > 0$ sufficiently small that for any $z \in U$, the set

$$U_z = \{\, z_1 \in U \mid d_U(z_1, z) < \varepsilon \,\}$$

is homeomorphic to a disk, and that p restricted to U_z is a homeomorphism to its image.

For any $z \in J_p$, define $U_z^0 = U_z$, and recursively set

$$U_z^n = U_z^{n-1} \cap p^{-n}(U_{p^{\circ n}(z)}).$$

It is easy to show that each U_z^n is homeomorphic to a disk.

Proposition 2.2. *We have*

$$\{z\} = \bigcap_n U_z^n.$$

Proof. Clearly z is in the intersection; the only problem is to show that the intersection is a single point. This follows from the strong expansion: if p expands by a factor of $K > 1$, then the diameter of U_z^n is at most ε / K^n. □

Sullivan defines a *p-telescope* to be a sequence of disks $W_0, W_1, \dots$ such that W_{n+1} is relatively compact in $p(W_n)$.

Example 2.3. If $z \in J_p$, the sequence of disks $U_z, U_{p(z)}, \dots$ is a telescope, and Proposition 2.2 says that a telescope defines a point. But clearly a telescope for p is also a telescope for a small perturbation of p, so that going from points to telescopes to points provides a conjugacy between the Julia set of a hyperbolic polynomial and that of a small perturbation. This is the idea behind Sullivan's proof.

Theorem 2.4. *For any neighborhood V of the Julia set of p, there exists a neighborhood of p in the C^1-topology such that any p_1 in that neighborhood is conjugate to p on a neighborhood of the Julia set.*

Sketch of proof. Define $\varphi : J_p \to V$ by

$$\varphi(z) = \bigcap_n p_1^{-n}(U_{p^{\circ n}(z)}).$$

Just as above, for p_1 sufficiently close to p in the C^1-topology, this intersection is a single point. A similar construction gives an inverse for φ on

$$J_{p_1} = \{\, z \in V \mid p_1^{\circ n}(z) \in V \text{ for all } n \,\}.$$

Thus $\varphi : J_p \to J_{p_1}$ is a homeomorphism conjugating p to p_1 on the Julia sets. We leave to the reader to verify that this homeomorphism can be extended to a neighborhood of J_p which still conjugates p to p_1. □

In one dimension, the mappings useful for structural stability are those which map a disk strictly outside another. In higher dimensions, we will be interested in bijective mappings defined on bidisks which map the "horizontal boundary" outside itself, and the inverses of which map the "vertical boundary" outside itself. Thus, they "look like" Figure 3.1.

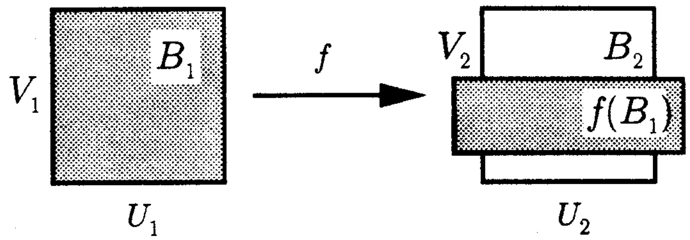

Figure 3.1: A 1-crossed mapping

We need to formalize what this means. Let $B_1 = U_1 \times V_1$ and $B_2 = U_2 \times V_2$ be bidisks.

Definition 3.2. A crossed mapping from B_1 to B_2 is a triple (W_1, W_2, f), where

(1) $W_1 \subset U_1' \times V_1$ where $U_1' \subset U_1$ is a relatively compact open subset,
(2) $W_2 \subset U_2 \times V_2'$ where $V_2' \subset V_2$ is a relatively compact open subset,
(3) $f : W_1 \to W_2$ is a holomorphic isomorphism, such that for all $y \in V_1$, the mapping

$$\mathrm{pr}_1 \circ f|_{W_1 \cap (U_1 \times \{y\})} : W_1 \cap (U_1 \times \{y\}) \to U_2$$

is proper, and the mapping

$$\mathrm{pr}_2 \circ f^{-1}|_{W_2 \cap (\{x\} \times V_2)} : W_2 \cap (\{x\} \times V_2) \to V_1$$

is proper.

To make the notation less cumbersome, we will often write $f : B_1 \underset{\times}{\to} B_2$ for a crossed mapping, leaving the precise W_1 and W_2 to be determined by the context.

Proposition 3.3. *If $f : W_1 \to W_2$ is a crossed mapping from B_1 to B_2, then all maps*

$$\mathrm{pr}_1 \circ f|_{W_1 \cap (U_1 \times \{y\})} : W_1 \cap (U_1 \times \{y\}) \to U_2$$

and

$$\mathrm{pr}_2 \circ f^{-1}|_{W_2 \cap (\{x\} \times V_2)} : W_2 \cap (\{x\} \times V_2) \to V_1$$

have the same degree, which will be called the degree of the crossed mapping.

Proof. Choose $x \in U_2$, and consider $Z_x = f^{-1}(W_2 \cap (\{x\} \times V_2))$, which is a closed analytic curve in B_1 (*i.e.*, a Riemann surface closed in B_1). The mapping $\mathrm{pr}_2 : Z_x \to V_1$ is proper, hence a finite ramified covering map, of some degree $k(x)$. For every $y \in V_1$, the line $U_1 \times \{y\}$ cuts Z_x in precisely $k(x)$ points, counted with multiplicity (where the multiplicity is almost by definition the local degree of the projection above). But for each such y, these $k(x)$ intersection points are mapped by f exactly to the intersections of $f(W_1 \cap (U_1 \times \{y\}))$ with the line $\{x\} \times V_2$; these count the degree of

$$\mathrm{pr}_1 \circ f|_{W_1 \cap (U_1 \times \{y\})} : W_1 \cap (U_1 \times \{y\}) \to U_2.$$

Thus these maps all have the same degree, and the same argument applied to f^{-1} shows that the maps

$$\mathrm{pr}_2 \circ f^{-1}|_{W_2 \cap (\{x\} \times V_2)} : W_2 \cap (\{x\} \times V_2) \to V_1$$

also all have the same degree; *i.e.*, $k(x)$ does not depend on x. It is clear from the proof that the two classes of mappings have the same degree. □

Figure 3.1 represents a 1-crossed mapping.

Proposition 3.4. *If $f : W_1 \to W_2$ is a crossed mapping from B_1 to B_2 and $X \subset B_1$ is an analytic curve such that $\mathrm{pr}_1 : X \to U_1$ is proper of degree l, then $\mathrm{pr}_1 \circ f : X \cap W_1 \to U_2$ is proper of degree kl.*

Remark 3.5. The case where X is a horizontal disk is part of the definition of a crossed mapping.

We require the following lemma, which is classical.

Lemma 3.6. *Let X and Y be curves in a bidisk $B = U \times V$ such that $\mathrm{pr}_1 : X \to U$ and $\mathrm{pr}_2 : Y \to V$ are proper of degrees k_X and k_Y, respectively. Moreover, suppose that $\mathrm{pr}_1(Y)$ is relatively compact in U. Then X and Y intersect in $k_X k_Y$ points counted with multiplicity.*

Proof of Proposition 3.4. For each $x \in U_2$, the curve X and the curve

$$Y_x = f^{-1}(W_2 \cap (\{x\} \times V_2))$$

satisfy the hypotheses of Lemma 3.6. So these curves intersect in kl points independent of x. But this means that every vertical line in B_2 intersects $f(X \cap W_1)$ in the same number of points. Since $f(X \cap W_1)$ is clearly closed in W_2, this shows that it maps by a proper map to U_2. □ (Proposition 3.4)

Proposition 3.7. *(a) Let $f : W_1 \to W_2$ be a crossed mapping from B_1 to B_2 of degree k. Then $f^{-1} : W_2 \to W_1$ is also a crossed mapping if all the coordinates are flipped.*
(b) If B_1, B_2, and B_3 are bidisks, $W_1 \subset B_1$, $W_2 \subset B_2$, $\tilde{W}_2 \subset B_2$, and $\tilde{W}_3 \subset B_3$, and $f_1 : W_1 \to \tilde{W}_2$ and $f_2 : W_2 \to \tilde{W}_3$ are k_1- and k_2-crossed mappings, then

$$f_2 \circ f_1 : W_1 \cap f_1^{-1}(W_2) \to \tilde{W}_3 \cap f_2(\tilde{W}_2)$$

is a $k_1 k_2$-crossed mapping from B_1 to B_3.

Proof. Part (a) is obvious. For (b), observe that the sets $S_1 = W_1 \cap f_1^{-1}(W_2)$ and $S_2 = \tilde{W}_3 \cap f_2(\tilde{W}_2)$ clearly satisfy conditions (1) and (2) of the definition; it remains to show (3). For any $y \in V_1$, the curve $X_y = f_1\left(W_1 \cap (U_1 \times \{y\})\right)$ satisfies the hypothesis of Proposition 3.4, with respect to the crossed mapping $f_2 : W_2 \to \tilde{W}_3$. So the projection $\mathrm{pr}_1 : f_2(W_2 \cap X_y) \to U_3$ is proper. Proposition 3.4 also shows that this proper projection has degree $k_1 k_2$. □

A bidisk $B = U \times V$ carries, like all bounded domains, the Kobayashi metric, which in this case is easy to describe: it is the product of the Poincaré metrics of U and of V. Crossed mappings of degree 1 have special expansion and contraction properties with respect to this metric. If $\xi \in T_x U$, we will denote by $|(x, \xi)|_U$ the length of the tangent vector for the (infinitesimal) Poincaré metric of U.

A first observation about this metric is the following:

Lemma 3.8. *A tangent vector $(\xi, \eta) \in T_{(x,y)}B$ is tangent to a disk which is the graph of an injective mapping $g : U \to V$ if and only if $|(x, \xi)|_U \geq |(y, \eta)|_V$. This mapping can be taken to have relatively compact image in V if and only if $|(x, \xi)|_U > |(y, \eta)|_V$.*

Proof. In one direction, this is Schwarz's Lemma: such a g contracts in the Poincaré metrics and contracts strictly if the image is relatively compact. In the other direction, by a biholomorphic isomorphism, we may suppose the bidisk is the standard bidisk $D \times D$, and that $(x, y) = (0, 0)$. Then the line containing (ξ, η) intersects the bidisk in an appropriate graph. □

This gives us the appropriate tool to study crossed mappings of degree 1. For any bidisk $U \times V$, consider the horizontal cone field $C_{(x,y)} \subset T_{(x,y)}B$ defined by

$$C_{(x,y)} = \left\{ \, (\xi, \eta) \in T_{(x,y)}B \, \middle| \, |(x, \xi)|_U \geq |(y, \eta)|_V \, \right\}.$$

Reversing the inequality gives the vertical cone field.

Proposition 3.9. *Let $f : W_1 \to W_2$ be a crossed mapping from B_1 to B_2 of degree 1. Then for all $(x, y) \in W_1$, we have*

$$d_{(x,y)} f(C_{(x,y)}) \subset C_{f(x,y)}.$$

Moreover, if $(\xi_1, \eta_1) \in C_{(x_1,y_1)}$*,* $f(x_1, y_1) = (x_2, y_2)$ *and* $d_{(x_1,y_1)} f(\xi_1, \eta_1) = (\xi_2, \eta_2)$*, then* $|(x_2, \xi_2)|_{U_2} > |(x_1, \xi_1)|_{U_1}$*.*

Remark 3.10. This proposition illustrates the principle that in complex analysis, inequalities often follow from topology. In the complex setting, the existence of an invariant cone-field is automatic; in the real it needs to be verified in each case [**Yoc1**], this is often quite difficult.

Proof of Proposition 3.9. A vector in the cone $C_{(x_1,y_1)}$ is tangent to a curve X in B_1 proper of degree 1 over U_1. The curve $f(X \cap W_1)$ is then proper of degree 1 over U_2 by Proposition 3.4. Thus the tangent to $f(X \cap W_1)$ at $f(x_1, y_1)$ is in the cone $C_{(x_2,y_2)}$. This proves the first part.

For the second part, observe that $\mathrm{pr}_1 \circ f : X \cap W_1 \to U_2$ is an isomorphism, so that

$$\mathrm{pr}_1 \circ (\mathrm{pr}_1 \circ f|_{X \cap W_1})^{-1} : U_2 \to U_1$$

is an analytic mapping with relatively compact image. Thus it strictly contracts Poincaré lengths, and its derivative maps ξ_2 to ξ_1. □ (Proposition 3.9)

We will refer to smooth curves and surface in a bidisk $B = U \times V$ as *horizontal-like* of *vertical-like* if their tangent spaces are in the horizontal or vertical cone respectively at each of their points.

Suppose $B_1, \dots, B_{n+1}$ are bidisks such that $B_i = U_i \times V_i$. Suppose also that $W_i \subset B_i$ $(i = 1, \dots, n)$ and $\tilde{W}_i \subset B_i$ $(i = 2, \dots, n+1)$ are open subsets so that $f_i : W_i \to \tilde{W}_{i+1}$ are crossed mappings of degree 1. Let

$$S_1 = W_1 \cap f_1^{-1}(W_2) \cdots \cap (f_1^{-1} \circ \cdots \circ f_{n-1}^{-1})(W_{n-1})$$

and

$$S_2 = \tilde{W}_{n+1} \cap f_n(\tilde{W}_n) \cap \cdots \cap (f_n \circ \cdots \circ f_2)(\tilde{W}_2)$$

so that Proposition 3.7 and an obvious induction shows that the restriction g of $f_n \circ \cdots \circ f_1$ to S_1 makes $g : S_1 \to S_2$ a crossed mapping of degree 1 from B_1 to B_{n+1}.

We want to say that horizontal disks in B_1 intersect S_1 in regions with small diameter. It is a bit harder to do this from Proposition 3.9 than one might expect: the contraction there is infinitesimal, and a domain $U' \subset U$ may have small diameter in U although the shortest curve joining points of U' in U' may still be long. We will take a different tack, using complex analysis and moduli of annuli. Note that such methods, in a more complicated context, have had great success recently in one-dimensional dynamics [**BH**], [**H**], [**Yoc2**].

Let U be a simply connected Riemann surface isomorphic to the disk, and U' a relatively compact open subset. Define the *size of* U' *in* U to be the $1/M$, where M is the largest modulus of an annulus separating U' from the boundary of U. We will really be interested in the case where U' is connected and simply connected; so that the size of U' in U is the inverse of the modulus of $U - \overline{U'}$.

Note that the size is related to the Poincaré diameter by a double inequality; after conformal mapping of U to the disk, the extreme cases for a given diameter are a line segment and a round disk.

Proposition 3.11. *Suppose that the size of the projection* $\mathrm{pr}_1(W_i)$ *in* U_i *is* $1/M_i$*. Then for any* $y \in V_1$*, the size in* U_1 *of* $S_1 \cap (U_1 \times \{y\})$ *is at least*

$$1 \Big/ \sum_1^{n-1} \frac{1}{M_i} \, .$$

Proof. Consider the subsets

$$W_1^j = f_1^{-1} \circ \cdots \circ f_j^{-1}(W_{j+1}),$$

which are nested so that

$$B_1 \supset W_1 = W_1^1 \supset W_1^2 \supset \cdots \supset W_1^n = S_1.$$

For any $y \in V_1$, the annuli $(V_1 \times \{y\}) \cap (W_1^j - W_1^{j+1})$ are disjoint nested annuli for $j = 1, \dots, n-1$, and the jth maps by $\mathrm{pr}_1 \circ f_j \circ \cdots \circ f_1$ to an annulus which contains $U_{j+1} - U'_{j+1}$, hence has modulus at least $1/M_{j+1}$. The result now follows from the additivity of moduli. $\square$

Corollary 3.12. *Let* $B_0 = U_0 \times V_0, B_1 = U_1 \times V_1, \dots$ *be an infinite sequence of bidisks, and* $f_i : B_i \underset{\times}{\to} B_{i+1}$ *be crossed mappings of degree 1, with* U'_i *of uniformly bounded size in* U_i*. Then the set*

$$W^S_{\{B_n, f_n\}} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in B_0 \;\middle|\; f_n \circ \cdots \circ f_0 \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) \in B_n \quad \textit{for all } n \right\}$$

is a vertical-like analytic disk in B_0*, which maps by* pr_2 *isomorphically to* V_0*, which we will call the stable disk of the sequence of crossed mappings.*

Similarly, when we have backwards sequence of crossed mappings

$$\dots \underset{\times}{\to} B_{-1} \underset{\times}{\to} B_0$$

with uniformly bounded sizes, it will have a *unstable disk*, which will be horizontal-like.

Remark 3.13. Rather than requiring that the sizes $1/M_i$ of the U'_i in U_i be uniformly bounded, it would be enough to require that $\sum M_i = \infty$.

Proof of Corollary 3.12. For any $u_m \in U_m$, we can consider the set

$$\Gamma_m = \{ (x,y) \in B_0 \mid f_{m-1} \circ \cdots \circ f_0(x,y) \in \{u_m\} \times V_m \} .$$

This is a vertical-like analytic disk, so that there exists an inverse $\gamma_m : V_0 \to B_0$ of pr_2 which parameterizes it.

If $u_0, u_1, \ldots$ is any sequence with $u_m \in U_m$, and $\gamma_m : V_0 \to B_0$ is constructed as above for each m, then Proposition 3.11 says that the sequence γ_m converges uniformly. Clearly the limit is a parameterization of an vertical-like analytic disk contained in $W^S_{\{B_n,f_n\}}$. Clearly by Corollary 3.12, it is all of $W^S_{\{B_n,f_n\}}$. □ (Corollary 3.12)

We will refer to $W^S_{\{B_n,f_n\}}$ as the stable set of the sequence of crossed mappings

$$B_0 \underset{\times}{\overset{f_0}{\to}} B_1 \underset{\times}{\overset{f_1}{\to}} B_2 \ldots$$

This vertical set only depends on the underlying bidisks in a fairly crude way, as the following Proposition shows.

Proposition 3.14. *If $U_m = U'_m \cup U''_m$ with $U'_m \cap U''_m \neq \emptyset$ and U'_m and U''_m homeomorphic to disks. Set $B'_m = U'_m \times V_m$ and $B''_m = U''_m \times V_m$. Suppose*

$$f_m : U_m \times V_m \to U_{m+1} \times V_{m+1}$$

is an analytic map defined on an appropriate subset, such that the restrictions

$$f'_m : B'_m \underset{\times}{\to} B'_{m+1} \quad \text{and} \quad f''_m : B''_m \underset{\times}{\to} B''_{m+1}$$

are crossed mappings of degree 1, then the stable sets of the sequences

$$B'_0 \underset{\times}{\overset{f'_0}{\to}} B'_1 \underset{\times}{\overset{f'_1}{\to}} B'_2 \ldots \quad \text{and} \quad B''_0 \underset{\times}{\overset{f''_0}{\to}} B''_1 \underset{\times}{\overset{f''_1}{\to}} B''_2 \ldots$$

coincide.

Proof. In the proof of Corollary 3.12 above, the sequence u_m could be chosen arbitrarily, in particular in $U'_m \cap U''_m$. □

Corollary 3.15. *Let*

$$\ldots B_{-1} = U_{-1} \times V_{-1}, B_0 = U_0 \times V_0, B_1 = U_1 \times V_1, \ldots$$

be a bi-infinite sequence of bidisks, and $f_i : B_i \underset{\times}{\to} B_{i+1}$ *be crossed mappings of degree 1, with* U_i' *of uniformly bounded size in* U_i*. Then for all* $m \in \mathbb{Z}$*,*
(1) the set

$$W_m^S = \{\, (x_m, y_m) \mid \textit{there exist } (x_n, y_n) \in B_n \\ \textit{for all } n \geq m \textit{ such that } f_n(x_n, y_n) = (x_{n+1}, y_{n+1}) \,\}$$

is a closed vertical-like Riemann surface in B_m*, and* $\operatorname{pr}_2 : W_m^S \to V_m$ *is an isomorphism;*
(2) the set

$$W_m^U = \{\, (x_m, y_m) \mid \textit{there exist } (x_n, y_n) \in B_n \\ \textit{for all } n < m \textit{ such that } f_n(x_n, y_n) = (x_{n+1}, y_{n+1}) \,\}$$

is a closed horizontal-like Riemann surface in B_m*, and* $\operatorname{pr}_1 : W_m^S \to U_m$ *is an isomorphism.*

(3) Moreover, the sequence

$$(x_m, y_m) := W_m^S \cap W_m^U, \quad m \in \mathbb{Z},$$

is the unique bi-infinite sequence with $(x_m, y_m) \in B_m$ *for all* $m \in \mathbb{Z}$*, and* $f_m(x_m, y_m) = (x_{m+1}, y_{m+1})$.

Proof. The first statement is immediate from Corollary 3.12, and the second also by considering the mappings $g_n = f_{n+1}^{-1}$, which also define a bi-infinite sequence of crossed mappings by Proposition 3.7(a). The third part follows immediately from the first two. □

4. Perturbations of hyperbolic polynomials

Let $p(z)$ be a hyperbolic polynomial of degree $k \geq 2$, which will be fixed for the next three sections. We will drop the subscript p, and write

$$H_a\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} p(x) - ay \\ x \end{bmatrix}.$$

Choose as in section 2 a neighborhood U of J_p such that $p : U' = p^{-1}(U) \to U$ is a covering map. Set $U'' = p^{-1}(U')$.

Recall that when $a = 0$, the Hénon mapping

$$H_0 : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} p(x) \\ x \end{bmatrix}$$

maps all of $\mathbb{C}^2$ to the curve C_p of equation $x = p(y)$, and reduces to $x \mapsto p(x)$ in the first coordinate. Thus we can think of the polynomial p as a mapping $C_p \to C_p$; when we think of U as a subset of C_p, we will denote it by $\tilde{U}$, and its projection onto the y-axis simply by U.

First let us recall the crudest properties of Hénon mappings; we will suppose $|a| \le 1$. If $p(z) = a_k z^k + \cdots + a_0 = a_k z^k + q(z)$, denote by $|q|(r) = |a_{k-1}|r^{k-1} + \cdots + |a_0|$, and let R be the largest root of the equation $|a_k|r^k - |q|(r) - 2r = 0$. We will call

$$B_R = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{C}^2 \;\middle|\; |x| < R, |y| < R \right\}.$$

All of the interesting dynamics of H_a occurs in B_R, because Figure 4.1 roughly describes the orbits of points.

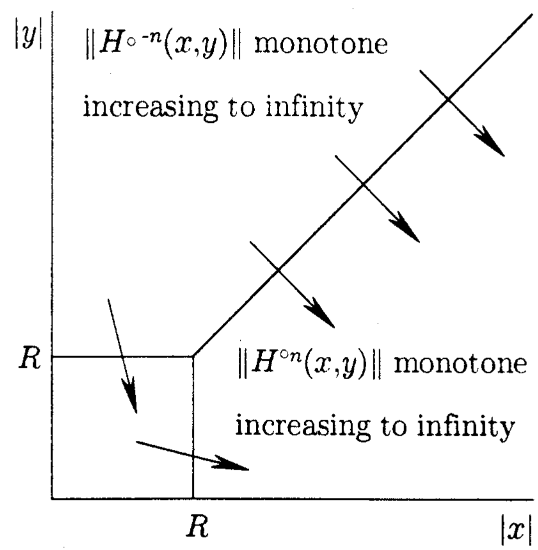

Figure 4.1: Crude picture of the dynamics of H_a.

Our construction will depend on two numbers $\delta > 0$ and $A > 0$, which will be chosen to satisfy Requirements 1, 2, 3, 4, and 5, which are given below. Consider the neighborhood

$$N_\delta = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{C}^2 \;\middle|\; |p(y) - x| < \delta \right\}$$

of C_p.

Our first requirement concerns only δ.

Requirement 1. The number $\delta > 0$ is sufficiently small that N_δ intersects the boundary of B_R only in the "vertical" boundary $|x| = R$, and moreover for any $x_0 \in U$, each component of the intersection $L_{x_0} \cap N_\delta$, where L_{x_0} is the vertical line of equation

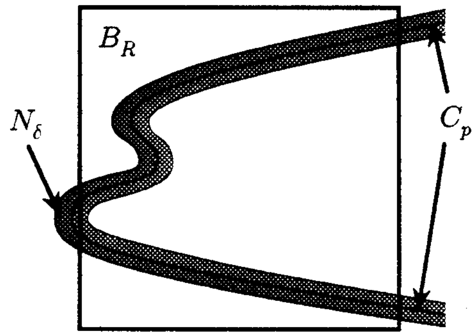

Figure 4.2: The curve C_p and its neighborhood N_δ, drawn in $\mathbb{R}^2$.

$x = x_0$ contains a unique point of C_p, which will belong to $\tilde{U}'$. We will further require that

$$\left| p\left(\zeta + \frac{z}{p'(\zeta)}\right) - (p(\zeta) + z) \right| < \frac{\delta}{2}$$

for all $\zeta \in J_p$ and $|z| < \delta$.

Choose, for the rest of the paper, a number δ satisfying Requirement 1.

Requirement 2. Now choose a number $\varepsilon > 0$ such that the sets $U_z, z \in U$ are all homeomorphic to disks, as in Section 2. In Section 6, we will require a bit more: for all $z \in J_p$, the image $p(U_z) \subset D_{\delta/2}(p(z))$, is contained in the Euclidean disk of radius $\delta/2$ centered at $p(z)$. This will clearly be the case if ε is sufficiently small.

Our next requirements all concern the size of $|a|$.

Requirement 3. We have $H_a(B_R) \subset N_\delta$ when $|a| < A$.

This will clearly be satisfied as soon as A is sufficiently small.

Let

$$V' = \mathrm{pr}_1^{-1}(U) \cap N_\delta.$$

be the union of these components. There is a well-defined function $u : V' \to U'$ given by $u(x, y) = p^{-1}(x)$, the branch of the inverse image being precisely the intersection with C_p above, which one can also understand as the branch "close to y".

The pair of functions $(u, v) : V \to \mathbb{C}^2$ given by the formulas

$$u(x, y) = p^{-1}(x), \quad v(x, y) = p(y) - x$$

parameterize V.

Requirement 4. We will require that H_a should map the vertical boundary of V' outside of V when $|a| \leq A$.

Again this will occur whenever $|a|$ is sufficiently small.

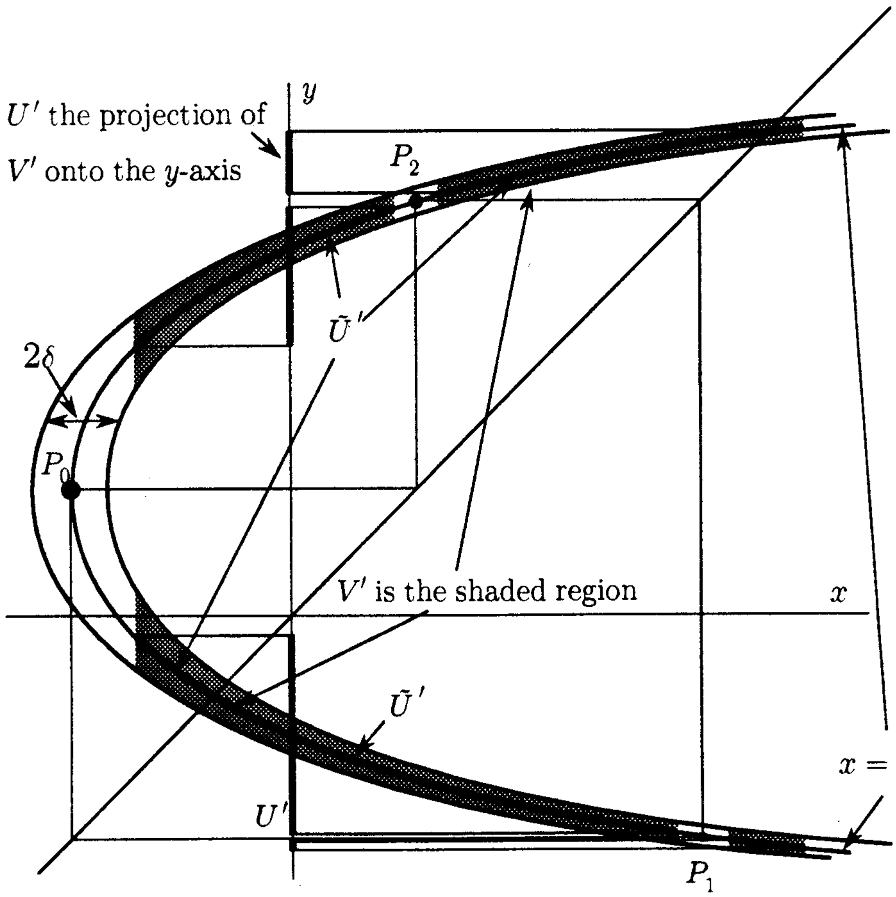

Figure 4.3: The neighborhood V' of the Julia set $\tilde{J}_p \subset C_p$

Proposition 4.4. *(a) For each attractive periodic point z_0 of p, there is an analytic function $z(a)$ defined for $|a| < A$, such that $z(0) = z_0$ and $z(a)$ is an attractive cycle of H_a.*
(b) The points of compact components of $\overline{N}_\delta - V'$ are attracted to these cycles, and the points of the unique non compact component iterate to infinity.

Proof. The union of the compact components of $\overline{N}_\delta - V'$ is mapped into itself by Requirement 3. Thus the sequence of iterates of H_a is normal. On the other hand any limit function has compact image. So the sequence of iterates is accumulating on finitely many attracting cycles. The proof shows that these depend analytically on a for $|a| < A$. □

For every $z \in U'$ consider the neighborhood

$$V_z = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in V' \,\middle|\, u(x,y) \in U_z \right\}$$

of the point $(p(z), z) \in \tilde{U}'$.

Lemma 4.5. *Under Requirement 3, the mapping*

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} u(x, y \\ v(x, y) \end{bmatrix}$$

is a biholomorphic isomorphism of V_z onto the bidisk $U_z \times D_\delta$.

The proof is left to the reader.

For all $z \in U''$, set

$$W_z = V_z \cap H_a^{-1}(V_{p(z)}) \quad \text{and} \quad \tilde{W}_z = V_{p(z)} \cap H_a(V_z).$$

Proposition 4.6. *There exists $A > 0$ such that if $|a| < A$, then for all $\zeta \in U''$, the mapping $H_a : W_z \to \tilde{W}_z$ is a crossed mapping $V_z \underset{\times}{\to} V_{p(z)}$ of degree 1.*

Proof. Choose $\zeta \in \mathbb{C}$ with $|\zeta| < \delta$, and consider

$$u \circ H_a : V_z \cap \{p(y) - x = \zeta\} \to \mathbb{C}.$$

The disk $V_z \cap \{p(y) - x = \zeta\}$ is parameterized by y, and when the Jacobian a of H_a is zero, this map is simply $\zeta \mapsto p(y)$, and in particular maps the boundary of U_z strictly outside $U_{p(z)}$, with degree 1. This remains true for a sufficiently small perturbation, in particular for $|a| < A$ when $A > 0$ is small enough, and it is easy to see that if a is sufficiently small, then this will be true for all $z \in U''$ and ζ with $|\zeta| \le \delta$. It follows that for such sufficiently small A, condition 1 of Definition 3.2 of a crossed mapping is satisfied, and the first half of condition 3.

For condition 2 and the second part of 3, we use the inverse mapping. For any fixed $z \in U''$ and $w \in p\left(U_{p(z)}\right)$, consider the vertical disk

$$\Delta_w = \left\{ \begin{bmatrix} w \\ y \end{bmatrix} \in V_{p(z)} \right\};$$

which the coordinate function v maps isomorphicaly to the disk of radius δ: on its boundary, we have $p(y) - w = \delta$. Let us compute:

$$v\left(H_a^{-1}\left(\begin{bmatrix} w \\ y \end{bmatrix}\right)\right) = p\left(\frac{p(y) - w}{a}\right) - x.$$

This takes $\partial\Delta_w$ to a large curve when $|a|$ is small, since $|(p(y) - w)/a| = \delta/|a|$ is large, and p takes large values there. □

Remark 4.7. We do not need to calculate the degree of this mapping restricted to $\tilde{W}_z \cap \Delta_w$. By Proposition 3.3, this must be one. It is not obvious from our computation: we might rather have expected $k = \deg p$. This is because the set

$$\left\{ \begin{bmatrix} w \\ y \end{bmatrix} \in \Delta_w \;\middle|\; |p(y) - w| < \delta \right\}$$

has k components, one for each inverse image of $p(z)$ under p. Figure 4.8 illustrates this phenomenon.

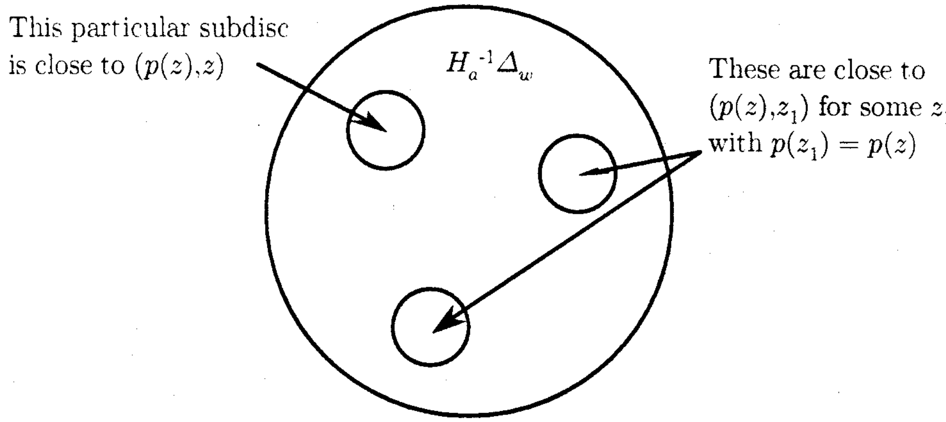

Figure 4.8: How the image of a vertical disc intersects N_δ

Requirement 5. The number A is sufficiently small that the conclusion of Proposition 4.6 is satisfied when $0 < |a| < A$.

For any point $\underline{z} = (\ldots, z_{-2}, z_{-1}, z_0) \in \hat{J}_p$, consider the bi-infinite family of crossed mappings which we will denote, by abuse of notation

$$\ldots \underset{\times}{\overset{H_a}{\to}} V_{z_{-1}} \underset{\times}{\overset{H_a}{\to}} V_{z_0} \underset{\times}{\overset{H_a}{\to}} V_{p(z_0)} \underset{\times}{\overset{H_a}{\to}} V_{p^{\circ 2}(z_0)} \underset{\times}{\overset{H_a}{\to}} \ldots.$$

We can now define the mapping Φ: by Proposition 3.11, there is a unique point $\Phi(\underline{z}) \in V_{z_0}$ such that

$$H_a^{\circ m}(\Phi(\underline{z})) \in \begin{cases} V_{z_{-k}} & \text{for } m \le 0 \\ V_{p^{\circ k}(z_0)} & \text{for } m > 0 \end{cases}.$$

Theorem 4.9. *The mapping $\Phi : \hat{J}_p \to \mathbb{C}^2$ is a homeomorphism onto J which conjugates $\hat{p}$ to H_a.*

Proof. The mapping Φ is obviously continuous. We will construct an inverse $\Psi : J \to \hat{J}_p$.

Observe first that $J \subset V$. Indeed, $J \subset B_R$, hence $J \subset N_\delta \cap B_R$. But by Proposition 4.4, we know that the points in $N_\delta \cap B_R$ and not in V tend to ∞ or attracting cycles, hence cannot be points of J.

Therefore a point $(x, y) \in J$ defines a bi-infinite p-telescope $U_{u(H_a^{\circ m}(x,y))}$. We have seen that such a bi-infinite telescope defines a point of $\underline{z} \in \hat{J}_p$. The mapping $\Psi : (x, y) \mapsto \underline{z}$ is obviously continuous. We leave it to the reader to check that it is an inverse of Φ. $\square$

This construction endows J with a "stable" and "unstable" manifold: set W^U (respectively W^S) to be the union of all the unstable (respectively, stable) disks of the families of 1-crossed mappings defining Φ.

Proposition 4.10. *(a) We have the equalities*

$$J_+ \cap V = \bigcap_{n \geq 0} H_a^{-n}(V) \quad \textit{and} \quad J_- \cap V = \bigcap_{n \geq 0} H_a^{n}(V).$$

(b) Moreover, the forward orbit of any point in J_+ is eventually contained in V, and the backwards orbit of any point in J_- is also eventually contained in V, except for the attracting cycles described in Proposition 4.4.

Proof. The inclusions

$$\bigcap_{n \geq 0} H_a^{-n}(V) \subset K_+ \cap V \quad \text{and} \quad \bigcap_{n \geq 0} H_a^{n}(V) \subset K_- \cap V = J_- \cap V$$

are obvious.

To see $\bigcap_{n \geq 0} H_a^{-n}(V) \subset J_+ \cap V$, we need to know that no interior point of K_+ can have its forward orbit entirely in V. This follows from Proposition 3.14. On such an open set, the sequence $H_a^{\circ n}, n \geq 0$ is normal, and we can extract a subsequence $H_a^{\circ n_i}$ convergent on compact subsets. Given any two points (x_1, y_1) and (x_2, y_2), and extracting a further subsequence if necessary, the infinite sequences of bidisks

$$V_{u(H_a^{\circ n_i}(x_1,y_1))} \quad \text{and} \quad V_{u(H_a^{\circ n_i}(x_2,y_2))}$$

connected by the mappings $H_a^{\circ(n_{i+1}-n_i)}$ are equivalent, hence define the same stable set, which is a vertical-like disk in $V_{u(H_a^{\circ n_1}(x_1,y_1))}$. This contradicts the assumption that (x_1, y_1) and (x_2, y_2) could be chosen in an open set.

To see the opposite inclusions, consider a point of V. If its forward orbit ever leaves V, we know what it does: it is either attracted to one of the attracting cycles described in Proposition 4.4, in which case it is in the interior of K_+, or it iterates to infinity, in which case it is not in K_+. Thus $J_+ \cap V = \bigcap_{n \geq 0} H_a^{-n}(V)$.

Now, suppose a point in V has an inverse image not in V. Then since $H_a(N_\delta \cap B_R) \subset N_\delta$, the inverse image is not in N_δ, and a further inverse image is not in B_R. By the discussion of Figure 4.1, this means that the backwards orbit of the point tends to infinity. This proves (a).

For (b), consider the forward image of any point in K_+. It will eventually be contained in $N_\delta \cap B_R$, and if it is not eventually in V, then it is attracted to an attracting cycle, by Proposition 4.4. Similarly, the backwards orbit of any point in J_- will eventually be contained in $N_\delta \cap B_R$. If it is in the basin of attraction of one of the attracting cycles of Proposition 4.4, then its backwards orbit will enter and remain in V unless it is one of the attracting cycles itself. □

Remark. Proposition 4.10 is where we eliminate the possibility of wandering domains. In one dimension, we have Sullivan's *No Wandering Domains* Theorem to eliminate this possibility, but this uses quasi-conformal mappings, and there does not appear to be an analog in several dimensions. For hyperbolic polynomials (or rational functions), one can also eliminate the existence of wandering domains by using the expanding metric on a neighborhood of the Julia set. The proof above is a natural extension of that proof.

5. Characterization of J_-

In this section we will prove the following result.

Theorem 5.1. *There is a homeomorphism*

$$\Phi_- : \hat{\mathbb{C}}_p \to J_-$$

which conjugates $\hat{p}$ to H_a. This homeomorphism coincides with $\Phi : \hat{J}_p \to J$ on $\hat{J}_p$.

Remark 5.2. This result can be slightly improved: Φ_- can be chosen analytic on $\hat{\mathbb{C}}_p - \hat{K}_p$. But it cannot be made analytic on $\hat{K}_p$.

Proof. We will begin by finding the restriction

$$\Phi_- \mid_{\hat{U}'} : \hat{U}' \to W^U$$

which conjugates $\hat{p}$ to H_a there, then we will extend it to the remainder of $\hat{\mathbb{C}}_p$. This requires the following.

Proposition 5.3. *There exists a mapping $\pi_{U'} : V' \to U'$ which semi-conjugates*

$$H_a : H_a^{-1}(V') \cap V' = V'' \to V' \quad \text{to} \quad p : U'' \to U'.$$

This mapping can be chosen so that for every $(x, y) \in V'$, we have

$$\pi_{U'}(x, y) \in U_{u(x,y)},$$

and so that the fibers are vertical-like.

Proof. We will begin by constructing our mapping on $V' - V''$.

Lemma 5.4. *There exists a continuous mapping $\pi_{U'} : \overline{V'} - V'' \to \overline{U'} - U''$ which semi-conjugates H_a to p as maps from the inner boundary to the outer boundary, and such that the fibers are vertical-like disks.*

We have found this lemma surprisingly difficult to prove. Note that $v : V' - V'' \to D_\delta$ is a locally trivial fibration, and our lemma says that there exists a trivialization with special properties. Of course trivializations exist, since the base is contractible. But the requirement that the induced sections be vertical-like does not seem to be accessible by topological techniques. Instead, we will use differential equations.

Before doing this, we require a fundamental statement about complex vector spaces.

Sublemma 5.5. *A real hyperplane F in a complex vector space E contains a unique complex hyperplane $[F] = F \cap iF$.*

The statement contains the proof.

Proof of Lemma 5.4. Consider the radial vector field $\xi = -r\partial/\partial r$ on the disk D_δ. This vector field lifts canonically to a vector field $\tilde{\tilde{\xi}}$ on the "vertical boundary" of $\partial^{\mathrm{ver}}(\bar{V}' - V'')$, both inner and outer. This boundary is a 3-dimensional real manifold in $\mathbb{C}^2$, so at each point

$$(x, y) \in \partial^{\mathrm{ver}}(\bar{V}' - V''),$$

the tangent space

$$T_{(x,y)}\partial^{\mathrm{ver}}(\bar{V}' - V'')$$

contains a unique complex line

$$[T_{(x,y)}\partial^{\mathrm{ver}}(\bar{V}' - V'')],$$

which maps isomorphically to $\mathbb{C}$ by the derivative $d_{(x,y)}v$. The vector field $\tilde{\tilde{\xi}}$ is the unique lift of ξ to this bundle of complex lines.

Let $\tilde{\xi}$ be a C^∞ lifting of ξ to $\bar{V}' - V''$, which extends $\tilde{\tilde{\xi}}$, and everywhere points into the vertical cone. Such a lifting exists, since local liftings exist, and can be patched together by partitions of unity.

Denote by $\varphi_{(x,y)}(t)$ the solution curve of the differential equation defined by $\tilde{\xi}$ with $\varphi_{(x,y)}(0) = (x, y)$.

Sublemma 5.6. *The lift $\tilde{\xi}$ can be chosen so that*
(a) The limit $w(x, y) := \lim_{t\to\infty} \varphi_{(x,y)}(t)$ of this solution exists.
(b) The set $w^{-1}(\tilde{z})$ is a vertical-like "disk" for all $\tilde{z} = (p(z), z) \in \tilde{U}' - \tilde{U}''$, although this disk is not an analytic disk in general unless z is in the boundary $\partial(\bar{U}' - U'')$.

Proof of Sublemma 5.6. We will work in the real oriented blow-up of V' along $\tilde{U}'$. This is a set in which every point of $\tilde{U}'$ is replaced by a circle. It is easy to describe in this case. Consider $\tilde{V}' = U' \times ([0, \delta) \times S^1)$; the map $\tilde{V}' \to V'$ which sends (z, r, θ) to the point $(x, y) \in V'$ with $u(x, y) = z$ and $v(x, y) = re^{i\theta}$ realizes $\tilde{V}'$ as such a blow-up. In $\tilde{V}'$, we can lift not just $-r\partial/\partial r$ but also $-\partial/\partial r$, and the existence and uniqueness theorem applies even to points on the boundary $U' \times \{0\} \times S^1$. Now our disks are precisely the solutions which end on a circle $\{z\} \times \{0\} \times S^1$. □ (Sublemma 5.6)

The disks which foliate the vertical boundary of $\bar{V}' - V''$ are collapsed to points under w, in fact the point at which such a disk intersects C_p.

Thus it is now enough to choose a homeomorphism of

$$\overline{\tilde{U}'} - \tilde{U}'' \quad \text{with} \quad \bar{U}' - U''$$

which extends the identity on the outer boundary, and maps a point (x, y) of the inner boundary to the point in $p^{-1}(u(H_a(x, y)))$ which is close to (x, y). □ (Lemma 5.4)

To prove Proposition 5.3, we need to extend $\pi_{U'}$, and there is an obvious way to do so on $V' - W^S$: define

$$\pi_{U'}(x, y) = p^{\circ -N}(\pi_{U'}(H_a^{\circ N}(x, y)))$$

where N is defined so that $H_a^{\circ N}(x, y) \in V' - V''$, and the branch of $p^{\circ -N}$ being chosen recursively so that $p^{\circ -m}(\pi_{U'}(H_a^{\circ N}(x, y)))$ is the inverse image of $p^{\circ -m+1}(\pi_{U'}(H_a^{\circ N}(x, y)))$ in $U_{p^{\circ N-m}(u(x,y))}$.

This defines $\pi_{U'}$ on $V' - W^S$. On W^S it is defined by the telescope construction. We still need to show that $\pi_{U'}$ is continuous; this is only an issue at points of W^S.

If (x_k, y_k) is a sequence in V' approaching $(x, y) \in W^S$, then the number N_k of moves it takes to escape V' tends to ∞. Then the point $\pi_{U'}(x_k, y_k)$ is in the intersection of the nested sequence

$$\bigcap_{m=0}^{N_k} p^{-m} U_{u(H_a^{\circ m}(x_k, y_k))}$$

which is a set with diameter tending to 0 as $k \to \infty$. Moreover, the sets

$$\bigcap_{m=0}^{N_k} p^{-m} U_{u(H_a^{\circ m}(x_k, y_k))} \quad \text{and} \quad \bigcap_{m=0}^{N_k} p^{-m} U_{u(H_a^{\circ m}(x, y))}$$

are close for k large (in the sense that the Hausdorff distance of their closures are close, for instance). Moreover, the second intersection converges $\pi_{U'}(x, y)$. □ (Proposition 5.3)

We can now construct our mapping

$$\Phi_- : \hat{U} \to W^U.$$

Given a point $(\ldots, z_{-1}, z_0) \in \hat{U}$, let us consider the intersection

$$\bigcap H_a^m(V_{z_{-k}}),$$

which we have seen in Corollary 3.15 is a Riemann surface isomorphic to a horizontal-like disk, in fact a section of the projection $v : V_{z_0} \to U_{z_0}$. This section will intersect $\pi_{U'}^{-1}(z_0)$ in a single point: this point is $\Phi_-(\ldots, z_{-1}, z_0)$.

With this definition, Φ_- is obviously continuous. And it is easy to construct an inverse: simply associate to $(x, y) \in W^U$ the sequence

$$z_{-n}(x, y) = \pi_{U'}(H_a^{\circ -n}(x, y)), \quad n \in \mathbb{N}.$$

This is clearly a point of $\hat{U}$, and $\Phi_-(\underline{z})(x, y) = (x, y)$.

Notice that the construction of Φ_- did not use the hypothesis that H_a was injective, and works even if $a = 0$. In fact, it is especially easy in that case, giving the canonical projection $\hat{U}' \to U'$, perhaps perturbed by a homeomorphism of U' which commutes with p. Only the construction of the inverse, *i.e.*, the statement that Φ_- is injective, really uses the fact that H_a is an automorphism.

Now to extend Φ_- to the remainder of $\hat{\mathbb{C}}_p$. Any sequence of inverse images under p tends to the Julia set, except for those finitely many periodic sequences consisting entirely of points of the attracting cycles. Take a sequence $(\dots, z_{-1}, z_0)$. If it is not one of these exceptional periodic histories, there exists N such that $(\dots, z_{-N-1}, z_{-N}) \in \hat{U}'$. Map such a point to

$$\Phi_-(\dots, z_{-1}, z_0) = H_a^{\circ N} \Phi_-(\dots, z_{-N-1}, z_{-N}).$$

If the sequence $(\dots, z_{-1}, z_0)$ is periodic, consisting of points of an attracting cycle, map it to the attracting periodic point of H_a corresponding to z_0.

We need to check that this is continuous at these exceptional points. Suppose that $(\dots, z_{-1}, z_0)$ is such an exceptional point, and that $(\dots, z'_{-1}, z'_0)$ is close to it. Let N be the first index such that $z'_{-N} \in U'$; the number N is large. The point $\Phi_-(\dots, z_{-N-1}, z_{-N})$ is in $V - V''$, and we have seen that such points have the same fate under H_a as z_{-N} has under p, whether tending to infinity or to an attracting cycle. In fact, we have seen further that the distance of $H_a^{\circ N}(\Phi_-(\dots, z_{-N-1}, z_{-N}))$ to the attracting cycle is arbitrarily small, by a quantity which depends only on N. This proves continuity.

The only thing remaining is to check that the mapping $\Phi_- : \hat{C}_p \to J_-$ is surjective. We invite the reader to check that if $H_a^{\circ -n}(x, y)$ is bounded as $n \to \infty$, then either the sequence is eventually contained in V', or it is the orbit of a repelling cycle for H_a^{-1}.

$\square$ (Theorem 5.1)

6. Characterization of J_+

We are going to show that for $|a|$ sufficiently small, the set J_+ is modeled on $\check{\mathbb{C}}_p$.

First, we need to know a little more about this space: $\check{\mathbb{C}}_p$ is foliated by surfaces, in fact by Riemann surfaces isomorphic to $\mathbb{C}$, each of which is dense. For each $\zeta \in J_p$, let L_ζ be the inductive limit of

$$\{\zeta\} \times D \overset{f_p}{\hookrightarrow} \{p(\zeta)\} \times D \overset{f_p}{\hookrightarrow} \{p^{\circ 2}(\zeta)\} \times D \overset{f_p}{\hookrightarrow} \dots$$

which is an increasing union of discs.

Proposition 6.1. *Each L_ζ is a Riemann surface isomorphic to $\mathbb{C}$, and is dense in $\check{\mathbb{C}}_p$. The foliation is compatible with the dynamics in the sense that*

$$\check{p}(L_\zeta) = L_{p(\zeta)}.$$

Proof. The space L_ζ is a Riemann surface since the inclusions

$$\{\zeta\} \times D \overset{f_p}{\hookrightarrow} \{p(\zeta)\} \times D \overset{f_p}{\hookrightarrow} \{p^{\circ 2}(\zeta)\} \times D \overset{f_p}{\hookrightarrow} \dots$$

are analytic inclusions; it is simply connected since it is a union of discs. The union is isomorphic to $\mathbb{C}$ by Proposition 7.3 of [**HO**]. The formula $\check{p}(L_\zeta) = L_{p(\zeta)}$ is obvious, and it implies the density as follows.

If $p^k(\zeta_1) = p^k(\zeta_2) = \zeta$, then $L_{\zeta_1} = L_{\zeta_2}$ since both are equal to $\check{p}^{-k}(L_\zeta)$. Thus given any point ζ, we see that $L_{\zeta'} = L_\zeta$ for all $\zeta' \in p^{-k}(p^{\circ k}(\zeta)), k = 1, 2, 3, \dots$. But such points ζ' are dense in J_p, so the leaf L_ζ is dense in $J_p \times D$, which it intersects in infinitely many discs $\{\zeta'\} \times D$. It is then easy to show that it is till dense in all the $((J_p \times D), n)$ in the inductive limit describing $\check{\mathbb{C}}_p$. □

Now, let us see the third part of Theorem 1.4.

Theorem 6.2. *(a) If p is a hyperbolic polynomial and $|a|$ is sufficiently small, there exists a homeomorphism*

$$\Phi_+ : \check{\mathbb{C}}_p \to J_+$$

conjugating $\check{p}$ to $H_a\,|_{J_+}$.
(b) The mapping Φ_+ maps the leaves of the foliation of $\check{\mathbb{C}}_p$ to Riemann surfaces isomorphic to $\mathbb{C}$ immersed in $\mathbb{C}^2$.

Proof. We will construct Φ_+ first on $J_p \times D = (J_p \times D) \times 0 \subset \check{\mathbb{C}}_p$, and even then we will define it on larger and larger parts. The restriction of Φ_+ to $J_p \times D$ and all its further restrictions will be called F_+.

The first step (and the hardest) is to get started. We have already constructed J and its stable manifold W^S.

Lemma 6.3. *There is a unique projection $\pi : W^S \to J_p$ such that the diagram*

$$\begin{array}{ccc} W^S & \xrightarrow{H_a} & W^S \\ \pi\big\downarrow & & \pi\big\downarrow \\ J_p & \xrightarrow{p} & J_p \end{array}$$

commutes, and the fibers of π are stable disks of the crossed mappings.

Proof of Lemma 6.3. This is precisely what telescopes are for. Given a point $(x, y) \in W^S$, we can consider the p-telescope $U_{u(H_a^{\circ k}(x,y))}$, which defines a unique point $\pi(x, y) \in J_p$.
□ Lemma 6.3

Proposition 6.4. *There exists a homeomorphism*

$$F_+ : J_p \times D \to W^S$$

such that the following diagram commutes.

$$\begin{tikzcd}
J_p \times D \arrow[rrr, "F_+"] \arrow[dr, "f_p"] \arrow[ddr, "\mathrm{pr}_1"'] & & & W^s \arrow[dr, "H"] \arrow[ddll, "\pi"] & \\
& J_p \times D \arrow[rrr, "F_+"] \arrow[ddr, "\mathrm{pr}_1"'] & & & W^s \arrow[ddll, "\pi"] \\
& J_p \arrow[dr, "p"'] & & & \\
& & J_p & &
\end{tikzcd}$$

This is a fairly difficult result and will require several steps, of which the first is the hardest.

Proposition 6.5. *There exists a homeomorphism*

$$F_+ : J_p \times \bar{D} - f_p(J_p \times D) \to \bar{W}^S - H_a(W^S)$$

such that the diagram above commutes on the boundary.

Proof of Proposition 6.5. Before embarking on the proof, we will want to write our mapping f_p in a different way, more convenient for the rather delicate argument in Lemma 6.10. Temporarily write

$$g_p(\zeta, z) = \left(p(\zeta), a\left(\zeta + \frac{z}{p'(\zeta)}\right)\right), \quad \text{where} \quad |z| < \delta.$$

This mapping is conjugate to our original one: set $w = Rz/\delta$, so that in the coordinate w we find

$$(\zeta, w) \mapsto \left(p(\zeta), \frac{aR}{\delta}\left(\zeta + \frac{\delta w}{Rp'(\zeta)}\right)\right).$$

Thus if we choose $R = \delta/a$, we find that this linear change of variables conjugates the two mappings.

Note that both $J_p \times D$ and W^S are trivial bundles of disks of radius δ over J_p. They are canonically homeomorphic, although that homeomorphism is not compatible with the dynamics and is not the one we are trying to find. But we will use it to define

$$F_+ : J_p \times \partial D \to \partial W^S$$

to be the "identity" (*i.e.*, the map which takes (ζ, z) to the unique point (x, y) above ζ such that $p(y) - x = z$). We can immediately define F_+ on the interior boundary

$$F_+ : f_p(J_p \times D) \to H_a(\partial W^S)$$

by the formula

$$F_+ = H_a \circ F_+ \circ f_p^{-1}.$$

To extend it to the region in between, we will use some heavy-duty topology. We define the space

$$\mathrm{Homeo}_{J_p}(J_p \times D - \check{p}(J_p \times D), W^S - H_a(W^S); F_+)$$

to be the fiber bundle over J_p, the fiber of which over $\zeta \in J_p$ is the space of homeomorphisms of the fiber of $J_p \times D - \check{p}(J_p \times D)$ above ζ to the fiber of $W^S - H_a(W^S)$ above ζ, which agree with F_+ (as defined so far) on the boundaries.

Remark 6.6. This space is pretty wild: it is a fiber bundle over J_p, the fiber of which is an infinite-dimensional space of homeomorphisms of a surface with boundary, in fact a disk with holes, to another. More precisely, the fiber of $J_p \times D - f_p(J_p \times D) \to J_p$ is the disk of radius δ with k disks of radius $a\delta$ removed:

$$D - \bigcup_{\zeta_1 \in p^{-1}(\zeta)} D_{a\delta}(a\zeta_1).$$

The fiber of $W^S - H_a(W^S) \to J_p$ is similar.

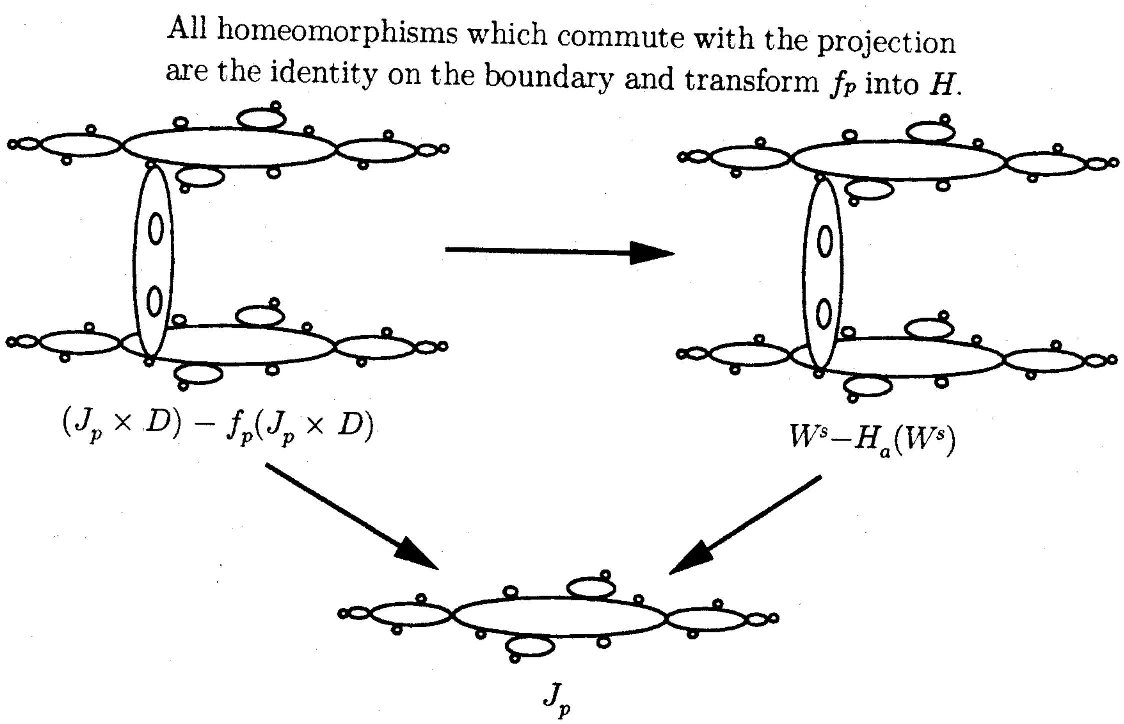

Figure 6.7: The two fiber bundles

The statement of Proposition 6.5 says exactly that the fiber bundle

$$\mathrm{Homeo}_{J_p}(J_p \times D - f_p(J_p \times D), W^S - H_a(W^S); F_+)$$

has a continuous section. As a first step to proving this, we will require the following Lemma.

Lemma 6.8. *The fibers of the fiber bundle*

$$\operatorname{Homeo}_{J_p}(J_p \times D - f_p(J_p \times D), W^S - H_a(W^S); F_+)$$

are contractible.

Proof of Lemma 6.8. This is an immediate consequence of a hard theorem from topology, due to Hamstrom [**Ha**]. This theorem asserts that if S is a compact surface with non-empty boundary, then the components of the group of homeomorphism which are the identity on the boundary are contractible. In our case, we can choose a homeomorphism

$$D - \bigcup_{\zeta_1 \in p^{-1}(\zeta)} D_{a\delta}(a\zeta_1) \to \pi_{U'}(\zeta) - H_a(W^S)$$

which extends F_+ on the boundary by the classification of surfaces. Then the space of homeomorphism homotopic to that one is acted on freely by the group of homeomorphisms of the domain which are the identity on the boundary. □ Lemma 6.8

Remark 6.9. We are making this assertion for homeomorphisms. It is also true of C^∞ diffeomorphisms [**EE**], and there might be good reasons to prefer them. We are going to pull back the mapping F_+ repeatedly by diffeomorphisms, and eventually extend to a Cantor set. The resulting mapping will not be differentiable on the Cantor set, but it will be quasi-conformal. The mapping F_+ would then be a quasi-conformal map from the leaves of the foliation of $\check{\mathbb{C}}_p$ to the leaves of J_+, showing that these Riemann surfaces are isomorphic to $\mathbb{C}$. We will get this result by other means; but it might be nice to know that as dynamical systems $\check{\mathbb{C}}_p$ and J_+ are quasi-conformally isomorphic, in the sense of Riemann Surface Laminations.

It follows that it is enough to show that there is section of the covering space of connected components of the fibers, or alternately, that it is enough to show that there is a preferred homotopy class of homeomorphisms from fibers to fibers, and there is: those which are homotopic to a homeomorphism close to the identity. This requires a bit of elaboration.

Lemma 6.10. *In the coordinates above, H_a is close to f_p when a is small.*

Proof of Lemma 6.10. Let us start with a point $(\zeta, z) \in J_p \times D$, and consider $H_a^{-1} \circ f_p(\zeta, z)$, where implicit in the composition is the parameterization of W^S by the (u, v)-coordinates. Let (x_1, y_1) be the point of W^S with $u(x_1, y_1) = p(\zeta)$ and $v(x_1, y_1) = a(\zeta + z/p'(\zeta))$.

This has the very pleasant consequence that

$$H_a^{-1}\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

where the $y2$-coordinate can be expressed exactly:

$$y_2 = \frac{1}{a}(p(y_1) - x_1) = \frac{1}{a}v(x_1, y_1) = \zeta + \frac{z}{p'(\zeta)}.$$

Thus

$$v(x_2, y_2) = p\left(\zeta + \frac{z}{p'(\zeta)}\right) - x_2 \approx z$$

and we need to evaluate how good the approximation is. There are two approximations to consider: $p(\zeta) = x_2 + \mathrm{err}_1$ and $p\left(\zeta + \frac{z}{p'(\zeta)}\right) = p(\zeta) + z + \mathrm{err}_2$.

We have that $|\mathrm{err}_2| < \delta/2$ by Requirement 1 of Section 4. Moreover, we have $u(x_2, y_2) \in U_\zeta$, so $x_2 \in U_{p(\zeta)}$, and by Requirement 2 of Section 4, this is smaller than $\delta/2$

$\square$ Lemma 6.10

Thus we can consider those homeomorphisms

$$h : J_p \times D - f_p(J_p \times D) \to W^S - H_a(W^S)$$

which coincide with F_+ on the boundary, and which are homotopic with boundaries fixed to homeomorphisms h' which move all points by at most $|a|\delta$. All such homeomorphisms are homotopic, and this set is not empty, so it does define a homotopy class of homeomorphisms as required. $\square$ Proposition 6.5

The next step is much easier.

Proposition 6.11. *The mapping F_+ extends to a homeomorphism*

$$F_+ : J_p \times D - \hat{J}_p \to W^S - J.$$

Proof of Proposition 6.11. Just map any point on $J_p \times D - \hat{J}_p$ backwards by f_p until it is in the region $J_p \times D - f_p(J_p \times D)$, then map over by F_+ and back inwards by iterating the Hénon H_a mapping the same number of times. $\square$ Proposition 6.11

Proposition 6.12. *The map F_+ defined above and the map Φ from Theorem 4.9 together give a homeomorphism*

$$F_+ : J_p \times D \to W^S.$$

Proof of Proposition 6.12. The only thing to prove is continuity, which is an issue only at points of $\hat{J}_p$. Suppose a sequence (ζ_n, z_n) in $J_p \times D$ converges to a point $(\zeta_0, z_0) \in \hat{J}_p$. Then the point ζ_0 comes with a history, namely the first coordinates of

$$f_p^{\circ -1}(\zeta_0, z_0), f_p^{\circ -2}(\zeta_0, z_0), \ldots$$

which determines the point (ζ_0, z_0). Clearly the points $F_+(\zeta_n, z_n)$ have long backwards orbits $H_a^{\circ -m}(F_+(\zeta_n, z_n))$ contained in W^S, and the histories

$$\pi_{U'}(H_a^{\circ -m}(F_+(\zeta_n, z_n))$$

are close to that of ζ_0.

The result follows from Proposition 3.11, which says that these long backwards orbits restrict z_n to a small disk, and Proposition 3.14, which says that this disc is close to z_0.

□ Proposition 6.12

We can now complete the proof Theorem 6.2. Any point $((\zeta, z), n) \in \check{\mathbb{C}}_p$ has a forward image in $J_p \times D$, namely

$$((\zeta, z), 0) = \check{p}^{\circ n}((\zeta, z), n).$$

So simply define

$$\Phi_+((\zeta, z), n) = H_a^{\circ -n}\left(F_+(\zeta, z)\right).$$

This clearly defines an injective mapping $\Phi_+ : \check{\mathbb{C}}_p \to J_+$. It is surjective because $J_+ \cap N_\delta = W^S$ by Proposition 4.10. □Theorem 6.2

Finally, we will prove the following result, promised in the introduction.

Proposition 6.13. *For all sufficiently small α_1 and α_2 and for all sufficiently large R_1 and R_2, there is a homeomorphism*

$$\psi : J_p \times D_{R_1} \to J_p \times D_{R_2}$$

conjugating f_{p,α_1,R_1} to f_{p,α_2,R_2}.

Proof. This is a consequence of Theorem 6.2: We proved that $H_a : W^S \to W^S$ is conjugate to $f_{p,\alpha,R} : J_p \times D_R \to J_p \times D_R$ independent of α and R, so long as they are respectively sufficiently small and sufficiently large. This certainly shows that $f_{p,\alpha_1,R_1} : J_p \times D_{R_1} \to J_p \times D_{R_1}$ and $f_{p,\alpha_2,R_2} : J_p \times D_{R_2} \to J_p \times D_{R_2}$ are conjugate to each other under the same requirements.

It is also possible to prove this directly, by a proof very analogous to that of Theorem 6.2. But we do not think that there is a much easier proof. Indeed, each of $f_{p,\alpha_i,R_i} : J_p \times D_{R_i} \to J_p \times D_{R_i}$ has infinitely many cycles, each of which has multipliers in the vertical direction which depend on α_i. Thus any conjugating map cannot be differentiable, so cannot be "given by a formula", and some construction involving infinite processes must occur. □

Examples of $\hat{\mathbb{C}}_p$.

It is rather hard to find any particular category that $\hat{\mathbb{C}}_p$ belongs to in general. However, when p is hyperbolic, it is not too difficult to understand its structure. Except at the periodic histories corresponding to the attracting cycles, the canonical projection $\hat{\mathbb{C}}_p \to \mathbb{C}$ is a "ramified covering lamination", with fibers homeomorphic to Cantor sets, and ramified above the forward orbits of the critical points. The exceptional points have neighborhoods homeomorphic to cones over solenoids.

The following statement is a first step towards seeing this.

Proposition 7.1. *If $\Omega \subset \mathbb{C}$ is a subset which does not intersect the orbit of the critical point, then $\pi_\Omega : \hat{\Omega} \to \Omega$ is a locally trivial fibration with fiber a Cantor set.*

Proof. Let $\hat{z} = (\dots, z_{-2}, z_{-1}, z_0) \in \hat{\Omega}$ and choose a connected, simply connected neighborhood Ω' of z_0 in Ω. Then for each i there exists a branch g_i of $p^{\circ -i}$ defined on Ω' with $g_i(z_0) = z_{-i}$. Then the mapping

$$(\hat{z}, v) \mapsto (\dots, g_2(v), g_1(v), g_0(v))$$

is a homeomorphism $\pi_\Omega^{-1}(z_0) \times \Omega' \to \hat{\Omega}'$. $\square$

Recall from the introduction that a Riemann surface lamination is a Hausdorff space locally isomorphic to a product of a topological space with a Riemann surface. So long as a critical point is not periodic, $\hat{\mathbb{C}}_p$ is still a Riemann surface lamination above the orbit. The only problem is that the projection to $\mathbb{C}$ is ramified there. Indeed, let z be any point not on an attracting cycle, and choose k such that all the points $y \in p^{-k}(z)$ contain no post-critical points. If p is hyperbolic, then such a k exists unless z is a point of an attracting cycle, since the orbits of the critical points are all attracted to the attracting cycles. Choose a connected, simply-connected neighborhood Ω of z such that there are no post-critical points in $\Omega - \{z\}$.

Now consider the following commutative diagram.

$$\begin{tikzcd}
\hat{\Omega} \arrow[r, "\hat{p}^{\circ -k}"] \arrow[d, "\pi_\Omega"'] & \displaystyle\bigcup_{\text{components of } p^{-k}(\Omega)} \hat{\Omega}' \arrow[d, "\bigcup \pi_{\Omega'}"] \\
\Omega & \displaystyle\bigcup_{\text{components of } p^{-k}(\Omega)} \Omega' \arrow[l, "p^{\circ k}"']
\end{tikzcd}$$

The top mapping is an isomorphism in the category of Riemann surface laminations, so at any history of a point except an attracting periodic point, the set $\hat{\mathbb{C}}_p$ is a Riemann surface lamination. And since

$$\pi_\Omega = p^{\circ k} \circ \bigcup \pi_{\Omega'} \circ \hat{p}^{\circ -k},$$

we see that π_Ω is a "ramified covering mapping" of Riemann surface laminations, ramified no worse than $p^{\circ k}$.

The space $\hat{\mathbb{C}}_p$ is *not* a Riemann surface lamination at the periodic histories of attracting periodic points.

First, let us examine the case $p_0 : z \mapsto z^n$. In that case, $\hat{\mathbb{C}}_{p_0}$ is the cone over the solenoid $\Sigma_n = \varprojlim(S^1, z \mapsto z^n)$. Indeed, in polar coordinates p_0 decouples, and the history of the radius contains no more information than the radius, since every positive number has a unique positive nth root. In particular, $\hat{\mathbb{C}}_{p_0}$ is not a Riemann surface lamination near the cone-point $\underline{0} = (\dots, 0, 0)$.

Next, let p be any polynomial with an attractive cycle; we will see that $\hat{\mathbb{C}}_p$ is not much nastier than the case above. Since $\hat{\mathbb{C}}_p$ and $\hat{\mathbb{C}}_{p^{\circ k}}$ are canonically isomorphic for any $k \geq 1$, we may assume that z is an attracting fixed point of p. Let Ω be the component of $\mathbb{C} - J_p$ containing z; and denote by n the degree of p mapping Ω to itself.

Let $\underline{z} = (\dots, z, z)$ be the fixed point of $\hat{p}$ corresponding to z.

The space $\hat{\Omega}$ is connected only if p has an attractive fixed point which attracts all the critical points of p. In that case J_p is a Jordan curve bounding Ω, as in the case of p_0 above. In general, $\hat{\Omega}$ is not connected, and the components of $\hat{\Omega}$ are labeled by the totally disconnected set (projective limit of finite sets)

$$\varprojlim_m(\pi_0(p^{-m}(\Omega)), \pi_0(p)),$$

where $\pi_0(X)$ (the zeroth homotopy set) is the set of connected components of X, and $\pi_0(p) : \pi_0(p^{-m}(\Omega)) \to (p^{-m+1}(\Omega))$ is the map induced by p. The component to which $\underline{w} = (\dots, w_{-1}, w_0) \in \hat{\Omega}$ belongs is indexed by

$$(\dots, [w_{-1}], [w_0]),$$

where the $[w_{-i}]$ denotes the component of $p^{-i}(\Omega)$ containing w_{-i}. There is a distinguished component

$$\hat{\hat{\Omega}} = (\dots, [z], [z]).$$

of $\hat{\Omega}$.

The same argument as above show that the other components are Riemann surface laminations, but $\hat{\hat{\Omega}}$ is not; in fact it is a cone over a solenoid analogous to the case of p_0. To state this precisely, let $\hat{\mathbb{D}} \subset \hat{\mathbb{C}}_{p_0}$ be the component of $\hat{\mathbb{C}}_{p_0} - \hat{J}_{p_0}$ consisting of the histories of points in the unit disk ($\mathbb{D}$ is the unit disk).

Proposition 7.2. *There is a homeomorphism*

$$\psi : \hat{\hat{\Omega}} \to \widehat{\mathbb{D}}$$

conjugating $\hat{p}$ to $\hat{p}_0$.

Remarks 7.3. (1) In particular, $\underline{z}$ cannot have a neighborhood which is a Riemann surface lamination.

(2) We must have $\psi(\underline{z}) = \underline{0}$ since these are the unique fixed points of $\hat{p}$ in $\hat{\hat{\Omega}}$ and of $\hat{p}_0$ in $\hat{D}$.

Proof of Proposition 7.2. We will first construct the homeomorphism ψ near the boundary of Ω. Take the circle Γ of radius R centered at z in Ω, for the Poincaré metric of Ω, with R sufficiently large that the disk bounded by Γ contains all the critical values in Ω. Let $\Gamma' = p^{-1}(\Gamma) \cap \Omega$, and let A_0 be the closed annular region between Γ and Γ'. Similarly, let $B_0 = \left\{ w \in \mathbb{C} \;\middle|\; r \leq |w| \leq r^{1/n} \right\}$ for some $0 < r < 1$.

Lemma 7.4. *There exists a homeomorphism $\bar{\psi}_0 : A_0 \to B_0$ conjugating p to p_0 on the inner boundaries.*

The proof is left to the reader; the details are given in [**DH1**].

Next extend $\bar{\psi}_0$ to $A_1 = p^{-1}(A_0)$, $A_2 = p^{-1}(A_1)$, etc., until it is defined on the part $A = \cup A_i$ of Ω outside of Γ. These extensions exist by the lifting criterion for covering spaces. This extended $\bar{\psi}$ now defines a homeomorphism $\psi : \hat{A}_p \to \hat{B}_{p_0}$.

Now take any point $\underline{w} \in \hat{\hat{\Omega}}$. If $\underline{w} \neq \underline{z}$, then for some $m > 0$ we will have $\hat{p}^{\circ -m}(\underline{w}) \in \hat{A}$, so we can define

$$\psi(\underline{w}) = \hat{p}_0^m \circ \psi \circ \hat{p}^{\circ -m}(\underline{w}).$$

Since the construction can be made reversing the roles of p and p_0, the map ψ above is a homeomorphism. □ (Proposition 7.2)

Examples of $\check{\mathbb{C}}_p$.

We showed, in Proposition 4.4 of [**HO**], that the inductive limit of

$$f_{p_0} : S^1 \times D \to S^1 \times D, \quad f_{p_0}(\zeta, z) = (\zeta^d, \zeta - \alpha \frac{z}{\zeta^{d-1}})$$

is a 3-sphere with a solenoid removed, and we identified the map $\check{p}_0 : \check{\mathbb{C}}_{p_0} \to \check{\mathbb{C}}_{p_0}$ as the restriction of a certain map $\tau_{d,0}$ from the 3-sphere to itself with two invariant solenoids, one attracting and one repelling.

Moreover, we showed in Theorem 3.11 of [**HO**] that the conjugacy class of any injective mapping from the solid torus to itself, with appropriate contraction and expansion properties depended only on its homotopy class. Thus we obtain the following result:

Proposition 7.5. *If p is a polynomial with an attractive fixed point which attracts all the critical points of p, then $\check{\mathbb{C}}_p$ is a 3-sphere with a solenoid removed, and $\check{p}$ is conjugate to $\tau_{d,0}$.*

This gives quite a complete understanding of J_+ for the small perturbations of such polynomials. For other polynomials, even hyperbolic, the situation is more complicated. However, there still are 3-spheres contained in $\check{\mathbb{C}}_p$.

Let p be a hyperbolic polynomial with an attracting cycle; as above, by considering an iterate of p, we may assume that all attracting periodic points of p are fixed. Let z be such an attractive fixed point, and let Ω be the component of $\mathbb{C} - J_p$ containing z. Then $p\,|_\Omega \colon \Omega \to \Omega$ is a proper mapping of some degree k.

The boundary $\partial\Omega$ is a Jordan curve, which is mapped to itself with degree k, and there exist (exactly $k-1$) homeomorphisms $\gamma_\Omega : S^1 \to \partial\Omega$ such that

$$\gamma(z^k) = p(\gamma(z)).$$

Call $f_{\gamma_\Omega} : S^1 \times D \to S^1 \times D$ the mapping

$$(\zeta, z) \mapsto \left(\zeta^k, \gamma_\Omega(\zeta) - \alpha \frac{z}{p'(\gamma_\Omega(\zeta))}\right)$$

where α is chosen as in Lemma 1.2.

Consider now the diagram

$$\begin{array}{ccccccc}
J_p \times D & \xrightarrow{f_p} & J_p \times D & \xrightarrow{f_p} & J_p \times D & \xrightarrow{f_p} & \dots \\
{\scriptstyle \gamma_\Omega \times \mathrm{id}}\big\uparrow & & {\scriptstyle \gamma_\Omega \times \mathrm{id}}\big\uparrow & & {\scriptstyle \gamma_\Omega \times \mathrm{id}}\big\uparrow & & \\
S^1 \times D & \xrightarrow{f_{\gamma_\Omega}} & S^1 \times D & \xrightarrow{f_{\gamma_\Omega}} & S^1 \times D & \xrightarrow{f_{\gamma_\Omega}} & \dots
\end{array}$$

Clearly this leads to an embedding

$$\varinjlim (S^1 \times D, f_{\gamma_\Omega}) \hookrightarrow \check{\mathbb{C}}_p.$$

Using Theorem 3.11 and Proposition 4.4 of [**HO**], we can easily see that

$$\varinjlim (S^1 \times D, f_{\gamma_\Omega})$$

is homeomorphic to the 3-sphere with a solenoid removed; call $(\partial\Omega)\check{}$ its image in $\check{\mathbb{C}}_p$. Just as $\partial\Omega$ is the boundary of the immediate basin of attraction of an attracting fixed point for the polynomial p, we would like to say that $\Phi_+((\partial\Omega)\check{})$ is the boundary of the basin of attraction of the corresponding attracting fixed point of H_a. This is nonsense: the basin of attraction is dense in J_+, and the boundary is all of J_+. This has been proved in full generality by Bedford and Smillie [**BS2**], and independently by Fornaess and Sibony [**FS**], and can easily be verified directly in our case.

But there is a refinement of the notion of boundary: the *accessible boundary*. Suppose U is an open subset of a topological space X. Then the accessible boundary of U in X is the set of $x \in X - U$ for which there exists a continuous mapping $\eta : [0,1] \to X$ such a $\eta(0) = x$ and $\eta(0,1] \subset U$.

Example 7.6. Let $X = [0, 1]$ and $U = X - C$, where C is the standard Cantor set. Of course, U is dense, so the boundary of U is all of C. But the accessible boundary is just the set of endpoints of intervals of U: any mapping η as above must have image in a single interval. These two "boundaries" are very different: for instance, the accessible boundary is countable and C is uncountable.

Theorem 7.7. *If $|a|$ is small as in the Requirements of Section 4, then $H_{p,a}$ has an attractive fixed point $z(a)$ corresponding to z, and the accessible boundary of its basin is $\Phi_+((\partial\Omega)\breve{})$.*

Proof. In N_δ there is a projection map $\pi : W^S \to J_p$. Our analysis in Sections 4 and 6 show that $\pi^{-1}(\partial\Omega)$ is the boundary of the immediate domain of $z(a)$ in N_δ, and this agrees with the accessible boundary of that component. But $\pi^{-1}(\partial\Omega)$ is the first stage in the construction of $\Phi_+((\partial\Omega)\breve{})$.

Now let $\eta : [0, 1] \to K_+$ be an access to a point $(x, y) = \eta(0) \in J_+$ with $\eta(0, 1]$ in the basin of $z(a)$. Since the image of η is compact, some forward image will be in $\mathbb{N}_\delta$, and some further forward image will be in the component basin in N_δ containing $z(a)$, hence in $\Phi_+((\partial\Omega)\breve{})$. But this set is stable under $H_{p,a}$, so $(x, y) \in \Phi_+((\partial\Omega)\breve{})$. $\square$

We will now describe what we know of the algebraic topology of $\check{\mathbb{C}}_p$. This is quite difficult when J_p is not a Jordan curve. For spaces like these, which are not locally contractible, the only really well behaved theory is Čech cohomology; unfortunately, we will see in Theorem 7.11 that $\check{H}^1(\check{\mathbb{C}}_p, G) = 0$ for all coefficients G, and Čech cohomology carries no information. Thus we are forced to consider homology; and there does not appear to be any really good homology theory. We will use singular homology "faute de mieux"; it may be that Čech or Steenrod homology are better behaved.

Example 7.8. Consider the "Hawaiian earring" space

$$X = \bigcup_{k \in \mathbb{N}} \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; x^2 + \left(y + \frac{1}{k}\right)^2 = \frac{1}{k^2} \right\}.$$

There is a canonical mapping $H_1(X, \mathbb{Z}) \to \mathbb{Z}^{\mathbb{N}}$ which associates to a cycle α the sequence $\alpha(n), n \in \mathbb{N}$ where $\alpha(n)$ is the number of times α turns around the nth circle. It is easy to show that the mapping is surjective, but it is apparently not injective. At least, we do not see how a loop

$$e_1 e_2 \ldots e_1^{-1} e_2^{-1} \ldots$$

in the fundamental group can be written as a commutator.

Suppose that p is hyperbolic, that J_p is connected, and that all the attractive cycles are fixed. Call $\mathcal{X} = \pi_0(\overset{\circ}{K}_p)$, and $p_* : \mathcal{X} \to \mathcal{X}$ the map induced by p. Let $\mathcal{X}_0 \subset \mathcal{X}$ be the

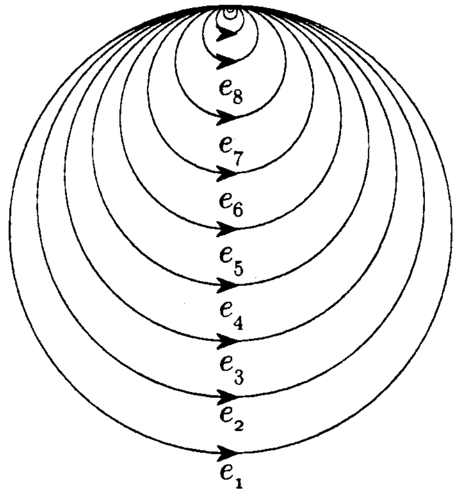

Figure 7.9: The Hawaiian earring

finite subset of components containing attractive fixed points. For each $X \in \mathcal{X}$ the integer $k(X)$ is the degree of p restricted to the component X.

The space J_p is very much like the Hawaiian earring, and there is an analogous mapping $H_1(J_p, \mathbb{Z}) \to \mathbb{Z}^X$ which is surjective but probably not injective; nevertheless we consider the kernel as pathological.

Recall from Corollary 4.5 of [**HO**] that when p is of degree d and J_p is a Jordan curve, so that $\mathcal{X} = \mathcal{X}_0$ has a single element, then $\check{\mathbb{C}}_p$ is homeomorphic to the complement of a solenoid in S^3, and that

$$H_1(\check{\mathbb{C}}_p, \mathbb{Z}) = \mathbb{Z}\left[\frac{1}{d}\right].$$

The same holds for the spaces $(\partial X)\check{}$ for all $X \in \mathcal{X}_0$, with d replaced by $k(X)$.

Theorem 7.10. *The inclusion*

$$\bigcup_{X \in \mathcal{X}_0} (\partial X)\check{} \to \check{\mathbb{C}}_p$$

induces a split injection

$$\bigoplus_{X \in \mathcal{X}_0} \mathbb{Z}\left[\frac{1}{k(X)}\right] \to H_1(\check{\mathbb{C}}_p, \mathbb{Z}).$$

This mapping is not surjective.

Proof. Consider the following diagram.

$$\begin{tikzcd}
H_1(J_p \times D) \arrow[rr, "(f_p)_*"] \arrow[dd] & & H_1(J_p \times D) \arrow[rr] \arrow[dd] & & \cdots \\
& \bigoplus_{X\in\mathcal{X}_0} H_1(S^1 \times D) \arrow[ul] \arrow[rr, "\oplus (f_{\gamma X})_*"] \arrow[dd, "\cong"] & & \bigoplus_{X\in\mathcal{X}_0} H_1(S^1 \times D) \arrow[ul] \arrow[rr] \arrow[dd, "\cong"] & & \cdots \\
\mathbb{Z}^X \arrow[rr, "p_*"] & & \mathbb{Z}^X \arrow[rr] & & \cdots \\
& \mathbb{Z}^{\mathcal{X}_0} \arrow[ul] \arrow[rr, "p_*"'] & & \mathbb{Z}^{\mathcal{X}_0} \arrow[ul] \arrow[rr] & & \cdots
\end{tikzcd}$$

For each face

$$\begin{array}{ccc}
\bigoplus_{X\in\mathcal{X}_0} H_1(S_1 \times D) & \xrightarrow{\bigoplus (f_{\gamma_X})_*} & H_1(J_p \times D) \\
\downarrow & & \downarrow \\
\mathbb{Z}^{\mathcal{X}_0} & \longrightarrow & \mathbb{Z}^X
\end{array}$$

the horizontal mappings are split injections. This is obvious for the bottom mapping, and for the top mapping the inclusion

$$J_p \to J_p \cup \bigcup_{X\in\mathcal{X}-\mathcal{X}_0} X$$

induces a splitting on the homology. The theorem follows by passing to the direct limit, since the direct limit is an exact functor, and the homology of the direct limit is the direct limit of the homology. $\square$

Finally, why is Čech cohomology useless in this setting?

Theorem 7.11. *If J_p is not a Jordan curve, all covering spaces of $\check{\mathbb{C}}_p$ are trivial.*

Remark 7.12. It might seem that this is just another way of saying that $\check{\mathbb{C}}_p$ is simply connected, which evidently contradicts Theorem 7.10. To resolve the apparent contradiction, notice that $\check{\mathbb{C}}_p$ is not locally simply connected, so it doesn't have a universal covering space, and the "singular" fundamental group defined using loops does not classify covers. On the other hand, its abelianization is the singular homology, so this fundamental group is enormous.

Proof of Theorem 7.11. A covering $Z \to \check{\mathbb{C}}_p$ restricts to covers $Z_i \to (J_p \times D) \times \{i\}$, together with inclusions $Z_i \subset Z_{i+1}$ such that the following diagram commutes.

$$\begin{array}{ccccc}
Z_0 & \longrightarrow & Z_1 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \\
(J_p \times D) \times \{0\} & \xrightarrow[f_p]{} & (J_p \times D) \times \{1\} & \xrightarrow[f_p]{} & \cdots
\end{array}$$

Of course, the cover Z_i of $J_p \times D$ restricts to a cover Y_i of J_p, and the diagram above gives covering homeomorphisms $\alpha_i : Y_i \to p^*Y_{i+1}$.

$$\begin{tikzcd}
Y_0 \arrow[rr, "\alpha_0", "\approx"'] \arrow[dr] & & p^*Y_1 \arrow[r] \arrow[dl] & Y_1 \arrow[rr, "\alpha_1", "\approx"'] \arrow[dr] & & p^*Y_2 \arrow[r] \arrow[dl] & \cdots \\
& J_p \arrow[rrr, "p"'] & & & J_p \arrow[rr] & & \cdots
\end{tikzcd}$$

Now consider the cover Y_0: it is ramified at most over finitely many of the Jordan curves $\partial X, X \in \mathcal{X}$. Indeed, you can find a cover $\mathcal{U} = (U_j)$ of J_p such that over every U_j,Y_0 is trivial, and since J_p is compact, we may take the cover finite. There then exists a number δ for every $\zeta \in J_p$, the δ-ball around ζ is contained in one U_j. Since all but finitely many $X \in \mathcal{X}$ have diameter smaller than δ, Y_0 is trivial over the boundaries of such components X.

Denote by $\mathcal{X}' \subset \mathcal{X}$ the set of components X such that Y_0 is trivial over ∂X; notice that there exists k_0 such that $p^{\circ k}(X') = X$ for all $k \geq k_0$.

Choose $k \geq k_0$, any component $X \in \mathcal{X}$, and $X' \in \mathcal{X}'$ such that $p^{\circ k}(X') = X$. If Y is ramified above ∂X, then $Y_0 \cong (p^{\circ k})^* Y_k$ is ramified above $\partial X'$. Since this is not the case, all Y_k are unramified above all $\partial X, X \in \mathcal{X}$. But this implies that Y_k is trivial for all k, hence that all Z_k are trivial, hence that Z is trivial. $\square$ (Theorem 7.11)

Remark 7.13. This proof actually shows more than we claimed: it show that all principal G-bundles over $\check{J}_p$ are trivial. Since the Čech cohomology $\check{H}^1(\check{\mathbb{C}}_p, G)$ classifies such principal bundles, this shows that $\check{H}^1(\check{\mathbb{C}}_p, G) = 0$ for all coefficient groups G (even non-abelian, if you know how to define such things).

It certainly seems remarkable that singular homology is picking up so much more than Čech cohomology; one might expect the opposite. The following example should help to explain how this happens, as well as give some insight into how J_+ is made.

Example 7.14. Consider a tube $[0, \infty) \times S^1$ embedded in $\mathbb{R}^3$ so that it spirals onto a circle, as suggested in Figure 7.15.

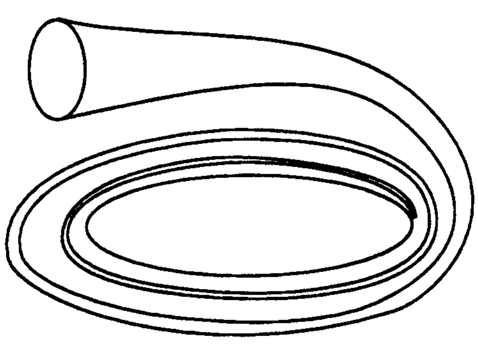

Figure 7.15: How a non-trivial circle may support no non-trivial covers

Let C be the union of the circle and the tube. Then the singular homology is $H_1(C, \mathbb{Z}) = \mathbb{Z}^2$, generated by the circle and the boundary of the tube. But no covering can be non-trivial over the boundary of the tube, because it would then also be non-trivial over tiny cross-sections of the tube near the circle, and such cross-sections will be contained within a single open set of any open finite cover.

This is the way $\check{\mathbb{C}}_p$ is made. There are big Jordan curves in J_p, but there are tubes in $\check{\mathbb{C}}_p$ joining them to all their inverse images in J_p, which become arbitrarily small.

8. Lakes of Wada in Dynamical Systems

A famous example in plane topology, due to Wada, is that there exist three bounded, connected and simply connected open sets in $\mathbb{R}^2$ such that $\partial U_1 = \partial U_2 = \partial U_3$. We wish to show that under appropriate circumstances the components of the basin of attraction of an attractive cycle for a Hénon mapping will form Lakes of Wada [**Y**].

The classical construction of Lakes of Wada illustrates the perils of philanthropy. Consider a circular island, inhabited, to the sorrow of the others, by three philanthropists. One has a lake of water, another of milk and a third of wine. The first, in a fit of generosity, decides to build a network of canals bringing water within 100 meters of every spot of the island. It is clearly possible to do this keeping the union of the original water lake and the water canals connected and simply connected, with closures disjoint from the other lakes.

Next the second, perhaps worried about child nutrition, decides to bring milk to within 10 m of every spot on the island, and builds a system of canals to that effect. She also keeps her milk locus connected and simply connected.

Not to be outdone, the purveyor of wine now decides to bring wine to within 1 m of every spot on the island. He finds his canal building rather more of an effort than the previous two, but being properly fortified, he carries it out.

In turn, each of the three philanthropists brings his or her product closer to the poor inhabitants. It should be clear that the construction can be continued, and that in the limit the construction achieves the desired result: each of the lakes, being an increasing union of connected, simply connected open sets, is a connected, simply connected set, and each point of the boundary of one is in the boundary of the other two.

We will show that under appropriate circumstances, the basins of attraction of attracting cycles form Lakes of Wada for Hénon mappings in $\mathbb{R}^2$. As it turns out, the "strategy" of these basins is remarkably similar to that of the philanthropists.

More specifically, we will work with *dense polynomials*. Let p be a real hyperbolic polynomial with connected Julia set, and suppose all the attracting cycles of $p^{\circ k}$ are real fixed points. We will say that p is dense if for each such fixed point x, its real domain of attraction $\Omega_x \cap \mathbb{R}$ is dense in $J_p \cap \mathbb{R}$.

There are lots of dense polynomials. The following lemma describes some of them in degree 2. We have found this lemma to be harder to prove than we had expected.

Lemma 8.1. *Let p be a real quadratic polynomial with an attracting cycle of period k, with k an odd prime. Then the k basins $U_1 = 0, \dots, U_{k-1}$ of the attracting fixed points of $p^{\circ k}$ in $\mathbb{R}$ are all dense in $J_p \cap \mathbb{R}$.*

Proof. Denote by I_0 the largest bounded interval invariant under the polynomial; it is bounded by the "external" fixed point and its inverse image. Without loss of generality we may assume that the critical point is periodic of period k; let $c_0, c_1, \dots, c_{k-1}, c_k = c_0$ be the critical orbit; all the interesting dynamics occurs in the interval $I = [c_1, c_2] \subset I_0$.

The polynomial p also has an "internal" fixed point $\alpha \in [c_0, c_1]$. If $J \subset I$ is any interval containing α, then $\cup p^{\circ n}(J) = I$. The alternative is that $\cup p^{\circ n}(J) = J_0$ is an interval in $[c_0, c_1]$ bounded by a cycle of period 2, and there are no such cycles in $[c_0, c_1]$ (here we are using that p is a polynomial, not just a unimodal map). It follows from this that each of the basins U_i accumulates at α.

Thus to prove the lemma, it is enough to show that the real inverse images of α are dense in the real Julia set $J_p \cap \mathbb{R}$. Let us denote by $V_0, \dots, V_{k-1}$ the immediate domains of attraction in $\mathbb{R}$. It is known that if k is an odd prime (or more generally simply odd) the V_i have disjoint closures; let $\mathcal{T} = \{T_1, \dots, T_{k-1}\}$ be the bounded components of $I - \cup V_i$.

Sublemma 8.2. *If there is an inverse image of α in each T_j, then p is a dense polynomial.*

Proof of 8.2. The Julia set is

$$J_p \cap \mathbb{R} = I_0 - \bigcup_{i=0}^{k-1} \bigcup_{n=0}^{\infty} p^{-n}(V_i).$$

If each component of

$$X_M = I_0 - \bigcup_{i=0}^{k-1} \bigcup_{n=0}^{M} p^{-n}(V_i)$$

contains an inverse image of α, then these inverse images will accumulate on all of $J_p \cap \mathbb{R}$. But if each component of X_M contains an inverse image of α, then this is also true of each component of X_{M+1}, since p maps each component of X_{M+1} to a component of X_M. Thus it is enough to start the induction, which is the hypothesis of the sublemma.

□ Sublemma 8.2

There is a repelling cycle Z of length k such that all endpoints of intervals $T \in \mathcal{T}$ are either in Z or in its inverse images. Let us denote $\mathcal{T}'$ those intervals for which at least one end-point is periodic, and $\mathcal{T}''$ the others. Moreover set

$$A = \bigcup_{T \in \mathcal{T}'} \bigcup_{n=0}^{\infty} p^n(T).$$

Now there are two possibilities:

(a) If $\alpha \in A$, there is an inverse image of α in some $T' \in \mathcal{T}'$. But then there must be an inverse image of α in every $T \in \mathcal{T}$, since each endpoint of T will eventually land on every point of Z, in particular on an end-point of T'; that iterate of T will cover T'. Then by sublemma 8.2, p is dense.

(b) If $\alpha \notin A$, then A is disconnected, and p permutes the components of A circularly, with period k' with $1 < k' < k$. This is because some interval $T \in \mathcal{T}'$ must have both endpoints in Z, as there is one more point in Z than there are intervals in $\mathcal{T}$. That interval must return to itself in fewer that k moves. Moreover k' divides k, since the map $Z \to \pi_0(A)$ is equivariant, *i.e.*, the following diagram commutes.

$$\begin{tikzcd} Z \arrow[r, "\psi"] \arrow[d, "p"'] & \pi_0(A) \arrow[d, "\pi_0(p)"] \\ Z \arrow[r, "\psi"'] & \pi_0(A) \end{tikzcd}$$

This cannot happen if k is prime. □ (Lemma 8.1)

Figure 8.3 should illustrate what is going on.

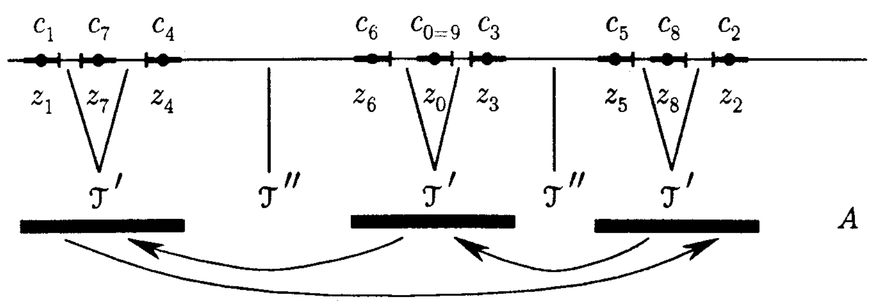

Figure 8.3: The polynomial $z^2 - 1.785866\ldots$, with an attractive cycle of length 9

For this polynomial, the critical point is periodic of period 9. We have used heavy lines to indicate the immediate basin, and the line segments pointing down form the repelling cycle $Z = \{z_0, \ldots, z_8\}$. The 8 intervals forming $\mathcal{T}$ break up into 6 in $\mathcal{T}'$, and two in $\mathcal{T}''$. The forward images of the intervals in $\mathcal{T}'$ form the set A which consists of 3 intervals which are permuted circularly. The point α is not in A, and this polynomial is not dense.

Remark 8.4. The proof above should shows that if a hyperbolic polynomial is not dense, then it is renormalizable in an appropriate sense. We could get necessary and sufficient conditions for a quadratic polynomial to be dense by pushing the argument a bit further.

Theorem 8.5. *If p is a dense polynomial and if $|a|$ is sufficiently small, then the Hénon mapping $H_{p,a}$ has attractive cycles close to those of p, and the boundaries of all the components of the basins coincide.*

Remark 8.6. General theorems of Bedford and Smillie [**BS3**], and independently by Sibony and Fornaess [**FS**], assert that for any saddle point of a Hénon mapping (and many other mappings besides), the stable manifold is dense in J_+. We will use an analogous statement, in the much more restricted class of mappings to which Theorem 6.3 applies. But Theorem 8.5 does not immediately follow from this density argument. For instance, the mapping

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x^2 - 1.05 - .38y \\ x \end{bmatrix}$$

has an attractive cycle of period 3 (as well as an attractive fixed point), and the basin of this cycle is bounded by the stable manifold of a cycle of period 3 which is a saddle. Of course, in $\mathbb{C}^2$, each path component of this stable manifold is dense in J_+, and in particular each path component accumulates onto the others. But not in $\mathbb{R}^2$: in the real, each of these path components accumulates exactly on the stable manifold of the saddle fixed point.

Proof. The proof is contained in Lemma 6.3, Proposition 6.1 and Proposition 7.7. Let us review how these fit together to give the result.

Notice that the proof of Lemma 6.3 is valid over the reals. Thus for $|a|$ sufficiently small, $\Phi_+ : \check{\mathbb{R}}_p \to J_+ \cap \mathbb{R}^2$ is a homeomorphism, where the space

$$\check{\mathbb{R}}_p = \varinjlim \left((J_p \cap \mathbb{R}) \times I, f_p \mid_{(J_p \cap \mathbb{R}) \times I} \right)$$

is obtained by the same inductive limit construction as in the complex. Figures 8.7, 8.8, 8.9and 8.10 illustrate this construction.

Moreover, Proposition 7.7 is also valid over the reals: if x is a fixed point of $p^{\circ k}$ with immediate basin Ω, the accessible boundary of each basin is

$$(\partial(\Omega \cap \mathbb{R}))\check{} = \varinjlim (\partial\Omega \times I, f_p^{\circ k}).$$

But $\Omega \cap \mathbb{R}$ is an interval, bounded by a repelling fixed point ξ of $p^{\circ k}$ and one of its inverse images ξ'. As such, the inductive limit above is a real line, which maps by Φ_+ to the the stable manifold of the fixed point $\xi(a)$ of $H_{p^{\circ k},a}$. Thus we understand exactly what the accessible boundary of each basin is, and what its inverse image by Φ_+ is. So far, none of this required that p be dense.

If p is dense, then every point of $J_p \cap \mathbb{R}$ can be approximated by inverse images $\xi_n \in p^{-nk}(\xi)$; the curves $\pi_{U'}^{-1}(\xi_n)$ are then part of $(\partial(\Omega \cap \mathbb{R}))\check{}$, by the argument of Proposition 6.1. Thus $(\partial(\Omega \cap \mathbb{R}))\check{}$ is dense in $(J_p \times I) \times \{0\}$, the first term in the inductive limit defining

$\check{\mathbb{R}}_p$, and by the argument of 6.1, this shows it is dense in all of $\check{\mathbb{R}}_p$. Thus the accessible boundary of each basin is dense in $J_+ \cap \mathbb{R}^2$, so they do have common boundary. □

The following pictures carry out the construction of $\check{\mathbb{R}}_p$ for p a real quadratic polynomial with an attractive cycle of period 3. It is of course easy to imagine the first step of the construction $(J_p \cap \mathbb{R}) \times I$, which is a product of a Cantor set by an interval.

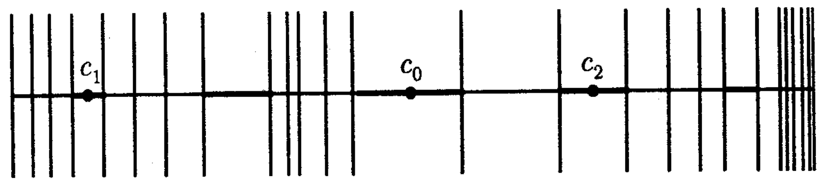

Figure 8.7: The set $(J_p \cap \mathbb{R}) \times I$; the first step in the construction

We have drawn a few genuine points of the Cantor set, and others "impressionistically".

How should we imagine the inclusion

$$((J_p \cap \mathbb{R}) \times I) \times \{0\} \hookrightarrow ((J_p \cap \mathbb{R}) \times I) \times \{1\}?$$

Note f_p maps the two intervals through the endpoints of the immediate basin of c_0 to two disjoint subintervals in the interval through the right endpoint of the immediate basin of c_1. Note also that the $p'(\zeta)$ in the denominator in the definition of f_p is essential for the orientations to be as indicated by the arrows in Figure 8.8.

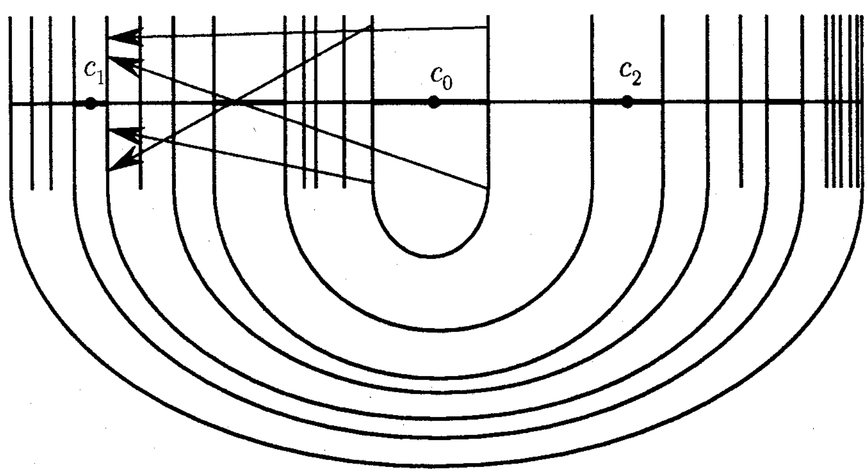

Figure 8.8: The set $((J_p \cap \mathbb{R}) \times I) \times \{1\}$; the second step in the construction

Thus in $((J_p \cap \mathbb{R}) \times I) \times \{1\}$ there must be an arc joining the two intervals above, so that these intervals and the arc will map to the interval where the arrows end. Similarly one sees that there must be an arc joining every pair of symmetric intervals.

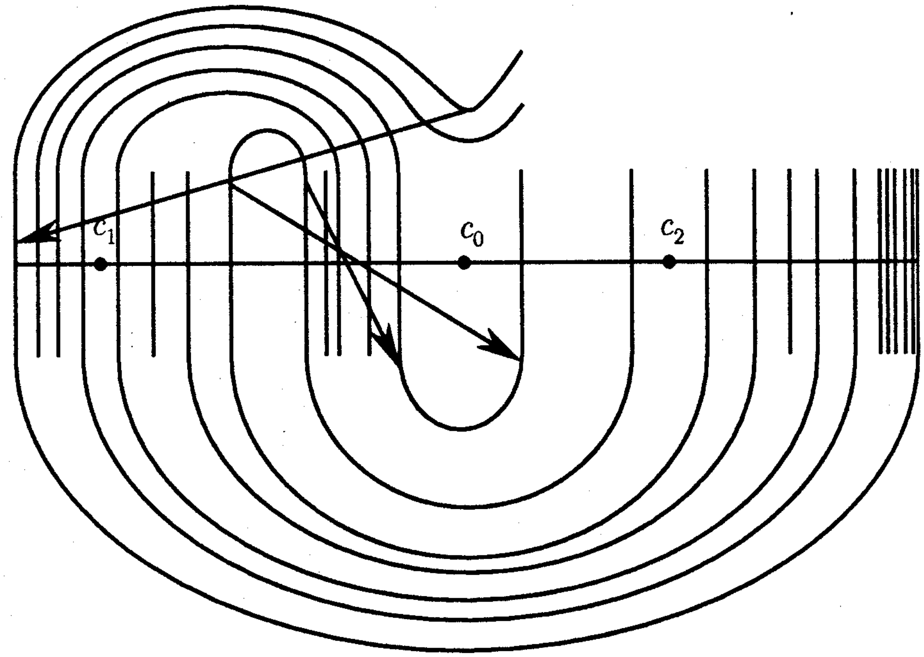

Figure 8.9: The set $((J_p \cap \mathbb{R}) \times I) \times \{2\}$; the third step in the construction

Figure 8.8 illustrates this construction. How should we continue the construction? In $((J_p \cap \mathbb{R}) \times I) \times \{1\}$ we need inverse images of the arcs added in the previous step; Figure 8.9 illustrates how this is to be done. Note that this time some of these arcs do not join intervals to intervals. This is because points to the left of c_1 have no inverse images in the Cantor set $J_p \cap \mathbb{R}$.

Making these pictures is a bit addictive, and if one gets carried away, the result may look like Figure 8.10.

Exercise for the Continuum Theorists. Let p be a dense quadratic polynomial, and denote by $X_{p,a}$ the one point compactification of $J_+ \cap \mathbb{R}^2$. Since this is naturally a subset of the sphere S^2, we can apply Alexander duality, to get

$$\check{H}^1(X_{p,a}, \mathbb{Z}) = \overline{H}^0(S^2 - X, \mathbb{Z}) \cong \mathbb{Z}^k$$

if p has an attracting cycle of period k. So for different k these sets are certainly not homeomorphic. What happens when p_1 and p_2 have the same period, but belong to different components of the Mandelbrot set? The first interesting case is $k = 5$, where there are 3 different candidates. We would speculate that the space $X_{p,a}$ are not homeomorphic, but it isn't clear what topological invariant distinguishes between them. Perhaps something could be made from the fact that in the construction of the basins, the sequence of lefts and rights which the "dead branches" make when they leave the "main branch" is different for these three polynomials, essentially reflecting the kneading sequences.

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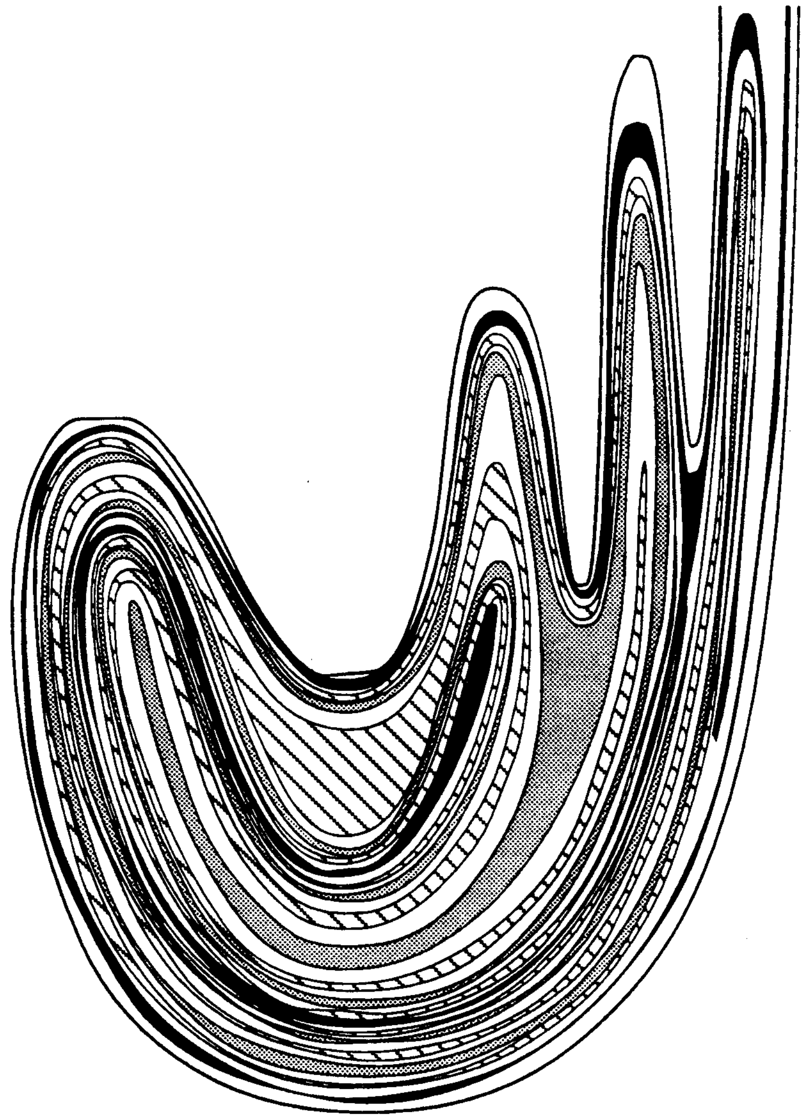

Figure 8.10: How the basins of the attractive cycle fit to form Lakes of Wada

Giarrusso, J. Mayer, C. McMullen, J. Milnor, J. Rogers, N. Sibony, J. Smillie, D. Sullivan, and J.-C. Yoccoz. We thank H. Smith for much computer experimentation over the years.

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