The Geometry of Symplectic Energy

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Introduction

One of the most striking early results in symplectic topology is Gromov's "Non-Squeezing Theorem" which says that it is impossible to embed a large ball symplectically into a thin cylinder of the form $\mathbf{R}^{2n} \times B^2$, where B^2 is a 2-disc. This led to Hofer's discovery of symplectic capacities, which give a way of measuring the size of subsets in symplectic manifolds. Recently, Hofer found a way to measure the size (or energy) of symplectic diffeomorphisms by looking at the total variation of their generating Hamiltonians. This gives rise to a bi-invariant (pseudo-)norm on the group $\mathrm{Ham}(M)$ of compactly supported Hamiltonian symplectomorphisms of the manifold M. The deep fact is that this pseudo-norm is a norm; in other words, the only symplectomorphism on M with zero energy is the identity map. Up to now, this had been proved only for sufficiently nice symplectic manifolds, and by rather complicated analytic arguments.

In this paper we consider a more geometric version of this energy, which was first considered by Eliashberg and Hofer in connection with their study of the extent to which the interior of a region in a symplectic manifold determines its boundary. We prove, by a simple geometric argument, that both versions of energy give rise to genuine norms on all symplectic manifolds. Roughly speaking, we show that if there were a symplectomorphism of M

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which had “too little” energy, one could embed a large ball into a thin cylinder $M \times B^2$. Thus there is a direct geometric relation between symplectic rigidity and energy.

The second half of the paper is devoted to a proof of the Non-Squeezing theorem for an arbitrary manifold M. We do not need to restrict to manifolds in which the theory of pseudo-holomorphic curves behaves well. This is of interest since most other deep results in symplectic topology are generalised from Euclidean space to other manifolds by using this theory, and hence are still not known to be valid for arbitrary symplectic manifolds.

1 The Main Results

In [6], Hofer defined the energy $\|\phi\|_H$ of a compactly supported Hamiltonian diffeomorphism $\phi : (M, \omega) \to (M, \omega)$ as follows:

$$\|\phi\|_H = \inf{}_H (\sup_{x,t} H(x,t) - \inf_{x,t} H(x,t)),$$

where $(x,t) \in M \times [0,1]$ and H ranges over the set of all compactly supported Hamiltonian functions $H : M \times [0,1] \to \mathbf{R}$ whose symplectic gradient vector fields generate a time 1 map equal to ϕ. It is easy to check that, for all ϕ, ψ,

- $\|\phi\|_H = \|\phi^{-1}\|_H$;
- $\|\phi \circ \psi\|_H \le \|\phi\|_H + \|\psi\|_H$; and
- $\|\psi^{-1} \circ \phi \circ \psi\|_H = \|\phi\|_H$.

Thus $\|\cdot\|_H$ is a symmetric and conjugation-invariant semi-norm on the group $\mathrm{Ham}(M)$ of all compactly supported Hamiltonian diffeomorphisms of M, and it follows that the associated function ρ_H given by:

$$\rho_H(\phi, \psi) = \|\phi^{-1}\psi\|_H,$$

is a bi-invariant pseudo-metric. However, it is harder to show that $\|\cdot\|_H$ is a norm, or, equivalently, that ρ_H is a metric. Hofer established this when M is the standard Euclidean space, using quite complicated analytical arguments. This norm is still rather little understood. A good introduction to its properties may be found in [7, 8].

In this paper, we will consider the generalized Hofer semi-norm $\|\cdot\|$ which is defined as follows. Let M be a symplectic manifold of dimension $2n$. If $\partial M \neq \emptyset$, define $\mathrm{Ham}(M)$ as the group of all compactly supported Hamiltonian diffeomorphisms which are the identity near the boundary. Consider embeddings Φ of the strip $M \times [0,1]$ in the product manifold $(M \times [0,1] \times \mathbf{R}, \omega + dt \wedge dz)$ which are trivial, i.e. equal to $(x,t) \mapsto (x,t,0)$, for t near 0 and 1 and for x outside some compact subset of $\mathrm{Int}\, M$, and are such that all leaves of the characteristic foliation on the hypersurface $Q = \mathrm{Im}\, \Phi$ beginning on $M \times \{(0,0)\}$ go through the hypersurface and reach $M \times \{(1,0)\}$. The induced diffeomorphism ϕ from $M = M \times \{(0,0)\}$ to $M = M \times \{(1,0)\}$ is the **monodromy** of Q. Further, the **energy** of Q is defined to be the minimum length of an interval I such that Q is a subset of the product $M \times [0,1] \times I$.

We define $\|\phi\|$ to be the infimum of the energy of all hypersurfaces Q with monodromy ϕ. Since ϕ^{-1} is the monodromy of the hypersurface Q when read in the opposite direction, $\|\cdot\|$ is symmetric. Further, because the time 1 map of the isotopy generated by the function $H(x,t)$ is exactly equal to the monodromy of the embedding:

$$(x,t) \mapsto (x,t,-H(x,t)),$$

we find that

$$\|\phi\| \leq \|\phi\|_H$$

for all $\phi \in \mathrm{Ham}(M)$. This semi-norm was first considered in [1]. It is relevant, for instance, when one is trying to understand the extent to which the boundary of a region is determined by its interior, since the boundary can always be C^0-approximated by a sequence of hypersurfaces lying inside the region: see [2]. Note that the two norms defined here might coincide, since no example is yet known where they differ.

As in Hofer's proof of the non-degeneracy of $\|\cdot\|_H$ in $\mathbf{R}^{2n}$, we will prove that $\|\cdot\|$ is a norm on $\mathrm{Ham}(M)$ by establishing an energy-capacity inequality which gives a lower bound for the disjunction energy of a subset in terms of its capacity. Since all our arguments will rely on properties of embedded balls, the appropriate capacity to use in the present context is Gromov's radius c. Thus for any subset $A \subset M$, we define

$$c(A) = \sup\{u : \text{there is a symplectic embedding } B^{2n}(u) \hookrightarrow \mathrm{Int}\, A\}.$$

Here, we use the notation $B^{2n}(u)$ to denote the standard ball in standard Euclidean space $(\mathbf{R}^{2n}, \omega_0)$ of capacity u and radius $\sqrt{u/\pi}$. Thus the capacity of a ball of radius r is πr^2. In order to distinguish the standard balls in $\mathbf{R}^{2n}$ from their images in M, we will reserve the dimensional upperscript to the former only.

The **disjunction** (or **displacement**) **energy** of $A \subset \operatorname{Int} M$ is defined to be:

$$\mathbf{e}(A) = \inf\{\|\phi\| : \phi \in \operatorname{Ham}(M),\ \phi(A) \cap A = \emptyset\}.$$

We will also need to consider maps ϕ which not only disjoin A, but also move A to a new position which is sufficently separated from the old one. This gives us the notion of **proper disjunction energy**. This is easiest to define for balls. A disjunction ϕ of $B(c)$ is said to be **proper** if (some parametrization of) $B(c)$ extends to a ball $B(2c)$ such that $\phi(B(c)) \cap B(2c) = \emptyset$, and the **proper disjunction energy** $\mathbf{e_p}(B(c))$ is the infimum of the energies of all proper disjunctions of $B(c)$. Similarly, ϕ is said to be a proper disjunction of A if each ball $B(c) \subset A$ may be extended to a ball $B(2c)$ such that

$$\phi(A) \cap (A \cup B(2c)) = \emptyset,$$

and the proper disjunction energy $\mathbf{e_p}(A)$ is the minimum energy of such a ϕ. ΠΠΠAs usual, if there are no (proper) disjunctions of A in M, we define its (proper) disjunction energy to be infinite.

Our main result is:

Theorem 1.1 *Let (M, ω) be any symplectic manifold, and A any compact subset of* $\operatorname{Int} M$*. Then*

(i) $\quad \mathbf{e_p}(A) \geq c(A), \quad$ *and*

(ii) $\quad \mathbf{e}(A) \geq \frac{1}{2} c(A).$

Corollary 1.2 *For any symplectic manifold M, $\|\cdot\|$ is a (non-degenerate) norm on Ham(M). Hence $\|\cdot\|_H$ is also non-degenerate.*

Remark 1.3 **(i)** It is very easy to see that the disjunction energy of a ball in $\mathbf{R}^{2n}$ is exactly equal to its capacity. Indeed, when $n = 1$, an open ball can be identified with a square and then disjoined by a translation of energy equal to its capacity. In higher dimensions, the result follows from this by considering the ball as a subset of a product of squares. $\square$

(ii) It is also easy to check that any disjunction of a ball in $\mathbf{R}^{2n}$ is a proper disjunction. (See the proof of Proposition 2.2.) However, very little is known about the space $Emb(B^{2n}(u), M)$ of balls of capacity u in an arbitrary symplectic manifold M. For example, if $n > 2$ it is not even known whether $Emb(B^{2n}(u), B^{2n}(u'))$ is path-connected when $u < u'$. Therefore, even if one restricts to balls B of capacity less than $c(M)/2$, it is not clear what the relation is between $\mathbf{e}(B)$ and $\mathbf{e_p}(B)$. $\square$

(iii) Our arguments actually prove more than what is stated above, because they are **local**: they use only the part of the hypersurface Q swept out by the characteristics emanating from A. We will say that a piece Q of hypersurface in $M \times [0,1] \times \mathbf{R}$ disjoins $A \subset M$ if, for all $x \in A$, the point $(x, 0, 0)$ is one end of a characteristic on Q, the other end of which is a point in $(M - A) \times \{(1, 0)\}$. Then we can define the energy $\mathbf{e}(Q)$ of Q to be the minimal length of an interval I such that $Q \subset M \times [0,1] \times I$, and our result reads:

$$\mathbf{e}(Q) \geq \tfrac{1}{2} c(A) \quad \text{if } Q \text{ disjoins } A\text{, and}$$
$$\mathbf{e}(Q) \geq c(B) \quad \text{if } Q \text{ properly disjoins the ball } B.$$

In a similar way, we can define and estimate the energy of a local Hamiltonian diffeomorphism of M. $\square$

(iv) Polterovich, using geometric arguments which are very similar in spirit to ours, established in [16] that $\|\cdot\|_H$ is a norm on rational symplectic manifolds which are tame at infinity. His result is not as sharp as ours, because he considered the disjunction energy of Lagrangian submanifolds, which are more unwieldly than balls. $\square$

Our methods also allow us to prove Gromov's Non-Squeezing Theorem in full generality.

Theorem 1.4 (Non-Squeezing Theorem) *Let (M, ω) be any symplectic manifold, and denote by $M \times B^2(\lambda)$ the product of M with the disc $B^2(\lambda)$ of area λ, equipped with the product form. Then*

$$c(M \times B^2(\lambda)) \leq \lambda.$$

Remark 1.5 This result was first proved for manifolds such as the standard $\mathbf{R}^{2n}$ and T^{2n} by Gromov in [5]. Its range of validity was extended by improvements in the understanding of the behavior of pseudo-holomorphic curves. However, this method has definite limitations and is not yet known to apply to all manifolds. (The best result which can be obtained in this way is described in §3.) We manage to overcome these limitations by using the techniques which we developed to prove Theorem 1.1. As we shall see in Remark 2.3 below, these two theorems are very closely connected, and we will, in fact, deduce Theorem 1.1 from Theorem 1.4. To the authors' knowledge, these are the first deep results in symplectic topology which have been established for all symplectic manifolds. Arnold's conjecture, for example, has still not been proved, even for all compact manifolds.

Finally, we observe that the methods developed here permit the construction of some new embeddings of ellipsoids into balls. In particular, it is possible to solve a problem posed by Floer, Hofer and Wysocki in [4]. This is discussed further in Remark 2.4.

Throughout this paper, all embeddings and isotopies will always be assumed to preserve the symplectic forms involved.

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2 The energy-capacity inequality in $\mathbf{R}^{2n}$.

This section presents a very simple proof of the energy-capacity inequality for subsets of $\mathbf{R}^{2n}$. The basic idea is that if a hypersurface Q of small energy disjoins a large ball in M, one can construct an embedded ball in the product $M \times B^2$ whose capacity is larger than the area of the B^2 factor. But the Non-Squeezing Theorem states that when $M = \mathbf{R}^{2n}$ this area is an upper bound for the capacity of embedded balls in $M \times B^2$.

We need an auxilliary lemma about decompositions of the ball, which was inspired by Traynor's constructions in [17]. Given any set A, we will write $\mathcal{N}(A)$ to denote some small neighborhood of it.

Lemma 2.1 *Suppose that $0 < c < C$, and let $Y = P_1 \cup L \cup P_2 \subset \mathbf{R}^2$ be the union of two rectangles, P_1 of area $C - c$ and P_2 of area c, joined by a line segment L. Further, let*

$$Z_{C,c} = B^{2n}(C) \times P_1 \ \cup \ B^{2n}(c) \times (L \cup P_2).$$

Then, there is a symplectic embedding $B^{2n+2}(C) \hookrightarrow \mathcal{N}(\,Z_{C,c})$ in any neighborhood of $Z_{C,c}$.

Proof: Let π be the projection $B^{2n+2}(C) \to B^2(C)$ which is induced by projection onto the last two coordinates. This represents $B^{2n+2}(C)$ as a kind of fibration over the disc, with fibers which are concentric balls of different capacities. Note that the set

$$\{\, x \in B^2(C) : c(\pi^{-1}(x)) \geq c\}$$

is exactly $B^2(C - c)$. It is easy to see that there is an area preserving embedding $g : B^2(C) \hookrightarrow \mathcal{N}(Y)$ which takes $B^2(C - c)$ into a neighborhood of P_1. In fact, we may choose g so that it sends an open neighbourhood of

$$B^2(C - c) \,\cup\, (\,(-\infty, 0) \times \{0\} \,\cap\, B^2(C)\,) \,\subset\, B^2(C)$$

into a neighborhood of P_1. Clearly, g is covered by the desired embedding of $B^{2n+2}(C)$ into $\mathcal{N}(Z_{C,c})$. $\square$

Proposition 2.2 *For any compact subset $A \subset \mathbf{R}^{2n}$,*

$$\mathbf{e}(A) \geq c(A).$$

Proof: Let Q be a hypersurface of energy e which disjoins A, and let $B \subset A$ be the image of a standard ball of capacity c. We must show that $e \geq c$. This will follow if, for any $\delta > 0$, we can find an embedding of the ball $B^{2n+2}(2c)$ of capacity $2c$ into the product $\mathbf{R}^{2n} \times B^2(e + c + \varepsilon)$, since the Non-Squeezing Theorem then tells us that

$$2c \leq e + c + \varepsilon.$$

By Lemma 2.1, it suffices to embed $Z_{2c,c}$ symplectically into $\mathbf{R}^{2n} \times X$, where X is an annulus of area $e + c$. By hypothesis, there is a rectangle R

in $[0,1] \times \mathbf{R}$ of area e such that $Q \subset \mathbf{R}^{2n} \times R$. Note that, by hypothesis, Q is flat near its ends, that is, that Q coincides with the hypersurface $\{z = 0\}$ near the boundary $t = 0, 1$. (Recall that we use the coordinates (t, z) on $[0,1] \times \mathbf{R}$.) Let R' be another rectangle in $\mathbf{R}^2$ of area c with one edge along $t = 1$, chosen so that $R \cup R'$ is a rectangle of area $e + c$ with one edge along $t = 0$ and another along $t = t_1 > 0$. Then form X by identifying these two edges.

Let $g : B^{2n}(c) \to B \subset A$ be a symplectic embedding, and extend g to an embedding, which we also call g, of $B^{2n}(2c)$ in $\mathbf{R}^{2n}$. This is possible because the space of embedded symplectic balls of any given radius in $\mathbf{R}^{2n}$ is path connected. (The space of embedded balls of variable radius in any manifold M is always connected, so long as M is connected, and, when $M = \mathbf{R}^{2n}$ we can fix the size of the radius by composing with appropriate homotheties. A similar argument shows that the space of symplectic embeddings of two balls in $\mathbf{R}^{2n}$ is path-connected.) This implies that any ball in $\mathbf{R}^{2n}$ is isotopic to a standard one and thus can be extended as much as we wish. Further, we may suppose that the ball $g(B^{2n}(2c))$ is disjoint from $\phi_Q(B)$, where ϕ_Q is the monodromy of Q. To see this, note that because the space of symplectic embeddings of two balls in $\mathbf{R}^{2n}$ is path-connected, there is a symplectomorphism τ which is the identity on B and which moves $\phi_Q(B)$ far away. Hence we may alter Q without changing its energy to a hypersurface with the conjugate monodromy $\tau^{-1} \circ \phi_Q \circ \tau$.

We now define the embedding $Z_{2c,c} \to \mathbf{R}^{2n} \times X$ as follows.

- $B^{2n}(2c) \times P_1$ goes to $\mathbf{R}^{2n} \times R'$ by $g \times i$, where $i : U_1 \to R'$.
- $B^{2n}(c) \times L$ maps to the hypersurface $Q \subset \mathbf{R}^{2n} \times R$ by a map which takes each line $\{x\} \times L$ to the corresponding flow line of the characteristic flow on Q.
- $B^{2n}(c) \times P_2$ goes onto $\phi_Q(B) \times R'$ by the map $(\phi \circ g) \times i$.

It is easy to check that this map preserves the symplectic form. Hence, the symplectic neighborhood theorem implies that it extends to the required symplectomorphism from $\mathcal{N}(Z_{2c,c})$ to $\mathbf{R}^{2n} \times B^2(e + c + \varepsilon)$. $\square$

Remark 2.3 The above argument clearly proves Part (i) of Theorem 1.1 for manifolds for which the Non-Squeezing theorem holds. For, suppose that the monodromy ϕ_Q of a hypersurface Q is a proper disjunction of A of energy e.

Then, for every ball $B(c) \subset A$, ϕ_Q disjoins B from a ball $\tilde{B}(2c)$ of twice the capacity and one can construct an embedding $Z_{2c,c} \hookrightarrow M \times B^2(e+c+\varepsilon)$ as above.

Similarly, one can prove Part (ii) of Theorem 1.1 in this case, by using a slightly different embedding. Note that the ball $B^{2n+2}(c)$ embeds into the set

$$W_{c,c} = B^{2n}(c) \times Y$$

where now P_1 and P_2 both have area $c/2$. This gives a ball of capacity c in $M \times B^2(e + c/2 + \varepsilon)$. Therefore, if the Non-Squeezing Theorem holds, we must have

$$c \leq e + c/2 + \varepsilon,$$

for all $\varepsilon > 0$ and hence $e \geq c/2$, as claimed. □

Remark 2.4 Let $E(c_1, c_2)$ denote the ellipsoid

$$\Sigma_i \pi(x_i^2 + y_i^2)/c_i \leq 1,$$

where $c_1 \leq c_2$. Floer, Hofer and Wysocki show in [4] that if $c_1 \geq 1/2$, $E(c_1, c_2)$ embeds symplectically in $B^4(1) = E(1,1)$ if and only if there is a symplectic linear embedding from $E(c_1, c_2)$ into $E(1,1)$, i.e. only if $a_2 \leq 1$. They asked whether this is sharp. In other words, if $c_1 < 1/2$, is there $c_2 > 1$ such that $E(c_1, c_2)$ embeds in $B^4(1)$? Our embedding methods allow us to answer this question in the affirmative.

The idea is as follows. As Traynor points out in [17], the ellipsoid $E(c_1, c_2)$ may be considered to be fibered over the disc $B^2(c_2)$, with fibers which are smaller by a factor of c_1/c_2 than those of the corresponding ball $B^4(c_2)$. If $c_1/c_2 < 1/2$, we can therefore fit two of these fibers in the corresponding fiber of the ball. From this, Traynor constructs a full filling of the ball $B^4(1)$ by two open ellipsoids.

Now, suppose that we split $E(c_1, c_2)$ into two, considering it to be contained in a neighborhood of a set Z such as $Z_{C,c}$ above. Then, it is not hard to construct the desired embedding $E(c_1, c_2) \to B^4(1)$ by folding the two parts of Z on top of each other. Details of this construction will be published elsewhere. □

3 The Non-Squeezing Theorem

In this section we will use the theory of J-holomorphic curves to prove the Non-Squeezing Theorem under certain hypotheses which look rather artificial. We will see in §4 how to construct families of embeddings which satisfy them.

A symplectic manifold (M, ω) is often said to be **rational** if the homomorphism induced by $[\omega]$ from $\pi_2(M)$ to $\mathbf{R}$ has discrete image. In this case and if this image is not $\{0\}$, we will call the positive generator of this image the **index of rationality** of (M, ω), denoted $r(M)$ or simply r. If the image is $\{0\}$, we set $r(M) = \infty$ and if M is not rational, this index is set equal to 0 (this does not quite follow the conventional definitions).

We will consider $M \times S^2(\lambda)$ with a product form $\Omega = \omega \oplus \lambda\sigma$ where σ is normalised to have total area 1, and will say that a ball in $M \times S^2(\lambda)$ is **standard** if it is the image of an embedding of the form

$$B^{2n+2}(c) \hookrightarrow B^{2n}(c) \times B^2(c) \hookrightarrow M \times S^2(\lambda)$$

where the first map is the obvious inclusion and the second is a product. Clearly the capacity of any standard ball in $M \times S^2(\lambda)$ is bounded above by λ. The main result of this section says that this remains true for any ball which is isotopic to a standard ball through large balls.

Proposition 3.1 *Let M be closed and have index of rationality $r > 0$, and suppose that $g_t : B^{2n+2}(c_t) \to M \times S^2(\lambda)$ is a family of symplectic embeddings such that g_0 is standard and*

$$c_t \geq \sup\{r, \lambda - r\},$$

for all t. Suppose also that the image of g_t misses one fiber $M \times pt$ for all t. Then

$$c_t < \lambda$$

for all t.

We will begin by explaining the usual proof of Gromov's Non-Squeezing theorem via pseudo-holomorphic curves. We assume that the reader is familiar with the basics of the theory of pseudo-holomorphic curves as explained in [5, 10] for example. The most general argument works when the manifold $(V,\Omega) = (M \times S^2(\lambda), \omega + \lambda\sigma)$ is semi-positive (or semi-monotone): see [12]. This condition says that, for generic tame J there are no J-holomorphic spheres with negative Chern number. It is satisfied by all manifolds of dimension ≤ 6 and by manifolds for which there is a constant $\mu \geq 0$ such that

$$c_1(\alpha) = \mu[\omega](\alpha), \quad \text{for all } \alpha \in \pi_2(M),$$

where c_1 is the first Chern class of (M,ω).

Lemma 3.2 *Let $g : B^{2n+2}(c) \to M \times \operatorname{Int} B^2(\lambda)$ be a symplectic embedding, and suppose that $(V,\Omega) = (M \times S^2(\lambda), \omega + \lambda\sigma)$ is semi-positive. Then*

$$c < \lambda.$$

Proof: Clearly, we may consider g as an embedding into V which misses one fiber. Let J_B be an almost complex structure tamed by Ω which extends the image by g of the standard complex structure on $\mathbf{R}^{2n+2}$. In order to show that $c \leq \lambda$, it is enough, by Gromov's monotonicity argument, to show that there exists a J_B-rational curve C of symplectic area smaller or equal to λ passing through the center $g(0) = p_0 = (q_0, z_0)$ of the ball $g(B^{2n+2}(c))$. The reason is that the part of this curve in $g(B^{2n+2}(c))$ pulls back to a holomorphic curve S through the center of $B^{2n+2}(c)$. Since S is minimal with respect to the usual metric on $B^{2n+2}(c)$, the monotonicity theorem implies that its area is at least c. Thus

$$c \leq \operatorname{area} S = \int_S \omega_0 < \int_C \Omega \leq \lambda,$$

as required.

To produce the required J_B-curve, we argue as follows. Let $\mathcal{J}$ denote the space of all C^∞-smooth almost complex structures tamed by Ω and, for $A \in H_2(V;\mathbf{Z})$, let $\mathcal{M}(A)$ be the space of all pairs (f,J) such that $f : \mathbf{C}P^1 \longrightarrow V$ is J-holomorphic, $[f] \in A$, and f does not factor through a self-covering. A basic fact in this respect is that the projection map $P_A : \mathcal{M}(A) \to \mathcal{J}$ is Fredholm with index $2(c_1(A) + n + 1)$, where c_1 is the first Chern class of the tangent bundle of (V,J). (For more details of this step see [10, 15].)

Take any almost complex structure J_1 tamed by ω on M, integrable in a neighbourhood U of q_0, take the usual complex structure J_0 on S^2, and denote by J_{spl} the split structure $J_1 \oplus J_0$. It is easy to see that for all $q \in U$ the rational curves $\{q\} \times S^2$ in class $A_0 = [\{pt\} \times S^2]$ are regular for this complex structure in the Fredholm sense. (This means that, given any holomorphic parametrization f of these curves, the points (f, J_{spl}) are regular points of P_{A_0}.) Note also that $P_{A_0}^{-1}(J_{spl}) = \{q \times S^2 : q \in M\}$, because any holomorphic map from $\mathbf{C}P^1$ to the split $M \times S^2$ in class A_0 induces, by projection on the first factor, a holomorphic map to M, which is null-homologous and hence constant. Obviously all points $p \in V$ that project to U are regular values of the evaluation map

$$ev : P_{A_0}^{-1}(J_{spl}) \times_G \mathbf{C}P^1 \longrightarrow V,$$

where G is the conformal group of $\mathbf{C}P^1$, and these p have pre-image $ev^{-1}(p)$ containing exactly one point. It follows that there exists a structure J' near J_{spl} which is generic, that is regular for all projections P_A, and is such that some point $p' \in U \times S^2$ is still a regular value for the evaluation map on $P_{A_0}^{-1}(J') \times_G \mathbf{C}P^1$, with exactly one point in its pre-image.

Now, let J'' be a generic almost complex structure in the neighbourhood of J_B and Γ a path in $\mathcal{J}$ from $\Gamma(0) = J'$ to $\Gamma(1) = J''$, transverse to all projections P_A. Then $P_{A_0}^{-1}(\Gamma)$ is a smooth manifold, and we consider a short path γ in V from $\gamma(0) = p'$ to a point $\gamma(1) = p''$ in a neighbourhood of p_0, such that the obvious evaluation map

$$ev_{\Gamma,A} : P_A^{-1}(\Gamma) \times_G \mathbf{C}P^1 \longrightarrow V \times [0,1]$$

is transverse to $\gamma \times id : [0,1] \to V \times [0,1]$ for all A. (Here $ev_{\Gamma,A}$ maps onto $[0,1]$ by projection through Γ.) Denote by N the one-dimensional submanifold $ev_{\Gamma,A_0}^{-1}(\gamma \times id)$. By construction, there is exactly one point of N which maps to $(\gamma(0), 0)$. Therefore, if N were compact, it would have at least one other point over $(\gamma(1), 1)$. In other words, there would exist a J''-rational curve in class A_0 passing through p''. And if this were also true for a sequence of paths in $\mathcal{J}$ whose end points converge to J_B and a sequence of paths in V whose end points converge to p_0, this would give, by Gromov's compactness theorem (see [5]), a sequence of holomorphic spheres (weakly) converging to a J_B-cusp-curve. The component of this cusp-curve passing through p_0 would have area smaller or equal to λ and so would be the desired J_B-curve.

Let us suppose now that one of these manifolds N is not compact. There would then exist a sequence of J_i-curves $f_i : \mathbf{C}P^1 \to V$ with $\{J_i\}$ converging to some $\overline{J} = \Gamma(t_0)$ and $\{f_i\}$ diverging. Since V is compact, the compactness theorem implies that some subsequence would converge weakly to a cusp-curve passing through $\overline{p} = \gamma(t_0)$. This cusp-curve would be a connected union of $\overline{J}$-curves in classes $A_1, \ldots, A_k$, where

$$A_0 = A_1 + \ldots + A_k.$$

Therefore, the proof may be finished if we put a hypothesis on M which ensures that a generic path (Γ, γ) does not meet any such cusp-curve. For example, since $\Omega(A_j) > 0$ for $j = 1, \ldots, k$, it is clearly enough to assume that $\pi_2(M) = 0$ or more generally that $\lambda = \Omega(A_0) \leq r$. The real trouble comes from the possible presence of multiply-covered curves of negative Chern number, and it is shown in [12, §4] that it suffices to assume that V is semi-positive. □

Proof of Proposition 3.1

There is no semi-positivity hypothesis here: we get around the problem caused by cusp-curves by considering a very special path Γ. First, let us consider a path from $J_0 = J_{spl}$ to J_1 such that, at each time t, J_t is equal to the push-forward by the embedding g_t of the standard structure on $\mathbf{R}^{2n+2}$. By assumption on the embeddings, we may also suppose that one fiber $M \times pt$ is J_t-holomorphic, for each t. Suppose that there were a J_t-holomorphic A-cusp-curve through the center $g_t(0)$ of the ball for some t with homology decomposition

$$A = A_1 + \ldots + A_k.$$

Let C be the component of this cusp-curve through $g_t(0)$. We may suppose that $[C] = A_1$. The argument at the beginning of Lemma 3.2 shows that $\Omega(A_1) = \int_C \Omega > c_t$. Hence,

$$\Omega(A_j) < \lambda - c_t \leq r, \quad \text{if} \quad j > 1.$$

Thus the classes $A_j, j > 1$, do not lie in $H_2(M)$. On the other hand, by positivity of intersections (see [5, 11]), the fact that a fiber is J_t-holomorphic implies that the intersection number $A_j \cdot [M]$ is ≥ 0 for all j. It follows that one of the A_j has the form $A - B_j$ for some $B_j \in H_2(M)$, and that the others are all elements of $H_2(M)$. Putting all this together, we see that the decomposition must have the form

$$A = B + (A - B),$$

for some $B \in H_2(M)$ such that $\Omega(B) > c_t$.

Note that neither component of this cusp-curve can be multiply- covered. For if the component in class B were a k-fold covering for some $k \geq 2$, we would have to have $\omega(B) > 2c_t$, which is impossible by assumption on c_t since we must have $\omega(B) < \lambda$. Further, the component in class $A - B$ cannot be multiply covered since $A - B$ is not a multiple class.

This argument shows that, for all the elements J_t in our special path, the only J_t-holomorphic A-cusp-curves are of type $(B, A-B)$. By the compactness theorem, a similar statement must hold for every J in some neighborhood of this path. Thus we may assume that the points on the regular path Γ considered above have this property. The arguments in [12] show that these cusp-curves are well-behaved, and fill out a subset of V of codimension at least 2. Therefore, a generic path (Γ, γ) will not meet these cusp-curves, and the argument may be finished as before. □

The next lemma is the key step in extending our results to non-compact manifolds.

Lemma 3.3 *Let M be a non-compact symplectic manifold of index of rationality $r > 0$. For each compact subset K of M, there is a number $\zeta > 0$ such that whenever $g_t : B^{2n+2}(c_t) \to K \times S^2(\lambda)$ is a family of symplectic embeddings missing one fiber $M \times pt$ for all t and beginning with a standard embedding g_0, then*

$$c_t \geq \lambda - \min(r, \zeta) \quad \textit{for all } t \quad \Rightarrow \quad c_t < \lambda \quad \textit{for all } t.$$

Proof: Let K_1 be a compact subset of M which contains K in its interior. In order to make the previous argument go through, we just have to ensure that the A-curves in N do not escape outside $K_1 \times S^2$. Suppose that $\partial K_1 = \Sigma$ is a smooth hypersurface and let U be a compact neighbourhood of Σ disjoint from K. Since all balls lie in K we may assume that the special almost complex structures J_t are all equal to the Ω-compatible split structure J_{spl} on U. Because U is compact, it is easy to see that there exists ζ, s both small enough so that, given any J_t-holomorphic curve C passing through some point $p \in \Sigma$, the Ω-area of $C \cap B_s(p)$ is larger than 3ζ. Choosing now, in the proof of Proposition 3.1, the generic path of almost complex structures so that each be sufficiently close to J_{spl} on U, we get a lower bound equal to 2ζ on the area of $C \cap B_s(p)$. If C is a curve in the path N, the Ω-area of the part of C which lies in the ball $\operatorname{Im} g_t$ must be at least c_t. But $c_t + 2\zeta > \lambda$, by hypothesis. Therefore, none of the A-curves in N meet Σ. $\square$

4 Embedding balls along monodromies

By Remark 2.3, all theorems will be proved if we show that the Non-Squeezing Theorem holds for all manifolds. Our tool to do this is Proposition 3.1. Thus, given a ball $B(c)$ in a cylinder $M \times B^2(\lambda)$ with $c > \lambda$, we aim to construct another ball $B''(c)$ of capacity c, which is contained in some cylinder $M'' \times B^2(\lambda'')$ with $c > \lambda''$ and which is isotopic to a standard ball through large balls. Proposition 3.1 then implies that the ball $B''(c)$ cannot exist, and it follows that $B(c)$ does not exist either.

It is quite a delicate matter to obtain a ball B'' with the required properties, and as our notation implies we do this by a two-step process, first constructing an intermediate ball B', and then using that to get B''. We begin by explaining the basic procedure which constructs these balls, and then will give the proof.

4.1 The N-fold wrapping construction

Because the original ball B may not extend to a ball of capacity $2c$, we must use a multiple wrapping process to maintain the capacity of our balls. Therefore, instead of using the set Y of Lemma 2.1, we will use the sets $Y_N \subset \mathbf{R}^2$ described below. It will be often convenient to use rectangles rather than discs. As always, the label of a set will indicate its capacity (or area), and as before we will distinguish the standard balls (the domain of our maps) from their images by reserving the dimensional upperscript to the former only.

Let V^{2m} be any symplectic manifold, $g : B^{2m}(\kappa) \to V$ a symplectic embedding of a ball of capacity κ, whose image is denoted $B(\kappa)$, and $\phi_s, 0 \leq s < \infty$, a diffeotopy of V such that ϕ_s is periodic in s with period 1. We assume that ϕ_s has a 1-periodic generating Hamiltonian $H : V \times [0, \infty) \to \mathbf{R}$ which satisfies

- H vanishes near any integral value of s;
- for each s, $\min_V H_s = 0$.

When referring in this section to the energy of such a diffeotopy ϕ_s, we will always mean the maximum over $s \in [0,1]$ of the total variation of H_s. Further, we will say that ϕ_s **strictly disjoins** a ball $B \subset V$ if the balls

$$B, \phi_1(B), \phi_2(B), \dots, \phi_k(B), \dots$$

are all disjoint.

Now choose any positive integer N, and, for $i = 1, \dots, N$ define $a_i, b_i \in \mathbf{R}$ by

$$a_i = (i-1)(1 + 1/N), \qquad b_i = a_i + 1/N.$$

Thus $a_{i+1} = b_i + 1$. Let $Y_N = Y_N(\kappa) \subset \mathbf{R}^2$ be the union of N rectangles $P_i, 1 \le i \le N$, of area κ/N with $N-1$ lines $L_i, 1 \le i \le N-1$, of length 1, where:

$$P_i = \{(u,v) : a_i \le u \le b_i,\ 0 \le v \le \kappa\},$$
$$L_i = \{(u,v) : b_i \le u \le a_{i+1},\ v = 0\}.$$

Observe that any embedding of $Y_N(\kappa)$ extends to some neighborhood $\mathcal{N}(Y_N(\kappa))$ and hence induces an embedding of the disc $B^2(\kappa)$, since this fits inside any neighborhood of $Y_N(\kappa)$. Therefore an embedding

$$G : B^{2m}(\kappa) \times Y_N(\kappa) \hookrightarrow V \times \mathbf{R}^2,$$

induces an embedding $G \circ i$ of the ball $B^{2m+2}(\kappa)$ by

$$B^{2m+2}(\kappa) \ \overset{i}{\hookrightarrow}\ B^{2m}(\kappa) \times \mathcal{N}(Y_N(\kappa)) \ \overset{G}{\hookrightarrow}\ V \times \mathbf{R}^2$$

where i is the obvious inclusion.

Now define $G : B^{2m}(\kappa) \times Y_N(\kappa) \hookrightarrow V \times \mathbf{R}^2$ by

$$\begin{aligned} G(p,u,v) &= (\phi_{i-1}(g(p)), u, v), \quad \text{when} \quad (u,v) \in P_i, \\ &= (\phi_{(i-1)+u-b_i}(g(p)), u, H(\phi_{(i-1)+u-b_i}(g(p)), u - b_i)), \quad \text{when}(u,v) \in L_i. \end{aligned}$$

It is easy to check that G is symplectic and so extends to a symplectic embedding (which we will also call G) of $B^{2m}(\kappa) \times N(Y_N(\kappa))$ into $V \times \mathbf{R}^2$.

The corresponding ball, which is the image of the composite

$$B^{2m+2}(\kappa) \hookrightarrow B^{2m}(\kappa) \times \mathcal{N}(Y_N(\kappa)) \overset{G}{\hookrightarrow} V \times \mathbf{R}^2$$

will be called the **unwrapped ball** associated to $B(\kappa) = \operatorname{Im} g$ and ϕ_s, and will be denoted $B(B(\kappa), \phi)$. Note that it exists for any isotopy ϕ_s, not only for disjoining isotopies. For the sake of clarity, we will sometimes write G_ϕ for the corresponding embedding G.

If e is the energy of ϕ_s, then H takes values in $[0, e]$ and the image $\operatorname{Im} G$ of $B^{2m}(\kappa) \times Y_N$ by G lies in the set

$$V \times \left(\coprod P_i \cup \coprod Q_i\right) = V \times S \ \subset \ V \times \mathbf{R}^2,$$

where $Q_i, i = 1, \ldots, N-1$ are the rectangles

$$Q_i = \{(u, v) : b_i \le u \le a_{i+1},\ 0 \le v \le e\}.$$

Denote by τ_S the translation of $V \times S$ by $1 + 1/N$ in the u-direction of $\mathbf{R}^2$. (We will make no distinction between the translation of $S \subset \mathbf{R}^2$ and its lift to the product $V \times S$.) This sends P_i to P_{i+1} and Q_i to Q_{i+1}, and it is easy to see that, if ϕ_s strictly disjoins $B(\kappa)$, then τ_S strictly disjoins $\operatorname{Im} G$, that is, all the balls

$$\operatorname{Im} G, \tau_S(\operatorname{Im} G), (\tau_S)^2(\operatorname{Im} G), \ldots$$

are disjoint. Thus, if X is the annulus obtained by quotienting S by the translation τ_S, X has area $A = \kappa/N + e$ and, as in Proposition 2.2, we get an embedded ball in $V \times B^2(A + \varepsilon)$ by the composite:

$$B^{2m+2}(\kappa) \hookrightarrow B^{2m}(\kappa) \times \mathcal{N}(Y_N(\kappa)) \stackrel{G}{\hookrightarrow} V \times S \to V \times X \hookrightarrow V \times B^2(A + \varepsilon).$$

This ball wraps N times round the annulus X, and will be called the **wrapped ball** $B_W(B(\kappa), \phi)$ generated by $B(\kappa)$ and ϕ_s.

Remark 4.1 Note that if $e < \kappa$, we may, by choosing N large enough, arrange that A be arbitrarily close to e. Thus, for sufficently small $\varepsilon > 0$, we get a ball $B_W = B_W(B, \phi)$ of capacity κ inside a cylinder in $V \times \mathbf{R}^2$ of area $A + \varepsilon < e + 2\varepsilon < \kappa$.

The next result is obvious.

Lemma 4.2 *If $B_t = \operatorname{Im}(f_t : B^{2m}(\kappa_t) \to V)$ and $\{\phi^t\}$ vary smoothly with respect to a parameter t, the corresponding wrapped ball $B_W(B_t, \phi^t)$ varies smoothly.*

Lemma 4.3 *The translation τ_S of $V \times S$ which disjoins the unwrapped ball $B(B, \phi)$ may be extended to a 1-periodic diffeotopy $\{\sigma_s\}_{0 \le s < \infty}$ of $V \times \mathbf{R}^2$ in such a manner that*

- $\sigma_1 = \tau_S$;
- *the diffeotopy σ_s strictly disjoins $B(B, \phi)$; and*
- *the energy of σ_s is $\le A + \varepsilon$.*

Proof: Suppose first that the rectangles P_i, Q_i in T all have the same v-height. Then S has the form $\mathbf{R} \times I$, and one can extend τ_S to have the form $(x, u, v) \mapsto (x, u + \beta(v), v)$ on $V \times \mathbf{R} \times I$, for some suitable bump function β which equals $1 + 1/N$ on the interval I. This map has the generating Hamiltonian $H(x, u, v) = \int^v \beta$ which has energy $\le \int \beta$. The general result follows because there is an area-preserving map which commutes with τ_S and takes S into a set S_0 of the form $\mathbf{R} \times I$ with area $S_0/\tau < A + \varepsilon$. $\square$

Of course, the wrapped ball $B_W(B, \phi)$ may also be strictly disjoined by a diffeotopy (a translation) of energy $< A + \varepsilon < \kappa$. But for the first step of our argument we consider instead $B(B, \phi)$ with disjoining isotopy σ_s since the latter is more flexible.

4.2 Regularity and the plan of the proof

We can now make more precise the plan briefly outlined in the introduction of the section. We fix a small constant $\varepsilon > 0$ of size to be determined later. We start with the ball

$$B^{2n+2}(c) \overset{g}{\hookrightarrow} B(c) \subset M \times B(\lambda) \subset M \times \mathbf{R}^2 = M'$$

(with $\lambda < c$) given in the statement of the Non-Squeezing Theorem. This is strictly disjoined by a diffeotopy ϕ_s of energy $< \lambda + \varepsilon$ which translates points in the $\mathbf{R}^2$ direction. We first construct the unwrapped ball $B'(c) = (B(c), \phi) \subset M' \times \mathbf{R}^2$, together with the disjoining diffeotopy σ_s. By Lemma 4.3 above, we may choose N so large that σ_s has energy $< \lambda + 2\varepsilon$. Thus the wrapped ball $B''(c) = B_W(B'(c), \sigma_s)$ of capacity c lies in a cylinder $C'' = M' \times \mathbf{R}^2 \times B^2(\lambda + 3\varepsilon)$.

Proposition 4.4 *The ball $B''(c)$ in $C'' = M' \times \mathbf{R}^2 \times B^2(\lambda + 3\varepsilon)$ is isotopic through balls of capacity $\ge \lambda$ to a standard ball.*

Corollary 4.5 *The Non-Squeezing Theorem holds for any symplectic manifold.*

Proof of the corollary: If (M,ω) is compact and rational with index of rationality r, choose $\varepsilon < r/3$. The Non-Squeezing Theorem then follows immediately from Propositions 4.4 and 3.1: simply apply Proposition 3.1 to the closed manifold $M\times T^2(\Lambda)\times T^2(\Lambda)$ which has the same index of rationality as M, where Λ is chosen large enough so that $M\times T^2(\Lambda)\times T^2(\Lambda)\times B^2(\lambda+3\varepsilon)$ contains the ball $B''(c)$. If (M,ω) is compact but not rational, one can slightly perturb both the form ω on M and the ball $B(c)$ in $M\times B^2(\lambda)$ to get a ball $\tilde{B}$ sitting inside $\tilde{M}\times B^2(\lambda)$, where $(\tilde{M},\tilde{\omega})$ is rational. Although $\tilde{B}$ may have capacity $\tilde{c}$ a little less than c, we may clearly arrange that if $c>\lambda$ then $\tilde{c}>\lambda$. Thus the Non-Squeezing Theorem holds true for all compact manifolds.

Suppose now that M is non-compact. Note that the initial ball $B(c)$ sits in a compact region of $M\times B^2(\lambda)$, so that we may assume both that M is a compact manifold with boundary and that it has positive index of rationality r. Then $B(c)\subset K\times B^2(\lambda)$, where K is a compact subset of M. Hence $B''(c)$ lies in $K\times T^2(\Lambda)\times T^2(\Lambda)\times S^2(\lambda+3\varepsilon)$ where Λ increases as ε decreases (and as N increases), because the N-fold wrapping construction does not move the ball in the M-direction. This ball is isotopic to a standard ball through balls in $K\times T^2(\Lambda)\times T^2(\Lambda)\times S^2(\lambda+3\varepsilon)$ of capacity equal to $\lambda+3\varepsilon$ up to a small quantity 3ε. But note that $r(M)=r(M\times T^2(\Lambda)\times T^2(\Lambda))$ and that the constraining number ζ of Lemma 3.3 depends only on K: it is independant of the size Λ of $T^2(\Lambda)$ as soon as Λ is large enough. Therefore, for $\varepsilon<\frac{\min(r,\zeta)}{3}$, the argument of Lemma 3.3 applies. $\square$

In order to explain our strategy for proving Proposition 4.4 it will be convenient to introduce the following definition.

Given $\delta>0$, we will say that an isotopy ϕ_s in some manifold is δ-**regular** on a ball $B(\kappa)$ if, for each $s\in[\delta,1]$, the balls

$$B(\kappa),\ \phi_s(B(s\kappa)),\ \phi_s^2(B(s\kappa)),\ldots,\ \phi_s^k(B(s\kappa)),\ldots$$

are all disjoint (note, in particular, that ϕ_s strictly disjoins $B(\kappa)$). Here $B(s\kappa)$ denotes a ball of capacity $s\kappa$ which is concentric with $B(\kappa)$. Notice that it is not so much the isotopy itself which is important but its relation to the concentric balls $B(\kappa)$. Further, all the balls $\phi_{ks}(B(s\kappa))$ are assumed to be disjoint from the whole initial ball $B(\kappa)$.

Regularity is really a 1-dimensional notion: any translation of $\mathbf{R}$ at constant speed which disjoins a given interval also disjoins subintervals within a time proportional to their lengths. The basic 2-dimensional example of a regular diffeotopy is a translation which disjoins a rectangle in $\mathbf{R}^2$, or its conjugate which disjoins a disc. (We will often call the latter a translation too.)

To be more precise, let $\phi_s, 0 \leq s < \infty$, be the translation of the strip $S = \mathbf{R} \times [0,h]$ in the u-direction at speed $\nu > 1$. (As above, we use the coordinates (u,v) on S.) Note that this isotopy is generated by a Hamiltonian which is a linear function of v on the strip, and vanishes outside some slightly larger strip. As in Lemma 4.3 above, its total variation may be taken arbitrarily close to $h\nu$. Then for any small $\delta > 0$ and any $\kappa < h$, consider an embedding of the standard disk $g : B^2(\kappa) \hookrightarrow S$ such that $B(s\kappa) = g(B^2(s\kappa))$ lies inside $[-s, 0] \times [0,h]$ for all $s \in [\delta, 1]$. (Such an embedding exists because our choice of constants $h > \kappa$, $\delta > 0$ leaves a little extra room.) It is easy to check that ϕ_s is regular on $\operatorname{Im} g$.

With this simple basic example, we can construct higher dimensional examples of regular disjoining diffeotopies since regularity is stable under products.

Lemma 4.6 *Given a δ-regular pair (f, ϕ) in $\mathbf{R}^2$ and g any symplectic embedding of a ball in any manifold, form the product $h = g \times f : B^{2n}(\kappa) \times B^2(\kappa) \to V = M \times \mathbf{R}^2$. Then the pull-back $\pi^*(\phi_s)$ of ϕ by the projection $\pi : M \times \mathbf{R}^2 \to \mathbf{R}^2$ is also δ-regular on the composite $h \circ i$ of h with the standard embedding $i : B^{2n+2}(\kappa) \hookrightarrow B^{2n}(\kappa) \times B^2(\kappa)$.*

Proof: This is clear because i sends each concentric subball of capacity κ' onto a ball in $B^{2n}(\kappa) \times B^2(\kappa)$ whose projection on the second factor is a concentric subdisc of same capacity κ'. □

The following lemmas form the heart of our argument. They are proved in the following section.

Lemma 4.7 *For every $\delta, \varepsilon > 0$, there is a 1-parameter family $B'_t(\kappa_t), \sigma^t_s, 0 \leq t \leq 1$, of balls and strict disjoining isotopies in $M' \times \mathbf{R}^2$ such that:*

- *$B'_0(\kappa_0) = B'(c)$ and $\sigma^0_s = \sigma_s$;*
- *the final isotopy σ^1_s is δ-regular over the ball $B'_1(\kappa_1)$;*

- *the isotopy σ_s^t has energy $< \lambda + 2\varepsilon$ for all t; and*
- *the balls B_t' have capacity $\kappa_t \geq \lambda$ for all t, and $\kappa_1 = \lambda$.*

Lemma 4.8 *Suppose given a ball B of capacity λ in V with δ-regular strictly disjoining isotopy σ_s of energy $< \lambda + 2\varepsilon$, and let B_W be the corresponding wrapped ball in the cylinder $C = V \times B^2(\lambda + 3\varepsilon)$. Then, if δ is sufficiently small, B_W is isotopic through balls in C of capacity λ to a standard ball.*

Proof of Proposition 4.4 By Lemmas 4.2 and 4.7, the wrapped ball $B''(c)$ is isotopic through balls of capacity $\kappa_t \geq \lambda$ which embed in C'' to the wrapped ball $B_1'' = B_W(B_1'(\kappa_1), \sigma^1)$. Since the isotopy σ_s^1 is regular, Lemma 4.8 shows that the wrapped ball B_1'' is isotopic in C'' to a standard ball. □

4.3 The Construction of Isotopies

Our first lemma constructs an isotopy of an unwrapped ball, and the second one for a wrapped ball. We begin with the proof of the second lemma.

Proof of Lemma 4.8 The wrapped ball B_W is the image of a composite

$$B^{2m+2}(\lambda) \hookrightarrow B^{2m}(\lambda) \times \mathcal{N}(Y_N(\lambda)) \overset{G}{\hookrightarrow} V \times S \to V \times B^2(\lambda + 3\varepsilon),$$

where the last map is essentially the quotient by the translation τ_S. The easiest way to describe the isotopy from B_W to a standard embedding is to construct an isotopy of the map

$$B^{2m+2}(\lambda) \hookrightarrow B^{2m}(\lambda) \times \mathcal{N}(Y_N(\lambda)) \overset{G}{\hookrightarrow} V \times S$$

whose image is disjoined by all iterates of τ_S.

The crucial observation is that, just as in Lemma 2.1, the ball $B^{2m+2}(\lambda)$ fits inside a neighborhood of the subset W of $B^{2m}(\lambda) \times Y_N(\lambda)$ which is defined as follows:

$$\begin{aligned} W = \Big(B^{2m}(\lambda) \times P_1\Big) \;\cup\; & \Big(B^{2m}(\lambda(1-1/N)) \times (L_1 \cup P_2)\Big) \;\cup\; \ldots \\ \cup\; & \Big(B^{2m}(\lambda(1-i/N)) \times (L_i \cup P_{i+1})\Big) \;\cup\; \ldots \\ & \qquad \cup\; \Big(B^{2m}(\lambda/N) \times P_N\Big). \end{aligned}$$

Thus the interval L_1 for example is longer than we need: G maps it into a hypersurface which disjoins the whole ball $B^{2m}(\lambda)$ while we only need the hypersurface to disjoin the subball $B^{2m}(\lambda(1-1/N))$ from $B^{2m}(\lambda)$. Because the hypersurface comes from the regular isotopy σ_s, we only need the part of it corresponding to the isotopy $\sigma_s, 0 \leq s \leq 1-1/N$, which sits over an interval of length $1-1/N$.

Here are the details. Let us change the proportions in the set Y_N by moving the points b_i, keeping the a_i fixed. Thus, for $0 \leq s \leq 1$, put $b_i^s = b_i + s$, and let

$$Y_N^s = P_1^s \cup L_1^s \cup \ldots \cup P_N^s,$$

where the rectangles P_i^s have area $(1/N+s)\lambda$ and the intervals L_i have length $1-s$. Then, for each s, $B^{2m+2}(\lambda)$ embeds in a neighborhood of

$$\begin{aligned} W^s = \Big(B^{2m}(\lambda) \times P_1^s\Big) \ \cup \ \Big(B^{2m}(\lambda(1-1/N-s)) \times (L_1^s \cup P_2^s)\Big) \ \cup \ \ldots \\ \cup \ \Big(B^{2m}(\lambda(1-i/N-is)) \times (L_i^s \cup P_{i+1}^s)\Big) \ \cup \ \ldots \\ \cup \ \Big(B^{2m}(\lambda(1/N-(N-1)s)) \times P_N^s\Big). \end{aligned}$$

Here, the convention is that $B^{2m}(\kappa)$ denotes a point when $\kappa = 0$ and is empty for $\kappa \leq 0$. Note that when $s = s_0 = 1-1/N$, W^{s_0} is just $B^{2m}(\lambda) \times P_1^{s_0}$ embedded by a standard embedding and the isotopy may be ended. The length of $L_i^{s_0}$ is then $\frac{1}{N} > \delta$: the lengths of L_i^s are thus larger than δ during the whole isotopy.

We now map W^s to $V \times S$ by the obvious map G^s, which, for each $0 \leq s \leq 1-1/N$ is constructed from the isotopy σ_u, $0 \leq u \leq 1-s$. It is not hard to check that the δ-regularity of σ_s implies that $\operatorname{Im} G^s$ is disjoined by the translation τ_S along S. Since we may choose everything to vary smoothly with s, the result follows. $\square$

Proof of Lemma 4.7 We construct a family of balls B_t' and isotopies σ_s^t from the unwrapped ball $B'(c) = B(B,\phi)$ to a ball and isotopy which are the lifts of a ball and strictly disjoining isotopy in $\mathbf{R}^2$. The first part of the isotopy is simple: we just decrease the capacity of the unwrapped ball B' from c to λ by shrinking the initial ball $B(c)$ while keeping the initial disjoining diffeotopy ϕ_s fixed. Note that τ_s does not change through this isotopy.

Consider now the ball $B'(\lambda)$ obtained at the end of this isotopy. It is clear by the definition of the unwrapped construction that the following diagram commutes:

$$\begin{array}{ccccc} B^{2n+4}(\lambda) & \overset{i}{\hookrightarrow} & B^{2n+2}(\lambda)\times\mathcal{N}(Y_N(\lambda)) & \overset{\tilde{G}_g}{\hookrightarrow} & M'\times\mathbf{R}^2 \\ & & \downarrow(\pi\circ g)\times id & & \downarrow(\pi\times id) \\ & & B^2(\lambda)\times\mathcal{N}(Y_N(\lambda)) & \overset{G_j}{\hookrightarrow} & \mathbf{R}^2\times\mathbf{R}^2 \end{array}$$

where g is the initial embedding $B^{2n}(\lambda)\hookrightarrow B'(\lambda)$, j is the inclusion $B^2(\lambda)\hookrightarrow \mathbf{R}^2$, π is the projection onto $\mathbf{R}^2$, and we write $\tilde{G}_g$, G_j for the corresponding maps which give the unwrapped balls. It follows that:

(i) any isotopy $G_t, 0\le t\le t_0$ beginning with $G_0=G_j$ lifts to an isotopy $\tilde{G}_t$ beginning with $\tilde{G}_0=\tilde{G}_g$.

(ii) If, for each t, ρ_s^t is a 1-periodic diffeotopy of $\mathbf{R}^4$ which strictly disjoins $\operatorname{Im}(G_t)$, then its pull-back $\tilde{\rho}_s^t$ to $M\times\mathbf{R}^4$ also strictly disjoins $\operatorname{Im}(\tilde{G}_t)$ and has the same energy.

(iii) Finally, if at time t_0, G_{t_0} is a product

$$f_1\times f_2 : B^2(\lambda)\times N(Y_N(\lambda))\hookrightarrow\mathbf{R}^2\times\mathbf{R}^2$$

and $\rho_s^{t_0}$ is (the pull-back of) a translation ψ_s in the second $\mathbf{R}^2$-factor such that the pair (f_2,ψ_s) is δ-regular, then $\tilde{G}_{t_0}$ is again a product whose second factor is δ-regular under ψ_s. It follows that the pair $(\tilde{G}_{t_0}\circ i,\tilde{\rho}_s^{t_0})$ is also δ-regular by Lemma 4.6.

This reduces the proof to the construction of the 4-dimensional isotopy G_t. This may be easily defined by using the fact that the translation ϕ_s which strictly disjoins $B^2(\lambda)$ in $\mathbf{R}^2$ is generated by an *autonomous* Hamiltonian H (of total variation $\lambda+\varepsilon$): for the moment, let us forget the (technical) requirement that our isotopies should be constant when the time s is near an integer value, and let us suppose that the isotopy σ_s is a 1-parameter group. Then, for each $0\le t\le 1$, set

$$\sigma_s^t=\sigma_{(1-t)s},$$

and consider the corresponding unwrapped ball $B(B^2(\lambda),\sigma^t)=\operatorname{Im}G_{\sigma^t}$. Because the energy $e(\sigma^t)$ of σ^t is $(1-t)e(\sigma)=(1-t)(\lambda+\varepsilon)$, the ball $B(B^2(\lambda),\sigma^t)$

sits over a strip $S^t = \coprod_i P_i \cup \coprod_i Q_i^t$ where the rectanges Q_i^t have area $(1-t)(\lambda+\varepsilon)$. Therefore the translation τ^t which moves S^t through the distance $1+1/N$ may be extended as in Lemma 4.3 to an isotopy τ_s^t of energy

$$\frac{\lambda}{N} + e(\sigma^t) + \varepsilon/2 = \frac{\lambda}{N} + (1-t)(\lambda+\varepsilon) + \varepsilon/2.$$

Note that τ^t does not disjoin $B(B^2(\lambda), \sigma^t)$ from itself, since σ^t does not disjoin $B^2(\lambda)$ at time $s=1$. However, if we follow τ^t (which is a movement in the second $\mathbf{R}^2$ factor) with the translation $\sigma_1^{t+1} = \sigma_{-t}$ in the first $\mathbf{R}^2$ factor, we do get a disjoining isotopy. Thus, the isotopy

$$\rho_s^t = \sigma_{-ts} \circ \tau_s^t$$

disjoins $B(B^2(\lambda), \sigma^t)$ from itself at time $s=1$, and may be extended to an isotopy of $\mathbf{R}^4$ with total energy

$$\begin{aligned} \frac{\lambda}{N} + e(\sigma^t) + e(\sigma^{t+1}) + \varepsilon &= \frac{\lambda}{N} + (1-t)(\lambda+\varepsilon) + t(\lambda+\varepsilon) + \varepsilon \\ &= \frac{\lambda}{N} + \lambda + 2\varepsilon. \end{aligned}$$

Taking $G^t = G_{\sigma^t}$, we can therefore satisfy (i) and (ii) above. Observe that when $t=1$, G_1 is simply the inclusion of $B^2(\lambda) \times N(Y_N(\lambda))$ and ρ_s^1 is (isotopic to) the disjoining translation $(\sigma_s)^{-1}$ in the first $\mathbf{R}^2$-direction. It is easy to see that there is a family of symplectomorphisms $\beta_t, 1 \le t \le 2$ of $\mathbf{R}^4$ which begins with the identity such that, for each $t \in [1,2]$, the conjugate isotopies $\beta_t^{-1} \circ \rho_s^1 \circ \beta_t$ strictly disjoin $\mathrm{Im}\,(G_1)$ and such that the final isotopy $\psi_s = \beta_2^{-1} \circ \rho_s^1 \circ \beta_2$ has the form of a translation in the u-direction

$$(x,y,u,v) \mapsto (x,y,u,v+\alpha_s(u))$$

that can be chosen δ-regular on the second factor of G_1. The pair (G_1, ψ_s) then satisfies the conditions in (iii) above.

It remains to take into account the requirement that our isotopies should be constant when the time s is near an integral value. But it is sufficient to choose the initial translation ϕ_s, that disjoins $B^2(\lambda)$ in $\mathbf{R}^2$, generated by an Hamiltonian of the form $H_s = hH$ where H is autonomous and $h : \mathbf{R} \to \mathbf{R}$ is a bump function equal to 1 everywhere except on a μ-neighborhood of each integer: choosing μ small enough with respect to δ, we may clearly arrange that the above argument holds. $\square$

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