Accessability of typical points for invariant measures of positive Lyapunov exponents for iterations of holomorphic maps

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Abstract. *We prove that if A is the basin of immediate attraction to a periodic attracting or parabolic point for a rational map f on the Riemann sphere, if A is completely invariant (i.e. $f^{-1}(A) = A$), and if μ is an arbitrary f-invariant measure with positive Lyapunov exponents on ∂A, then μ-almost every point $q \in \partial A$ is accessible along a curve from A. In fact we prove the accessability of every "good" q i.e. such q for which "small neighbourhoods arrive at large scale" under iteration of f.*

This generalizes Douady-Eremenko-Levin-Petersen theorem on the accessability of periodic sources.

We prove a general "tree" version of this theorem. This allows to deduce that on the limit set of a geometric coding tree (in particular on the whole Julia set), if diameters of the edges converge to 0 uniformly with the number of generation converging to ∞, every f-invariant probability ergodic measure with positive Lyapunov exponent is the image through coding with the help of the tree, of an invariant measure on the full one-sided shift space.

The assumption that f is holomorphic on A, or on the domain U of the tree, can be relaxed and one does not need to assume f extends beyond A or U.

Finally we prove that in the case f is polynomial-like on a neighbourhood of $\overline{\mathbb{C}} \setminus A$ every "good" $q \in \partial A$ is accessible along an external ray.

Introduction.

Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map of the Riemann sphere $\overline{\mathbb{C}}$. Let $J(f)$ denote its Julia set. We say a periodic point p of period m is attracting (a sink) if $|(f^m)'(p)| < 1$, repelling (a source) if $|(f^m)'(p)| > 1$ and parabolic if $(f^m)'(p)$ is a root of unity. We say that $A = A_p$ is the immediate basin of attraction to a sink or a parabolic point p if A is a component of $\overline{\mathbb{C}} \setminus J(f)$ such that $f^{nm}|_A \to p$ as $n \to \infty$ and $p \in A_p$ in the case p is attracting, $p \in \partial A$ in the case p is parabolic.

We call $q \in \partial A$ *good* if there exist real numbers $r > 0, \kappa > 0, \delta : 0 < \delta < r$ and an integer $\Delta > 0$ such that for every n large enough

$$\sharp\{ \text{ good times } \}/n \geq \kappa \tag{0.0}$$

We call here $\overline{n} : 0 \leq \overline{n} \leq n$ a *good time* if for each $0 \leq l \leq \overline{n} - \Delta$ the component $B_{\overline{n},l}$ of $f^{-(\overline{n}-l)}(B(f^{\overline{n}}(q), r)$ containing $f^l(q)$ satisfies:

$$B_{\overline{n},l} \subset B(f^l(q), r - \delta) \tag{0.1}$$

* supported by Polish KBN Grants 210469101 "Iteracje i Fraktale" and 210909101 "...Uklady Dynamiczne".

In the definition of *good* q we assume also that

$$\lim_{\overline{n}\to\infty} \operatorname{diam}(B_{\overline{n},0}) \to 0 \tag{0.2}$$

lim taken over *good* $\overline{n}$'s.

Finally in the definition of *good* q we assume about each *good* $\overline{n}$ that

$$f^{-\overline{n}}(A) \cap B_{\overline{n},0} \subset A. \tag{0.3}$$

We shall prove the following

Theorem A. Every good $q \in \partial A$ is accessible from A, i.e. there exists a continuous curve $\gamma : [0,1] \to \overline{\mathbb{C}}$ such that $\gamma([0,1)) \subset A$ and $\gamma(1) = q$.

Theorem A generalizes Douady-Eremenko-Levin-Petersen theorem on the accessability of periodic sources. Remark that in the case of periodic sources one obtains curves along which periodic q is accessible, of finite lengths, see Section 1. Condition (0.1) holds in the case q is a periodic source for all $\overline{n}$'s. Condition (0.3) is true if A is the basin of attraction to ∞ for f a polynomial, and more generally if A is completely invariant, i.e $f^{-1}(A) = A$.

Condition (0.3) in the case of a source is equivalent to Petersen's condition [Pe].

Under the assumption of the complete invariance of A μ-almost every point for μ an invariant probability measure with positive Lyapunov exponents is *good* hence accessible, cf. Corollary 0.2.

In fact we shall introduce in Section 2 a weaker definition of *good* q and prove Theorem A with that weaker definition. In that weaker definition parabolic periodic points in ∂A are good. The traces of *telescopes* built there can sit in an arbitrary interpetal, so one obtains the accessability in each interpetal. One obtains in particular Theorem 18.9 in [Mi1].

Remark that the above conditions of being *good* are already quite weak. In particular we do not exclude critical points in $B_{\overline{n},l}$.

For example every point in ∂A is *good* if A is the basin of attraction to ∞ for a polynomial $z \mapsto z^2 + c$ which is non-renormalizable, c outside the "cardioid". This is Yoccoz-Branner-Hubbard theory, see [Mi2]. (In this case however theorem A is worthless because one proves directly the local connectedness of ∂A.)

Remark that complete invariance of A, a basin of attraction to a sink, does not imply that f on a neighbourhood of $\overline{\mathbb{C}} \setminus A$ is polynomial-like. (Polynomial-like maps were first defined and studied in [DH].) In [P4] an example of degree 3, of the form $z \to z^2 + c + \frac{b}{z-a}$, with a completely invariant basin of attraction to ∞, not simply-connected, with only 2 critical points in the basin, is described.

We prove in the paper a theorem more general than Theorem A, namely a theorem on the accessability along branches of a *geometric coding tree*. We recall now basic definitions from [P1, P2, PUZ, PS].

Let U be an open connected subset of the Riemann sphere $\overline{\mathbb{C}}$. Consider any holomorphic mapping $f : U \to \overline{\mathbb{C}}$ such that $f(U) \supset U$ and $f : U \to f(U)$ is a proper map. Denote $\mathrm{Crit}(f) = \{z : f'(z) = 0\}$. This is called the set of critical points for f. Suppose that $\mathrm{Crit}(f)$ is finite. Consider any $z \in f(U)$. Let $z^1, z^2, ..., z^d$ be all the f-preimages of z in U where $d = \deg f \geq 2$. (Pay attention that we consider here, unlike in the other papers, only the full tree i.e. not only some preimages but all preimages of z in U.)

Consider smooth curves $\gamma^j : [0,1] \to f(U)$, $j = 1, ..., d$, joining z with z^j respectively (i.e. $\gamma^j(0) = z, \gamma^j(1) = z^j$), such that there are no critical values for iterations of f in $\bigcup_{j=1}^d \gamma^j$, i.e. $\gamma^j \cap f^n(\mathrm{Crit}(f)) = \emptyset$ for every j and $n > 0$. We allow self-intersections of each γ^j.

Let $\Sigma^d := \{1, ..., d\}^{\mathbb{Z}^+}$ denote the one-sided shift space and σ the shift to the left, i.e. $\sigma((\alpha_n)) = (\alpha_{n+1})$. We consider the standard metric on Σ^d

$$\rho((\alpha_n),(\beta_n)) = \exp -k((\alpha_n),(\beta_n))$$

where $k((\alpha_n),(\beta_n))$ is the least integer for which $\alpha_k \neq \beta_k$.

For every sequence $\alpha = (\alpha_n)_{n=0}^\infty \in \Sigma^d$ we define $\gamma_0(\alpha) := \gamma^{\alpha_0}$. Suppose that for some $n \geq 0$, for every $0 \leq m \leq n$, and all $\alpha \in \Sigma^d$, the curves $\gamma_m(\alpha)$ are already defined. Suppose that for $1 \leq m \leq n$ we have $f \circ \gamma_m(\alpha) = \gamma_{m-1}(\sigma(\alpha))$, and $\gamma_m(\alpha)(0) = \gamma_{m-1}(\alpha)(1)$.

Define the curves $\gamma_{n+1}(\alpha)$ so that the previous equalities hold by taking respective f-preimages of curves γ_n. For every $\alpha \in \Sigma^d$ and $n \geq 0$ denote $z_n(\alpha) := \gamma_n(\alpha)(1)$.

For every $n \geq 0$ denote by $\Sigma_n = \Sigma_n^d$ the space of all sequences of elements of $\{1, ..., d\}$ of length $n+1$. Let π_n denote the projection $\pi_n : \Sigma^d \to \Sigma_n$ defined by $\pi_n(\alpha) = (\alpha_0, ..., \alpha_n)$. As $z_n(\alpha)$ and $\gamma_n(\alpha)$ depends only on $(\alpha_0, ..., \alpha_n)$, we can consider z_n and γ_n as functions on Σ_n.

The graph $\mathcal{T} = \mathcal{T}(z, \gamma^1, ..., \gamma^d)$ with the vertices z and $z_n(\alpha)$ and edges $\gamma_n(\alpha)$ is called a *geometric coding tree* with the root at z. For every $\alpha \in \Sigma^d$ the subgraph composed of $z, z_n(\alpha)$ and $\gamma_n(\alpha)$ for all $n \geq 0$ is called a *geometric branch* and denoted by $b(\alpha)$. The branch $b(\alpha)$ is called *convergent* if the sequence $\gamma_n(\alpha)$ is convergent to a point in $\mathrm{cl} U$. We define the *coding map* $z_\infty : \mathcal{D}(z_\infty) \to \mathrm{cl} U$ by $z_\infty(\alpha) := \lim_{n\to\infty} z_n(\alpha)$ on the domain $\mathcal{D} = \mathcal{D}(z_\infty)$ of all such α's for which $b(\alpha)$ is convergent.

In Sections 1-3, for any curve (maybe with self-intersections) $\gamma : I \to \overline{\mathbb{C}}$ where I is a closed interval in $\mathbb{R}$, we call γ restricted to J a subinterval (maybe degenerated to a point) of I a *part* of γ. Consider γ on $J_1 \subset [0,1]$ and γ' on $J_2 \subset [0,1]$ either both γ and γ' being parts of one $\gamma_n(\alpha)$, $J_1 \cap J_2 = \emptyset$, J_1 between 0 and J_2 , or γ a part of $\gamma_{n_1}(\alpha)$ and γ' a part of γ_{n_2} where $n_1 < n_2$. Let $\Gamma : [0, n_2 - n_1 + 1] \to \overline{\mathbb{C}}$ be the concatenation of $\gamma_{n_1}, \gamma_{n_1+1}, ..., \gamma_{n_2}$. We call the restriction of Γ to the convex hull of $J_1 \subset [0,1]$ and $J_2 \subset [n_2 - n_1, n_2 - n_1 + 1]$ (we identified here $[0,1]$ with $[n_2 - n_1, n_2 - n_1 + 1]$) *a part of* $b(\alpha)$ *between* γ *and* γ' .

For every continuous map $F : X \to X$ of a compact space X denote by $M(F)$ the set of all probability F-invariant measures on X. In the case X is a compact subset of the

Riemann sphere $\overline{\mathbb{C}}$ and F extends holomorphically to a neighbourhood of X and $\mu \in M(F)$ we can consider for μ-a.e. x Lyapunov characteristic exponent

$$\chi(F,x) = \lim_{n\to\infty} \frac{1}{n} \log |(F^n)'(x)|.$$

If μ is ergodic then for μ-a.e. x

$$\chi(F,x) = \chi_\mu(F) = \int \log(F')d\mu.$$

In this paper where we shall discuss properties of μ-a.e. point, it is enough to consider only ergodic measures, because by Rochlin Decomposition Theorem every $\mu \in M(F)$ can be decomposed into ergodic ones.

Denote

$$M_e^{\chi+}(F) = \{\mu \in M(F) : \mu \text{ ergodic } \chi_\mu(F) > 0\}$$

$$M_e^{\mathrm{h}+}(F) = \{\mu \in M(F) : \mu \text{ ergodic } \mathrm{h}_\mu(F) > 0\}$$

where $\mathrm{h}_\mu(F)$ denotes measure-theoretic entropy.

From Ruelle Theorem it follows that $\mathrm{h}_\mu(F) \le 2\chi_\mu(F)$ see [R], so $M_e^{\mathrm{h}+}(F) \subset M_e^{\chi+}(F)$.

The basic theorem concerning convergence of geometric coding trees is the following:

Convergence Theorem. 1. Every branch except branches in a set of Hausdorff dimension 0 in the metric ρ on Σ^d, is convergent. (i.e $\mathrm{HD}(\Sigma^d \setminus \mathcal{D}) = 0$). In particular for every $\nu \in M^{\mathrm{h}+}(\sigma)$ we have $\nu(\Sigma^d \setminus \mathcal{D}) = 0$, so the measure $(z_\infty)_*(\nu)$ makes sense.

2. For every $z \in \mathrm{cl}U$, $\mathrm{HD}(z_\infty^{-1}(\{z\})) = 0$. Hence for every $\nu \in M(\sigma)$ we have for the entropies: $\mathrm{h}_{\nu_\varphi}(\sigma) = \mathrm{h}_{(z_\infty)_*(\nu_\varphi)}(\overline{f}) > 0$, (if we assume that there exists $\overline{f}$ a continuous extension of f to $\mathrm{cl}U$).

The proof of this Theorem can be found in [P1] and [P2] under some assumptions on a slow convergence of $f^n(\mathrm{Crit}(f)$ to γ^j for $n \to \infty$) and in [PS] in full generality (even with $f^n(\mathrm{Crit}(f)) \cap \gamma^j \ne \emptyset$ allowed).

Let $\hat{\Lambda}$ denote the set of all limit points of $f^{-n}(z), n \to \infty$. Analogously to the case $q \in \partial A$ we say that $q \in \hat{\Lambda}$ is *good* if f extends holomorphically to a neighbourhood of $\{f^n(q), n = 0, 1, ...\}$ (we use the same symbol f to denote the extension) and conditions (0.0'), (0.1'), (0.2') and (0.3') hold. These conditions are defined similarly to (0.0)-(0.3), with A replaced by U and ∂A replaced by $\hat{\Lambda}$.

Again pay attention that we shall give a precise weaker definition of q *good* in Section 2. and prove Theorem B with that weaker definition. That definition will not demand f extending beyond U.

Theorem B. Let f be a holomorphic mapping $f : U \to \overline{\mathbb{C}}$ and $\mathcal{T}$ be a geometric coding tree in U as above. Suppose

$$\mathrm{diam}(\gamma_n(\alpha)) \to 0 \text{ as } n \to \infty \qquad (0.4)$$

uniformly with respect to $\alpha \in \Sigma^d$.

Then every good $q \in \hat{\Lambda}$ is a limit point of a branch $b(\alpha)$.

Using a lemma belonging to Pesin Theory (see Section 2) we prove that μ-a.e.q below is good and easily obtain the following

Corollary 0.1. Let f be a holomorphic mapping $f : U \to \overline{\mathbb{C}}$ and $\mathcal{T}$ be a geometric coding tree in U such that the condition (0.4) holds. If μ is a probability measure on $\hat{\Lambda}$ and the map f extends holomorphically from U to a neighbourhood of suppμ so that $\mu \in M_e^{\chi+}(f)$, then for μ-almost every $q \in \hat{\Lambda}$ satisfying (0.3') there exists $\alpha \in \Sigma^d$ such that $b(\alpha)$ converges to q. In particular μ is a $(z_\infty)_*$-image of a measure $m \in M(\sigma)$ on Σ^d.

Remark that Corollary 0.1 concerns in particular every μ with $\mathrm{h}_\mu(f) > 0$. Assuming that f extends holomorphically to a neighbourhood of $\hat{\Lambda}$ and refering also to Convergence Theorem we see that $(z_\infty)_*$ maps $M_e^{\mathrm{h}+}(\sigma)$ onto $M_e^{\mathrm{h}+}(f|_{\hat{\Lambda}})$ preserving entropy.

The question whether this correspondence is *onto* is stated in [P3]. Thus Corollary 0.1 answers this question in positive under additional assumptions (0.3') and (0.4).

We do not know whether this correspondence is finite-to-one except measures supported by orbits of periodic sources for which the answer is positive, see Proposition 1.2.

Two special cases are of particular interest. The first one corresponds to Theorem A:

Corollary 0.2. Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational mapping and A be a completely invariant basin of attraction to a sink or a parabolic point. Then for every $\mu \in M_e^{\chi+}(f|_{\partial A})$ μ-a.e. $q \in \partial A$ is accessible from A.

Corollary 0.3. Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational mapping, $\deg f = d$, and $\mathcal{T} = \mathcal{T}(z, \gamma^1, ..., \gamma^d)$ be a geometric coding tree. Assume (0.4). Let $\mu \in M_e^{\chi+}(f)$. Then for μ-a.e. q there exists $\alpha \in \Sigma^d$ such that $b(\alpha)$ converges to q.

In Theorem A and Corollary 0.2 in the case f is a polynomial (or a polynomial-like map) and A is the basin of attraction to ∞, the accessability of a point along a curve often implies automathically the accessability along an external ray. In the case A is simply-connected this follows from Lindelöf's Theorem. External rays are defined as images under standard Riemann map of rays $t\zeta, \zeta \in \partial\mathbb{D}, 1 < t < \infty$.

In the case A is not simply-connected one should first define external rays in the absence of Riemann map. This is done in [GM] and [LevS] in the case of f a polynomial and in [LevP] in the polynomial- like situation. We recall these definitions in Section 3.

We prove in Section 3 the following

Theorem C. Let $W_1 \subset W$ be open, connected, simply-connected domains in $\overline{\mathbb{C}}$ such that cl$W_1 \subset W$ and $f : W_1 \to W$ be a polynomial-like map. denote $K = \bigcap_{n \geq 0} f^{-n}(W)$. Then every good $q \in \partial K$ is accessible along an external ray in $W \setminus K$.

An alternative way to prove the accessability along an external ray is to use somehow, as in the simply-connected case, Lindelöf's Theorem. This is performed in [LevP]. It is proved there that if q is accessible along a curve in $W \setminus K$ and q belongs to a periodic or preperiodic component $K(q)$ of K then it is accessible along an external ray.

Pay attention also that for any $q \in \partial K$ if $K(q)$ is one point then q is accessible along an external ray. This is easy, see [GM, Appendix] and [LevP].

Remark 0.4. (Proof of Theorem A from B and Corollary 0.2 from 0.1). We do not know how to get rid of the assumption (0.4) in Theorem B and Corollary 1. In Theorem A and Corollary 2 this condition is guaranteed automathically. More precisely to deduce Theorem A from B and Corollary 2 from 1 we consider an arbitrary tree $\mathcal{T} = \mathcal{T}(z, \gamma^1, ..., \gamma^d)$ in A, where $d = \deg(f|_A)$, so that $\gamma^j \cap \bigcup_{n>0} f^n(\mathrm{Crit}(f)) = \emptyset$ and $p \notin \bigcup_{j=1,...,d} \gamma^j$. Only critical points in A account here. Forward orbits of these critical points converge to p hence the following condition holds:

$$\Big(\bigcup_{j=1,...,d} \gamma^j\Big) \cap \mathrm{cl}\Big(\bigcup_{n>0} f^n(\mathrm{Crit}(f))\Big) = \emptyset \tag{0.5}$$

Hence we can take open discs $U^j \supset \gamma^j$ such that

$$\bigcup_{j=1,...,d} U^j \cap \mathrm{cl}\Big(\bigcup_{n>0} f^n(\mathrm{Crit}(f))\Big) = \emptyset$$

and consider univalent branches $F_n(\alpha)$ of f^{-n} mapping respective γ^j to $\gamma_n(\alpha)$. $\{F_n(\alpha)\}_{\alpha,n}$ is a normal family of maps. If it had a non-constant limit function G then we would find an open domain V such that $F_{n_t}(V) \subset U$ as $n_t \to \infty$. If we assumed $p \notin U^j$ we arrive at a contradiction. This proves (0.4). Finally by the complete invariance of A we have $\hat{\Lambda} = \partial A$.

In Corollary 0.3 to find $\mathcal{T}$ such that (0.4) holds it is enough to assume that the forward limit set of $f^n(\mathrm{Crit}(f))$ does not dissect $\overline{\mathbb{C}}$, because then we find $\mathcal{T}$ so that (0.5) holds.

We believe however that in Proof of Corollary 3 we can omit (0.4), or maybe often find a tree such that (0.3) holds.

Remark 0.5. Observe that there are examples where (0.4) does not hold. Take for example z in a Siegel disc or z being just a sink. Even if $J(f) = \overline{\mathbb{C}}$ one should be careful: for M. Herman's examples $z \mapsto \lambda z \frac{z-a}{1-\overline{a}z} / \frac{z-b}{1-\overline{b}z}$, $|\lambda| = 1, a \neq 0 \neq b, a \approx b$,see [H1], the unit circe is invariant and for a branch in it (0.4) fails. These examples are related with the notion of neutral sets, see [GPS].

Remark 0.6. The assumption f is holomorphic on U (or A) can be replaced by the assumption f is just a continuous map, a branched cover over $f(U) \supset U$.

However without the holomorphy of f we do not know how the assumption (0.4) could be verified.

Remark 0.7. The fact that in, say, Theorem A we do not need to assume that f extends holomorphically beyond the basin A suggests that maybe the assumption (0.3) is substantial and without it the accessability in Theorem A is not true. We have in mind here an analogous situation of a Siegel disc with the boundary not simply-connected, where the map is only smooth beyond it, see [H2]. Accessability of periodic sources in the boundary of A in the absence of the assumption (0.3) is a famous open problem and we think that if the answer is positive one should substantially use in a proof the holomorphy of f outside A.

The paper is organised as follows: in Section 1 we prove theorem B for q a periodic source, in Section 2 we deal with the general case. The case of sources was known in the polynomial-like and parabolic p situations [D], [EL], [Pe]. The general case contains the case of sources but it is more tricky (though not more complicated) so we decided to separate the case of sources to make the paper more understandable. Section 3 is devoted to Theorem C.

Section 1. Accessability of periodic sources.

Theorem D. Let $f : U \to \overline{\mathbb{C}}$ be a holomorphic map and $\mathcal{T}(z, \gamma^1, ...\gamma^d)$ be a geometric coding tree in U, $d = \deg f|_U$. Assume (0.4). Next assume that f extends holomorphically to a neighbourhood of a family of points $q_0, ..., q_{n-1} \in \hat{\Lambda}$ so that this family is a periodic repelling orbit for this extension (the extension is also denoted by f).

Assume finally that there exists V a neighbourhood of q on which f^n is linearizable and if F is its inverse on V such that $F(q) = q$ then

$$F(V \cap U) \subset U \tag{1.1}$$

Then there exists a periodic $\alpha \in \Sigma^d$ such that $b(\alpha)$ is convergent to q. Moreover the convergence is exponential, in particular the curve being the body of $b(\alpha)$ is of finite length.

Proof of Theorem D. As usually we can suppose that q is a fixed point by passing to the iterate f^n if $n > 1$.

Assume that $q \neq z$. We shall deal with the case $q = z$ later.

Let h denote the linearizing map i.e. a map conjugating f on a neighbourhood of cl$V$ to $z \to \lambda z$ with $\lambda = f'(q)$, mapping q to $0 \in \mathbb{C}$.

Replace if necessary the set V by a smaller neighbourhood of q so that $z \notin V$ and $\partial V = h^{-1} \exp\{\Re\xi = a\}$ for a constant $a \in \mathbb{R}$.

For every set $K \subset \mathrm{cl}V \setminus \{q\}$ consider its diameter in the radial direction (with origin at q) in the logarithmic scale, namely the diameter of the projection of the set $\log h(K)$ to the real axis. This will be denoted by $\mathrm{diam}_{\Re\log}(K)$.

For every $m \geq 0$ write

$$R_m := h^{-1} \exp(\{\zeta \in \mathbb{C} : a - (m+1)\log|\lambda| < \Re\zeta < a - m\log|\lambda|\})$$

and

$$V_m := h^{-1} \exp(\{\zeta \in \mathbb{C} : \Re\zeta < a - m\log|\lambda|\}).$$

Observe the following important property of $\gamma_n(w)$'s, $n \geq 0, w \in \Sigma^d$:

For every $\varepsilon > 0$ there exists $N(\varepsilon)$ such that if a component γ of $\gamma_n(w) \cap R_m$ satisfies

$$\operatorname{diam}_{\Re \log}(\gamma) > \varepsilon \log|\lambda| \text{ and } z_n(w) \in V_m \tag{1.2}$$

then

$$0 < n - m < N(\varepsilon) \tag{1.3}$$

Indeed, by (1.2) for every $t = 0, 1, ..., m$ we have $f^t(z_n(w)) \in V_{m-t}$ so $f^t(z_n(w)) \neq z$. Hence $n > m$. On the other hand we have

$$\varepsilon \leq \operatorname{diam}_{\Re \log}(\gamma) = \operatorname{diam}_{\Re \log}(f^m(\gamma) \leq \text{Const diam}(f^m(\gamma)$$

So from (0.3) and from the estimate $\operatorname{diam} f^m(\gamma_n(w)) = \operatorname{diam}\gamma_{n-m}(\sigma^m(w)) \geq \varepsilon$, we deduce that $n - m$ is bounded by a constant depending only on ε. This proves (1.3).

Fix topological discs $U^1, ..., U^d$ being neighbourhoods of $\gamma^1, ..., \gamma^d$ respectively such that $\bigcup_{i=1}^{N(\varepsilon)} f^i(\text{Crit}(f)) \cap U^j = \emptyset$ for every $j = 1, ..., d$.

(There is a minor inaccuracy here because this concerns the case the curves γ^j are embedded. If they have self-intersections we should cover them by families of small discs and later lift them by branches of f^{-t} one by one along the curves.)

For every γ being a part of $\gamma_n(w)$ satisfying (1.2) we can consider

$$W_1 = F_{n-(m-1)}(\sigma^{m-1}(w))(U^j)$$

which is a neighbourhood of $f^{m-1}(\gamma_n(w))$. We used here the notation $F_t(v)$ for the branch of f^{-t} mapping γ^j to $\gamma_t(v), v \in \Sigma^d$. Here $j = v_t$.

Next consider the component W_2 of $W_1 \cap V$ containing $f^{m-1}(\gamma)$. Using Koebe's Bounded Distortion Theorem we can find a disc

$$W(\gamma) = B(x, \text{Const}\varepsilon\lambda^{-m}) \tag{1.4}$$

in $F^{m-1}(W_2)$ with $x \in \gamma$ such that f^n maps $W(\gamma)$ univalently into U^j. We take Const such that

$$\operatorname{diam}_{\Re \log} W(\gamma) < \frac{1}{2}\log|\lambda|. \tag{1.5}$$

(Remark that this part is easier if (0.5) is assumed. Then we just consider U^j's disjoint with $\text{cl}\bigcup_{n=1}^{\infty} \text{Crit}(f)$.)

By the definition of $\hat{\Lambda}$ there exist $n_0 \geq 0$ and $\alpha \in \Sigma^d$ such that $\gamma_{n_0}(\alpha) \cap V \neq \emptyset$. By (1.1) there exist $\beta_1, \beta_2, ...$ each in $\{1, ..., d\}$ such that for each $k \geq 0$ we have

$$F^k(b(\alpha)) = b(\beta_k, \beta_{k-1}, ..., \beta_1, \alpha).$$

More precisely we consider an arbitrary component $\hat{\gamma}$ of $\gamma_{n_0}(\alpha) \cap V$ and extend F^k from it holomorphically along $b(\alpha)$.

Denote for abbreviation $\beta_k, \beta_{k-1}, ..., \beta_1, \alpha$ by $k]\alpha$.

Denote also $F^k(\hat{\gamma})$ by $\hat{\gamma}_{k]}$ and the part of $\gamma_{n_0+k}(k]\alpha)$ between $\hat{\gamma}_{k]}$ and $z_{n_0+k-1}(k]\alpha)$ by $\gamma_{k]}$.

For each $k \geq 0$ denote by $\mathcal{N}_k$ the set of all pairs of integers (t, m) such that $t : 0 \leq t \leq k + n_0, 0 < m < k$ and $\gamma_t(k]\alpha)$ satisfies (1.2) for a curve γ being a part of $\gamma_t(k]\alpha)$ or a part of $\gamma \subset \gamma_{k]}$ if $t = k + n_0$ and for the integer m and additionally

$$\{ \text{ the part of } b(\alpha) \text{ between } \gamma \text{ and } \hat{\gamma}_{k]} \} \subset V_m. \tag{1.6}$$

We write in this case $W(\gamma) = W_{k,t,m}$ and $\gamma = \gamma_{k,t,m}$. Figure 1 illustrates our definitions:

Figure 1.

We have now two possibilities:

1. For every $k_2 > k_1 \geq 0,\ 0 < m_1 < k_1, 0 < m_2 < k_2$ and $0 \leq T \leq k_2 + n_0$ such that $(T, m_1) \in \mathcal{N}_{k_1}$, $(T, m_2) \in \mathcal{N}_{k_2}$, supposed the equality of the T-th entries $(k_1]\alpha)_T = (k_2]\alpha)_T$, we have

$$W_{k_1,T,m_1} \cap W_{k_2,T,m_2} = \emptyset.$$

(The equality of the T-th entries means that $f^T(W_{k_1,T,m_1}), f^T(W_{k_2,T,m_2})$ are in the same U^j.)

2. The case 1. does not hold, what implies obviously the existence of T and the other integers as above such that $\pi_T(k_1]\alpha) = \pi_T(k_2]\alpha)$, (i.e. the blocks of $k_1]\alpha$ and $k_2]\alpha$ from 0 to T are the same).

Later we shall prove that the case 1. leads to a contradiction. Now we shall prove that the case 2. allows to find a periodic branch convergent to q what proves our Theorem.

Denote $K = k_2 - k_1$. Repeat that we have

$$\pi_T(\sigma^K(k_2]\alpha)) = \pi_T(k_1]\alpha) = \pi_T(k_2]\alpha).$$

Denote $k_2]\alpha$ by ϑ. We get by the above:

$$f^K(z_{T+K}(\vartheta)) = z_T(\vartheta).$$

or writing this with the help of F which is the inverse of f on V so that $F(q) = q$ we have $F^K(z_T(\vartheta)) = z_{T+K}(\vartheta)$. We know also that $\gamma := \bigcup_{t=T+1}^{T+K} \gamma_t(\vartheta)$ being a curve joining $z_T(\vartheta)$ with $z_{T+K}(\vartheta))$ is contained in V (even in $V_{m(k_2,t)}$) by (1.4).

Hence the curve $\Gamma := \bigcup_{n\geq 0} F^{nK}(\gamma)$ is the body of the part starting from the T-th vertex of the periodic branch $(\vartheta_0, ..., \vartheta_{K-1}, \vartheta_0, ..., \vartheta_{K-1}, \vartheta_0, ...)$.

To finish Proof of Theorem D we should now eliminate the disjointness case 1. We shall just prove there is not enough room for that.

Denote for every $k \geq 0$

$$A_k^+ := \{m : 0 < m < k, \text{ there exists } \ t \text{ such that } (t,m) \in \mathcal{N}_k\}$$

Let $A_k^- := \{1, ..., k-1\} \setminus A_k^+$.

As $\gamma_{k+n_0}(k]\alpha)$ intersects V_k (at $\hat{\gamma}_{k]}$), each $0 < m \leq k-1$ is fully intersected by the curve built from the curves $\gamma_t(k]\alpha), t = 0, ..., k+n_0-1$ and $\gamma_{k]}$.

Hence

$$\sharp A_k^- \log|\lambda| \leq \sum_{m\in A_k^-} \Big(\sum_{0\leq s\leq n_0+k} \operatorname{diam}_{\Re\log}(\gamma_s(k]\alpha) \cap R_m) \Big) \leq 2(k+n_0+1)\varepsilon \log|\lambda|.$$

The coefficient 2 takes into account the possibility that one $\gamma_s(k]\alpha)$ intersects R_m and R_{m+1}, where $m, m+1 \in A_k^-$ (it cannot intersect more than two R_m's because $\operatorname{diam}_{\Re\log}(\gamma_s(k]\alpha) \cap R_m) < \varepsilon$).

Hence

$$\sharp A_k^- \leq 2(k+n_0+1)\varepsilon.$$

So

$$\sharp A_k^+ \geq k - 2(k+n_0+1)\varepsilon - 1 \geq k(1-3\varepsilon) \tag{1.7}$$

for k large enough.

Fix from now on $\varepsilon = 1/4$. Fix an arbitrary large k_0. Let $\mathcal{N}^+ = \bigcup_{0\leq k\leq k_0}(k,\mathcal{N}_k)$. Observe that each point $\xi \in V$ belongs to at most

$$4dN(1/4) \tag{1.8}$$

sets $W(k,t,m)$ where $(k,(t,m)) \in \mathcal{N}^+$.

Indeed if $W(k_1,t_1,m_1) \cap W(k_2,t_2,m_2) \neq \emptyset$ then $|m_1 - m_2| \leq 1$ by (1.5), and by (1.3) we have

$$|m_i - t_i| < N(1/4), \quad i = 1,2$$

hence

$$|t_1 - t_2| < 2N(1/4).$$

(In the case $t_i = k_i + n_0$ for $i = 1$ or 2 we cannot in fact refer to (1.3). The trouble is with its $n - m > 0$ part, because we do not know whether $z_{k_i+n_0} \in V_{m_i}$. But then directly $m_i < k_i \leq t_i$.)

But we assumed (this is our case 1.) that for every t, m and j all the sets $W(k,t,m)$ with the t-th entry of $k]\alpha$ equal to j, variable k, are pairwise disjoint. This finishes the proof of the estimate (1.8).

The conclusion from (1.8) and (1.4) is that because of the lack of room $\sharp\mathcal{N}^+ < \mathrm{Const}\, k_0$. This contradicts (1.7) for $\varepsilon = 1/4$ and k_0 large enough.

The disjointness case 1. is eliminated. Theorem D in the case $z \neq q$ is proved.

Consider the case $z = q$. Then, unless $\gamma^j \equiv q$ in which case Theorem is trivial, the role of z in the above proof can be played by arbitrary $z^j \in \gamma^j \setminus \{q\}$. Formally on the level 0 we have now d^2 curves joining each z^j with preimages of z^i in $\gamma_1((i,j))$. ♣

Remark 1.1. Under the assumption $z \neq q$ and moreover $q \notin \bigcup_{j=1,\dots,d} \gamma^j$ (which is the case when we apply Theorem B to prove theorem A) observe that there exists a constant M such that for every $n \geq 0$ and $\vartheta \in \Sigma^d$ we have $\mathrm{diam}_{\Re \log}\gamma_n(\vartheta) < M$.

Indeed let $m = m_1 \geq 0$ be the smallest integer such that $\gamma_n(\vartheta)$ intersects R_m and let m_2 be the largest one. Suppose that $m_2 - m_1 > 1$. Then by (1.3) $n < m_1 + 1 + N(1)$ and $m_2 < n$. (The role of $z_n(\vartheta)$ in the proof of this part of (1.3) is played by $V_{m_2} \cap \gamma_n(\vartheta)$.) Thus $m_2 - m_1 < N(1)$.

This observation allows to modify (simplify) slightly Proof of Theorem B. One does not need (1.6) then .

Proposition 1.2. Every branch $b(\alpha)$ convergent to a periodic source q is periodic (i.e α is periodic). There is only a finite number of α's such that $b(\alpha)$ converges to q.

Proof. Suppose $z \neq q$ and $b(\alpha)$ converges to q. We can take V, a neighbourhood of q, arbitrarily small. Then the constant n_0 will depend on it. However the above proof shows that we obtain the equality

$$\pi_T(k_1]\alpha) = \pi_T(k_2]\alpha)$$

for $k_1 - k_2$ bounded by a constant independent of n_0. $z \neq q$ implies that $T \to \infty$ as V shrinks to q. So there exists a finite block of symbols β such that $\alpha = \beta\beta\beta...\beta\alpha'$ (α' infinite) with arbitrarily many b's. So α is periodic. This consideration gives also a bound for the period of α hence it proves finitness of the set of α's with $b(\alpha)$ convergent to q. ♣

Remark that with some additional effort we could obtain an estimate for the number of branches convergent to q. In the case q is in the boundary of a basin of attarction to a sink this estimate should give so called Pommerenke-Levin-Yoccoz inequality (see for example [Pe]).

Section 2. Theorem B and Corollary 0.1.

Given $f : U \to \overline{\mathbb{C}}$ a holomorphic map and $\mathcal{T} = \mathcal{T}(z, \gamma^1, ..., \gamma^d)$ a geometric coding tree in U as in Introduction we shall give a definition of $q \in \hat{\Lambda}$ *good* more general then in Introduction.

Let us start with some preliminary definitions:

Definition 2.1. $D \subset U$ is called n_0-*significant* if there exists $\alpha \in \Sigma^d$ and $0 \leq n \leq n_0$ such that $\gamma_n(\alpha) \cap D \neq \emptyset$.

Definition 2.2. For every $\delta, \kappa > 0$ and integer $k > 0$ a pair of sequences $(D_t)_{t=0,1,...,k}$ and $(D_{t,t-1})_{t=1,...,k}$ is called a *telescope* or a (δ, κ, k)-*telescope* if each D_t is an open connected subset of U, there exists a strictly increasing sequence of integers $0 = n_0, n_1, ..., n_k$ such that each $D_{t,t-1}$ is a nonempty component of $f^{-(n_t - n_{t-1})}(D_t)$ contained in D_{t-1} (of course $f^{n_t - n_{t-1}}$ can have critical points in $D_{t,t-1}$),

$$t/n_t > \kappa \quad \text{for each} \quad t, \tag{2.0}$$

and else

$$\text{dist}(\partial_U^{\text{ess}} D_{t,t-1}, \partial_U D_{t-1}) > \delta. \tag{2.1}$$

Here the subscript U means the boundary in U and the essential boundary $\partial_U^{\text{ess}} D_{t,t-1}$ is defined as $\partial_U D_{t,t-1} \setminus \bigcup_{n=1}^{n_t - n_{t-1}} f^{-n}(\partial U)$.

Definition 2.3. A (δ, κ, k)-telescope is called n_0-*significant* if D_k is n_0-significant.

Definition 2.4. For any (δ, κ, k)-telescope we can choose inductively sets $D_{t,l}$, where $l = t-2, t-3, ..., 0$ by choosing $D_{t,l-1}$ as a component of $f^{-(n_l - n_{l-1}}(D_{t,l})$ in $D_{t-1,l-1}$. We call the sequence

$$D_{k,0} \subset D_{k-1,0} \subset ... \subset D_{1,0} \subset D_0$$

a *trace* of the telescope.

Definition 2.5. We call $q \in \hat{\Lambda}$ *good* if there exist $\delta, \kappa > 0$ an integer $n_0 \geq 0$ and a sequence of n_0-significant telescopes $\mathrm{Tel}^k, k = 1, 2, ...,$ where Tel^k is a (δ, κ, k)-telescope, with traces $D^k_{k,0}, D^k_{k-1,0}, ..., D^k_0$ respectively (to the notation of each object related to the telescope Tel^k we add the superscript k) such that

$$D^k_{l,0} \to q \quad \text{as } l \to \infty \quad \text{uniformly over } k. \tag{2.2}$$

Remark 2.6. $q \in \hat{\Lambda}$ good in the sense of Introduction (conditions (0.0')-(0.3') satisfied) is of course good in the above sense. Indeed we choose each Δ's good time and denote these times by $n_0, n_1, ...,$ of course then κ in (2.0) is κ/Δ for the old κ from (0.0).

For each k we define a telescope Tel^k by taking as D^k_k an arbitrary n_0-significant component of $B(f^{n_k}(q), r)$. Such a component exists with n_0 depending only on r because the set all vertices of the tree $\mathcal{T}$ is by definition dense in $\hat{\Lambda}$. Then inductively for each $0 \leq t < k$ we choose as D^k_t a component of $B(f^{n_t}(q), r) \cap U$ containing a component $D^k_{t+1,t}$ of $f^{-(n_{t+1}-n_t)}(D^k_{t+1})$, (such a component $D^k_{t+1,t}$ exists by (0.3'). By (0.2') an arbitrary choice of traces will be OK.

Of course in the case of $U = A$ a basin of immediate attraction to a sink or a parabolic point one can build telescopes with $D^k_{t,l}$ not containing critical points, but there is no reason for that to be possible in general.

Proof of Theorem B. Let $q \in \hat{\Lambda}$ be a *good* point according to the definition above. Fix constants δ, κ and n_0 and a sequence of δ, κ, k telescopes and their traces $,k = 0, 1, ...$ as in Definition 2.5.

We can suppose that $z \notin D^k_0$ or at least that each $\gamma^j, j = 1, ..., d$ has a point outside D^k_0. If it is not so then either there exists l such that each γ^j has a point outside $D^k_{l,0}$ for every k in which case in the considerations below we should consider $m \geq l$ rather than $m > 0$ or else there exists j such that $\gamma^j \equiv q$ in which case obviously $b(j, j, j, ...)$ converges to q.

Denote $D^k_{m,0} \setminus D^k_{m+1,0}$ by R^k_m for $m = 0, 1, ..., k-1$ and $D^k_{k,0}$ by R^k_k. These sets replace rings from Section 1.

Choose for each k a curve $\gamma_{n(k)}(\alpha^k)$ for $\alpha^k \in \Sigma^d$ and $n(k) \leq n_0$ intersecting D^k_k. Choose a part $\hat{\gamma}^k$ of $\gamma_{n(k)}(\alpha^k)$ in this intersection.

As in Section 1 there exists $k]\alpha^k = \beta^k_0 \beta^k_1 ... \beta_{n^k_k - 1} \alpha^k \in \Sigma^d$ such that $\gamma_{n(k)+n^k_k}(k]\alpha^k)$ intersects $D^k_{k,0}$ and moreover it contains a part $\hat{\gamma}_{k]}$ which is a lift of $\hat{\gamma}^k$ by f^{n_k}. Denote the part of $\gamma_{n(k)+n^k_k}(k]\alpha^k)$ between $z_{n(k)+n^k_k-1}(k]\alpha^k)$ and $\hat{\gamma}_{k]}$ by $\gamma_{k]}$

Fix an integer $E > 0$ to be specified later.

Define $\mathcal{N}_k$ as the set of such pairs (t, m) that $0 < m < k, 0 \leq t \leq n^k_k + n(k)$, there exist integers $E_1, E_2 \geq 0,\ E_1 + E_2 < E$ such that $\gamma_{t+E_2}(k]\alpha^k) \cap R^k_{m+1} \neq \emptyset,\ \gamma_{t-E_1}(k]\alpha^k) \cap R^k_{m-1} \neq$

$\emptyset$ and there exists a part $\gamma(t,m)$ of $\gamma_t(k]\alpha^k)$ in R^k_m, or of $\gamma_{k]}$ if $t = n^k_k + n(k)$, such that

$$\{ \text{ the part of } b(k]\alpha) \text{ between } \gamma(t,m) \text{ and } \hat{\gamma}_{k]}\} \subset D^k_{m,0} \tag{2.3}$$

analogously to (1.6), see Figure 2.

Figure 2

We claim that analogously to the right hand inequality of (1.3) we have for $(t,m) \in \mathcal{N}_k$

$$t \le n^k_{m+1} + E + N(\delta/E) \tag{2.4}$$

where $N(\varepsilon) := \sup\{n :$ there exists $\alpha \in \Sigma^d$ such that $\operatorname{diam}(\gamma_n(\alpha) \ge \varepsilon\}$. (The number $N(\varepsilon)$ is finite by (0.4).)

Indeed, denote the part of the curve being the concatenation of $\gamma_l(k]\alpha^k), l = t - E_1, ..., t + E_2$ in R^k_m joining R^k_{m-1} with R^k_{m+1} by Γ. suppose that $t - E_1 \ge n^k_m$ (otherwise the claim is proved). Then $f^{n^k_m}(\Gamma)$ joins a point $\xi \in \partial_U D^k_{m+1,m}$ in a curve

$$f^{n^k_m}(\gamma_t(k]\alpha^k) = \gamma_{t'+n^k_m}(\sigma^{n^k_m}(k]\alpha^k)), \quad t - E_1 \le t' \le t + E_2$$

with ∂D^k_m.

If $\xi \notin \partial^{\text{ess}}_U D^k_{m+1,m}$, then

$$t' < n^k_{n_{m+1}}$$

Otherwise there exists $n \le n^k_{m+1}$ such that $f^n(\xi) \in \partial U$. This is already outside U so the trajectory of ξ hits $\bigcup \gamma^j$ before the time n^k_{m+1} comes.

If $\xi \in \partial^{\text{ess}}_U D^k_{m+1,m}$ then by (2.1) at least one of the curves $f^{n^k_m}(\gamma_l(k]\alpha^k), t - E_1 \le l \le t + E_2$ has the diameter not less than δ/E. Hence

$$l - n_m \le N(\delta/E).$$

In both cases (2.4) is proved.

Define

$$A^+_k := \{m : 0 < m < n^k_k, \text{ there exists } t \text{ such that } (t,m) \in \mathcal{N}_k\}$$

and

$$A_k^- := \{1, ..., k-1\} \setminus A_k^+ \ .$$

As each set R_m^k for $m \in A_k^-$ is crossed by a part of $b(k]\alpha^k)$ between γ^j for respective j and $\hat{\gamma}_{k]}$ consisting of at least E edges and one edge cannot serve for more then two R_m^k's we obtain similarly to (1.7):

$$E \cdot \sharp A_k^- \leq 2(n_k^k + n(k) + 1)$$

Hence using (2.0) we obtain

$$\sharp A_k^+ \geq k - 1 - \frac{2}{E}(n(k) + n_k^k + 1) \geq k(1 - \frac{3}{E\kappa}) \tag{2.5}$$

Fix from now on $E > 3/\kappa$ and denote $\eta = 1 - \frac{3}{E\kappa} > 0$.

For every $0 < M \leq k$ define

$$A_k^+(M) := \{m \in A_k^+ : m < M\}$$

We claim that there exists $M_0 > 0$, not depending on k such that for every $M \geq M_0, \ M \in A_k^+$ we have

$$\sharp A_k^+(M) \geq \eta M. \tag{2.6}$$

This means that the property (2.6) true for $M = k$, see (2.5), extends miraculously to every $M \in A_k^+$ large enough. The proof of this claim is the same as for A_k^+:

Indeed $M \in A_k^+$ implies the existence of t such that $(t, M) \in \mathcal{N}_k$. By (2.3) $t \leq n_{M+1}^k + E + N(\frac{\delta}{E})$. Next we estimate $\sharp A_k^+(M)$ similarly as we estimated $\sharp A_k^+$ with $n_k^k + n(k) + 1$ replaced by $n_{M+1}^k + E + N(\frac{\delta}{E})$. We succeed for all M large enough.

Now we can conclude our Proof of Theorem B: Let $M_n := (\frac{1}{2}\eta)^{-n} M_0$. By (2.6) for every $k \geq 0$ and $n \geq 0$ there exists $m \in A_k^+$ such that $M_n \leq m < M_{n+1}$.

For each $n = 0, 1, ...$ there is only a finite number of blocks of symbols of the form $\pi_t(k]n_k)$ such that $(t, m) \in \mathcal{N}_k, \ m < M_{n+1}$. This is so by (2.4).

So there are constants $t_0 \geq 0$ and $\mathcal{D}_0 \in \Sigma_{t_0}$ and an infinite set

$$\begin{aligned} K_0 = &\{k \geq 0 : \text{ there exists } m \text{ such that} \\ &M_0 \leq m < M_1, \ (t_0, m) \in \mathcal{N}_k, \ \pi_{t_0}(k]\alpha^k) = \mathcal{D}_0\} \end{aligned}$$

In K_0 we find an infinite K_1 etc. by induction. For every $n > 0$ we obtain infinite $K_n \subset K_{n-1}$ and constants $t_n, \mathcal{D}_n$ such that

$$\begin{aligned} K_n = &\{k \in K_{n-1} : \text{ there exists } m \text{ such that} \\ &M_n \leq m < M_{n+1}, \ (t_n, m) \in \mathcal{N}_k, \ \pi_{t_n}(k]\alpha^k) = \mathcal{D}_n\} \end{aligned}$$

For $\alpha \in \Sigma^d$ such that $\pi_{t_n}(\alpha) = \pi_{t_n}(k]\alpha^k)$, we have that $b(\alpha)$ converges to q.

We assumed here that $t_n \to \infty$ as $n \to \infty$. If $\sup t_n = t_* < \infty$ then also $\mathcal{D}_n$ stabilize at $\mathcal{D}_*$ and by (2.3) $z_{t_*}(\mathcal{D}_*) = q$. Moreover there exists a sequence of integers $j_1, j_2, ... \in \{1, ..., d\}$ such that $\gamma_t(\mathcal{D}_*, j_1, j_2, ...) \equiv q$ for all $t \geq t_*$ so $b(\mathcal{D}_*, j_1, j_2, ...)$ converges to q.

(This is not an imaginary case. Consider a source $f(q) = q \in U$ and a tree $\mathcal{T}(q, \gamma^1, \gamma^2)$ such that $\gamma_1 \equiv q$ and γ_2 joins q with $q' \in f^{-1}(q), q \neq q'$. Then the above proof gives $b(2, 1, 1, ...)$ the branch for which $\gamma_n((2, 1, 1, ...)) =\equiv q'$ for every $n \geq 1$. ♣

Remark 2.1. It is curious that we did not need in the above proof neither the left hand side inequality (1.3): $t \geq m - \text{Const}$ for $(t, m) \in \mathcal{N}_k$, nor the sets $W(k, t, m)$. As mentioned already in Introduction no distortion estimates , i.e. no holomorphy was needed. The holomorphy of f is useful only to verify (0.4).

Proof of Corollary 0.1. This follows immediately from Theorem B and the following fact belonging to Pesin Theory:

Let X be a compact subset of $\overline{\mathbb{C}}$ and F be a holomorphic mapping on a neighbourhood of X such that $F(X) = X$. Let $\mu \in M_e^{\chi+}(F)$. Let $(\tilde{X}, \tilde{F}, \tilde{\mu})$ be a natural extension (inverse limit) of (X, F, μ). Denote by π the projection to the 0 coordinate, $\pi : \tilde{X} \to X$ and by π_n the projection to an arbitrary n-th coordinate.

Then for $\tilde{\mu}$-a.e. $\tilde{x} \in \tilde{X}$ there exists $r = r(\tilde{x}) > 0$ such that univalent branches F_n of F^{-n} on $B(\pi(x), r)$ for $n = 1, 2, ...$ such that $F_n(\pi(x)) = \pi_{-n}(x))$, exist. Moreover for an arbitrary $l : \exp(-\chi_\mu) < \lambda < 1$ (not depending on $\tilde{x}$) and a constant $C = C(\tilde{x}) > 0$

$$|F_n'(\pi(x))| < C\lambda^n \quad \text{and} \quad \frac{|F_n'(\pi(x))}{|F_n'(z)|} < C$$

for every $z \in B(\pi(x), r)$, $n > 0$, (distances and derivatives in the Riemann metric on $\overline{\mathbb{C}}$).

Moreover r and C are measurable functions of $\tilde{x}$.

To prove Corollary (0.1) observe that the above fact implies the existence of numbers $r, C > 0$ and a set of positive measure $\tilde{\mu}$: $\tilde{Y} \subset \tilde{X}$ such that the above properties hold for every $\tilde{x} \in \tilde{Y}$ and for these r and C. Ergodicity of μ implies ergodicity of $\tilde{\mu}$. So by Birkhoff Ergodic Theorem there exists a set $\tilde{Z} \subset \tilde{X}$ of full measure $\tilde{\mu}$ such that for each point $\tilde{x} \in \tilde{Z}$ its forward orbit by $\tilde{F}$ hits $\tilde{Y}$ at the positive density number of times. These are *good times* and $\pi(\tilde{x})$ is a *good* point in the sense of Introduction (provided they satisfy (0.3')). ♣

Section 3. External rays.

Let $W_1 \subset W$ be open, connected, simply-connected bounded domains in the complex plane $\mathbb{C}$ such that $\text{cl} W_1 \subset W$. Let $f : W_1 \to W$ be a holomorphic proper map "onto" W of degree $d \geq 2$. We call such a map f a *polynomial-like map*. Denote $K = \bigcap_{n\geq 0} f^{-n}(W)$. This set K is called a filled in Julia set [DH]. We can assume that ∂W is smooth. Let M be an arbitrary smooth function on a neighbourhood of $\mathrm{cl}W \setminus W_1$ not having critical points, such that $M|_{\partial W} \equiv 0$ and $M|_{\partial W_1} \equiv 1$ and $M \circ f = M - 1$ whereever it makes sense. Extend M to $W \setminus K$ by $M(z) = M(f^n(z)) + n$ where n is such that $f^n(z) \in W \setminus W_1$.

Fix $\tau : 0 < \tau < \pi$ and consider curves $\gamma : [0, \infty) \to \mathrm{cl}W \setminus K$, intersecting lines of constant M at the angle τ , (this demands fixing orientations), not containing critical points for M with $\gamma(0) \in \partial W$ and converging to K as the parameter converges to ∞. One can change the standard euclidean metric on $\mathbb{C}$ so that τ is the right angle and think about gradient lines in the new metric. We call such a line a *smooth τ-ray.* Instead of parametrizing such a curve with the gradient flow time we parametrize it by the values of M. Limits of smooth τ-rays are called *τ-rays.* They can pass through critical points of M. (Such a τ-ray enters a critical point along a stable separatrix and leaves it along an unstable one, the closest clockwise or counter-clockwise. If it hits again a critical point for the first time it leaves it along an unstable separatrix on the same side from which it came to the previous critical point, see [GM] and [LevP] for the more detailed description. See Fig. 3:

Figure 3.

Proof of Theorem C. Divide each τ-ray γ into pieces γ_n, $n \geq 1$ each joining $f^{-n}(\partial W)$ with $f^{-n-1}(\partial W)$.

One easily proves the fact corresponding to (0.4):

$$\mathrm{length}(\gamma_n) \to 0 \text{ as } n \to \infty \tag{3.0}$$

uniformly over τ-rays γ.

The proof is the same as that of the implication (0.5) $\Rightarrow$ 0.4) in Remark 0.4. We have univalent branches of f^{-k} for all k on neighbourhoods of γ_n for external rays γ, neighbourhoods not depending on k, for n large enough, because then critical points of f in $W \setminus K$ do not interfere. There is finite number of them and their forward trajectories escape out of W.

For our q find *significant* telescopes Tel^k as in Section 2, where n_0-significant means here that D^k_k intersects $\gamma^k_{n(k)}$ for a τ-ray γ^k and $n(k) \le n_0$ a constant independent of k. This is possible by (3.0).

Denote by $\gamma^{k]}$ the τ-ray containing a point of $f^{-n_k}(\gamma^k)$ being in $D^k_{k,0}$.

We consider, similarly to Section 2, (2.3), the set

$$\mathcal{N}_k = \{(t,m):\ \text{the same conditions as in Section 2, in particular } \gamma^{k]}_t \cap R_m \neq \emptyset\}$$

Similarly we define A^+_k and $A^+_k(M), M \le k$.

The same miracle that

$$\sharp A^+_k(M) \ge \eta M$$

takes place for $M \ge M_0,\ M \in A^+_k$.

To get it we prove and use the estimate $t \le m+\mathrm{Const}$ for $(t,m) \in \mathcal{N}_k$.

Because for $M_0 \le m < M_1, (t,m) \in \mathcal{N}_k$ integers t are uniformly bounded over all k by say T_0, the parts $\gamma'^{k]} = \bigcup_{l=1}^{T_0} \gamma^{k]}_l$ of respective τ-rays $\gamma^{k]}$ have a convergent subsequence and the limit ray (joining levels 0 and T_0) intersects R_m.

Choosing consecutive subsequences we find a limit ray converging to q. ♣

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