Quasiconformal mappings, operators on Hilbert space, and local formulae for characteristic classes

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dedicated to Bill Browder on his 60th birthday

Abstract. We give local formulae for the characteristic classes of a quasiconformal manifold using the subspace of exact forms in the Hilbert space of middle dimensional forms. The method applies to combinatorial manifolds and all topological manifolds except certain ones in dimension four.

By rolling, or better pressing, a sphere $S^{2t}$ all around the manifold $M^{2t}$, we will construct bounded operators on the space of $L^2$ middle dimensional forms of $M^{2t}$ analogous to the Ahlfors-Beurling operator on the Riemann sphere $\hat{\mathbb{C}}$

(1) \[ \varphi(z, \bar{z})dz = \eta \mapsto S\eta = \left( \frac{1}{2\pi i} \int_{\hat{\mathbb{C}}} \frac{\varphi(\zeta, \bar{\zeta})d\zeta}{(z - \zeta)^2} \right) dz. \]

The kernel of this operator is the biform $\frac{dz \cdot d\zeta'}{(z - \zeta')^2}$ on $\hat{\mathbb{C}} \times \hat{\mathbb{C}}$. Using these operators and their kernels we obtain explicit local cycle representatives of the Hirzebruch-Thom characteristic classes for any quasiconformal manifold. At the end of the introduction we explain how the construction applies to topological manifolds. This answers a question raised by Bill Browder in lectures at Princeton in 1964.

Generalizations of (1) to all even spheres $S^{2t}$ with any bounded measurable pointwise * operator on middle forms were constructed for the quasi conformal Yang Mills theory [1] by explicit formulae. For example, here we write

(2) \[ S_\mu = (1 + \mu)^{-1} (\mu + S) S(\mu + S)^{-1} (1 + \mu) \quad , \quad |\mu| < 1, \]
for the standard sphere $S^{2\ell}$ can be written out on $R^{2\ell}$ as a quadratic expression in Riesz transforms see [2]. When $\ell$ is even, these bounded operators could be called \textit{conformal signature operators}, being essentially the phases of the usual signature operators when the $*$ is smooth (see the proof of theorem 2 part 4). When $\ell$ is odd they are generalizations of the Ahlfors-Beurling operator to general “curved” measurable conformal structures.

For a general even dimensional quasi conformal manifold provided with a bounded measurable conformal structure or more generally a bounded measurable $*$ on middle forms we do the following:

i) locally copy via charts the structure on $M^{2\ell}$ by structures on $S^{2\ell}$,

ii) get locally defined operators on $M^{2\ell}$ from the operators on $S^{2\ell}$,

iii) collect these together on $M^{2\ell}$ using a partition of unity to construct global operators $S$ on the middle dimensional forms (§1).

Let $I(\ell)$ denote the ideal of compact operators $A$ on Hilbert space satisfying $\mu_n = O(n^{-1/2\ell})$ where $\mu_n = \text{distance in norm between } A$ and rank $n$ operators. The reader may recall that any degree one smoothing operators in dimension $2\ell$ e.g. the Poincaré lemma operator, belongs to the ideal $I(\ell)$. The standard notation for $I(\ell)$ is $L^{2\ell, \infty}$ (§1).

\textbf{Theorem 1.} Given quasi conformal $M^{2\ell}$ with a bounded measurable conformal structure or more generally a bounded measurable $*$-structure on the $\ell$-forms, the local construction yields an operator $S$ which is determined by $*$ up to the ideal $I(\ell)$. Moreover any such $S$ satisfies i) $S$ agrees mod $I(\ell)$ with the identity on exact $\ell$-forms ii) $S$ anticommutes modulo $I(\ell)$ with the involution $\gamma$ associated to $*$ ($\gamma = *$ if $\ell$ is even, $\gamma = i*$ if $\ell$ is odd).

Let us say an operator $S$ on the Hilbert space $\mathcal{H}$ of middle dimensional forms satisfying i) and ii) of Theorem 1 belongs to the \textit{Hodge class}. It is immediate (§1) that given $*$ any two Hodge class operators differ by a compact operator in $I(\ell)$. Note also that i) and ii) imply iii) $S^2 \equiv I \bmod I(\ell)$.

There is a canonical non-local Hodge class operator $S_*$ for the pair $(M, *)$. It is the involution defined up to finite rank by $S_*$ is the identity on exact forms and $S_*$ anticommutes with $\gamma$. The projectors associated with $S_*$ are compatible with the usual Hodge decomposition of middle dimensional forms. Examples are the $S_\mu$ defined above for $S^{2\ell}$.

An interesting analytical consequence of the formula (2) in the proof of Theorem 1 is the

\textbf{Corollary 2.} Any Hodge class operator $S$ on $\mathcal{H}$ defines a Fredholm module in the precise sense that for the sup norm dense subalgebra of continuous functions on $M^{2\ell}$ satisfying $\int_M |df|^{2\ell} < \infty$, the commutators $[S, f]$ where $f$ denotes the multiplication operator associated to $f$, belong to the ideal $I(\ell)$ of compact operators.

The theorem and corollary answers anew the question of Singer [10] about “constructing the operator” cf. [6], [19].
To any Hodge class operator we apply the algebraic procedure of [3] to construct
a refined Hodge class operator $H$ satisfying the further conditions a) $H^2 = I$ on $\mathcal{H}$ b)
$H\gamma + \gamma H$ is trace class. This algebraic process Theorem 2 (§2) preserves locality.

The trace class operator $L = (H \gamma + \gamma H) H = H \gamma H + \gamma$ can be used in a simple way
to construct representations of the Hirzebruch-Thom characteristic classes or rather their
Poincaré dual homology classes. Assume $M^{2\ell}$ is oriented, then the kernel of $L$, $L(x, x')$
is a biform on $M \times M$ of bigree $(\ell, \ell)$. The support of $L(x, x')$ is near the diagonal if we
started with a locally constructed Hodge class operator. The trace $L(x)$ on the diagonal
is a $2\ell$ form, or since $M$ is oriented it is a measure in the Lebesgue measure class.

More generally, consider the cyclic expression \{trace $L(x_0, x_1) L(x_1, x_2) \ldots L(x_{2q}, x_0)$\}
which can be considered as either a top dimensional form or as a measure on $M \times M \times \ldots \times M$
($2q + 1$ factors), supported near the diagonal.

**Theorem 3.** The cyclic expressions \{trace $L(x_0, x_1) L(x_1, x_2) \ldots L(x_{2q}, x_0)$\} when
considered as measures on $M^{2q+1}$ near the diagonal define Alexander Spanier cycles. If $\ell$
is even, these cycles for $q$ even and if $\ell$ is odd, these cycles for $q$ odd represent the dual
Hirzebruch-Thom characteristic homology classes times $2^{q+1} (2\pi i)^{-q} q! / 2q!$. In particular
if $\ell$ is even $L(x) = \text{trace} L(x, x)$ is a locally constructed measure whose total mass is twice
the signature of $M^{2\ell}$.

The algebraic construction used above to refine Hodge class operator, $S \rightarrow H$, and
the check that the odd cyclic expressions in the kernel $L(x,y)$ of $L = H \gamma H + \gamma$ define
cycles cover only a page or two. Behind this calculation is the idea that any Hodge duality
operator defines a $K$-homology element because of the corollary, and we know since [4] and
[6] what this element and its Chern character should be. We also know from [5] an explicit
construction of the Chern character starting from $K$-theory of an algebra and arriving in
the cyclic cohomology of the algebra which for a manifold is related to Alexander Spanier
[3]. Concretizing these ideas is the page or two. The connection with Hirzebruch-Thom
classes is filled in using the index theorem as in [6], [1] and [3].

The local construction of the operators and the analytical content of the corollary are
easy consequences of the formula (2) on spheres relating the (canonical non local) Hodge
operator $S_\mu$ for any measurable $\ast$ to the (canonical non local) Hodge operator $S$ for the
standard structure on $S^{2\ell}$. This formula also relates to the other discussions.

On the Riemann sphere one knows the remarkable measurable Riemann mapping
theorem that any bounded measurable conformal structure is related by a quasiconformal
homeomorphism $w = \varphi(z)$ to the standard one. Since $S_\mu$ is given by the kernel $\frac{dw \, dw'}{(w-w')^2}$
on $(0, 1)$ forms while $S$ is given by the kernel $\frac{dz \, dz'}{(z-z')^2}$ on $(0, 1)$ forms our basic formula (2)
has a direct relationship with the measurable Riemann mapping theorem. Namely, we can
calculate $w = \varphi(z)$ from the kernel of $S_\mu$ by expanding out the formula (2).

In dimension 4 the operators $S_\mu$ were used in the analytical underpinnings of the Yang-
Mills discussion. The formulae (2) show since $|\mu| < 1$ that $S_\mu$ determines isomorphisms on
$L^p$-forms for $p$ a little greater than two. Applied to the curvature 2-forms this gives the
extra regularity to get past the critical Sobolev exponent for the Yang-Mills connections and gauge transformations. A corollary of this theory [1] was that some closed $M^4$ have infinitely many distinct quasiconformal structures, and that some topological $M^4$ have no quasiconformal structure.

Thus our local constructions for characteristic classes based on (2) are higher dimensional relatives of the measurable Riemann mapping in dimension two and the Yang-Mills theory in dimension four.

Outside dimension four there is a proof [7] independent of the theory of [12] that stable topological manifolds have a quasiconformal structure unique up to isotopy. Here stable refers to the pseudogroup of homeomorphisms of $R^n$ in the connected components of the identity or a reflection [9]. Thus this paper defines local characteristic classes for stable topological manifolds independent of Novikov’s theory [12].

We can also apply our constructions or those of [7] and [8] to general topological manifolds but this uses Kirby’s result on the stable homeomorphism conjecture [9]. The proof in [9] properly contains Novikov’s theory needed for his original proof of the topological invariance of rational Pontryagin classes.

**Historical Remark.** If $g_{\alpha\beta}$ denotes the Jacobian matrices of the overlap homeomorphisms of charts covering a manifold $M^n$, the curving or non-flatness of $M^n$ is measured by $\theta_{\alpha\beta} = \{g^{-1}_{\alpha\beta} d g_{\alpha\beta}\}$ which is a Cech 1-cocycle with values in matrices of 1-forms. Thus if the overlap homeomorphisms have Lipschitz derivatives (or even 2nd derivatives in $L^n$) there is a Chern-Weil type construction of characteristic forms by forming products and traces.

By considering normal bundles to smooth foliations and the Bott vanishing theorem [15] one finds serious obstructions to the possibility of reducing this smoothness requirement and staying in the context of differential forms.

In our context of Lipschitz or quasiconformal manifolds we have exactly one less derivative than required above.

It seems natural then try to interpret $g^{-1} d g$ as a distribution or as an Alexander Spanier cochain. This was attempted in 1976 when the possibility of having Lipschitz or quasiconformal coordinates appeared. However, the distribution idea founders because of the impossibility of forming products. This difficulty is removed in Alexander Spanier at the expense of non-commuting products. But then the trace step in the classical Chern-Weil procedure becomes problematical. In other words there is an analytical qua algebraic barrier to copying the pointwise “curvature” route to characteristic classes for quasiconformal or Lipschitz charts.

In this paper these difficulties are surmounted by using trace ideals of operators on Hilbert space [5] and an algebraic addition to the Chern-Weil algorithm coming from cyclic cohomology [5]. The quasiconformal charts provide enough analysis to “quantize the manifold” in the sense of constructing a Hilbert space and a relevant operator replacing curvature.
This “quantized curvature” is then treated algebraically in a manner guided by the formulae of cyclic cohomology. Perhaps the essence of the latter point is that the cocycles in the cyclic context are just those multilinear functionals which when applied to (projector, projector, ...) remain constant when the projector is varied by a homotopy.

The reader may recall that exactly this kind of consideration appears classically when showing the Chern-Weil forms are cohomology invariants.

In summary we have treated a problem with one missing derivative in a classical context using the ideas and tools of “non-commutative geometry” [16].

1. Preliminaries on quasiconformal geometry

A quasiconformal (qc) homeomorphism \( h \) between two open domains \( \Omega_1, \Omega_2 \) in \( \mathbb{R}^n \) is a homeomorphism with the property that relative distances are boundedly distorted, i.e. for each \( x \) in \( \Omega_1 \),

\[
\lim_{\varepsilon \to 0} \frac{\max \{|h(x) - h(y)|; |x - y| = r\}}{\min \{|h(x) - h(y)|; |x - y| = r\}} = K(x) \leq K < \infty.
\]

We also assume the analogous statement for \( h^{-1} \).

Gehring [15] proved that when \( n > 1 \) a qc-homeomorphism is a.e. differentiable; moreover, the first order partial derivatives of the component functions of \( h \) belong to the Banach space \( L^{n+\varepsilon}_{\text{loc}} \), where \( \varepsilon = \varepsilon(K) > 0 \). It follows that \( h \) is a Hölder continuous function with exponent \( \varepsilon \), \( h \) is non-singular with respect to the Lebesgue measure class, and the best \( K \) that works in (1) almost everywhere for \( h \) also works for \( h^{-1} \).

Let \( g \) be an arbitrary Euclidean metric on the tangent space to \( \mathbb{R}^n \) at some of its points \( x_0 \). Recall that the metric \( g \) and all other similar metrics \( rg \), where \( r \) is an arbitrary positive real number, define the conformal class \([g]\) of the metric \( g \).

If \([g_0]\) and \([g_1]\) are two conformal structures, the conformal distance between them is by definition

\[
d([g_0], [g_1]) = \log \frac{\max \{|v|_{g_1}; |v|_{g_0} = 1\}}{\min \{|v|_{g_1}; |v|_{g_0} = 1\}},
\]

From now on we suppose that the dimension \( n = 2\ell = \) even, and we choose the standard orientation on \( \mathbb{R}^n \).

Let \( \Lambda \) denote the vector space of all differential forms of degree \( \ell \) at \( x_0 \). For any Euclidean metric \( g \) as above, the Hodge star operator \( *_g \) associated to \( g \) defines an endomorphism

\[*_g : \Lambda \to \Lambda\]

with
\[ *_g^2 = (-1)^\ell. \]

The main property of the operator \(*_g\) acting on \(\Lambda\) is that it remains unchanged under dilations of the metric \(g\), i.e. it depends only on the conformal class of \(g\).

We let \(\gamma_g\) be the involution of the complexification \(\Lambda_C\) of \(\Lambda\) given by

\[ \gamma_g = i^\ell *_g. \]

We let \(\Lambda^\pm(g)\) be the \(\pm 1\) eigenspaces of \(\gamma_g\). These subspaces are maximal definite subspaces for the quadratic form

\[ \omega \rightarrow \omega \wedge \omega. \]

The conformal distance between two conformal classes \([g_0]\) and \([g]\) may be estimated in terms of the relative position of the eigenspaces \(\Lambda^\pm\). Indeed, there exists a unique linear mapping

\[ \mu : \Lambda^-(g_0) \rightarrow \Lambda^+(g_0) \]

with the property that the graph of \(\mu\) is precisely \(\Lambda^-(g)\). The operator norm of \(\mu\) relative to the metric \(g_0\) satisfies

\[ |\mu|_{g_0} < 1 \]

and

\[ \frac{1}{2} \log \frac{1 + |\mu|_{g_0}}{1 - |\mu|_{g_0}} \leq d([g_0],[g_1]) \leq \log \frac{1 + |\mu|_{g_0}}{1 - |\mu|_{g_0}}. \]

A field \(c(x)\) of conformal structures over a domain \(U\) in \(\mathbb{R}^n\) is called a bounded measurable (bm) conformal structure on \(U\) if there exists a Riemannian metric \(g\) over \(U\), whose components are measurable functions, such that for any \(x\) in \(U\), \(c(x) = [g(x)]\), and

\[ d([g(x)],[e]) \leq C < \infty, \]

where \(e\) denotes the standard Euclidean metric.

Equivalently, \(c\) is a bounded measurable conformal structure iff the corresponding field of endomorphisms \(\mu\), relative to \(e\), is a matrix field with measurable entries, and

\[ \|\mu\|_{\infty} = \sup |\mu(x)| < 1. \]
If \( c \) is a bounded measurable conformal structure on \( U \), then \( \gamma_c \) is a field of matrices with bounded measurable entries. If \( c \) is a bounded measurable conformal structure on \( V \) and \( h: U \to V \) is a qc-homeomorphism, then \( h^*c \) is a bounded measurable conformal structure on \( U \) because a qc-homeomorphism induces uniformly quasi-homotheties on the tangent spaces, a.e.

A qc-manifold is a topological manifold equipped with an atlas whose changes of coordinates are qc-homeomorphisms. It possesses a well defined measure class, the Lebesgue measure class, since qc-homeomorphisms are absolutely continuous. The tangent bundle of a qc-manifold is a measurable real vector bundle.

A bounded measurable conformal structure on a qc-manifold is a field of conformal structures on its tangent spaces whose restriction to any qc-chart is bounded measurable. Any paracompact qc-manifold has such structures.

On a compact smooth manifold, a conformal structure is bounded iff the conformal distance (defined point by point) between it and the underlying conformal structure of a smooth Riemannian metric, is a bounded function.

On a compact qc-manifold \( M \) the space \( L^{n/r}(M, \wedge^r) \) of \( r \)-forms with coefficients in \( L^{n/r}, n = \dim M \), is well defined. Any bounded measurable conformal structure specifies a Banach space norm on \( L^{n/r} \) by:

\[
(||\omega||)^{n/r} = \int_M |\omega|^{n/r}.
\]

Given \( \omega_1 \in L^{n/r}(M, \wedge^r), \omega_2 \in L^{n/r+1}(M, \wedge^{r+1}) \) we write \( d\omega_1 = \omega_2 \) iff this holds, in the sense of distributions, in any qc local chart. This yields [17] a densely defined closed operator \( d: L^{n/r} \to L^{n/r+1} \) which commutes with qc homeomorphisms. We let \( \text{Im}d \) be the image of \( d \), it is closed in \( L^{n/r+1} \) provided \( r \geq 1 \) [17] [1].

The underlying topological vector spaces only depend on the quasiconformal structure.

2. Statement of the main result

Let \( M \) be a compact oriented quasiconformal manifold of even dimension \( 2\ell \). Let \( [g] \) be a bounded measurable conformal structure on \( M \) and \( \gamma, \gamma^2 = 1 \), the associated \( \ast \) operator in the Hilbert space \( \mathcal{H} = L^2(M, \wedge^{\ell}) \) of square integrable forms of degree \( \ell \) on \( M \).

Given an open neighborhood \( U \) of the diagonal in \( M \times M \) and any bounded operator \( \mathcal{T} \) in \( \mathcal{H} = L^2(M, \wedge^{\ell}) \), we say that Support \( (\mathcal{T}) \subset U \) iff the following holds for any open \( V \subset M \)

\[
\omega \in \mathcal{H}, \text{ Support } \omega \subset V \Rightarrow \text{ Support } (\mathcal{T}\omega) \subset U \circ V
\]

where \( U \circ V = \{ x \in M ; \exists y \in V , (x, y) \in U \} \).

For any \( p \in [1, \infty[ \) the following conditions define two sided ideals of compact operators in Hilbert space. We let for any compact operator \( \mathcal{T} \) in \( \mathcal{H} \), \( \mu_n(\mathcal{T}) \) be the \( n \)-th characteristic
value of \( T \), i.e. the \( n \)-th eigenvalue of \( |T| = (T^*T)^{1/2} \), or the distance in operator norm between \( T \) and rank \( n \) operators.

\[
\mathcal{L}^p(\mathcal{H}) = \left\{ T \text{ compact} ; \sum_{1}^{\infty} \mu_n(T)^p < \infty \right\}
\]

(3) \[
\mathcal{L}^{(p,\infty)}(\mathcal{H}) = \{ T \text{ compact} ; \mu_n(T) = O(n^{-1/p}) \}.
\]

We can now define the following key notion:

**Definition 1.** Let \( U \) be an open neighborhood refining that of Hodge class operators in the introduction of the diagonal in \( M \times M \). A \( U \)-local Hodge decomposition is a bounded operator \( H \) in \( L^2(M, \wedge^t T^*_c) \) such that:

\( \alpha \) \( H^2 = 1 \)

\( \beta \) Support \( H \subset U \)

\( \gamma \) \( (H - 1)/\text{Im} \ d \in \mathcal{L}^{(2t,\infty)} \) (= \( I(\ell) \) of the introduction)

\( \delta \) \( H \gamma + \gamma H \in \mathcal{L}^1 \) (= trace class by definition).

Giving \( H \) is the same as giving the decomposition of \( \mathcal{H} = L^2(M, \wedge^t T^*_c) \) as the linear sum of the two definition closed subspaces:

\[
\{ \xi \in \mathcal{H} ; \ H \xi = \pm \xi \}.
\]

We shall now explain how to construct for each \( q \in \{0, 1, \ldots, \ell\} \) an Alexander Spanier cycle on \( M \) from a \( U \)-local Hodge decomposition. To define Alexander Spanier homology on a compact space \( X \) we consider for each integer \( d \) the linear space \( A_d \) of totally antisymmetric measures \( \sigma \) on \( X^{d+1} \). Such a measure \( \sigma \) is uniquely determined by the value of \( \sigma(\varphi) = \int \varphi d\sigma \) on bounded borel antisymmetric functions \( \varphi \) on \( X^{d+1} \). We let \( \delta : A_d \to A_{d-1} \) be the boundary operator given by the equality:

\[
(\delta \sigma)(\varphi) = \sum_{0}^{d} (-1)^j \int \varphi(x_0, \ldots, x_{j}, \ldots, x_d) \ d\sigma \quad \forall \varphi.
\]

Let \( U \) be a neighborhood of the diagonal in \( X \times X \). We shall say that \( \sigma \in A_d \) is \( U \)-local iff:

\[
\text{Support} \sigma \subset \{(x_j) \in X^{d+1} ; \ (x_i, x_j) \in U \quad \forall i, j \in \{0, 1, \ldots, d\}\}.
\]
One checks that condition (6) is preserved by $\delta$. This defines the complex $(A_U, \delta)$ of $U$-local elements of $(A, \delta)$. The Alexander Spanier homology $H_*(X, \mathbb{R})$ is obtained as the projective limit $\lim \ H^*(A_U, \delta)$, when $U$ runs through all open neighborhoods of the diagonal.

Given a measure space $(X, \nu)$ and a measurable hermitian vector bundle $\Lambda$ on $X$, the Hilbert-Schmidt operators in $\mathcal{H} = L^2(X, \Lambda)$ are all given [18] by measurable kernels,

$$k(x, y) \in \text{Hom}(\Lambda_y, \Lambda_x) \quad x, y \in X$$

such that:

$$\int_X \text{trace}(k(x, y)^* \ k(x, y)) \ d\nu(x) \ d\nu(y) < \infty.$$  \hspace{1cm} (5)

In particular, for any such kernel $k$ the following expression defines a measure $\sigma$ on $X^{d+1}$ for any $d \geq 1$:

$$\sigma(\varphi) = \int_{X^{d+1}} \text{trace}(k(x_0, x_1) \ k(x_1, x_2) \ldots k(x_d, x_0)) \ \varphi(x_0, \ldots, x_d) \prod d\nu(x_i)$$

as follows from the inequality:

$$|\sigma(\varphi)| \leq \|k\|^{d+1}_2 \ |\varphi|_\infty \quad \forall \varphi \in L^\infty(X^{d+1}, \nu^{d+1}).$$  \hspace{1cm} (7)

We shall use the notation:

$$\text{trace}(\wedge^{d+1} k) = \text{Total antisymmetrisation of } \sigma$$  \hspace{1cm} (8)

(where $\sigma$ is associated to $k$ by (6)).

For $d = 0$ this formula continues to make sense provided the operator in $\mathcal{H} = L^2(X, \Lambda)$ associated to $k$ is of trace class [18].

We can now state the main result of this paper.

**Theorem 2.** Let $M$ be a compact oriented quasiconformal manifold of even dimension $2\ell$, $\gamma$ the $\mathbb{Z}/2$ grading of $\mathcal{H} = L^2(M, \wedge^\ell T^\ell_M)$ associated to a measurable bounded conformal structure [9] on $M$ and $U$ a neighborhood of the diagonal in $M \times M$.

1) There exists a locally constructed $U$-local Hodge decomposition $H$.

2) Let $H$ be a $U$-local Hodge decomposition, $k = H \gamma H + \gamma$ and $d$ an even integer. Then the measure $\sigma = \text{trace}(\wedge^{d+1} k)$ in a $U^d$ local Alexander Spanier cycle of dimension $d$.

3) The homology class of $\sigma$ among $U^q$ local cycles, $q = d(6\ell + 2)$ is independent of the choice of $H$.

4) The homology class of $\sigma$ is equal to $\lambda_d(L_{2\ell-d} \cap [M])$ where $L$ is the Hirzebruch-Thom $L$-class and where $\lambda_d = 2^{2m+1} (2\pi i)^{-m} \frac{m!}{(2m)!}$, $2m = d$.  

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3. Local construction of a $U$-local Hodge decomposition

Let $M$ be a quasiconformal manifold and $[g]$ a bounded measurable conformal structure on $M$. In this section we shall show how to construct local Hodge decompositions $H$ using a covering of $M$ by domains of $qc$ local charts:

$$\rho_\alpha : V_\alpha \to S^{2t}.$$  

The obtained formula for $H$ will be algebraic in terms of the following ingredients:
1) A partition of unity subordinate to the covering $(V_\alpha)$ of $M$.
2) The pull back by $\rho_\alpha$ of Hodge decompositions on $S^{2t}$ associated to a bounded measurable conformal structure which agrees with $\rho_\alpha[g]$ on $\rho_\alpha(V_\alpha)$.

We shall begin by describing the canonical Hodge decomposition on $S^{2t}$ associated to a bounded measurable conformal structure.

$\alpha$) Canonical Hodge decomposition on $S^{2t}$

Let $[g_0]$ be the standard conformal structure on the sphere $S^{2t}$, and $[g]$ an arbitrary bounded measurable conformal structure on $S^{2t}$. Let $\gamma_0$, $\gamma$ be the corresponding $\ast$ operations in the vector bundle $\Lambda$ of middle dimensional forms. If we let $\Lambda_\pm$ be the two eigenspaces of $\gamma_0$ we get two subbundles of $\Lambda$ and a unique measurable bundle homomorphism:

$$\mu_+ : \Lambda_- \to \Lambda_+$$

whose graph at each point $p \in S^{2t}$ gives the subspace:

$$\{ \omega \in \Lambda_p \ ; \ \gamma \omega = -\omega \}.$$  

We endow the vector bundle $\Lambda$ with the metric associated to the standard conformal structure $[g_0]$. The boundedness of the measurable conformal structure $[g]$ then means that:

$$||\mu_+||_\infty = \text{Sup}_{S^{2t}} |\mu_+(p)| < 1.$$  

Let $\mu = \begin{bmatrix} 0 & \mu^+ \\ \mu^*_+ & 0 \end{bmatrix}$ viewed as an endomorphism of the vector bundle $\Lambda$. One has:

(1)  
$$||\mu^+||_\infty = \text{Sup}_{S^{2t}} |\mu^+(p)| < 1.$$  

(2)  
$$\mu \gamma_0 = -\gamma_0 \mu , \ \mu = \mu^*$$  

(3)  
$$\gamma = (1 + \mu) \gamma_0 (1 + \mu)^{-1}.$$  

Indeed $\gamma^2 = 1$ and if $\gamma_0 \xi = -\xi$ one has $\gamma(1 + \mu)\xi = -(1 + \mu)\xi$ as desired.
Finally on the vector bundle $\Lambda$ the metric associated to the conformal structure $[g]$ is given by:

\[
(4) \quad \langle \omega_1, \omega_2 \rangle_g = \langle \omega_1, \gamma_0 \gamma \omega_2 \rangle_{g_0} = \left\langle \omega_1, \left( \frac{1 - \mu}{1 + \mu} \right) \omega_2 \right\rangle_{g_0}
\]

Note that since $\|\mu\| < 1$ the operator $(1 - \mu)(1 + \mu)^{-1}$ is positive. We now consider the Hilbert space $\mathcal{H}_0 = L^2(S^{2t}, \Lambda)$ with the inner product given by $[g_0]$. We view all the above endomorphisms of the vector bundle $\Lambda$ as operator in $\mathcal{H}_0$. The equalities (2) and (3) continue to hold.

The standard Hodge decomposition on $S^{2t}$ decomposes $\mathcal{H}_0$ as the direct sum of two orthogonal subspaces, the exact forms and the coexact forms. Let $H_0$ be the linear operator such that $H_0 \omega = \omega$ for any exact form and $H_0 \omega = -\omega$ for any coexact form. One has:

\[
(5) \quad H_0 = H_0^*, \quad H_0^2 = 1
\]

\[
(6) \quad H_0 \gamma_0 = -\gamma_0 H_0.
\]

Moreover since $H_0$ is a standard singular integral operator of order 0 the following subalgebra $A(S^{2t})$ of the algebra of continuous functions $C(S^{2t})$ contains all smooth functions $[2]$ and is therefore norm dense:

\[
(7) \quad A(S^{2t}) = \{ f \in C(S^{2t}) ; \ [H_0, f] \in L^{(2t, \infty)} \}
\]

where $f \in C(S^{2t})$ is considered as a multiplication operator in $L^2(S^{2t}, \Lambda)$ and recall $L^{(2t, \infty)}$ is the two sided ideal of compact operators in $\mathcal{H}_0$ given by the condition:

\[
(8) \quad T \in L^{(2t, \infty)} \Leftrightarrow \mu_n(T) = O(n^{-1/2t})
\]

where $\mu_n(T)$ is the $n$-th characteristic value of $T$.

Let us now consider on the locally convex vector space $\mathcal{H}_0$ the new inner product given by the metric $g$, using (4) this can be expressed by:

\[
(9) \quad \langle \omega_1, \omega_2 \rangle = \left\langle \omega_1, \frac{1 - \mu}{1 + \mu} \omega_2 \right\rangle_0.
\]

The Hodge decomposition on $S^{2t}$ relative to the bounded measurable conformal structure $[g]$ is given by:
Proposition 1. a) The orthogonal complement for the inner product (9) of $\text{Im} \, d = \{\omega \mid H_0 \, \omega = \omega\}$ is equal to $\gamma(\text{Im} \, d)$.

b) Let $H$ be the linear operator equal to 1 on $\text{Im} \, d$ and to $-1$ on $\gamma(\text{Im} \, d)$, then:

$$H = (1 - \mu)^{-1} (H_0 - \mu) H_0 (H_0 - \mu)^{-1} (1 - \mu).$$

Proof. Since $H_0^2 = 1$ and $\|\mu\| < 1$ the operator $H_0 - \mu$ is invertible. Let us consider the operator $T = (1 - \mu)^{-1} (H_0 - \mu) H_0 (H_0 - \mu)^{-1} (1 - \mu)$. It is conjugate to $H_0$ so that $T^2 = 1$. Its eigenspaces are obtained from those of $H_0$ by applying $(1 - \mu)^{-1} (H_0 - \mu)$. Hence $\{\xi \mid T \xi = \xi\} = \text{Im} \, d$. One has $\gamma = (1 - \mu)^{-1} \gamma_0 (1 - \mu)$ and hence $\gamma T = -T \gamma$ which shows that $\{\xi \mid T \xi = -\xi\} = \gamma(\text{Im} \, d)$. The orthogonality of $\text{Im} \, d$ with $\gamma_0(\text{Im} \, d)$ for the inner product $(\cdot)_0$ implies the orthogonality of $\text{Im} \, d$ with $\gamma(\text{Im} \, d)$ for the inner product (9), using $\gamma_0 \gamma = \frac{1 - \mu}{1 + \mu}$. Thus we have shown a) and b).

Corollary 2. For any $f \in L^\infty(S^{2\ell})$ and any two sided ideal $J$ of operators in $\mathcal{H}_0$ one has:

$$[H_0, f] \in J \iff [H, f] \in J.$$ 

Proof. Since $\mu$ commutes with $f$ and $[H, \cdot]$ satisfies Leibniz rule, direct calculation yields

$$[H, f] = -(1 + \mu)(H_0 - \mu)^{-1} [H_0, f](H_0 - \mu)^{-1} (1 - \mu).$$ (10)

Corollary 3. For any two sided ideal $J$ of operators in Hilbert space the class of functions $f \in L^\infty(S^{2\ell})$ such that $[H_0, f] \in J$ is invariant under qc-homeomorphisms.

Proof. Let $\varphi$ be a qc-homeomorphism of $S^{2\ell}$ then $\varphi$ is a.e. differentiable and it defines a bounded operator $U(\varphi)$ in $\mathcal{H}_0 = L^2(S^{2\ell}, \Lambda)$ by the formula:

$$U(\varphi) \omega = (\varphi^{-1})^* \omega \quad \forall \omega \in \mathcal{H}_0.$$ (11)

This yields a bounded operators in $\mathcal{H}_0$ such that

$$U(\varphi) f U(\varphi)^{-1} = f \circ \varphi^{-1} \quad \forall f \in L^\infty(S^{2\ell})$$ (12)

$$U(\varphi) H_0 U(\varphi)^{-1} = H_g$$ (13)

where $[g]$ is the mb conformal structure $(\varphi^{-1})^* [g_0]$. To prove (13) note first that $U(\varphi) \text{Im} \, d = \text{Im} \, d$ (cf. [1]), while $(U(\varphi) \omega_1, U(\varphi) \omega_2)_0 = (\omega_1, \omega_2)_g \quad \forall \omega_1, \omega_2 \in H_0$. Thus $U(\varphi) H_0 U(\varphi)^{-1}$
is equal to 1 on \( \text{Im} \, d \) and to -1 on its orthogonal complement for the inner product \( \left( \cdot, \cdot \right)_{g} \). Hence \( U(\varphi) \, H_0 \, U(\varphi)^{-1} = H_g \). Let then \( f \in L^\infty(S^{2\ell}) \), if \( [H_0, f] \in J \) then 
\[ [U(\varphi) \, H_0 \, U(\varphi)^{-1}, \, U(\varphi) \, f \, U(\varphi)^{-1}] \in J \text{ and } [H_g, f \circ \varphi^{-1}] \in J \] 
so that \( [H_0, f \circ \varphi^{-1}] \in J \) by the above corollary.

We shall now see that, modulo the ideal \( J = L^{(2\ell, \infty)} \), the class of the operator \( H_g \) is locally determined by the bounded measurable conformal structure \([g]\).

**Proposition 4.** Let \( U \subset S^{2\ell} \) be an open subset and \( f_1, f_2 \) be continuous functions with support in \( U \). Let \( g_1, g_2 \) be two bm conformal structures on \( S^{2\ell} \) which agree on \( U \). Then \( f_1(H_{g_1} - H_{g_2}) f_2 \in L^{(2\ell, \infty)} \).

**Proof.** We can replace \( f_i \) by smooth functions equal to 1 on the support of the previous ones. Thus we can assume that \( f_i \in A(S^{2\ell}) \). By proposition 1 both operators \( H_{g_i} \) are the sum of a geometrically norm convergent series:

\[
H_{g_i} = (1 - \mu_i)^{-1} \sum_{0}^{\infty} (\mu_i \, H_0)^n (1 - \mu_i)
\]

whose terms are monomials of the form

\[
a_{1,i} \, H_0 \, a_{2,i} \, H_0 \ldots a_{n,i} \, H_0 \, a_{n+1,i} = T_{n,i}
\]

where \( a_{k,i} \) belongs to the commutant of \( L^\infty(S^{2\ell}) \) in \( \mathcal{H}_0 \) and where \( f_1 \, a_{k,1} = f_1 \, a_{k,2} \) for all \( k \). It follows using \([H_0, f_1] \in L^{(2\ell, \infty)} \) that

\[
f_1(T_{n,1} - T_{n,2}) \in L^{(2\ell, \infty)}
\]

One thus expresses \( f_1(H_{g_1} - H_{g_2}) \) as the sum of a series convergent in the Banach space \( L^{(2\ell, \infty)} \), and the conclusion follows.

\( \beta \) The class of \( H \) modulo \( L^{(2\ell, \infty)} \)

Let now \( M \) be a quasiconformal manifold of dimension \( 2\ell \).

Let \( A(M) \) be the subalgebra of \( C(M) \) of functions \( f \) such that for any qc local chart: \( V \xrightarrow{\rho} S^{2\ell} \) and any \( h \in C_c(\rho(V)) \cap A(S^{2\ell}) \) one has \( h \circ \rho^{-1} \in A(S^{2\ell}) \). By corollary 3 one has \( \rho^*(A(S^{2\ell}) \cap C_c(\rho(V))) \subset A(M) \) for any qc local chart \( V \xrightarrow{\rho} S^{2\ell} \). It follows that \( A(M) \) is a norm dense subalgebra of \( C(M) \) and that it has partitions of unity subordinate to any finite open covering of \( M \).

Let \([g]\) a bounded measurable conformal structure on \( M \).
Let \((V_\alpha)\) be a finite open cover of \(M\) by domains of qc local charts \(\rho_\alpha : V_\alpha \to S^{2t}\) and \(g_\alpha\) a bounded measurable conformal structure on \(S^{2t}\) which agrees with \(\rho_\alpha[g]\) on \(\rho_\alpha(V_\alpha)\). For each \(\alpha\) let \(H_\alpha = H_{g_\alpha}\) be the corresponding Hodge decomposition on \(S^{2t}\). Let \((\varphi_\alpha)\) be a partition of unity, \(\varphi_\alpha \in A(M)\), support \(\varphi_\alpha \subset V_\alpha\) and for each \(\alpha\) let \(\psi_\alpha \in A(M)\) be equal to 1 in a neighborhood of Support \(\varphi_\alpha\), with Support \(\psi_\alpha \subset V_\alpha\). We let as above \(\Lambda\) be the measurable vector bundle \(\Lambda^t T^*_C\) on \(M\) and we consider the following locally constructed operator in \(L^2(M, \Lambda)\):

\[(17)\]

\[S = \sum \psi_\alpha (H_\alpha)^{\rho_\alpha} \varphi_\alpha\]

where we used \(\rho_\alpha\) to let \(H_\alpha\) act in \(L^2(M, \Lambda)\).

**Proposition 5.** The class of \(S\) modulo \(\mathcal{L}^{(2t, \infty)}\) only depends upon the bm conformal structure \([g]\) on \(M\) and one has:

1) \(S\gamma + \gamma S \in \mathcal{L}^{(2t, \infty)}\)
2) \([S, f] \in \mathcal{L}^{(2t, \infty)}\) \(\forall f \in A(M)\)
3) \(S^2 - 1 \in \mathcal{L}^{(2t, \infty)}\)
4) \((S - 1)/\text{Im } d \in \mathcal{L}^{(2t, \infty)}\).

**Proof.** We first need to show that \(S - S' \in \mathcal{L}^{(2t, \infty)}\) for any two operators \(S, S'\) constructed by formula 17. Using the compactness of \(M\) it is enough to show that for any \(x \in M\) there exists \(f \in A(M)\), \(f(x) \neq 0\) such that \((S - S')f \in \mathcal{L}^{(2t, \infty)}\). Using a local qc chart the proof follows from proposition 4. To check 1) note that any of the operators \(S\) given by 17 satisfies \(S\gamma + \gamma S = 0\).

The condition 2) follows from Corollary 3.

To check 3) we take a representative \(S = \sum \psi_\alpha H_\alpha \varphi_\alpha\) constructed from a covering \((V_\alpha)\) by domains of qc charts, such that for any \(\alpha\) the open set \(\cup V_\beta, V_\beta \cap V_\alpha \neq \emptyset\) is the domain of a qc chart. The result then follows from corollary 3 and proposition 4.

Let us check 4). On the sphere \(S^{2t}\) the operators \(H, g\) a bounded measurable conformal structure, are equal to 1 on \(\text{Im } d\) (proposition 1). Thus, using corollary 3, the operator \(S = \sum \psi_\alpha H_\alpha \varphi_\alpha\) satisfies \((S - 1)/E_\alpha \in \mathcal{L}^{(2t, \infty)}\) where \(E_\alpha\) is the closure in \(L^2(M, \Lambda)\) of \(\{d\omega ; \omega \in L^{2t/t - 1}(V_\alpha, \Lambda)\}\). Thus 4) follows if we show that the map \((\omega_\alpha) \in \oplus E_\alpha \to \sum \omega_\alpha \in L^2(M, \Lambda)\) is surjective on \(\text{Im } d\). Let \(P_\alpha\) (resp. \(Q_\alpha = \gamma P_\alpha\)) be the orthogonal projection on \(E_\alpha\) (resp. \(\gamma E_\alpha\)). For any \(\alpha, \beta\) the closed subspaces \(E_\alpha \subset \text{Im } d\) and \(\gamma E_\beta \subset \gamma(\text{Im } d)\) are orthogonal. Thus it is enough to show that the following operator is equal to \(1+\) compact:

\[T = \sum (P_\alpha + Q_\alpha) \varphi_\alpha.\]

(Its range is then closed by the Fredholm theory.)

Let us show that for each \(\alpha\) the operator \((P_\alpha + Q_\alpha) \varphi_\alpha - \varphi_\alpha\) is compact. We can assume that \(M = S^{2t}\) is the sphere. By proposition 1 a) one has the orthogonal decomposition
\( L^2(S^{2\ell}, \Lambda) = \text{Im } d \oplus \gamma \text{ Im } d \), where \( d : L^{2\ell/\ell-1}(S^{2\ell}, \Lambda^{\ell-1}) \rightarrow L^2(S^{2\ell}, \Lambda^\ell) \) has closed range. (We assume \( \ell \neq 1 \).)

Using a compact operator \( R : L^2(S^{2\ell}, \Lambda) \rightarrow L^{2\ell/\ell-1}(S^{2\ell}, \Lambda^{\ell-1}) \) such that \( dR = 1 \) on \( \text{Im } d \) we thus get a pair of compact operators \( R_j : L^2(S^{2\ell}, \Lambda) \rightarrow L^{2\ell/\ell-1}(S^{2\ell}, \Lambda^{\ell-1}) \) such that \( dR_1 + \gamma dR_2 = 1 \). The conclusion then follows using the following equality, with \( \psi \) a smooth function with support in \( V_\alpha \) and equal to 1 in a neighborhood of \( \text{Support } \varphi_\alpha \):

\[
\psi \omega = d(\psi \omega_1) + \gamma d(\psi \omega_2) - d \psi \wedge \omega_1 - \gamma(d \psi \wedge \omega_2) \quad \omega_i = R_i \omega.
\]

4. Proof of Theorem 2.

**Proof of 1).** First choose a neighborhood \( V \) of the diagonal such that \( V^{2q} \subset U, q = 6\ell + 2 \). Next (proposition 5) let \( S \) be an operator such that:

- Support \( (S) \subset V \)
- \( S^2 - 1 \in \mathcal{L}^{(2\ell, \infty)} \)
- \( S\gamma = -\gamma S \)
- \( (S - 1)/\text{Im } d \in \mathcal{L}^{(2\ell, \infty)} \).

Let then \( \theta = S^2 - 1 \). By construction \( \theta \) commutes with \( S \), it also commutes with \( \gamma \) by c). Let \( q(t) \) be the unique polynomial of degree \( 2\ell \) such that:

\[(1) \quad (1 + t)^{-1/2} = q(t) + O(t^{2\ell+1}) \quad \text{(for } t \text{ small).}\]

Let \( p(t) \) be the polynomial given by:

\[(2) \quad p(t) = (1 + t) q(t)^2 - 1.\]

We shall define an operator \( H \) by the formula:

\[(3) \quad H = \gamma p(\theta) + \left( 1 - \left( \frac{1 + \gamma}{2} \right) p(\theta) \right) q(\theta) S.\]

First note that \( \theta \) is \( V^{02} \) local, thus since \( q \) (resp. \( p \)) has degree \( 2\ell \) (resp. \( 4\ell + 1 \)) we see that \( T \) is \( U \) local as required. Next, as \( \theta \in \mathcal{L}^{(2\ell, \infty)} \) by b), and as \( p(t) = O(t^{2\ell+1}) \) we see that \( p(\theta) \in \mathcal{L}^1 \). We thus have:

\[(4) \quad H \gamma + \gamma H = 2p(\theta) \in \mathcal{L}^1\]

where we used c) to get the equality.

Since \( p(\theta) \in \mathcal{L}^1 \) and \( \theta \in \mathcal{L}^{(2\ell, \infty)} \) we have \( S - H \in \mathcal{L}^{(2\ell, \infty)} \) and hence, using d),
\( (H - \text{id})/ \text{Im } d \in \mathcal{L}^{(2\ell, \infty)} \).

It remains to check that \( H^2 = 1 \). Note first that the two terms \( \gamma \ p(\theta), H - \gamma \ p(\theta) \) anticommute so that \( H^2 \) is the sum of their squares:

\[
H^2 = (\gamma \ p(\theta))^2 + q(\theta)^2 \left( 1 - \frac{1 + \gamma}{2} \ p(\theta) \right) \left( 1 - \frac{1 - \gamma}{2} \ p(\theta) \right) S^2
= p(\theta)^2 + q(\theta)^2 (1 - p(\theta)) (1 + \theta) = 1.
\]

Before we begin the proof of 2) we recall that the cyclic complex \( (C^*_\lambda, b) \) of an algebra \( \mathcal{A} \) is given by:

\[
(6) \quad C^*_\lambda(\mathcal{A}) = \{ \text{multilinear forms } \tau \text{ on } \mathcal{A} \times \ldots \times \mathcal{A} \text{ (} n + 1 \text{ times)} \}
\text{such that } \tau(a^1, \ldots, a^n, a^0) = (-1)^n \tau(a^0, \ldots, a^n) \quad \forall a^j \in \mathcal{A}
\]

\[
(7) \quad (b\tau)(a^0, \ldots, a^{n+1}) = \sum_{0}^{n} (-1)^j \tau(a^0, \ldots, a^j a^{j+1}, \ldots, a^{n+1})
+ (-1)^{n+1} \tau(a^{n+1} a^0, \ldots, a^n) \quad \forall a^j \in \mathcal{A}.
\]

We note also that if \( J \) is an ideal in a larger algebra \( \mathcal{A} \) and if \( B \subset \mathcal{A} \) is a subalgebra such that \( J \cap B = \{0\} \), then the natural extension by 0 on \( B \) of the cochains \( \tau \in C^*_\lambda(J) \) satisfying:

\[
(8) \quad \tau(a^0, a^1, \ldots, a^j \delta, a^{j+1}, \ldots, a^n) = \tau(a^0, a^1, \ldots, a^i, \delta a^{j+1}, \ldots, a^n)
\forall a^i \in \mathcal{A}, \forall j, \forall \delta \in B.
\]

commutes with the coboundary \( b \) of \( C^*_\lambda(J + B) \).

**Proof of 2**. It follows from [3] lemma 2. But we shall give the details here. By construction \( \sigma = \text{trace}(\wedge^{d+1} k) \) is \( U^{0d} \) local. We need to show that \( b\sigma = 0 \). We shall first show that, with \( J \) the algebra of trace class operators in \( \mathcal{H} \), the following formula defines a morphism of complexes: \( (A^*, \delta) \rightarrow (C^*_\lambda(J), b) \), where \( (A^*, \delta) \) is the complex of bounded measurable totally antisymmetric functions (straight cochains) on \( M \).
(9) \[ \tau_\varphi(k^0, \ldots, k^n) = \]
\[ (-1)^n \int_{M^{n+1}} \text{trace} \left( k^0(x_0, x_1) k^1(x_1, x_2) \ldots k^n(x_n, x_0) \varphi(x_0, \ldots, x_n) \right) \prod d\nu(x_i) \]
\[ \forall \varphi \in A^n. \]

The coboundary \( \delta \) in \((A^*, \delta)\) is given by:

(10) \[ \delta \varphi = \sum_{j=0}^{n+1} (-1)^j \varphi_j, \quad \varphi_j(x_0, \ldots, x_{n+1}) = \varphi(x_0, \ldots, \overline{x}_j, \ldots, x_{n+1}). \]

Next, when one computes \( \tau_{\varphi_j} \), \( j \geq 1 \), the variable \( x_j \) does not occur in \( \varphi_j \) and thus one gets:

(11) \[ \tau_{\varphi_j}(k^0, \ldots, k^{n+1}) = -\tau_\varphi(k^0, \ldots, k^{j-1} k^j, \ldots, k^{n+1}). \]

For \( j = 0 \) it is the variable \( x_0 \) which does not occur in \( \varphi_0 \) and using the cyclicity of the trace one gets, using the antisymmetry of \( \varphi \):

(12) \[ \tau_{\varphi_0}(k_0, \ldots, k^{n+1}) = (-1)^{n+1} \tau_\varphi(k^{n+1}, k^0, \ldots, k^n). \]

Thus using 10 we get:

(13) \[ \tau_{\delta \varphi} = b \tau_\varphi \quad \forall \varphi \in A^n. \]

Next note that the compatibility condition (8) is fulfilled by any element of the algebra \( B = \{ \lambda_0 + \lambda_1 \gamma ; \lambda_j \in C \} \) generated by the operator \( \gamma \) in \( \mathcal{H} \). We shall still denote by \( \tau \) the extension of the above morphism of complexes to \((C^*_\chi(A), b), A = J + B. \)

Let us now check that \( b \sigma = 0 \), i.e. that \( \sigma(\delta \varphi) = 0 \) for any \( \varphi \in A^d. \) Using (13) we know that the cyclic cocycle \( \tau_{\delta \varphi} \) is a coboundary and thus that it vanishes when evaluated on an idempotent:

(14) \[ \tau_{\delta \varphi}(P, P, \ldots, P) = 0 \quad \forall P, \ P^2 = P, \ P \in A. \]

Applying this to \( P = H \quad \frac{1 - \gamma}{2} \quad H = \frac{1}{2} \left( \frac{1 + \gamma}{2} \right) + \frac{1}{2} k \) gives the desired result.

**Proof of 3.** Let \( H_0, H_1 \) be two \( U \)-local Hodge decompositions and \( Y = H_1 - H_0. \) We have:
(15) \[ Y/\text{Im} \, d \in \mathcal{L}^{(2t,\infty)} \]

(16) \[ Y\gamma + \gamma Y \in \mathcal{L}^1. \]

As \( \text{Im} \, d \oplus \gamma(\text{Im} \, d) \) is of finite codimension in \( \mathcal{H} \) it follows that:

(17) \[ Y \in \mathcal{L}^{(2t,\infty)}. \]

Let then \( H_t = H_0 + tY \). We have:

(18) \[ \text{Support} \, H_t \subset U \]

(19) \[ H_t^2 - 1 \in \mathcal{L}^{(2t,\infty)} \]

(20) \[ (H_t - 1)/\text{Im} \, d \in \mathcal{L}^{(2t,\infty)} \]

(21) \[ H_t \gamma + \gamma H_t \in \mathcal{L}^1. \]

It follows that \( S_t = \frac{1}{2}(H_t - \gamma H_t \gamma) \) anticommutes with \( \gamma \) and satisfies (18) (19) (20). Thus with \( q \) and \( p \) as in (1) (2) we get a family \( H_t' \) of \( U^n \) local Hodge decompositions:

(22) \[ H_t' = \gamma \, p(\theta_t) + \left( 1 - \left( \frac{1 + \gamma}{2} \right) \, p(\theta_t) \right) \, q(\theta_t) \, S_t \quad \theta_t = S_t^2 - 1. \]

For \( t = 0 \) (or for \( t = 1 \)), the operator \( \theta_0 = S_0^2 - 1 \) belongs to \( \mathcal{L}^1 \) we can thus, keeping the relation (2), replace the polynomial \( q \) by \( 1 + \lambda(q - 1) \), \( \lambda \in [0,1] \) and still get a family of \( U^n \) local Hodge decompositions joining \( H_0' \) (for \( \lambda = 1 \)) with

(23) \[ H_0'' = \gamma \, \theta_0 + \left( 1 - \left( \frac{1 + \gamma}{2} \right) \, \theta_0 \right) \, S_0 \quad \theta_0 = S_0^2 - 1, \quad S_0 = \frac{1}{2} \, (H_0 - \gamma \, H_0 \, \gamma). \]

Since the obtained path of idempotents \( (H_t' \left( \frac{1-\lambda}{2} \right) H_t') \) in \( \mathcal{A} \) are piecewise polynomial it follows that the corresponding straight cycles are \( U^n \) homologous. This is, using (13), a
restatement of the homotopy invariance of the pairing of $K$-theory with cyclic cohomology ([5]).

It remains to compare in the same way $H''_0$ with $H_0$. One has by hypothesis $S_0 - H_0 \in \mathcal{L}^1$, $\theta_0 \in \mathcal{L}^1$ thus:

\begin{equation}
H''_0 - H_0 \in \mathcal{L}^1.
\end{equation}

It follows that the idempotents

\[ e = H_0 \left( \frac{1 - \gamma}{2} \right) H_0, \quad e'' = H''_0 \left( \frac{1 - \gamma}{2} \right) H''_0 \]

satisfy the equality $e'' = W e W^{-1}$ where $W = H''_0 H_0 \in 1 + J$ as well as $W^{-1}$, while $W$ is $U^2$ local as well as $W^{-1}$. One then considers the following smooth path of idempotents $e(t) \in M_2(\mathcal{A})$ connecting $\begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}$ with $\begin{bmatrix} 0 & 0 \\ 0 & e'' \end{bmatrix}$ and with support in $U^3$:

\[ e(t) = \begin{bmatrix} 1 & 0 \\ 0 & W \end{bmatrix} R_t \begin{bmatrix} e & 0 \\ 0 & 1 \end{bmatrix} R^{-1}_t \begin{bmatrix} 1 & 0 \\ 0 & W^{-1} \end{bmatrix} \]

where $R_t$ is a rotation matrix.

Using again the homotopy invariance of the pairing of $K$-theory with cyclic cohomology one gets the desired result.

**Proof of 4).** We shall first check directly that $\int_M k(x, x) = 2 \text{Sign}(M)$. Since the Alexander Spanier cocycle given by the constant function $\varphi(x^0) = 1$ is everywhere defined, the proof of 3) shows that $\int_M k(x, x)$ is independent of the choice of the Hodge decomposition $H$, without any $U$-locality hypothesis. We can thus choose $H = 2P - 1$ where $P$ is the orthogonal projection on the closed subspace $\text{Im} d$. One has $P + \gamma P \gamma + K = 1$ where $K$ is the harmonic projection ([1]). Thus one gets:

\[ H \gamma H + \gamma = 2 K \gamma \]

\[ \text{Trace} (H \gamma H + \gamma) = 2 \text{Trace} (K \gamma) = 2 \text{Sign}(M). \]

To compute the other homology classes let us first assume that $M$ is a smooth manifold. By [3] these classes $\omega_{2q}$ represent $2q+1$ times the Chern character of the $K$-homology class of the operator $H = 2P - 1$, with $P$ as above.

Thus it is enough to show that the $K$-theory class of the symbol of $H$, $[\sigma(H)] \in K^0(T^*M)$ is the same as the $K$-theory class of the signature operator. The latter is given by the odd endomorphism $s(x, \xi) = e_\xi + i_\xi$, $\xi \in T_x^*(M)$ of the pull back of $\wedge^* T^*_\mathbb{C}$ (oriented by $\gamma = i^* (-1)^{\frac{p(p-1)}{2}}$) to $T^*M$. (Here $e_\xi, i_\xi$ are respectively exterior and interior multiplication by $\xi$.) The symbol $\sigma(H)$ of $H$ is the same as the symbol of $2P - 1$
where $P$ is the orthogonal projection on the image of $d$. Its restriction to the unit sphere $\{ \xi \in T^*V, \| \xi \| = 1 \}$ is thus given by:

$$
\sigma(x, \xi) = e_\xi i_\xi - i_\xi e_\xi \text{ acting on } \wedge^\ell T^*_C.
$$

Let $u(x, \xi) = \frac{1}{\sqrt{2}} (1 - e_\xi + i_\xi)$, then for $\| \xi \| = 1$, it is an invertible operator in $\wedge^* T^*_C$ with inverse $u^{-1}(x, \xi) = \frac{1}{\sqrt{2}} (1 + e_\xi - i_\xi)$. One has

$$(usu^{-1})(x, \xi) = e_\xi i_\xi - i_\xi e_\xi \text{ acting on } \wedge^* T^*_C.$$

By construction $u$ commutes with $\gamma$. This shows that the $K$-theory class $[\rho] - [\sigma]$ is given by the symbol:

$$
\rho(x, \xi) = e_\xi i_\xi - i_\xi e_\xi \text{ acting on } \wedge^* T^*_C \oplus \wedge^\ell T^*_C,
$$

with the $\mathbb{Z}/2$ grading $\gamma$. But using the canonical isomorphism, for $p \neq \ell$,

$$
\wedge^p T^*_C \simeq (\wedge^p T^*_C \oplus \wedge^{2\ell-p} T^*_C)_{\pm}: \omega \rightarrow \frac{1}{2}(\omega \pm \gamma \omega)
$$

one checks that the class of $\rho$ is equal to 0.

This completes the proof the classes we construct agree with Hirzebruch-Thom classes in the smooth case. The extension from the smooth case to the qc case can be done using cobordism as in [6]. One can also use $K$-theory as in [21].

**Remarks.** 1) Let $M$ be a compact quasiconformal manifold, one can characterize the subalgebra $A(M)$ of $C(M)$ (section 3) by the following simple criterion

$$
f \in A(M) \iff df \in L^{2\ell}(M, T^*_C).
$$

A proof of $\Rightarrow$ follows from [25] and the converse from [21] plus an identification of the Besov space there with this Sobolev space. It is sufficient by §3 to work with the standard singular integral operator on the sphere. For a more complete discussion see the Appendix.

2) Let $M$ be a polyhedron of dimension $2\ell$ with homological properties to be specified. Let $\mathcal{H} = \bigoplus \mathcal{H}_\alpha$ be the direct sum of the Hilbert spaces $\mathcal{H}_\alpha$ of square integrable $\ell$ forms on the $2\ell$ dimensional simplices. For each vertex $v$ let $E_v$ be the closed subspace of $\mathcal{H}$ of boundaries of forms with support in the double star of $v$. Let $\gamma$ be the * operation on $\mathcal{H}$ given by the canonical flat metric on each $2\ell$-simplex (with equilateral length). Let $b_v$ be the barycentric coordinate assigned to the vertex $v$. Then let us consider the following operator:

$$
\sum_v (P_v - \gamma P_v \gamma) b_v = S
$$

where $P_v$ is the orthogonal projection on $E_v$. It is clear that $S$ is localised, is equal to 1, modulo $L^{(2\ell, \infty)}$, on the image of $d$ and anticommutes with $\gamma$. Now we assume that $E_v$ and $\gamma(E_v)$ are a local decomposition, up to compacts i.e. $(1 - P_v - \gamma P_v \gamma)/ * \bigoplus h_\alpha$

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is compact of forms on the (first star). These are the prescribed homological conditions. Then \( S^2 - 1 \) is in \( \mathcal{L}^{(2,\infty)} \) and one can use the above formulae to define characteristic homology classes. The involved Hilbert space theoretical data are of the same nature as those appearing in transfer matrix theory of statistical mechanics and suggest a purely combinatorial approach to the \( K \)-orientation of [4] in the extended context of spaces with singularities.

**Appendix.** (with Stephen Semmes) An interesting analytical point.

There is an interesting issue concerning operator theory and classical analysis which is related to the topics of this paper and which does not seem to have been treated in the literature. For the sake of clarity we discuss this issue in a restricted setting. Let \( T \) be a \( 0 \)-th-order pseudodifferential operator on \( \mathbb{R}^d \), \( d > 1 \), which we also assume to be translation and dilation invariant and nonzero. Thus \( T \) could be represented by a Fourier multiplier which is homogeneous of degree 0, and \( T \) is a bounded linear operator on \( \mathcal{H} = L^2(\mathbb{R}^d) \).

Under what conditions on a function \( f(x) \) on \( \mathbb{R}^d \) is it true that \( [f,T] \in \mathcal{L}^{(d,\infty)}(\mathcal{H}) \)? (Recall that \( \mathcal{L}^{(d,\infty)}(\mathcal{H}) \) denotes the space of compact operators \( A \) on \( \mathcal{H} \) such that \( \mu_n(A) = O(n^{-\frac{1}{2}}) \), where \( \mu_n(A) \) is the \( n \)-th eigenvalue of \( (A^*A)^{\frac{1}{2}} \).) This type of question has been studied extensively (see [26], for instance), but this particular case involves a critical index and has some special features. It follows from [27] that

\[
[f, T] \in \mathcal{L}^{(d,\infty)}(\mathcal{H}) \Leftrightarrow f \in \text{Osc}^{d,\infty}(\mathbb{R}^d) \quad \text{when } d > 1,
\]

where \( \text{Osc}^{d,\infty}(\mathbb{R}^d) \) is a variant of a Besov space whose definition will be reviewed soon. For many purposes it would be preferable to work with a Sobolev space instead of \( \text{Osc}^{d,\infty} \). It was observed in (Theorem 2.2 on p 228 in) [28] that

\[
W^{1,d}(\mathbb{R}^d) \subseteq \text{Osc}^{d,\infty}(\mathbb{R}^d) \quad \text{when } d > 1,
\]

where \( W^{1,d}(\mathbb{R}^d) \) is the Sobolev space of locally integrable functions on \( \mathbb{R}^d \) whose distributional first derivatives all lie in \( L^{d}(\mathbb{R}^d) \). In fact we have the following

**Theorem.** When \( d > 1 \), \( W^{1,d}(\mathbb{R}^d) = \text{Osc}^{d,\infty}(\mathbb{R}^d) \), and so \( [f, T] \in \mathcal{L}^{(d,\infty)}(\mathcal{H}) \) if and only if \( f \in W^{1,d}(\mathbb{R}^d) \).

It is very important here that the dimension \( d \) is the same as the exponent in the function spaces; otherwise this theorem would not work. This theorem is rather surprising from the perspective of classical analysis, because Sobolev spaces normally coincide with Besov-type spaces only when the exponent is 2. Indeed, the second half of the theorem had been conjectured by Jan Krzysztof.  

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We need only show that the inclusion opposite to (2) holds. Our original proof of this was obtained by understanding the Dixmier trace of $|[f, T]|^d$, which in fact reduces to $\int_{\mathbb{R}^d} |\nabla f(x)|^d \, dx$ for certain $T$. We shall sketch a more direct proof below.

Let us recall the definition of $\text{Osc}^{d, \infty}(\mathbb{R}^d)$. This space is a little bit nonstandard; at this critical index, the standard Besov space is distinct from this one.

Given $(x, t) \in \mathbb{R}^d \times \mathbb{R}_+ = \mathbb{R}^{d+1}_+$, let $B(x, t)$ denote the ball with center $x$ and radius $t$. Let $m_{x, t}(g)$ denote the average of the locally-integrable function $g(y)$ on $\mathbb{R}^d$. For such a function we let $\Theta(x, t)$ denote its average oscillation on $B(x, t)$, which is given by

$$
\Theta(x, t) = m_{x, t}(|g - m_{x, t}(g)|).
$$

We shall define $\text{Osc}^{d, \infty}(\mathbb{R}^d)$ in terms of a global measurement of these localized oscillation quantities.

Let $\{(x_j, t_j)\}_j$ denote a reasonably thick hyperbolic lattice in $\mathbb{R}^{d+1}_+$: we require that every point in $\mathbb{R}^{d+1}_+$ be no further than $10^{-2}$ from some $(x_j, t_j)$ in the hyperbolic metric and that no pair of the $(x_j, t_j)$'s are closer to each other than $10^{-3}$. Thus the numbers $\Theta(x_j, t_j)$ measure the average oscillations of $g$ at all possible locations and scales.

Let $\theta_n, n = 1, 2, 3, \ldots$, denote the $n^{th}$ largest value of $\Theta(x_j, t_j)$. In other words, we reorder the $\Theta(x_j, t_j)$'s in decreasing size. Then

$$
g \in \text{Osc}^{d, \infty}(\mathbb{R}^d) \iff \theta_n = O(n^{-\frac{1}{2}}).
$$

This definition does not depend on the particular choice of the lattice $(x_j, t_j)$. (The point of (1), incidentally, is that the $\theta_n$'s for $f$ can be related to the $\mu_n([f, T])$'s.)

The main step in the proof of the theorem is to show that if $g$ is smooth then

$$
\left( \int_{\mathbb{R}^d} |\nabla g(x)|^d \, dx \right)^{\frac{1}{d}} \leq C \limsup_{n \to -\infty} n^{\frac{1}{2}} \theta_n
$$

(3)

for some constant $C$ which does not depend on $g$. Once we know this, then we know that the $\text{Osc}^{d, \infty}(\mathbb{R}^d)$ norm controls the $W^{1, d}(\mathbb{R}^d)$ norm for smooth functions, and the proof of the theorem can be finished with a standard approximation argument (which we omit). (The main point is that if you convolve $g \in \text{Osc}^{d, \infty}(\mathbb{R}^d)$ with a function in $L^1(\mathbb{R}^d)$ with bounded norm, then you get a function in $\text{Osc}^{d, \infty}(\mathbb{R}^d)$ with bounded norm.) The proof of (3) that follows is pretty sketchy, but it would not be too difficult to give a detailed argument.

Let us first try to understand the right side of (3) better. For each $\lambda > 0$ let $N(\lambda)$ be the number of $j$'s such that $\Theta(x_j, t_j) > \lambda$. Thus $\theta_{N(\lambda)} > \lambda$, and so

$$
\limsup_{n \to -\infty} n^{\frac{1}{2}} \theta_n \geq \limsup_{\lambda \to 0} N(\lambda)^{\frac{1}{2}} \lambda.
$$

(4)

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Now let us try to understand the left side of (3). For the time being we shall work on some fixed large cube $K$. Let $\{Q_\ell\}$ be a partition of $K$ into tiny cubes of sidelength $s$. If $s$ is small enough, then $\nabla g$ will be almost constant on each $Q_\ell$, because $g$ is smooth. Let $G_\ell$ denote the approximate value of $|\nabla g|$ on $Q_\ell$.

Set $\tilde{Q}_\ell = Q_\ell \times (0, s) \subseteq \mathbb{R}^{d+1}$. If $(x, t) \in \tilde{Q}_\ell$, then $\Theta(x, t)$ is approximately $tG_\ell$. For each $\ell$ let $N_\ell(\lambda)$ denote the number of $j$'s such that $(x_j, t_j) \in \tilde{Q}_\ell$ and $\Theta(x_j, t_j) > \lambda$. If $sG_\ell < \lambda$ then we should have that $N_\ell(\lambda)$ is approximately 0, while if $sG_\ell > \lambda$ then $N_\ell(\lambda)$ should be approximately the same as the number of $j$'s such that $(x_j, t_j) \in \tilde{Q}_\ell$ and $t_j > \lambda G_\ell^{-1}$. Simple considerations of hyperbolic geometry imply that $N_\ell(\lambda)$ is roughly proportional to $(sG_\ell)^d$ in this case. This is also about the same as what we got in the first case.

Since $N(\lambda) \geq \sum_\ell N_\ell(\lambda)$, we conclude that $N(\lambda)$ should dominate $\lambda^{-d} \sum_\ell (sG_\ell)^d$, modulo controllable errors. Form (4) we get that

$$\limsup_{n \to \infty} n^{\frac{d}{d+1}} \theta_n \text{ is roughly larger than } \left( \sum_\ell (sG_\ell)^d \right)^{\frac{1}{d}}.$$\[On the other hand, \sum_\ell (sG_\ell)^d \text{ is just a Riemann sum for } \int_K |\nabla g(x)|^d \, dx. In the limit we get that\]

$$\left( \int_K |\nabla g(x)|^d \, dx \right)^{\frac{1}{d}} \leq C \limsup_{n \to \infty} n^{\frac{d}{d+1}} \theta_n,$$\[where $C$ does not depend on $g$ or $K$. This implies (3), and proves the theorem.\]

Notice, incidentally, that the proof of (3) works also when $d = 1$. However, the approximation argument that gives $Osc^{d, \infty}(\mathbb{R}^d) \subseteq W^{1,d}(\mathbb{R}^d)$ when $d > 1$ gives only $Osc^{1, \infty}(\mathbb{R}) \subseteq BV(\mathbb{R})$, where $BV(\mathbb{R})$ denotes the space of functions on $\mathbb{R}$ of bounded variation (i.e., whose distributional derivatives are finite measures). The reciprocal inclusion is false, because jump discontinuities are bad for $Osc^{1, \infty}(\mathbb{R})$.\]
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