Distribution of Periodic Points of Polynomial Diffeomorphisms of $\mathbf{C}^2$

E. Bedford, M. Lyubich and J. Smillie

§1. Introduction.

This paper deals with the dynamics of a simple family of holomorphic diffeomorphisms of $\mathbf{C}^2$: the polynomial automorphisms. This family of maps has been studied by a number of authors. We refer to [BLS] for a general introduction to this class of dynamical systems. An interesting object from the point of view of potential theory is the equilibrium measure μ of the set K of points with bounded orbits. In [BLS] μ is also characterized dynamically as the unique measure of maximal entropy. Thus μ is also an equilibrium measure from the point of view of the thermodynamical formalism. In the present paper we give another dynamical interpretation of μ as the limit distribution of the periodic points of f.

Fix a polynomial automorphism f. A point p is periodic if $f^n p = p$ for some $n > 0$, and the smallest positive n for which this equation holds is the period of p. We let Fix_n denote the set of fixed points of f^n, and Per_n denote the set of points of period exactly n. Thus $\mathrm{Fix}_n = \bigcup \mathrm{Per}_k$, where the union is taken over all k which divide n. The map f has a dynamical degree d which we assume to be larger than 1. By [FM] f^n has exactly d^n fixed points counted with multiplicities. Since the multiplicity of each fixed point of f^n is positive we conclude that $\#\mathrm{Fix}_n \le d^n$.

A periodic point p is called a saddle point if the eigenvalues $\lambda^s(p)$ and $\lambda^u(p)$ of $Df^n(p)$ satisfy $|\lambda^s(p)| < 1 < |\lambda^u(p)|$. We let SPer_n denote the saddle points with period exactly n, so that

$$\mathrm{SPer}_n \subset \mathrm{Per}_n \subset \mathrm{Fix}_n \tag{1}$$

$$\#\mathrm{SPer}_n \le \#\mathrm{Per}_n \le \#\mathrm{Fix}_n \le d^n. \tag{2}$$

Theorem 1. *If P_n denotes any of the three sets in (1), then*

$$\lim_{n\to\infty} \frac{1}{d^n} \sum_{a\in P_n} \delta_a = \mu.$$

There seems to be a general tendency in many classes of dynamical systems for periodic points to be equidistributed with respect to the measure of maximal entropy. Indeed, this is true for subshifts of finite type and their smooth counterparts, axiom A diffeomorphisms (Bowen [B1]). This is also true for rational endomorphisms of the Riemann sphere [L]. In the special case of polynomial maps of the complex plane this result can also be derived by Brolin's methods [Br] (see Sibony [Si] and Tortrat [T].) For polynomial diffeomorphisms of $\mathbf{C}^2$ the equidistribution property was conjectured by N. Sibony. In [BS1], this was proven with the additional hypothesis f is hyperbolic (but without the a priori assumption that the periodic points are dense in the Julia set).

Brolin uses potential theory to analyze polynomials of one variable, where the harmonic measure is the unique measure of maximal entropy. In the case of (non-polynomial) rational maps there is a unique measure of maximal entropy but it is not in general related to the harmonic measure. Lyubich's proof [L] in the case of rational maps uses the balanced property of the unique measure of maximal entropy together with a Shadowing Lemma. For polynomial automorphisms of $\mathbf{C}^2$ the analogue of the balanced property is the product structure as described in [BLS]. In this paper we use the product structure and the Shadowing Lemma. This argument is of quite a general nature and we expect it to be useful outside the holomorphic setting. From our point of view the product structure of the measure of maximal entropy is the underlying reason for the equidistribution property in the known cases.

According to Newhouse there exist polynomial automorphisms with infinitely many sink orbits. The next result shows that the majority of orbits are saddle orbits.

Corollary 1. *Most periodic points are saddle points in the sense that*

$$\lim_{n\to\infty} \frac{1}{d^n} \#\mathrm{SPer}_n = 1.$$

A weaker asymptotic formula

$$\limsup_{n\to\infty} \left(\#\mathrm{SPer}_n\right)^{1/n} = d$$

follows from a theorem of Katok [K] and the entropy formula $h(f) = \log d$ ([FM] and [S]).

The following corollary answers a question of [FM]:

Corollary 2. *f has points of all but finitely many periods.*

Given a point x, let us define *the Lyapunov exponent* at x as

$$\chi(x) = \lim_{n\to+\infty} \frac{1}{n} \log \|Df^n(x)\|,$$

provided the limit exists. For example, if p is a saddle point of period n, then $\chi(p) = \frac{1}{n}\log\lambda^u(p)$. For any ergodic invariant measure ν the function $\chi(x)$ is constant ν almost everywhere. This common value is called the Lyapunov exponent of f with respect to ν. The harmonic measure μ is ergodic, and we denote by Λ the Lyapunov exponent of f with respect to μ. We have an alternate description of Λ as:

$$\Lambda = \lim_{n\to+\infty} \frac{1}{n} \int \log ||Df^n||\mu.$$

The following result allows us to compute Λ by averaging the Lyapunov exponents of the saddle points.

Theorem 2. *For any polynomial automorphism f we have:*

$$\Lambda = \lim_{n\to\infty} \frac{1}{nd^n} \sum_{p\in\mathrm{SPer}_n} \chi(p).$$

The statement is equally true with the set SPer_n replaced by Fix_n. We note (from [BS3]) that $\Lambda \geq \log d$, so the previous theorem gives a lower bound for the average exponent for periodic points.

For quadratic maps in one dimension the Lyapunov exponent with respect to the harmonic measure is closely related to the Green function of the Mandelbrot set. In particular the dynamical behavior of f is reflected in the behavior of this function. It is interesting to investigate the relation between the behavior of $\Lambda = \Lambda(f)$ and the dynamics of f for automorphisms of $\mathbf{C}^2$. Consider a holomorphic parameterization $c \mapsto f_c$. It is shown in [BS3] that $\Lambda(f_c)$ is subharmonic for all values of c. The following result shows that non-harmonicity of $\Lambda(f_c)$ corresponds to the creation of sinks (in the dissipative case) or Siegel balls (in the volume-preserving case) for nearby maps. (A Siegel ball is the analogue for volume-preserving maps of a Siegel disk, see [BS2], [FS].)

Theorem 3. *Consider a family of $\{f_c\}$ depending holomorpically on a parameter c in the disk. Assume that $c \mapsto \Lambda(f_c)$ is not harmonic at $c = c_0$. If the maps f_c are dissipative, then there is a sequence $c_i \to c_0$ defined for $i \geq N$ so that f_{c_i} has a sink of period i. If the maps f_c are volume preserving, then there is a sequence c_i defined for $i \geq N$ so that f_{c_i} has a Siegel ball of period i.*

It is shown in [BS3] that $c \mapsto \Lambda(f_c)$ is a harmonic function for the values of the parameter for which f_c is hyperbolic. In this context hyperbolicity implies structural stability which implies that the topological conjugacy class of the map is locally constant. The proof of Theorem 3 shows that structural stability directly implies the harmonicity of Λ:

Theorem 4. *If the maps f_c are topologically conjugate to one another in a neighborhood of $c = c_0$, then the function $c \mapsto \Lambda(f_c)$ is harmonic at c_0.*

§2. Lyapunov charts, Pesin boxes and the Shadowing Lemma.

Topological bidisks. Let us take a pair $\mathcal{D}^s$ and $\mathcal{D}^u$ of standard disks $\{z : |z| \leq 1\}$, and consider the standard bidisk $\mathcal{D} = \mathcal{D}^u \times \mathcal{D}^s$. We refer to the sets $\mathcal{D}^s(b) = \mathcal{D}^s \times \{b\}$ as *stable cross sections* and $\mathcal{D}^u(a) = \{a\} \times \mathcal{D}^u$ as *unstable cross sections*. Also, the boundary of $\mathcal{D} = \mathcal{D}^s \times \mathcal{D}^u$ will be partitioned into $\partial\mathcal{D}^s \cup \partial\mathcal{D}^u$, where we set $\partial^s\mathcal{D} := \mathcal{D}^s \times \partial\mathcal{D}^u$ and $\partial^u\mathcal{D} := \partial\mathcal{D}^s \times \mathcal{D}^u$.

Let us define a *topological bidisk* B as a compact set homeomorphic to $\mathcal{D}$ together with a homeomorphism $h : B \to \mathcal{D}^2$. Then B inherits the structure of the stable/unstable (s/u) cross sections $B^s(x)$ and $B^u(x)$, and the partition of the boundary into $\partial^s B$ and $\partial^u B$, which is induced by this homeomorphism. We note that the stable/unstable cross sections are not going to be stable or unstable manifolds, but rather an approximation to them. If h is affine we can talk about the *affine bidisk*.

Let us say that a two dimensional manifold Γ *u-overflows* a bidisk (B, h) if $h(\Gamma \cap B) \subset \mathcal{D}$ is the graph of a funcion $\phi : \mathcal{D}^u \to \mathcal{D}^s$. If additionally $\Gamma \subset B$ we also say that Γ is *u-inscribed* in B. By the slope of Γ in B we mean $\max \|D\phi(z)\|$. Similarly we can define the dual concept related to the stable direction.

We will say that a topological bidisk B_1 *u-correctly intersects a topological bidisk* B_2 (or "the pair (B_1, B_2) intersects u-correctly") if every unstable cross-section $B_1^u(x)$ u-overflows B_2 and every stable cross section $B_2^s(x)$ s-overflows B_1. If the pair (B_1, B_2) intersects u-correctly, then the intersection $B_1 \cap B_2$ becomes a topological bidisk, if we give it the unstable cross sections from B_1 and the stable cross sections from B_2. (In other words, the straightening homeomeorphism is $h : x \mapsto (h_1^s(x), h_2^u(x))$ where $h_i = (h_i^s, h_i^u) : B_i \to \mathcal{D}$ are straightening homeomorphisms for B_i). In the case when $B_1 \subset B_2$ the u-overflowing property is equivalent to $\partial^s B_1 \subset \partial^s B_2$. Then we also say that B_1 is *u-inscribed* in B_2.

Of course, we have the corresponding dual s-concepts.

Adapted Finsler metric and Lyapunov charts. We will use the exposition of Pugh and Shub [PS] as the reference to the Pesin theory. Below we will adapt our presentation of the theory to our specific goals. Let us consider a holomorphic diffeomorphism $f : \mathbf{C}^2 \to \mathbf{C}^2$. Let μ be an invariant, ergodic, hyperbolic measure (the latter means that it has non-zero characteristic exponents, $\chi^s < 0 < \chi^u$). Let $\mathcal{R}$ denote the set of Oseledets regular points for f. For $x \in \mathcal{R}$ there exists an invariant splitting of the tangent space $T_x\mathbf{C}^2 = E_x^s \oplus E_x^u$ into a contracting direction E_x^s and an expanding direction E_x^u which depend measurably on x. Further, for any $0 < \chi < \min(|\chi^s|, \chi^u)$ there is a measurable function $C(x) = C_\chi(x) > 0$ such that

$$\left|Df^n|_{E_x^s}\right| \le C(x)e^{-n\chi} \quad \text{and} \quad \left|Df^{-n}|_{E_x^u}\right| \le C(x)e^{-n\chi} \tag{2}$$

for $n = 1, 2, 3, \ldots$ Set $Q_C = \{x : \ C(x) \le C\}$. These sets exhaust a set of full μ measure as $C \to \infty$. Let us fix a big C and refer to Q_C as Q.

A key construction of the Pesin theory is a measurable change of the metric which makes f uniformly hyperbolic. Namely, for $\theta \in (0, \chi)$ and $x \in \mathcal{R}$, set

$$|v|^* = \sum_{n\ge 0} e^{n\theta}|Df^{-n}v| \quad \text{for} \quad v \in E_x^u, \tag{3^u}$$

$$|v|^* = \sum_{n\ge 0} e^{n\theta}|Df^{n}v| \quad \text{for} \quad v \in E_x^s. \tag{3^s}$$

For arbitrary tangent vector $v \in T_x$ set $|v|^* = \max(|v^s|^*, |v^u|^*)$ where v^s and v^u are its stable and unstable components. This metric is called *adapted Finsler* or *Lyapunov*. Observe that

$$\frac{1}{2}|v| \le |v|^* \le B(x)|v| \tag{4}$$

with a measurable function $B(x) = C(x)/(1 - e^{\theta-\chi})$. Hence the adapted Finsler metric is equivalent to the Euclidean metric on the set Q.

Let $r(x) > 0$ be a measurable function. Then taking advantage of natural identification of $\mathbf{C}^2$ with $T_x(\mathbf{C}^2)$, we can consider a family of affine bidisks

$$L(x) = \{x + v : |v|^* < r(x)\}. \tag{5}$$

By the *inner size* of $L(x)$ we mean $\min\{|v^s|, |v^u| : x+v \in L(x)\}$. The Pesin theory provides us with a choice of the "size-function" $r(x)$ with the following properties.

(L1) The inner size of Lyapunov charts is bounded away from 0 on the set Q. Indeed, by (4) on this set the adapted Finsler metric is equivalent to the Euclidean metric. But then $r(x)$ stays away from 0 as one can see from its explicit definition in [PS], p.13.

For a complex one-dimensional manifold $\Gamma \subset L(x)$, let us define its *cut-off iterate*, $f_x\Gamma$, as $f\Gamma \cap L(fx)$.

(L2) There is a measurable function $\kappa(x) > 0$ with the following property. Denote by $\mathcal{G}_x^u$ the family of complex one-dimensional manifolds which are u-inscribed into $L(x)$ and have the slope less than $\kappa(x)$. Then the operation of cut-off iteration, f_x, transforms $\mathcal{G}_x^u$ into $\mathcal{G}_{fx}^u$. In particular, the topological bidisk $fL(x)$ correctly intersects $L(x)$ (where the bidisk structure on $fL(x)$ comes from $L(x)$).

This allows us to repeat the cut-off process for iterates of f. Let f_x^n denotes the composition of the cut-off iterates at $x, fx, \dots, f^{n-1}x$. The hyperbolicity of μ implies:

(L3) For any two manifolds $\Gamma_i \in \mathcal{G}_x^u$, their cut-off iterates are getting close exponentially fast:

$$C^1\text{−dist}(f_x^n\Gamma_1, f_x^n\Gamma_2) \le A(x)e^{-\theta n}$$

with a measurable $A(x)$. Moreover, $A(x)$ is bounded on Q.

(L4) Let $p = f^{-n}x$. Then for any manifold $\Gamma \in \mathcal{G}_p^u$, and for any two points $y, z \in f_p^n Gamma$,

$$\text{dist}(f_p^{-n}y, f_p^{-n}z) \le A(x)\text{dist}(y, z)e^{-\theta n}$$

with $A(x)$ as above.

(In the stable direction we of course have the corresponding properties with the reversed time.) Such a family of affine bidisks $L(x)$ will be called a family of *Lyapunov charts.*

Stable and unstable manifolds. Let us consider the push-forward of a Lyapunov chart and trim it down by intersecting with the image chart. Since by (L2) the pair $(fL(x), L(fx))$ intersects u-correctly, the set $L_1^u(fx) = L(fx) \cap fL(x)$ a topological bidisk u-inscribed into $L(fx)$. Similarly, $L_1^s(x) = L(x) \cap f^{-1}L(fx)$ is a topological bidisk s-inscribed into $L(x)$, and $f : L_1^s(x) \to L_1^u(fx)$ is a bidisk diffeomorphism.

Property (L2) allows us continue inductively by noting that the pair $(L(f^n x), fL_{n-1}^u)$ intersects u-correctly, and we can consider topological bidisks

$$\begin{gathered} L_n^u(f^n x) = L(f^n x) \cap fL_{n-1}^u(f^{n-1}(x) = \\ f^n\{x : x \in L(x), fx \in L(fx), ..., f^n x \in L(f^n x)\}, \end{gathered} \tag{6a}$$

and

$$L_n^s(x) = f^{-n}L_n^u(f^n x) = \{x : x \in L(x), fx \in L(fx), ..., f^n x \in L(f^n x)\}. \tag{6b}$$

Observe that $L_n^s(x)$ can also be defined as the connected component of $L(x) \cap f^{-n}L(f^n x)$ containing x.

Furthemore, $L_n^s(x)$ is s-inscribed into $L(x)$, and by the dual property to (L3) the sizes of $L_n^s(x)$ in the stable direction shrink down to zero. Hence the intersection

$$W_{loc}^s(x) = \bigcap_{n\ge0} L_n^s(x) = \{x : f^n x \in L(f^n x),\ n = 0, 1, ...\} \tag{7}$$

is a manifold of the family $\mathcal{G}^u_x$ s-inscribed into $L(x)$. This is exactly the local stable manifold through x. By the dual to (L4), for any $y, z \in W^s(x)$

$$\mathrm{dist}(f^n y, f^n z) \leq A(x) e^{-\theta n}$$

with a measurable function $A(x)$.

Reversing the time we obtain local unstable manifolds as well.

Pesin boxes. Recall that $Q = Q_C \subset \mathcal{R}$ denotes the set on which $C(x) \leq C$ in (2). This set is compact, and it may be shown that $Q \ni x \mapsto E^{s/u}_x$ is continuous. Further, $W^s(x)$ and $W^u(x)$ intersect transverally at x, and $W^{s/u}_{loc}(x)$ varies continuously with $x \in Q$. Thus if $x, y \in Q$ are sufficiently close to each other, then there is a unique point of intersection $[x, y] := W^s_{loc}(x) \cap W^u_{loc}(y)$. It follows that $[\,,] : Q \times Q \to \mathbf{C}^2$ is defined and continuous near the diagonal. Given a small $\alpha > 0$, let us set

$$\overline{Q} = \overline{Q}_{C,\alpha} = \{[x, y] : \ x, y \in Q, \ \mathrm{dist}(x, y) < \alpha\}.$$

This is a closed subset of $\mathcal{R}$ on which $C(x) < C/2$ in (2) (provided α is sufficiently small), and which is locally closed with respect to $[\,,]$-operation. By (L1) the inner size of the Lyapunov charts $L(x)$ stays away from 0 for $x \in \overline{Q}$. Let $\rho > 0$ be a lower bound for this size.

We say that a set P has *product structure* if P is closed under $[\,,]$. If in addition P is (topologically) closed and has positive measure, then it will be called a *Pesin box*. Let us cover $\overline{Q}$ with countably many closed subsets X_i of positive measure with diameter $< \alpha$. Taking the $[\,,]$-closure of X_i, we obtain a covering of $\overline{Q}$ with a countably many Pesin boxes $P_i \subset \overline{Q}$. Since the μ almost whole space can be exhausted by the sets Q_C, we conclude that it can be covered by countably many Pesin boxes of arbitrarily small diameter. Our next goal is to make these boxes disjoint (at expense of losing a set of small measure).

Lemma 1. *For any $\epsilon > 0$ and $\eta > 0$ there is a finite family of disjoint Pesin boxes P_i such that* $\mathrm{diam}(P_i) < \eta$ *and* $\mu(\bigcup P_i) > 1 - \epsilon$.

Proof. As we can cover a set of full measure by countably many Pesin boxes, we can cover a set X of measure $1 - \epsilon$ by finitely many boxes. Now let us make them disjoint by the same procedure as used by Bowen (cf. [B2], Lemma 3.13). Namely, if two boxes P_1 and P_2 intersect, subdivide each of them into four sets $P^j_i \subset P_i$ with product structure in the following way:

$$P^1_1 = \{x \in P_1 : W^s(x) \cap P_2 \neq \emptyset, W^u(x) \cap P_2 \neq \emptyset\},$$
$$P^2_1 = \{x \in P_1 : W^s(x) \cap P_2 = \emptyset, W^u(x) \cap P_2 \neq \emptyset\},$$
$$P^3_1 = \{x \in P_1 : W^s(x) \cap P_2 \neq \emptyset, W^u(x) \cap P_2 = \emptyset\},$$
$$P^4_1 = \{x \in P_1 : W^s(x) \cap P_2 = \emptyset, W^u(x) \cap P_2 = \emptyset\}.$$

Repeating this procedure, we can cover the set X by a finite number of disjoint sets $Q_i, i = 1, \ldots, N$, with product structure. However, Q_i need not be closed. To restore this property, let us consider closed sets $K_j \subset Q_j$ such that $\mu(Q_j - K_j) < \epsilon/N$. Completing these sets with respect to the product structure, we obtain a suitable family of Pesin boxes. □

A common chart. Our goal is to have a common Lyapunov chart for all points returning to a Pesin box.

Observe that the family of Lyapunov charts can be reduced in size by a factor $t \leq 1$. Then the manifolds of the family $\mathcal{G}_x^u$ (correspondingly $\mathcal{G}_x^s$) become almost parallel within the scaled charts. In particular, the stable cross sections of the bidisks $L_n^s(x)$, as well as the local stable manifolds truncated by the scaled family of charts become almost parallel.

Let ρ be the lower bound of the size of the Lyapunov charts $L(x),\ x \in Q$. Let us take a Pesin box P of size $\eta < \rho/8$. Let $a \in P$, and let $B^s(a,r) \subset E_a^s$, $B^u(a,r) \subset E_a^u$ denote the Euclidean disks of radius r centered at a in the corresponding subspaces. Let us consider an affine bidisk $B = B^s(a,\rho/2) \times B^u(a,\rho/2)$. Since the stable/unstable directions through the points $x \in P$ are almost parallel, we have the inclusions:

$$P \subset B \subset \bigcap_{x \in P} L(x).$$

For $x \in P \cap f^{-n}P$ let $T = B_{n,x}^s$ be the component of $B \cap f^{-n}B$ containing x. Let also $y = f^n x$, $R = f^n T = B_{-n,y}^u$.

Lemma 2. *The set T/(respectively R) is a topological bidisk which is s/u-correctly inscribed in B. The s/u-cross sections of T/R belong to the families $\mathcal{G}^{s/u}$ respectively. Moreover, T s-correctly intersects R.*

Proof. For $z \in T$ let $T^u(z) = T \cap (z + E_a^u)$, and similarly for $\zeta \in R$ let $R^s(\zeta) = R \cap (\zeta + E_a^s)$. The first claim will follow from the following dual statements:

(i) $f_x^n T^u(z)$ is a topological disk u-correctly inscribed into B;

(ii) $f_y^{-n} R^s(\zeta)$ is a topological disk which is s-correctly inscribed into B.

Let us prove (i). Let us consider the Lyapunov chart $L(x) \supset B$. Then the disk $K = L(x) \cap (z + E_a^u)$ belongs to the family of graphs $\mathcal{G}_x^u$. Hence its cut-off iterate $f_x^n K \in \mathcal{G}^u(y)$ overflows $L(y)$. Since it is almost parallel to E_a^u, $f_x^n T^u(z) = f_x^n K \cap B$ is u-correctly inscribed into B. This proves the first two claims.

The last one now follows from the transversality of the families $\mathcal{G}^u$ and $\mathcal{G}^s$. □

Set $W_B^{u/s}(x) = W_{loc}^{u/s}(x) \cap B$.

Lemma 3. *Under the above circumstances we have*

$$W_B^s(x) \subset T \tag{8}$$

and

$$W_{loc}^u(x) \cap T \subset f^{-n} W_{loc}^u(y). \tag{9}$$

Proof. Since $f^n W_B^s(x)$ has an exponentially small size, it is contained in B. So $W_B^s(x) \subset B \cap f^{-n}B$. Since $W_B^s(x)$ is connected, (8) follows. □

In order to get (9), observe that $T \subset L_n^s(x)$. But we know by the construction of the stable/unstable manifolds that $W_{loc}^u(x) \cap L_n^s(x) = f^{-n} W_{loc}^u(y)$. □

Observe that by definition the sets $B^u_{n,x}$ are either disjoint or coincide. So, for each n we can consider the following equivalence relation on the set $P\cap f^{-n}P$ of returning points: $x\sim y$ if $B^u_{n,x}=B^u_{n,y}$.

Lemma 4. *The equivalence classes have a product structure. Moreover, if $x,z\in P\cap f^{-n}P$ are equivalent then $f^n[x,z]=[f^nx,f^nz]$.*

Proof. Denote $q=[x,z]$. Clearly, $f^nq\in W^s_{loc}(f^nx)$. Further, by (8), $q\in T$. Hence by (9) $f^nq\in W^u_{loc}(f^nz)$, and the property $f^n[x,z]=[f^nx,f^nz]$ follows. Moreover, it follows that $q\in T\cap P\cap f^{-n}P$, so that it is equivalent to x,z. □

The Shadowing Lemma. *For any $x\in P\cap f^{-n}P$, $y=f^nx$, there is a unique saddle point $\alpha\in B^s_{n,x}\cap B^u_{-n,y}$ of period n. Moreover, $\mathrm{dist}(\alpha,P)\le Ce^{-n\theta}$.*

Proof. Let us consider the family $\mathcal{G}^u$ of manifolds u-inscribed into B with the slope $\le\kappa\le\inf_{x\in P}\kappa(x)$ (with respect to the decomposition $E^u_a\oplus E^s_a$) . Take a returning point $x\in P\cap f^{-n}P$, and let $y=f^nx$. Then we can consider the cut-off iterate $\Phi_u:\Gamma\mapsto f^n_x\Gamma\cap B$. Since its image $\Phi^u\Gamma$ is close to the unstable manifold $W^u_B(y)$, it is almost parallel to E^u_a. It follows that for sufficiently big n, Φ_u maps $\mathcal{G}^u$ into itself. Moreover, this transformation is contracting according to (L3).

Hence there is a unique Φ_u-invariant manifold $G^u\in\mathcal{G}^u$. This manifold is u-inscribed into $B^u_{-n,y}$, and can be characterized as the set of all points non-escaping $B^u_{-n,y}$ under backward iterates of f^n. Similarly, we can define a transformation Φ_s corresponding to the return of y back to P under f^{-n}, and find a unique Φ_s-invariant manifold $\Gamma^s\in\mathcal{G}^s$, $\Gamma^s\subset B^s_{n,x}$. This manifolds intersect transversally at a unique point α which is the desired periodic point.

There are no other periodic ponts of period n in $B^u_{x,n}$. Indeed, all points escape $B^u_{x,n}$ either under forward or backward iterates of f^n. Finally since the bidisks $B^s_{n,x}$ and $B^u_{-n,y}$ are exponentially thin, the point α is exponentially close to $[x,y]\in P$. □

§3. Proofs of the theorems.

According to the Shadowing Lemma, to each returning point $x\in P\cap f^{-n}P$ we can assign a periodic point $\alpha=\alpha(x)\in B^s_{n,x}\cap B^u_{n,x}$ of period n. Given such an α, set

$$T(\alpha)=\{x\in P\cap f^{-n}P:\alpha(x)=\alpha\}=P\cap f^{-n}P\cap B^s_{n,\alpha},$$

where $B^s_{n,\alpha}=B^s_{n,x}$ is the component of $B\cap f^{-n}B$ containing x. The sets $T(\alpha)$ actually coincide with the equivalence classes introduced above.

In what follows we assume that μ is the measure of maximal entropy of the polynomial diffeomorphism f. Since P has the product structure, it is homeomorphic to $P^s\times P^u$, were P^s and P^u are cross sections. It was proved in [BLS] that μ is a product measure with respect to this topological structure, i.e. $\mu\llcorner P=\mu^-|_{P^s}\otimes\mu^+|_{P^u}$.

Lemma 5. $\mu(T(\alpha))\le\mu(P)d^{-n}$.

Proof. By Lemma 4, T is naturally homeomorphic to $T^s\times T^u$, where $T^s=T^s(x)$ and $T^u=T^u(x)$ be the cross sections of $T=T(\alpha)$ through the point $x\in T(\alpha)$. By the product property of μ,

$$\mu(T)=\mu^-(T^s)\mu^+(T^u).\tag{10}$$

Since $T^s \subset P^s(x)$,
$$\mu^-(T^s) \leq \mu^-(P^s(x)) = \mu^-(P^s). \tag{11}$$
By (9), $f^n T^u \subset W^u_{loc}(f^n x) \cap P = P^u(f^n x)$. By the transformation rule for μ^+ we conclude that
$$\mu^+(T^u) \leq d^{-n}\mu^+(P^u). \tag{12}$$
Now the result follows from (10)–(12). $\square$

In the following proof we let SFix_n be the set of saddle periodic points of period dividing n.

Lemma 6. *For any $\epsilon > 0$, there exists $C > 0$ depending on the Pesin box P such that*
$$\liminf_{n\to\infty} \frac{1}{d^n} \#\{\alpha \in \mathrm{SPer}_n : \mathrm{dist}(\alpha, P) < Ce^{-n\theta}\} \geq \mu(P).$$

Proof. In our work above, we have assigned to any returning point $x \in P \cap f^{-n}P$ a periodic point $\alpha(x) \in \mathrm{SFix}_n$ exponentially close to P. Let A_n denote the set of periodic points obtained in this way. Hence, by the Lemma 5,
$$\mu(P) d^{-n} \# A_n \geq \sum_{\alpha \in A_n} \mu(T(\alpha)) = \mu(P \cap f^{-n}P)$$
and by the mixing property of μ [BS],
$$\frac{\# A_n}{d^n} \geq \frac{\mu(P \cap f^{-n}P)}{\mu(P)} \to \mu(P)$$
as $n \to \infty$.

Thus we have shown that
$$\liminf_{n\to\infty} \frac{1}{d^n} \#\{\alpha \in \mathrm{SFix}_n : \mathrm{dist}(\alpha, P) < Ce^{-n\theta}\} \geq \mu(P).$$
It remains to show that SFix_n can be replaced by SPer_n. For this, we note that $\{k : k|n, k < n\} \subset \{k \leq n/2\}$. Since $\#\mathrm{Fix}_n \leq d^n$, it follows that
$$\#\mathrm{Fix}_n - \#\mathrm{Per}_n \leq \frac{n}{2} d^{n/2}.$$
Since this is $o(d^n)$, we see that almost all points of SFix_n have period precisely n. $\square$

Proof of Theorem 1. Let $\nu_n = d^{-n}\sum_{a \in \mathrm{Per}_n} \delta_a$. Consider any limit measure $\nu = \lim \nu_{n(i)}$ of a subsequence of these measures. It follows from Lemma 6 that for any Pesin box, $\nu(P) \geq \mu(P)$. Now let G be any open set. Consider a compact set $K \subset G$ such that $\mu(G - K) < \epsilon$. Let $\eta = \mathrm{dist}(K, \partial G)$. By Lemma 1 we can cover all but a set of measure ϵ by disjoint Pesin boxes P_i of size less than η. Let I denote the set of those Pesin boxes which intersect K. Then
$$\nu(G) \geq \sum_{i \in I} \nu(P_i) \geq \sum_{i \in I} \mu(P_i) \geq \mu(K) - \epsilon \geq \mu(G) - 2\epsilon.$$
Hence $\nu \geq \mu$, and as the both measures are normalized, $\nu = \mu$. $\square$

Now we turn to the Lyapunov exponent $\Lambda(f)$. For $x \in \mathcal{R}$ we have

$$\lim_{n\to\infty} \frac{1}{n} \log ||Df^n(x)|| = \lim_{n\to\infty} \frac{1}{n} \log \left|Df(x)|_{E^u_x}\right| = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \left|Df(f^j x)|_{E^u_{f^j x}}\right|.$$

Thus with the notation

$$\psi(x) = \log \left|Df(x)|_{E^u_x}\right| \tag{13}$$

we have

$$\Lambda(f) = \int \psi(x)\mu(x). \tag{14}$$

Lemma 7. *Let P denote a Pesin box, and let $\epsilon > 0$ be given. Then there exists N sufficiently large that the periodic saddle point $\alpha(x)$ generated by a returning point $x \in P \cap f^{-n}P$ with $n \geq N$ satisfies $dist(E^u_{\alpha(x)}, E^u(x)) < \epsilon$.*

Proof. By the construction above, the unstable manifold $W^u(\alpha(x))$ goes across the topological bidisk B, and $W^u(\alpha(x)) \subset B^u_n$. Since the stable cross section of B^u_n is exponentially small in n, we have the desired estimate on the distance of the tangent spaces. □

Proof of Theorem 2. The measures $d^{-n}\sum_{a\in P_n} \delta_a$ converge to μ. To evaluate the integral in (14), we may work on a countable, disjoint family of Pesin boxes P_j, with the property that Df and the tangent spaces E^u, and thus the expression ψ in (13), vary by no more than ϵ on each P_j. By Lemma 7, as $n \to \infty$, the tangent spaces $E^u_{\alpha(a)}$ converge to within ϵ of the tangent spaces E^u given by the Oseledec Theorem. Thus $\psi(\alpha(x))$ is within ϵ of the value of ψ on P_j. It follows that the expression in Theorem 2 will be within ϵ of the integral (14). □

Corollary 3. *If P_n denotes any of the sets defined in (1), then*

$$\Lambda(f) = \lim_{n\to\infty} d^{-n} \sum_{p\in P_n} \psi(p).$$

Proof. Since K is compact, it follows that $||Df_x|| \leq M$ for all $x \in K$. And since $P_n \subset K$, it follows that that $\psi(p) \leq M$. Further, the Jacobian of f is a constant a, so even at a sink orbit of order n, we have $\psi(p) \geq \log\sqrt{|a|}$. Thus we may use any of the three sets P_n in (1) in defining the limit. □

Proof of Theorem 3. The conclusion of the theorem is a consequence of the following statement: For any neighborhood U of c there is a number N_U so that for each $n \geq N_U$ there is a $c \in U$ so that f_c has a sink of period n. We will show that if the previous assertion does not hold then $\Lambda(f_c)$ is harmonic. Thus assume that there exists a neighborhood U and an infinite sequence n_i such that f_c has no sink of period n_i for any $c \in U$ and any i.

Fix an $n = n_i$. Let $V = \{(c,p) \in U \times \mathbf{C}^2 : f^n_c(p) = p\}$. The set V is a one dimensional analytic variety with a projection onto U. We can remove from V a discrete set of points $\{(c_1,p_1),(c_2,p_2),\ldots\}$ corresponding to "bifurcations" where either V is singular or the projection onto the first coordinate has a singularity. Let $V' = V - \{(c_1,p_1),(c_2,p_2),\ldots\}$

and let $U' = U - \{c_1, c_2, \ldots\}$. On V' the period of a periodic point is constant on each component. Remove components for which the period is less than n. Call the resulting set V''.

Assume first that we are in the dissipative case, and there are no sinks of period n. Since there is no sink, the modulus of the larger eigenvalue must be at least as large as 1, and since it is dissipative, the modulus of the smaller eigenvalue must be no larger than $\sqrt{|a|} < 1$. Thus at each point (c, p) in V' there is a unique largest eigenvalue for $Df_c^n(p)$ call it $\lambda^+(c, p)$. The function λ^+ is a continuously chosen root of the characteristic equation of $Df_c^n(p)$ so it is holomorphic. In particular the function $\log|\lambda^+(c, p)|$ is harmonic on V' and the function

$$U' \ni c \mapsto \frac{1}{d^n} \sum_{p \in P_n} \frac{1}{n} \log|\lambda^+(c, p)| = \frac{1}{d^n} \sum_{p \in P_n} \psi(p)$$

is harmonic on U'. This function extends to a harmonic function Λ_n on U. Now Theorem 2 implies that Λ_n converges pointwise to Λ. Since each function Λ_n is harmonic we conclude that Λ is harmonic. This proves our assertion in the dissipative case.

Assume now that we are in the volume preserving case. Note that since the function $c \mapsto \det f_c$ is holomorphic and of constant norm it is actually constant. Assume that there are no Siegel balls of period n. By the two-dimensional version of the Siegel linearization theorem (see [Z]), this implies that there are no elliptic periodic points of period n for which the eigenvalues satisfy certain Diophantine conditions. In particular if at any elliptic periodic point the eigenvalues are not constant as a function of c then we can vary the parameter so that λ_1 varies through an interval on the unit circle. This implies that there is some parameter value at which the Diophantine conditions are verified for the eigenvalues λ_1 and $\lambda_2 = const/\lambda_1$. We conclude that at any elliptic point (c, p) of period n the eigenvalues are locally independent of c. This implies that the eigenvalues are constant on the component of V' which contains the point (c, p).

As before we remove from V' the components on which the period is less than n. In addition we remove components on which both eigenvalues are constant and have modulus 1. The remaining variety V'' consists of saddles. Arguing as before we see that Λ_n is harmonic hence Λ is harmonic. This completes the proof of the theorem. □

Proof of Theorem 4. Let J^* denote the support of μ, which is the closure of the saddle points (cf. [BS3]). If the maps $f_c|_{J^*}$ are topologically conjugate, then the conjugacy preserves the set of periodic points, so that each periodic point p is part of a selection $c \mapsto p(c)$. As in the proof of Theorem 3, each function Λ_n is harmonic; hence Λ is harmonic. □

References

[BLS] E. Bedford, M. Lyubich, and J. Smillie, Polynomial diffeomorphisms of $\mathbf{C}^2$. IV: The measure of maximal entropy and laminar currents. Invent. Math., to appear.

[BS1] E. Bedford and J. Smillie, Polynomial diffeomorphisms of $\mathbf{C}^2$: Currents, equilibrium measure and hyperbolicity. Invent. Math. 87, 69–99 (1990)

[BS2] E. Bedford and J. Smillie, Polynomial diffeomorphisms of $\mathbf{C}^2$ II: Stable manifolds and recurrence. J. AMS 4, 657–679 (1991)

[BS3] E. Bedford and J. Smillie, Polynomial diffeomorphisms of $\mathbf{C}^2$ III: Ergodicity, exponents and entropy of the equilibrium measure. Math. Ann. 294. 395–420 (1992)

[B1] R. Bowen, Periodic points and measures for axiom A diffeomorphisms. Trans. AMS. 154, 377-397 (1971).

[B2] R. Bowen, Equilibrium states and the ergodic theory of Anosov diffeomorphisms, (Lect. Notes, Math., vol 470) Berlin Heidelberg New York: Springer 1975

[Br] H. Brolin, Invariant sets under iteration of rational functions. Ark. Mat. 6, 103–144 (1965)

[FS] J.-E. Fornæss and N. Sibony, Complex Hénon mappings in $\mathbf{C}^2$ and Fatou Bieberbach domains. Duke Math. J. 65, 345–380 (1992)

[K] A. Katok, Lyapunov exponents, entropy and periodic orbits for diffeomorphisms. Publ. Math. Inst. Hautes Etud. Sci. 51, 137–174 (1980)

[L] M. Lyubich, Entropy of analytic endomorphisms of the Riemannian sphere. Funct. Anal. Appl. 15, 300–302 (1981)

[PS] C. Pugh and M. Shub, Ergodic attractors. Trans. AMS 312, 1–54 (1989)

[Si] N. Sibony, Iteration of polynomials, U.C.L.A. course lecture notes.

[S] J. Smillie, the entropy of polynomial diffeomorphisms of $\mathbf{C}^2$. Ergodic Theory Dyn. Syst. 10, 923–827 (1990)

[T] P. Tortrat, Aspects potentialistes de l'itération des polynômes. In: Séminaire de Théorie du Potentiel Paris, No. 8 (Lect. Notes Math., vol. 1235) Berlin Heidelberg New York: Springer 1987.

[Z] E. Zehnder, A simple proof of a generalization of a Theorem by C. L. Siegel. Geometry and Topology (J. Palis and M. do Carmo, eds.), Lecture Notes in Math., vol. 597, Springer-Verlag, New York, 855–866 (1977)

Indiana University, Bloomington, IN 47405

SUNY at Stony Brook, Stony Brook, NY 11794

Cornell University, Ithaca, NY 14853