Section 5: Newton's Method

Bad Polynomials for Newton's Method

Scott Sutherland

Newton's method for solving $f(z) = 0$ corresponds to iteration of $z \mapsto z - f(z)/f'(z)$, which is a degree d rational map of $\overline{\mathbb{C}}$ in the case where f is a polynomial of degree d with distinct roots. Newton's method has long been an important source of examples and theorems in complex dynamical systems (for example, the work of Schröder [Sch, Sch1], Fatou [Fa], and more recently Douady and Hubbard [DH]), as well as being one of the most commonly used numerical schemes for approximating roots. See [HP] and [Sm] for an introduction to the dynamics of Newton's Method.

Describing the set of polynomials for which the corresponding Newton's method has periodic sinks which are not roots is an important open problem, (problem 6 of [Sm]). We shall refer to such polynomials as "bad polynomials". This question is essentially answered for cubic polynomials by the work of Tan Lei [Ta] and Janet Head [He], in which the more comprehensive task of giving a combinatorial description of the parameter space for Newton's method is undertaken. A complete description of the parameter space for higher degrees still seems some way off, however.

In order to answer Smale's question for higher degree polynomials, it may be helpful to consider the relationship between the "relaxed Newton's method"

$$N_{h,f}(z) = z - h\frac{f(z)}{f'(z)}$$

and the "Newton Flow" $\mathcal{N}_f$ given by the ordinary differential equation

$$\dot{z} = -\frac{f(z)}{f'(z)}.$$

One sees immediately that the map is an Euler approximation to the flow using step size h. The attractors of $\mathcal{N}_f$ are sinks located at the zeros of $f(z)$, ∞ is the only source, and the other fixed points are at the singularities corresponding to the critical points of f. We can rescale time for $\mathcal{N}_f$ to obtain $\dot{z} = -f(z)\overline{f'(z)}$ (or alternatively $\dot{z} = -\nabla||f(z)||^2$), from which we can easily see that these singularities are hyperbolic saddles. Furthermore, solution curves of $\mathcal{N}_f$ are mapped by f to straight lines emanating from the origin. Thus, if f has two critical values with the same argument, then the flow $\mathcal{N}_f$ is degenerate in the sense that there are solution curves which begin at one singularity and terminate at another. Refer to [JJT], [Sa], [STW], [Sm], and [Su] for more details about $\mathcal{N}_f$.

We propose the following conjecture (for which we have some numerical evidence) relating the degenerate flows and bad polynomials. This basically says that one can connect a polynomial which is bad for Newton's method to one which is bad for the flow.

Conjecture 1. Let f_1 be a bad polynomial of degree d, that is one for which Newton's method has an attractor which is not a root of f. Then there is a one-parameter family of polynomials $\{f_h\}_{0<h\leq 1}$ which are bad for the relaxed Newton's method N_{h,f_h}. Furthermore, as $h \to 0$, the corresponding flow $\mathcal{N}_{f_h}$ tends to a flow $\mathcal{N}_{f_0}$ which is degenerate.

This conjecture is consistent with the following, as explained below.

Conjecture 2. Let f be a polynomial of degree d with all its roots in the unit disk, let α be a root of multiplicity m for f, and let $A_h^*(\alpha)$ be the immediate attractive basin of α for the map $N_{h,f}$. Then the intersection of the set

$$\mathcal{A} = \bigcap_{0<h\leq m} A_h^*(\alpha)$$

with any circle of radius $R \geq 3$ contains arcs whose total length is at least $\frac{2\pi R}{cd}$, where c is a constant not depending on α, f, or d.

This second conjecture says that there is a definite neighborhood of the singular trajectories of $\mathcal{N}_f$ in which the Julia set of $N_{h,f}$ must be contained for all $h \in (0, m]$. Since the periodic orbits for $N_{h,f}$ which are not roots must be contained in the complement of $\bigcup_{f(\alpha)=0} A_h^*(\alpha)$, conjectures 1 and 2 taken together give some idea of the structure of the parameter space for $N_{h,f}$.

Conjecture 2 has been partially established by Benzinger [Be] (for all h sufficiently near 0), and is a generalization of the main result of [Su], which shows this for $h = 1$. I believe that with slight modifications, the proof in [Su] can be made to work for $0 < h \leq m$, which should nearly complete the proof of conjecture 2.

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