Section 4: Iterates of Entire Functions

Open Questions in Non-Rational Complex Dynamics

Robert Devaney

The dynamics of complex analytic functions have been studied by many authors during the past decade. Much of this work has been confined to the study of either rational or polynomial maps. The study of other analytic functions is still in its infancy and there are many unsolved problems in this area. In this note we describe a few of these problems.

1. Entire functions. The dynamics of entire functions are quite different from the dynamics of rational maps, mainly because of the essential singularity at infinity. By the Picard theorem, any neighborhood of this singularity is mapped infinitely often over the entire plane missing at most one point. This injects considerable hyperbolicity into the map and often causes the topology of the Julia set of the map to be vastly different from that of a rational map. In addition, the No Wandering Domains Theorem of Sullivan does not hold for this class of maps, so there may be both wandering domains and domains at infinity in the stable sets.

There is one class of entire maps whose dynamics are fairly well understood, namely the entire maps that have finitely many asymptotic and critical values (maps of finite type). With few exceptions (notably examples of Baker [B], Herman [H], and Eremenko and Lyubich [EL], most work has centered around this class of maps. Extending the study to a wider class of maps is an important problem.

Problem: Find a collection of representative examples of entire maps whose dynamics may be understood.

As a starting point, one might ask

Problem: What are the dynamics of maps of the form $\lambda e^z \sin z$ or $\lambda e^z \cos z$?

2. Entire functions of finite type. Most of the work thus far on the dynamics of entire maps has been concentrated on the class of finite type maps. These are the maps which have only finitely many singular (i.e., critical and asymptotic) values. This class includes λe^z, $\lambda \sin z$, and $\lambda \cos z$. It is known [GK,EL] that the No Wandering Domains theorem holds for this class, and that the Julia sets of these maps often contain Cantor bouquets [DT].

3. The exponential map. Of all entire maps, the exponential family $E_\lambda(z) = \lambda e^z$ has received the most attention. This is natural since E_λ, like the well-studied quadratic family $Q_c(z) = z^2 + c$, has only one singular value, the asymptotic value at 0. Thus this family is a "natural" one parameter family.

The parameter space for E_λ has been studied in [DGK]. However, there remain significant gaps in this picture. It is known that there exists Cantor sets of curves (called hairs) in the parameter plane for which the corresponding exponential maps have Julia sets that are the whole plane.

Problem: Describe completely the set of λ-values for which the Julia set of E_λ is $\mathbf{C}$.

Problem: Many of these λ-values lie on curves or hairs. Are these hairs C^∞? Analytic? Where and how do they terminate?

There are some interesting topological structures embedded in the dynamics of the exponential that warrant further study. For example, it is known that for $\lambda > 1/e$, $J(E_\lambda) = \mathbf{C}$. However, if $\lambda, \mu > 1/e$, then E_λ and E_μ are not topologically conjugate [DG]. If one looks at the invariant set consisting of $\{z \| 0 \leq \mathrm{Im} E_\lambda^n(z) \leq \pi \text{ for all } n\}$, it is known that this set is a Knaster-like continuum.

Problem: Are each of these Knaster-like continua homeomorphic? (for any $\lambda, \mu > 1/e$)

4. The Trigonometric Functions. The parameter spaces for families such as $S_\lambda(z) = \lambda \sin z$ or $C_\lambda(z) = \lambda \cos z$ also deserve special attention. They also contain curves on which the Julia set is the entire plane. The fundamental difference here is that C_λ and S_λ have no finite asymptotic values (only critical values), whereas the opposite is true for E_λ.

Problem: Describe the structure of the parameter space for C_λ and S_λ.

One fundamental difference between the trigonometric and exponential families is the following. Both maps are known to possess Cantor bouquets [DT] in their Julia sets. And any two planar Cantor bouquets are homeomorphic [AO]. Finally, McMullen [Mc] has shown that these Cantor bouquets always have Hausdorff dimension 2. However, the Lebesgue measure of these bouquets is quite different: they always have measure zero in the exponential case, but infinite measure in the trigonometric case.

Problem: What is the measure and dimension of the hairs i the parameter space for E_λ, S_λ and C_λ.

5. Other families of non-rational maps. Newton's method applied to non-rational maps offers a fertile area for further investigation. Outside of the work of Haruta [Ha] and van Haesler and Kriete [HK], there is little that is known. So a general problem is:

Problem: Describe the dynamics of Newton's method applied to general classes of entire functions?

This, of course, immediately leads to the question of iteration of meromorphic functions. Some work has been done here in case the map has polynomial Schwarzian derivative [DK] or when the map has finitely many singular values [BK]. But not much else is known.

Finally, there is an intriguing object called the tricorn introduced by Milnor [M] as one of his basic slices of parameter space for higher dimensional maps. This object arises as the analogue of the Mandelbrot set for the anti-holomorphic family $A_c(z) = \overline{z}^2 + c$. It is known [La] that the tricorn is not locally connected, but it also contains smooth arcs in the boundary (with no decorations attached) [W]. As this object arises in slices of the cubic connected locus, it certainly warrants further study. Winters [W] also has introduced a family of fourth degree polynomials whose parameter space is "naturally" $\mathbf{R}^3$ and which contains perpendicular slices given by the Mandelbrot set and the tricorn. Winters suggests that this family can model cubics since there are only two critical orbits.

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