Section 3: Measurable Dynamics

Measure and Dimension of Julia Sets

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Problem 1. Can it happen that a nowhere dense Julia set has positive Lebesgue measure?

The corresponding Ahlfors problem in Kleinian groups is also still unsolved.

So far it is known that the Julia set has zero measure in the following cases:

(i) hyperbolic, subhyperbolic and parabolic cases [DH], [L1].

(ii) a cubic polynomial with one simple non-escaping critical point and with a "non-periodic tableaue" (McMullen, see [BH]);

(iii) a quadratic polynomial which is only finitely renormalizable and has no neutral irrational cycles (Lyubich [L2] and Shishikura (unpublished)).

Let us say that a polynomial with one non-escaping critical point c is *renormalizable* if there is a quadratic-like map $f^n : U \to V,\ c \in U \subset V\ n > 1$, with connected Julia set. It corresponds to the case of periodic tableaue. Cases (i) and (ii) can be generalized in the following way:

(iv) a polynomial of any degree but with only one non-escaping critical point which does not have irrational neutral points and which is only finitely renormalizable.

In higher degrees one can describe a wide class of combinatorics for which the Julia set has zero measure (non-recurrent and "reluctantly recurrent" cases). The basic examples for which the answer is still unclear are

1. The Feigenbaum quadratic polynomial.

2. The Fibonacci polynomial $z \mapsto z^d + c$ with $d > 2$ (see [BH] or [LM] for the definition of the Fibonacci polynomial).

3. A polynomial with a Cremer point or Siegel disk (see the disciussion in Milnor's notes).

In the case when the Julia set coincides with the whole sphere the corresponding question is the following.

Problem 2. Is it true for all f with $J(f) = \bar{\mathbf{C}}$ that the following hold?

(i) $\omega(z) = \bar{\mathbf{C}}$ for almost all $z \in \bar{\mathbf{C}}$?

(ii) f is conservative with respect to the Lebesgue measure? (Conservativity means that the Poincaré Return Theorem holds).

Note that for the interval maps (replacing $\bar{\mathbf{C}}$ by an interval on which f is topologically mixing) (i) and (ii) are equivalent [BL2]. Moreovere, both of them hold for the quadratic-like maps of the interval [L3].

Problem 3. Let again $J(f) = \bar{\mathbf{C}}$. Is it true that f is ergodic with respect to the Lebesgue measure? Is it at least true that it has at most $2\ \deg f - 2$ ergodic components?

The answer to the first question is yes for a large set of rational maps [R]. The answer to the second one is yes for interval maps [BL1].

The discussed problems are closely related to the deformation theory of rational maps. The link between them is given by the notion of *measurable invariant line field on the Julia set* (see [MSS]). Each such field generates a quasi-conformal deformation of f supported on the Julia set. There is a series of Lattes examples having an invariant line field on the Julia set, and in these examples $J(f) = \bar{\mathbf{C}}$. Such a phenomenon is impossible at all for finitely generated Kleinian groups [S].

Problem 4. (Sullivan) Are the Lattes examples the only ones having measurable invariant line fields on the Julia sets?

Let us consider now an analytic family A of rational maps, and denote by $Q \subset A$ the set of J-unstable maps.

A recent remarkable result by Shishikura [Sh] says that in the quadratic family $z \mapsto z^2 + c$ there are a lot of Julia sets with Hausdorff dimension 2.

Problem 5 Find an explicit example of a Julia set of Hausdorff dimension 2. What is a natural geometric measure in the case when $J(f)$ has Hausdorff dimension 2 but zero Lebesgue measure?

A more general program is to develop an appropriate Thermodynamical Formalism in non-hyperbolic situations.

Problem 6. (i) What is the Lebesgue measure of Q?

(ii) Is the Hausdorff dimension of Q equal to $\dim A$? The answer is yes in the quadratic case [Sh]

Mary Rees proved that the Lebesgue measure of Q is positive [R] in the case when A is the whole space of rational maps of degree d. On the other hand, Shishikura claims that in the quadratic family $z \mapsto z^2 + c$ the measure of the set of only finitely renormalizable points in Q is equal to zero (here Q is just the boundary of the Mandelbrot set). How do these results fit?

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