Section 2: Geometry of Julia Sets

Geometry of Julia sets
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The geometry of connected Julia sets for hyperbolic quadratic polynomials is now well understood. Bounded components of the Fatou set are quasi-circles while the unbounded component is a John domain.

The geometry of a flower (for a rational fixed point) is also known. If the flower has more than one petal, each component is a quasi-disk. The 1-petal flower is a John domain (see a forthcoming paper by P. Jones and L. Carleson in Boletin de Brasil).

For Siegel disks $S$, a basic result by M. Herman is that the critical point belongs to the boundary of $S$ if the rotation number $\lambda = e^{2\pi i \theta}$ is a Siegel number, i.e.

$$|\theta - \frac{p}{q}| > \frac{c}{q^n}$$

for some $c > 0$, $n < \infty$, or more generally when the arithmetic condition of J.-C. Yoccoz's global theorem on conjugacy of analytic diffeomorphisms of the circle is satisfied.

Another remarkable result of M. Herman is that when the critical point belongs to $\partial S$, then $S$ is a quasi-disk if and only if $\lambda$ is of bounded type, i.e.

$$|\theta - \frac{p}{q}| > \frac{c}{q^2}.$$

With J.-C. Yoccoz he has also proved that $\partial S$ is a Jordan curve for almost all $\theta$. It is not known which arithmetic condition implies this. E.g., is there $\gamma_0 > 2$ so that $|\theta - p/q| > C/q^\gamma$ implies that $S$ is a Jordan domain for $\gamma < \gamma_0$ but not for $\gamma > \gamma_0$?

A particularly interesting question concerns the geometry of $\partial S$ at the critical point. Computer experiments show that in many cases $\partial S$ has an angle of about $120^\circ$ opening at the critical point. Prove this at least for $\theta = \theta_0 = (\sqrt{5} - 1)/2$. For this value there should also exist a renormalization at the critical point.
There is also a very interesting regularity of the Taylor coefficients of the conjugating map. Consider more generally the family $P_{\rho}(z)$, $\text{Re}(\rho) > 0$, with

$$P'_{\rho} = \lambda(1 - z)^\rho, \quad P_{\rho}(0) = 0$$

so that $\rho = 1$ corresponds to $\lambda(z - z^2/2)$. Let $h(\zeta)$ be the conjugating map in $|\zeta| < 1$ with $h(1) = 1$. (For general $\rho$ the proof that $1 \in \partial S$ is not known, but should be rather similar to the case $\rho = 1$). Form

$$f(\zeta) = \frac{h'(\zeta)}{1 - h(\zeta)} = \sum_{\nu} a_\nu \zeta^\nu.$$ 

Then

$$f' - f^2 = f\rho \sum_{\nu} \left( \frac{1}{2} + \frac{i}{2} \cot(\nu + 1)\pi \theta \right) a_\nu \zeta^\nu.$$ 

If the imaginary part in the parenthesis is dropped we obtain

$$f'_{0} = (1 + \rho/2) f^2_{0}, \quad f_{0} = \frac{1}{(1 + \rho/2)(1 - z)}, \quad h_{0} = (1 - z)^{2/(\rho + 2)}.$$ 

Computer experiments indicate for $\theta = \theta_0$, $\rho = 1$

$$|a_\nu - \frac{2}{3}| < 0.1 \quad \text{(say) for all } \nu,$$

where $2/3$ corresponds to $f_0$. It would be interesting to make the approximation rigorous at least for small $\rho$.

In the non-hyperbolic case very little is known (and very little can be probably said in general). The simplest case of a strictly preperiodic critical point leads to John domains (the Julia set is called a dendrite). It should be possible to analyse the general Misiurewicz case when the critical point never returns close to itself. In the case of $1 - a z^2$, $a$ is real, this condition is equivalent to the Fatou set being a John domain. To which extent does this hold for general Misiurewicz points?