Section 2: Geometry of Julia Sets

Geometry of Julia sets

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The geometry of connected Julia sets for hyperbolic quadratic polynomials is now well understood. Bounded components of the Fatou set are quasi-circles while the unbounded component is a John domain.

The geometry of a flower (for a rational fixed point) is also known. If the flower has more than one petal, each component is a quasi-disk. The 1-petal flower is a John domain (see a forthcoming paper by P.Jones and L.Carleson in Boletin de Brasil).

For Siegel disks S, a basic result by M. Herman is that the critical point belongs to the boundary of S if the rotation number $\lambda = e^{2\pi i\theta}$ is a Siegel number, i.e.

$$|\theta - \frac{p}{q}| > \frac{c}{q^n} \text{ for some } c > 0,\ n < \infty,$$

or more generally when the arithmetic condition of J.-C. Yoccoz's global theorem on conjugacy of analytic diffeomorphisms of the circle is satisfied.

Another remarkable result of M.Herman is that when the critical point belongs to ∂S, then S is a quasi-disk if and only if λ is of bounded type, i.e.

$$|\theta - \frac{p}{q}| > \frac{c}{q^2}.$$

With J.-C. Yoccoz he has also proved that ∂S is a Jordan curve for almost all θ. It is not known which arithmetic condition implies this. E.g., is there $\gamma_0 > 2$ so that $|\theta - p/q| > C/q^\gamma$ implies that S is a Jordan domain for $\gamma < \gamma_0$ but not for $\gamma > \gamma_0$?

A particularly interesting question concerns the geometry of ∂S at the critical point. Computer experiments show that in many cases ∂S has an angle of about $120°$ opening at the critical point. Prove this at least for $\theta = \theta_0 = (\sqrt{5} - 1)/2$. For this value there should also exist a renormalization at the critical point.

There is also a very interesting regularity of the Taylor coefficients of the conjugating map. Consider more generally the family $\mathcal{P}_\rho(z)$, $\mathrm{Re}(\rho) > 0$, with

$$\mathcal{P}_\rho' = \lambda(1-z)^\rho, \quad \mathcal{P}_\rho(0) = 0$$

so that $\rho = 1$ corresponds to $\lambda(z - z^2/2)$. Let $h(\zeta)$ be the conjugating map in $|\zeta| < 1$ with $h(1) = 1$. (For general ρ the proof that $1 \in \partial S$ is not known, but should be rather similar to the case $\rho = 1$). Form

$$f(\zeta) = \frac{h'(\zeta)}{1 - h(\zeta)} = \sum_0^\infty a_\nu \zeta^\nu.$$

Then

$$f' - f^2 = f\rho \sum_0^\infty (\frac{1}{2} + \frac{i}{2}\cot(\nu+1)\pi\theta) a_\nu \zeta^\nu.$$

If the imaginary part in the parenthesis is dropped we obtain

$$f_0' = (1 + \frac{\rho}{2}) f_0^2\ , \quad f_0 = \frac{1}{(1+\rho/2)(1-z)}, \ \ h_0 = (1-z)^{2/(\rho+2)}.$$

Computer experiments indicate for $\theta = \theta_0,\ \rho = 1$

$$|a_\nu - \frac{2}{3}| < 0.1 \quad \text{(say) for all} \ \ \nu,$$

where $2/3$ corresponds to f_0. It would be interesting to make the approximation rigorous at least for small ρ.

In the non-hyperbolic case very little is known (and very little can be probably said in general). The simplest case of a strictly preperiodic critical point leads to John domains (the Julia set is called a dendrite). It should be possible to analyse the general Misiurewicz case when the critical point never returns close to itself. In the case of $1 - az^2$, a is real, this condition is equivalent to the Fatou set being a John domain. To which extent does this hold for general Misiurewicz points?