Section 1: Quasiconformal Surgery and Deformations

Questions in Quasiconformal Surgery

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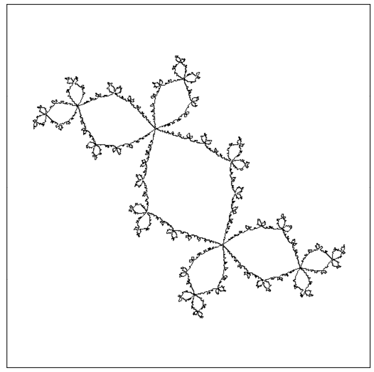
It is possible to investigate rational functions using the technique of quasiconformal surgery as developed in [**DH**2], [**BD**] and [**S**]. There are various methods of gluing together polynomials via quasiconformal surgery to make new polynomials or rational functions. The idea of quasiconformal surgery is to cut and paste the dynamical spaces for two polynomials so as to end up with a branched map whose dynamics combines the dynamics of the two polynomials. One then tries to find a conformal structure that is preserved under this branched map of the sphere to itself, so that using the Ahlfors-Bers theorem the map is conjugate to a rational function. There are several topological surgeries which experimentally seem to exist, but for which no one has yet been able to find a preserved complex structure.

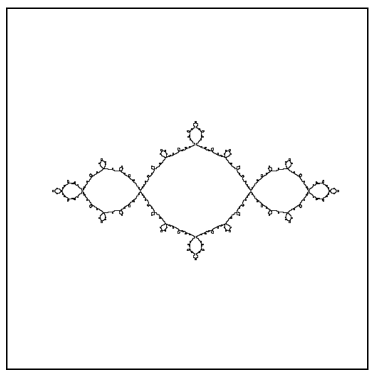
The first such kind of topological surgery is **mating** of two monic polynomials with the same degree. (Compare [**TL**].) The first step is to think of each polynomial as a map on a closed disk by thinking of infinity as a circle worth of points, one point for each angular direction. The obvious extension of the polynomial at the circle at infinity is $\theta \mapsto d\theta$ where d is the degree of the polynomial. Now glue two such polynomials together at the circles at infinity by mapping the θ of the first polynomial to $-\theta$ in the second. Finally, we must shrink each of the external rays for the two polynomials to a single point. The result should be conjugate to a rational map of degree d. (Surprisingly this construction sometimes seems to make sense even when the filled Julia sets for both polynomials have vacuous interior.)

For instance we can take the rabbit to be the first polynomial, that is $z^2 + c$ where the critical point is periodic of period 3 ($c \sim -.122561 + .744862i$). The Julia set appears in the following picture.



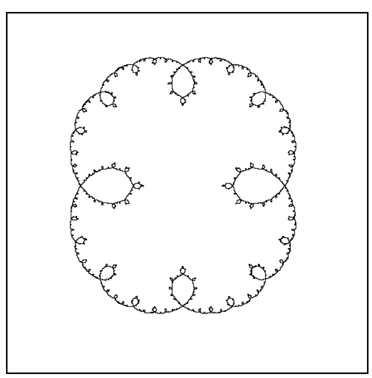
The rabbit

Then for the second polynomial we could take the basilica, that is $z^2 - 1$ (it is named after the Basilica San Marco in Venice. One can see the basilica on top and its reflection in the water below). The Julia set for the basilica appears in the following figure.



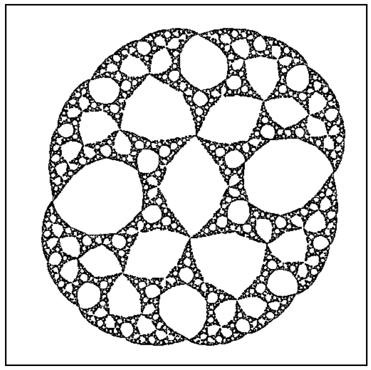
The basilica

Next we show the basilica inside-out ($\frac{z^2}{z^2-1}$) which is what we will glue to the rabbit.



The inside-out basilica

And finally we have the Julia set for the mating ($\frac{z^2+c}{z^2-1}$ where $c = \frac{1+\sqrt{-3}}{2}$).



The basilica mated with the rabbit

Question 1. Which matings correspond to rational functions? There are some known obstructions. For example, Tan Lei has shown that matings between postcritically finite quadratic polynomials can exist only if and only if they do not belong to complex conjugate limbs of the Mandelbrot set.

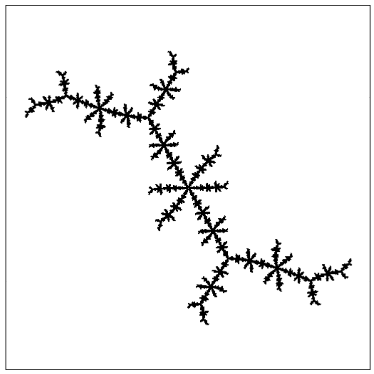
Question 2. Can matings be constructed with quasiconformal surgery? Tan Lei uses Thurston's topological characterization of rational maps to do this. It would be nice to have a cut and paste type of construction, giving results for the case when the orbit of critical points is not finite.

Question 3. If one polynomial is held fixed and the other is varied continuously, does the resulting rational function vary continuously? Is mating a continuous function of two variables?

The second type of topological surgery is **tuning**. First take a polynomial P_1 with a periodic critical point ω of period k, and assume that no other critical points are in the entire basin of this superattractive cycle. Let P_2 be a polynomial with one critical point whose degree is the same as the degree of ω. We also assume that the Julia sets of P_1 and P_2 are connected. We assume the closure $\bar{B}$ of the immediate basin of ω is homeomorphic to the closed unit disk $\bar{D}$, and that the Julia set for P_2 is locally connected. Now, P_1^k maps $\bar{B}$ to itself by a map which is conjugate to the map $z \mapsto z^d$ of $\bar{D}$, where d is the degree of the critical point. (In fact, if $d > 2$, then there are $d-1$ possible choices for the conjugating homeomorphism, and we must choose one of them.) Intuitively the idea is now the following. Replace the basin B by a copy of the dynamical plane for P_2, gluing the "circle at infinity" for this plane onto the boundary of B so that external angles for P_2 correspond to internal angles in $\bar{B}$. Now shrink each external ray for P_2 to a point. Also, make an analogous modification at each pre-image of B. The map from the modified B to its image will be given by P_2, and the map on all other inverse images of the modified B will be the identity. The result, P_3, called P_1 tuned with P_2 at ω, should be conjugate to a polynomial having the same degree as P_1. Conversely P_2 is said to be obtained from P_3 by renormalization.

In the case of quadratic polynomials, the tunings can be made also in the case when P_2 is not locally connected.

As an example we can take P_1 to be the rabbit polynomial. Then we can take $P_2(z) = z^2 - 2$ which has the closed segment from -2 to 2 as its Julia set. The following figure shows the resulting quadratic Julia set tuning the rabbit with the segment ($z^2 + c$ where $c \sim -.101096 + .956287i$).



The rabbit tuned with the segment

In the picture we see each ear of the rabbit replaced with a segment.

Question 4. Does the tuning construction always give a result which is conjugate to a polynomial? This is true when P_1 and P_2 are quadratic.

Question 5. Can tunings be constructed with quasiconformal surgery?

Question 6. Does the resulting polynomial vary continuously with P_2? This is true when P_1 and P_2 are quadratic [**DH**2].

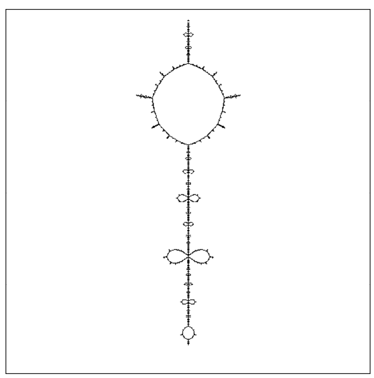
Question 7. Does the resulting tuning vary continuously with P_1? (here we consider only polynomials P_1 of degree greater than 2 with a superstable orbit of fixed period.)

Question 8. Let $P_{1,k}$ be a sequence of polynomials with a superstable orbit whose period tends to infinity. If $P_{1,k}$ tends to a limit $P_{1,\infty}$, do the tunings of P_2 with $P_{1,k}$ also tend to $P_{1,\infty}$?

The third kind of surgery is **intertwining surgery**.

Let P_1 be a monic polynomial with connected Julia set having a repelling fixed point x_0 which has a ray landing on it with combinatorial rotation number p/q. Look at the cycle of q rays which are the forward images of the first. Cut along these rays and we get q disjoint wedges. Now let P_2 be a monic polynomial with a ray of the same combinatorial rotation number landing on a repelling periodic point of some period dividing q (such as 1 or q). Slit this dynamical plane along the same rays making holes for the wedges. Fill the holes in by the corresponding wedges above making a new sphere. The new map will be given by P_1 and P_2 except on a neighborhood of the inverse images of the cut rays where it will have to be adjusted to make it continuous. This construction should be possible to do quasiconformally using the methods in [**BD**] together with Shishikura's new (unpublished) method of presurgery in the case where the rays in the P_2 space land at a repelling orbit. This construction doesn't seem to work when the rays land at a parabolic orbit.

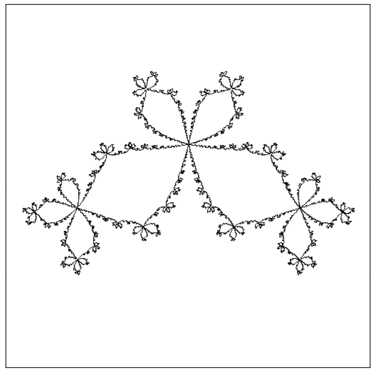
For instance we can take $P_1(z) = z^2$ and $P_2(z) = z^2 - 2$. The Julia set for P_1 is the unit circle with repelling fixed point at 1 and the ray at angle 0 lands on it with combinatorial rotation number 0. The Julia set for P_2 is the closed segment from -2 to 2 with repelling fixed point 2 and the ray at angle 0 lands on it with combinatorial rotation number 0. We cut along the 0 ray in both cases. Opening the cut in the first dynamical space gives us one wedge. The space created by opening the cut in the second space is the hole into which we put the wedge. The resulting cubic Julia set is shown in the following picture (the polynomial is $z^3 + az$ where $a \sim 2.55799i$).



A circle intertwined with a segment

We see in the picture the circle and the segment, and at the inverse image of the fixed point on the segment we see another circle. At the other inverse of the fixed point on the circle we see a segment attached. All the other decorations come from taking various inverses of the main circle and segment.

As a second example we can intertwine the basilica with itself. The ray 1/3 lands at a fixed point and has combinatorial rotation number 1/2. The following is the Julia set for the basilica intertwined with itself (the polynomial here is $z^3 - \frac{3}{4}z + \frac{\sqrt{-7}}{4}$).



A basilica intertwined with itself

Question 9. When does an intertwining construction give something which is conjugate to a polynomial?

Question 10. Can intertwinings be constructed with quasiconformal surgery?

Question 11. Does the resulting polynomial vary continuously in P_2?

References:

[**BD**] B. Branner and A. Douady, *Surgery on Complex Polynomials*, Proc. Symp. of Dynamical Systems Mexico (1986).

[**DH**2] A. Douady and J.H. Hubbard, *On the Dynamics of Polynomial Like Mappings*, Ann. Sc. E.N.S., 4ème Séries **18** (1966).

[**S**] M. Shishikura, *On the Quasiconformal Surgery of Rational Functions*, Ann. Sc. E.N.S., 4ème Séries **20** (1987).

[**STL**] M. Shishikura and Tan Lei, *A Family of Cubic Rational Maps and Matings of Cubic Polynomials*, preprint of Max-Plank-Institute, Bonn **50** (1988).

[**TL**1] Tan Lei, *Accouplements des polynômes quadratiques complexes*, CRAS Paris (1986), 635-638.

[**TL**2] Tan Lei, *Accouplements des polynômes complexes*, Thèse, Orsay (1987).

[**W**] B. Wittner, *On the Bifurcation Loci of Rational Maps of Degree Two*, Ph.D thesis, Cornell Univ., Ithaca N.Y. (1986).