The Teichmüller space of the standard action of $SL(2, Z)$ on T^2 is trivial

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1 Introduction

The group $SL(n, \mathbf{Z})$ acts linearly on $\mathbf{R}^n$, preserving the integer lattice $\mathbf{Z}^n \subset \mathbf{R}^n$. The induced (left) action on the n-torus $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$ will be referred to as the "standard action". It has recently been shown that the standard action of $SL(n, \mathbf{Z})$ on $\mathbf{T}^n$, for $n \geq 3$, is both topologically and smoothly rigid. [Hu], [KL], [HKLZ]. That is, nearby actions in the space of representations of $SL(n, \mathbf{Z})$ into $\mathrm{Diff}^+(\mathbf{T}^n)$ are smoothly conjugate to the standard action. In fact, this rigidity persists for the standard action of a subgroup of finite index. On the other hand, while the $\mathbf{Z}$ action on $\mathbf{T}^n$ defined by a single hyperbolic element of $SL(n, \mathbf{Z})$ is topologically rigid, an infinite dimensional space of smooth conjugacy classes occur in a neighborhood of the linear action. [C], [L], [MM1], [MM2] The standard action of $SL(2, \mathbf{Z})$ on $\mathbf{T}^2$ forms an intermediate case, with different rigidity properties from either extreme. One can construct continuous deformations of the standard action to obtain an (arbitrarily near) action to which it is not topologically conjugate. [Hu] [Ta] The purpose of the present paper is to show that if a nearby action, or more generally, an action with some mild Anosov properties, is conjugate to the standard action of $SL(2, \mathbf{Z})$ on $\mathbf{T}^2$ by a homeomorphism h, then h is smooth. In fact, it will be shown that this rigidity holds for any non-cyclic subgroup of $SL(2, \mathbf{Z})$.

A smooth action of a group Γ on a manifold M is called Anosov if there is a least one element $\gamma \in \Gamma$ which acts by an Anosov diffeomorphism. We define the C^r Teichmüller space of an Anosov action $F : \Gamma \times M \to M$ to

be the space of "marked" C^r smooth structures preserved by the underlying topological dynamics of the action, and such that Anosov group elements for the action F define Anosov diffeomorphisms with respect to the new smooth structure as well. The precise definition is given in the next section.
The main theorem is:

Theorem 1 *Let $\Gamma \subset SL(2, \mathbf{Z})$ be a subgroup containing two hyperbolic elements. We assume that the (four) eigenvectors of these two elements are pairwise linearly independent. Let $0 < \alpha < 1$. Then the $C^{1+\alpha}$ Teichmüller space* $\mathrm{Teich}^{1+\alpha}(F : \Gamma \times \mathbf{T}^n \to \mathbf{T}^n)$ *is trivial, where F is the standard action.*

Remark. Any non-cyclic subgroup satisfies the hypotheses of the Theorem.

An immediate corollary of Theorem 1 is:

Theorem 2 *Let $0 < \alpha < 1$. The $C^{1+\alpha}$ Teichmüller space* $\mathrm{Teich}^{1+\alpha}(F : SL(2, \mathbf{Z}) \times \mathbf{T}^2 \to \mathbf{T}^2)$ *is trivial, where F is the standard action.*

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2 The Teichmüller space of an Anosov action

We consider Γ, a locally compact, second countable group, and a C^r left action

$$F : \Gamma \times M \to M$$

of Γ on a C^r manifold M, $1 \leq r \leq \omega$. The action is called Anosov if at least one element $\gamma \in \Gamma$ acts by an Anosov diffeomorphism. That is, there is a continous splitting of the tangent bundle $TM = E^s \bigoplus E^u$, invariant under $D\gamma$, and such that vectors in E^s are exponentially contracted, and vectors in E^u are exponentially expanded, by iterates of $D\gamma$.

The Teichmüller space of F is defined as follows. Consider triples (h, N, G) where N is a C^r manifold, $h : M \to N$ is a homeomorphism, and

$$G : \Gamma \times N \to N$$

is a C^r action conjugate to F by h. That is, $G(x,\gamma) = h(F(h^{-1}(x),\gamma))$, for every $\gamma \in \Gamma$. We assume in addition that if $F(\gamma,\cdot)$ is Anosov, then $G(\gamma,\cdot)$ is also Anosov. Such a triple is called a *marked Anosov action modeled on F*. Two triples (h_1, N_1, G_1) and (h_2, N_2, G_2) are *equivalent* if the homeomorphism $s : N_1 \to N_2$ defined by $s \circ h_1 = s_2$ is a C^1 diffeomorphism.

3 The main argument

The subbundles E^s and E^u in the definition of an Anosov map are integrable, and the corresponding foliations are called the stable, respectively unstable, foliations of the map. The diffeomorphism is called codimension-one if either the stable or the unstable foliation is codimension-one. If the diffeomorphism is $C^{1+\alpha}$, i.e. the derivative is Hölder continuous with exponent $0 < \alpha < 1$, then the codimension-one foliation has $C^{1+\beta}$ transverse regularity for some $0 < \beta < 1$.[Ho], [Mn] (A codimension-k Anosov foliation is transversely absolutely continuous, and the holonomy maps have Hölder Jacobian. [An]) An Anosov diffeomorphism of $\mathbf{T}^2$ has simultaneous foliation charts

$$\phi : D^1 \times D^1 \to U \subset \mathbf{T}^2$$

where D^1 is the one dimensional disk. The intersection of a leaf of the unstable foliation $\mathcal{W}^u$ with the neighborhood U is a union of horizontals $\phi(D^1 \times y)$, and that of a leaf of the stable foliation $\mathcal{W}^s$ is a union of verticals $\phi(x \times D^1)$. Moreover, we can choose ϕ to be smooth along, say $x_0 \times D^1$ and $D^1 \times y_0$, for some $(x_0, y_0) \in D^1 \times D^1$. A simple but important observation is the following: since both foliations have $C^{1+\beta}$ transverse regularity, these charts belong to the $C^{1+\beta}$ smooth structure on $\mathbf{T}^2$ preserved by the diffeomorphism. Therefore the smooth structure is determined (up to $C^{1+\beta}$ equivalence) by the pair of transverse smooth structures.

We give the main argument of the proof of Theorem 1. The lemmas used here are proved in the next section. Let $\gamma_1,\ \gamma_2 \in SL(2,\mathbf{Z})$ be hyperbolic elements such that the eigenvectors $\{v_1^s, v_1^u, v_2^s, v_2^u\}$ are pairwise linearly independent. Here v_i^s, respectively v_i^u, is the contracting (or stable), respectively expanding (or unstable), eigenvector of γ_i, for $i = 1, 2$. Let

$$F : \Gamma \times \mathbf{T}^2 \to \mathbf{T}^2$$

be the standard linear action. Define

$$f_i(\cdot) = F(\gamma_i, \cdot)$$

for $i = 1, 2$. The lines in $\mathbf{R}^2$ parallel to v_i^s project to $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$ to give the stable foliation $\mathcal{W}_i^{s,F}$, for $i = 1, 2$. The unstable foliation $\mathcal{W}_i^{u,F}$ is obtained analogously.

Consider a segment τ contained in a leaf $W \in \mathcal{W}_1^{u,F}$. Then τ is a transversal to both $\mathcal{W}_1^{s,F}$ and $\mathcal{W}_2^{s,F}$. There is a *locally defined* holonomy from τ to itself, "generated" by the pair of foliations as follows. Slide τ a small distance along the leaves of $\mathcal{W}_1^{s,F}$, remaining transverse to both stable foliations. Then slide τ up the leaves of $\mathcal{W}_2^{s,F}$, returning to the original leaf W of $\mathcal{W}_1^{u,F}$. This is all done inside a single foliation chart for $\mathcal{W}_1^{u,F}$. The resulting motion is simply rigid translation in the leaf W. In other words, the translation group of W, restricted to small translations defined on the segment τ, can be canonically factored into a composition of holonomy along $\mathcal{W}_1^{s,F}$, followed by holonomy along $\mathcal{W}_2^{s,F}$.

Let

$$G : \Gamma \times \mathbf{T}^2 \to \mathbf{T}^2$$

define a point in the $C^{1+\alpha}$ Teichmüller space of the standard action of Γ on $\mathbf{T}^2$. In other words, G is a $C^{1+\alpha}$ action, and there is a homeomorphism $h : \mathbf{T}^2 \to \mathbf{T}^2$ such that

$$G(\gamma, x) = h(F(\gamma, h^{-1}(x)))$$

for all $\gamma \in \Gamma$. Let

$$g_i(\cdot) = G(\gamma_i, \cdot)$$

for $i = 1, 2$. Then g_1 and g_2 are Anosov. Let $\mathcal{W}_i^{s,G}$ and $\mathcal{W}_i^{u,G}$ be the stable and unstable foliations respectively of g_i. We have $h(\mathcal{W}_i^{s,F}) = \mathcal{W}_i^{s,G}$ and $h(\mathcal{W}_i^{u,F}) = \mathcal{W}_i^{u,G}$. We consider $W' = h(W)$, and $\tau' = h(\tau)$. The homeomorphism h conjugates the small translations on $\tau \subset W$ to an action on $\tau' \subset W'$. The main claim is that this action, denoted

$$S : [0, \epsilon] \times \tau' \to \tau',$$

is smooth. More precisely,

Lemma 1 *$\frac{\partial S}{\partial y}(t, y)$ is continuous.*

Figure 1: Translations along W are compositions of holonomy.

The following rigidity property of the translation pseudogroup of the line implies that h must be C^1 along τ.

Proposition 1 *Let $T : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be the rigid translations, namely $T(t,x) = x + t$. Let $S : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be an action conjugate to T by a homeomorphism $h : \mathbf{R} \to \mathbf{R}$. So $S(t,y) = h(T(t,h^{-1}(y)))$. If S is smooth in a weak sense, namely if $\frac{\partial S}{\partial y}(t,y)$ is continuous, then in fact h is a C^1 diffeomorphism.*

To prove Lemma 1, we need the following important fact.

Lemma 2 *The foliations $\mathcal{W}_1^{s,G}$, $\mathcal{W}_1^{u,G}$, $\mathcal{W}_2^{s,G}$, $\mathcal{W}_2^{u,G}$ are pairwise transverse.*

Of course we know $\mathcal{W}_1^{s,G}$and $\mathcal{W}_1^{u,G}$are transverse, and similarly $\mathcal{W}_2^{s,G}$and $\mathcal{W}_2^{u,G}$are transverse. But one doesn't know a priori that the homeomorphism h did not introduce tangencies between, say, $\mathcal{W}_1^{u,G}$ and $\mathcal{W}_2^{u,G}$.

Remark. Lemma 2 is automatically satisfied for actions *sufficiently near* to the standard action, since the transversality property is stable. The fact that this can be *proved* without any nearness assumption is the ingredient that globalizes the rigidity result.

Proof of Lemma 1, assuming Lemma 2. The conjugacy h preserves the foliations. Thus we note that the action S on τ' can be factored into holonomy along $\mathcal{W}_1^{s,G}$ followed by holonomy along $\mathcal{W}_2^{s,G}$, simply by applying h to the corresponding decomposition for the rigid translations on τ. The foliations $\mathcal{W}_1^{s,G}$ and $\mathcal{W}_2^{s,G}$ are transversely $C^{1+\beta}$. The leaves of $\mathcal{W}_1^{u,G}$ form a $C^{1+\beta}$ family of transversals to $\mathcal{W}_1^{s,G}$ and $\mathcal{W}_2^{s,G}$. The claimed regularity of S follows.

$\square$

The argument up to now shows that the transverse smooth structure to the unstable foliation $\mathcal{W}_1^{u,G}$ is C^1 equivalent by the conjugacy h to the tranverse smooth structure to $\mathcal{W}_1^{u,F}$, the corresponding foliation for the standard action. Similarly we see that the transverse smooth structure to the stable foliation $\mathcal{W}_1^{s,G}$ is C^1 equivalent by h to the linear one. But the simultaneous foliation charts are charts in the $C^{1+\beta}$ smooth structure, hence we conclude that h is C^1.

Remark. It can be shown that h is in fact as smooth as the mapping G.[L]

4 The translation pseudogroup of the real line is rigid

We prove Proposition 1. Let $T : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be the rigid translations, $T(t,x) = x+t$. Let $S : \mathbf{R}\times\mathbf{R} \to \mathbf{R}$ be conjugate to T by the homeomorphism h. We assume that $\frac{\partial S}{\partial y}(t,y)$ is continuous. We will show that S is C^1 conjugate to a rigid translation $L : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, $L(t,z) = z + \alpha t$, where α is a fixed real number. Since L is affinely conjugate to T, we will conclude that h is C^1.

Pick $y_0 \in \mathbf{R}$. Let $t(y)$ be defined by $S(t(y), y) = y_0$. Define $g : \mathbf{R} \to \mathbf{R}$

$$g(y) = \int_{y_0}^{y} \frac{\partial S}{\partial y'}(t(y'), y')dy'.$$

Conjugate the action of S by the homeomorphism g to obtain an action L,

$$L(t,z) = g(S(t, g^{-1}(z))).$$

One sees that $\frac{\partial L}{\partial z}(t,z) = 1$. Hence $L(t,y) = y + k(t)$, for some function $k(t)$. But $L(t+s,y) = L(s, L(y,t)) = L(s, y+k(t)) = y+k(t)+k(s)$, so $k(t) = \alpha t$ for some α.

5 The foliations remain transverse

In this section we prove Lemma 2. There is an equivalence relation naturally associated to an Anosov diffeomorphism, defined by the pair of foliations $\mathcal{W}^s$ and $\mathcal{W}^u$. Namely, x is equivalent to y if $x \in W^s(y) \cap W^u(y)$. This equivalence relation is generated by a pseudogroup of local homeomorphisms, defined in simultaneous foliation charts as the product of holonomy along $\mathcal{W}^u$ by holonomy along $\mathcal{W}^s$.

Lemma 3 *Suppose $\mathcal{W}_1^{u,G}$ is tangent to $\mathcal{W}_2^{s,G}$ at z. The $\mathcal{W}_1^{u,G}$ is tangent to $\mathcal{W}_2^{s,G}$ at every $z' \in W_1^{u,G}(z) \cap W_1^{s,G}(z)$.*

Assuming this, we can prove Lemma 2. The set of $z' \in W_1^{u,G}(z) \cap W_1^{s,G}(z)$ is dense in $\mathbf{T}^2$, and the distributions $E_1^{u,G}$ and $E_2^{s,G}$ are continuous. But the foliations $\mathcal{W}_1^{u,G}$ and $\mathcal{W}_2^{s,G}$ do not coincide, since they are homeomorphic images of the transverse foliations $\mathcal{W}_1^{u,F}$ and $\mathcal{W}_2^{s,F}$. Therefore there can be no tangencies, and Lemma 2 is proved.

Proof of Lemma 3. Let $z' \in W_1^{u,G}(z) \cap W_1^{s,G}(z)$. We denote by $W_1^{u,G}(z,\epsilon)$ a small neighborhood of z in the leaf of $\mathcal{W}_1^{u,G}$ containing it. Let

$$hol_s : W_1^{u,G}(z,\epsilon) \to W_1^{u,G}(z',\epsilon)$$

be the map defined by the holonomy of $W_1^{s,G}$. Similarly, let

$$hol_u : W_1^{s,G}(z,\epsilon) \to W_1^{s,G}(z',\epsilon)$$

be the holonomy of $\mathcal{W}_1^{u,G}$. We can represent $W_2^{s,G}(z,\epsilon)$ as the graph of a map

$$\theta_z : W_1^{u,G}(z,\epsilon) \to W_1^{s,G}(z,\epsilon).$$

Similarly we can represent $W_2^{s,G}(z',\epsilon)$ as the graph of a map

$$\theta_{z'} : W_1^{u,G}(z',\epsilon) \to W_1^{s,G}(z',\epsilon).$$

We observe that

$$\theta_{z'} = hol_s \circ \theta_z \circ hol_u^{-1}.$$

To see this, consider the corresponding graphs and holonomy for the linear map. The corresponding equation holds there. This is a consequence of a non-obvious feature of the foliations in the linear case: the local pseudogroup of homeomorphisms that generates the Anosov equivalence relation described above *preserves the contracting (or expanding) foliation of any other hyperbolic element*. Since the equation is defined by conjugacy invariant objects, it must hold for the non-standard action as well. Since hol_s and hol_u are $C^{1+\beta}$ diffeomorphisms, we see that a tangency at z forces a tangency at z'.

$\square$

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