# Hyperbolic components in spaces of polynomial maps.

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With an Appendix by A. Poirier.

**Abstract.** *We consider polynomial maps* $f : \mathbf{C} \to \mathbf{C}$ *of degree* $d \geq 2$ *, or more generally polynomial maps from a finite union of copies of* $\mathbf{C}$ *to itself which have degree two or more on each copy. In any space* $\mathcal{P}^S$ *of suitably normalized maps of this type, the post-critically bounded maps form a compact subset* $\mathcal{C}^S$ *called the* **connectedness locus**, *and the hyperbolic maps in* $\mathcal{C}^S$ *form an open set* $\mathcal{H}^S$ *called the* **hyperbolic** *connectedness locus. The various connected components* $H_\alpha \subset \mathcal{H}^S$ *are called* **hyperbolic components**. *It is shown that each hyperbolic component is a topological cell, containing a unique post-critically finite map which is called its* **center point**. *These hyperbolic components can be separated into finitely many distinct "types", each of which is characterized by a suitable* **reduced mapping schema** $\bar{S}(f)$ *. This is a rather crude invariant, which depends only on the topology of* $f$ *restricted to the complement of the Julia set. Any two components with the same reduced mapping schema are canonically biholomorphic to each other. There are similar statements for real polynomial maps, or for maps with marked critical points.*

## §1. Introduction.

First some mild generalizations of standard concepts. (See for example [D1], [DH1], [M2].) Let $M$ be the disjoint union of finitely many copies of the complex numbers $\mathbf{C}$ , and let $f : M \to M$ be a proper holomorphic map, of degree at least two on each component. The *filled Julia set* $K(f) \subset M$ is the set of all points whose forward orbit is contained in some compact subset of $M$ . Such a map is *post-critically bounded* if every critical point is contained in $K(f)$ , and is *post-critically finite* if the orbit of every critical point is finite. Define the *fully invariant Julia*[1] *set* $J(f)$ to be the complement of the region in $M$ where the collection of iterates of $f$ forms a normal family. Extending the classical theory of Fatou and Julia, it is easy to check that $J(f)$ coincides with the topological boundary $\partial K(f)$ . Furthermore, the intersection of $K(f)$ or $J(f)$ with each component of $M$ is connected if and only if $f$ is post-critically bounded. Such a map is *hyperbolic* (ie., hyperbolic on its Julia set) if and only if every critical orbit converges to an attracting cycle. Here a cycle (=periodic orbit) is called *attracting* if and only if its *multiplier* $\lambda$ , the first derivative around the orbit, satisfies $|\lambda| < 1$ , so that it attracts all orbits in a neighborhood.

First consider the classical and most important case $M = \mathbf{C}$ . A polynomial map $f : \mathbf{C} \to \mathbf{C}$ of degree $d$ is *monic* and *centered* if it has the form

$$f(z) \ = \ z^d + a_{d-2} z^{d-2} + \cdots + a_1 z + a_0 \ .$$

[1] This form of the definition is actually due to Fatou rather than Julia. In our mildly generalized context, note that the closure of the repelling periodic set as considered by Julia may well be strictly smaller than this fully invariant set $J(f)$ .

For $d \geq 2$, let $\mathcal{P}^{d-1}$ be the complex $(d-1)$-dimensional affine space consisting of all polynomial maps from $\mathbf{C}$ to itself which are monic and centered of degree $d$. (**Caution:** It might seem more natural to index such a space by the degree $d$. However, for our purposes it will be much more convenient to index by the total number of critical points, which is $d-1$.) Note that every polynomial map from $\mathbf{C}$ to itself is conjugate under an affine change of variable to a map $f$ which is monic and centered. Furthermore, this $f$ is uniquely determined up to the action of the group $G(d-1)$ of $(d-1)$-st roots of unity, where each $\eta \in G(d-1)$ acts on $f \in \mathcal{P}^{d-1}$ by the transformation

$$f(z) \mapsto f(\eta z)/\eta .$$

Compare 2.7 below.

By definition, the *connectedness locus* $\mathcal{C}^{d-1} \subset \mathcal{P}^{d-1}$ is the compact set consisting of all polynomials $f \in \mathcal{P}^{d-1}$ for which $K(f)$ is connected, or equivalently contains all critical points. We define the *hyperbolic connectedness locus* $\mathcal{H}^{d-1} \subset \mathcal{C}^{d-1}$ to be the open set consisting of all $f \in \mathcal{P}^{d-1}$ for which the orbits of all critical points are not only bounded, but actually converge to attracting periodic orbits.

**Remarks.** The connectedness locus $\mathcal{C}^1$ for quadratic maps, better known as the *Mandelbrot set*, has been extensively studied by Douady and Hubbard. The locus $\mathcal{H}^1$, consisting of hyperbolic maps in $\mathcal{C}^1$, was considered somewhat earlier by Brooks and Matelski. The *cubic connectedness locus* $\mathcal{C}^2$ has been studied by Branner and Hubbard. In both the quadratic and the cubic cases, an important result is that this connectedness locus is *cellular*, ie., equal to the intersection of a strictly nested family of closed topological cells. The corresponding statement for maps of higher degree has been proved by Lavaurs. In all cases, it is conjectured that $\mathcal{H}^{d-1}$ is everywhere dense in $\mathcal{C}^{d-1}$, and coincides with the interior of $\mathcal{C}^{d-1}$; however these statements have not been proved even in the simplest case $d = 2$.

Each connected component $H_\alpha \subset \mathcal{H}^{d-1}$ is called a *hyperbolic component* in $\mathcal{C}^{d-1}$. For degree $d = 2$, each map $f \in \mathcal{H}^1$ has a unique attracting periodic orbit. Let $\lambda_f$ be its multiplier. Douady and Hubbard [D1], [DH1] have shown that every hyperbolic component $H_\alpha \subset \mathcal{H}^1$ is canonically biholomorphic to the open unit disk $D$ under the correspondence

$$f \;\mapsto\; \lambda_f \in D .$$

Rees [R1] has studied hyperbolic components for quadratic rational maps, and McMullen [McM] has implicitly made a corresponding study for rational maps of arbitrary degree with connected Julia set. This paper is concerned with the special case of polynomial maps.

Section 2 will begin by defining the *reduced mapping schema*

$$\bar{S}(f) \;=\; (|\bar{S}|\,,\,\bar{F}\,,\,\bar{w})$$

associated with any hyperbolic polynomial map $f$. This consists of a finite set $|\bar{S}|$ having one point $v$ for each component $W_v$ of the interior of $K(f)$ which contains critical points, together with a "first return map" $\bar{F}$ from $|\bar{S}|$ to itself, and an integer valued *critical weight function*, where $\bar{w}(v) \geq 1$ is the number of critical points in $W_v$, counted with multiplicity. To each reduced mapping schema $S$ there is associated an affine space $\mathcal{P}^S$ of polynomial mappings whose *principal hyperbolic component* $H_0^S$ provides a standard model for hyperbolic components with schema $S$. This standard model has a finite group $\bar{G}(S) \cong \mathrm{Aut}(H_0^S)$ of automorphisms.

Sections 3 and 4 show that each hyperbolic component $H_\alpha$, in any space $\mathcal{P}^{S_1}$, is a topological cell, and that it contains a unique post-critically finite map, called its *center* point $f_\alpha$. (Compare [McM], [R1].) The proof, using Douady-Hubbard surgery, shows also that every hyperbolic component with reduced schema isomorphic to $S$ is canonically diffeomorphic to a standard model $B(S)$, which consists of certain collections of Blaschke products. The diffeomorphism is unique up to composition with an element of the group $\bar{G}(S)$, which acts on $B(S)$. In particular, any two hyperbolic components with the same reduced schema are canonically diffeomorphic to each other. Section 5 sharpens this statement by showing that they are canonically biholomorphic. Section 6 discusses analogous results for polynomial mappings with real coefficients, or more generally for "real forms" of complex polynomial mappings. Section 7 studies polynomial mappings which have been *critically marked* by specifying an ordered list of their critical points. It is shown that all of the principal results carry over to the critically marked case. The Appendix, by Alfredo Poirier, shows that every possible reduced mapping schema actually occurs as the schema associated with some critically finite hyperbolic map $f : \mathbf{C} \to \mathbf{C}$.

**Remark.** Both the statement that each hyperbolic component is a topological cell, and the statement that it has a preferred center point, are strongly dependent on the fact that we consider only maps with connected Julia set. In the case of quadratic rational maps, Rees shows that the unique hyperbolic component consisting of maps with disconnected Julia set has a more complicated topology. (Compare [M4].) For polynomial maps outside the connectedness locus, Blanchard, Devaney and Keen describe a very rich topology within the open set consisting of hyperbolic maps for which all critical orbits escape to infinity.

The present work is a fairly straightforward extension of ideas originated by Douady, Hubbard, McMullen, Rees and others. I am particularly grateful to Branner and Douady for their considerable help.

## §2. The Mapping Schema of a Hyperbolic Component.

**Definition 2.1.** By a *mapping schema* $S = (|S|\,,\,F\,,\,w)$ we mean:

(1) a finite set $|S|$ of points, together with

(2) a function $F = F_S$ from $|S|$ to itself, and also

(3) a "weight function" $w = w_S$ which assigns an integer $w(v) \ge 0$ called the *critical weight* to each $v \in |S|$.

Equivalently, such a mapping schema can be represented by a finite graph with one vertex for each $v \in |S|$, and with exactly one directed edge $e_v$ leading out from each vertex $v$ to a vertex $F(v)$. By definition, the *degree* associated with the edge $e_v$ (or with the vertex $v$ ) is the integer $d(v) = w(v) + 1 \ge 1$. The possibility that $v = F(v)$, so that $e_v$ is a closed loop, is not excluded. The vertex $v$ is called a "critical point" if $w(v) \ge 1$, and a "multiple critical point" if $w(v) \ge 2$. The sum $w(S) = \Sigma_{v \in |S|}\, w(v)$ is called the *total critical weight* of the schema $S$.

Such a mapping schema is *reduced* if every vertex is critical. Suppose that we start with a not necessarily reduced mapping schema $S$ which satisfies the following very mild condition: *Every cycle in $S$ must contain at least one critical point.* Then there is an *associated reduced schema* $\bar{S}$ which is obtained from $S$ simply by discarding all vertices of weight zero and shrinking every edge of degree one to a point. (Compare Figure 1.) Note that $S$ and $\bar{S}$ have the same critical weight. In the special case where $S$ is itself a reduced mapping schema, evidently $S = \bar{S}$ .

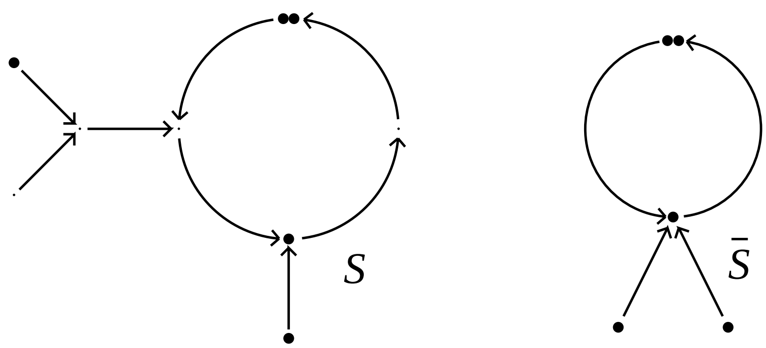



**Figure 1. A mapping schema of total weight five, and the associated reduced schema of weight five. The heavy dots represent critical points, and the double heavy dots represent critical points of weight two.**

Each mapping schema can be conveniently described by a symbol as follows. The symbol $(w)$ stands for the graph with a single vertex of weight $w$ and a single edge which is a closed loop of degree $w+1$ , while $(w_1, \ldots, w_k)$ , or any cyclic permutation of this symbol, stands for a graph with $k$ vertices of weights $w_1, \ldots, w_k$ arranged in a cycle. Note that every connected mapping schema contains exactly one such cycle. To indicate that additional vertices with weights $w'_1, \ldots, w'_m$ map to the vertex of weight $w_i$ , we insert the expression $\{w'_1, \ldots, w'_m\}$ immediately in front of the symbol $w_i$ . Continuing this construction inductively, we obtain an appropriate symbol for any connected mapping schema. As an example, the schema $S$ of Figure 1 has symbol $\big(\{\{1,0\}0\}0\,,\,\{1\}1\,,\,0\,,\,2\big)$ up to cyclic permutation, and the associated reduced schema $\bar{S}$ has symbol $(\{1,1\}1\,,\,2)$ . Finally, using the notation $S+S'$ for a disjoint union, we obtain symbols for disconnected schemata also.

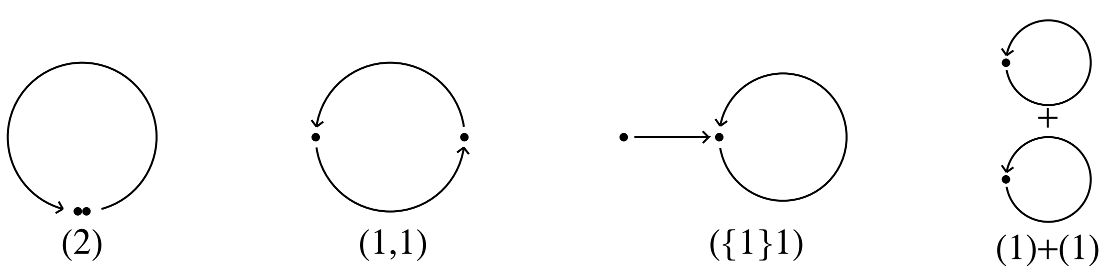



**Figure 2. The reduced schemata of weight two.**

Up to isomorphism, there is just one reduced mapping schema of total weight one, with symbol $(1)$; while there are four reduced schemata $(2)$, $(1,1)$, $(\{1\}1)$, and $(1)+(1)$ of weight two. (Compare [R1]. In [M3] the letters $A, B, C, D$ were used for these four types.) I don't know any general formula for the number $N(w)$ of distinct reduced schemata of weight $w$. However, for $n \le 5$ this number can be computed as follows:

$$
\begin{aligned}
N(1) &= 1 \\
N(2) &= 4 = 3 + 1 \\
N(3) &= 12 = 8 + 3 + 1 \\
N(4) &= 42 = 24 + 14 + 3 + 1 \\
N(5) &= 138 = 72 + 48 + 14 + 3 + 1 .
\end{aligned}
$$

Here the first summand gives the number of connected schemata, the next gives the number with two components, and so on. This can be proved simply by constructing an exhaustive list.

Any hyperbolic polynomial map $f$ from $\mathbf{C}$ to $\mathbf{C}$, or from a finite union of copies of $\mathbf{C}$ to itself of degree two or more on each copy, gives rise to an *associated full mapping schema* $S = (|S|, F, d)$ by the following construction. Let $W(f)$ be the union of the basins of attraction for all attracting periodic orbits of $f$ in $\mathbf{C}$. (Equivalently, since $f$ is hyperbolic, $W(f)$ is the interior of the filled Julia set.) Thus $W(f)$ is the bounded open set consisting of all points $z$ whose forward orbit converges to a attracting periodic orbit. It is not difficult to check that:

(a) *each connected component of* $W(f)$ *is simply connected, and hence is conformally isomorphic to the open unit disk; and*

(b) *the map* $f$ *carries each component* $W_\alpha$ *of* $W(f)$ *properly onto some component* $f(W_\alpha) = W_\beta$ *of* $W(f)$ *by a map of degree* $d_\alpha \ge 1$.

Furthermore, $f$ is bijective on a given component $W_\alpha$ if and only if $d_\alpha = 1$, or in other words if and only if there are no critical points in $W_\alpha$. Let $W^{\mathcal{PC}}$ be the union of those components $W_v$ of $W(f)$ which contain *post-critical* points of $f$, that is points which are either critical or belong to the forward orbit of some critical point.

**Definition 2.2: The full mapping schema of $f$.** This schema $S(f)$ has one vertex $v$ corresponding to each post-critical component $W_v \subset W^{\mathcal{PC}}$. The *weight* $w(v)$ is defined to be the number of critical points in $W_v$, counted with multiplicity. Each vertex $v$ is joined to a vertex $F(v)$ by an edge $e_v$ of degree $w(v)+1$, where $F(v)$ is the vertex associated with the component $f(W_v) = W_{F(v)} \subset W^{\mathcal{PC}}$.

**Definition 2.3: The reduced schema of $f$.** In practice, we will always work with the associated reduced mapping schema $\bar{S}(f)$, which can be constructed from $S(f)$ as above, or can be constructed directly as follows. *Let* $W^{\mathcal{C}} \subset W^{\mathcal{PC}}$ *be the union of the* *critical components* *of* $W(f)$, *that is the union of those components* $W_v$ *which contain critical points of* $f$, *and let* $f^{\mathcal{C}} : W^{\mathcal{C}} \to W^{\mathcal{C}}$ *be the first return map.* In other words, $f^{\mathcal{C}}(z)$ is equal to the first of the points

$$f(z), \quad f^{\circ 2}(z) = f(f(z)), \quad f^{\circ 3}(z) = f(f(f(z))), \quad \cdots$$

which belongs to a critical component of $W(f)$ . It follows from the classical theory of Fatou and Julia that every attracting periodic orbit contains at least one point of $W^{\mathcal{C}}$ , so that $f^{\mathcal{C}}$ is defined. By definition, the schema $\bar{S}(f)$ has one vertex $v$ for each critical component $W_v \subset W^{\mathcal{C}}$ , with a directed edge leading from each vertex $v$ to the vertex $\bar{F}(v)$ where the first return map $f^{\mathcal{C}}$ on $W^{\mathcal{C}}$ carries $W_v$ onto $W_{\bar{F}(v)}$ . The weight $w(v)) \geq 1$ is equal to the number of critical points of either $f^{\mathcal{C}}$ or $f$ in $W_v$ , counted with multiplicity.

Thus the total critical weight $w$ of $S(f)$ or of $\bar{S}(f)$ is equal to the total number of critical points of $f$ , counted with multiplicity. We will see in the Appendix that all possible reduced schemata arise in this way: *Given any reduced mapping schema* $S_0$ *with total critical weight* $w$ *there exists a polynomial map* $f : \mathbf{C} \to \mathbf{C}$ *of degree* $d = w + 1$ *which is hyperbolic, with reduced schema* $\bar{S}(f)$ *isomorphic to* $S_0$ . We will say briefly that such an $f$ has ***type*** $S_0$ .

If we allow mappings from a finite union of copies of $\mathbf{C}$ to itself, then such a map $f$ can be constructed trivially as follows.

**Definition 2.4. The universal polynomial model space.** To each reduced mapping schema $S = (|S|, F, w)$ there is associated a complex affine space $\mathcal{P}^S$ of polynomial maps as follows. Form the disjoint union $|S| \times \mathbf{C}$ of $n$ copies of the complex numbers $\mathbf{C}$ , where $n$ is the number of vertices of $S$ . In other words, replace each vertex of $S$ by a copy of $\mathbf{C}$ . Let $\mathcal{P}^S$ be the space consisting of all maps $f$ from $|S| \times \mathbf{C}$ to itself such that the restriction of $f$ to each component $v \times \mathbf{C}$ is a monic centered polynomial map of degree $d(v) = w(v) + 1$ , taking values in $F(v) \times \mathbf{C}$ . Note that the complex dimension of this affine space $\mathcal{P}^S$ is equal to the total critical weight $w(S)$ .

**Remark.** The principle that the dimension of a suitably normalized space of holomorphic maps is equal to the (generic) number of critical points seems to be true in a number of interesting cases. Here are two further examples. For polynomials of degree $d \geq 2$ with just one multiple critical point, there is a normal form $z \mapsto z^d + c$ depending on one parameter. On the other hand, the space of all holomorphic conjugacy classes of rational maps $f : \mathbf{C} \cup \infty \to \mathbf{C} \cup \infty$ of degree $d \geq 2$ has complex dimension $2d - 2$ , and each such map has exactly $2d - 2$ critical points counted with multiplicity.

In the special case where $S$ is the schema $(w) = (d-1)$ with just one vertex, clearly the space $\mathcal{P}^{(w)}$ of 2.4 coincides with the space $\mathcal{P}^{d-1}$ described in §1. For any $S$ , the connectedness locus $\mathcal{C}^S \subset \mathcal{P}^S$ and the hyperbolic set $\mathcal{H}^S \subset \mathcal{C}^S$ can be defined just as in the special case $S = (w)$ . By definition, a map $f \in \mathcal{P}^S$ belongs to the connectedness locus $\mathcal{C}^S$ if and only if its filled Julia set $K(f) \subset |S| \times \mathbf{C}$ intersects each component $v \times \mathbf{C}$ in a connected set, or if and only if every critical point has bounded orbit. Each hyperbolic map $f \in \mathcal{H}^S$ will have its own reduced schema $\bar{S}(f)$ . In general, $\bar{S}(f)$ will not be isomorphic to the given schema $S$ , although it will have the same total critical weight. (Compare 2.11.)

There is a preferred base point $f_0$ in $\mathcal{H}^S$ , namely the unique map

$$f_0(v, z) = (F(v), z^{d(v)})$$

whose constituent polynomials are all monic monomials. Clearly the mapping schema

$S(f_0) = \bar{S}(f_0)$ associated with this central point can be identified with the given reduced schema $S$. The hyperbolic component $H_0^S$ containing $f_0$ will be called the *principal hyperbolic component* of $\mathcal{C}^S$. We will use $H_0^S$ as a standard model for hyperbolic components with reduced schema isomorphic to $S$.

**Definition 2.5. The symmetry groups $G(S) \to \bar{G}(S)$.** By the group $\mathrm{Aut}(S)$ of *automorphisms* of the schema $S = (|S|, F, w)$, we mean the finite group consisting of all permutations of the set $|S|$ of vertices which commute with the map $F$ and preserve the weight $w(v)$. As examples, for the graphs shown in Figures 1b, 2b, and 2d, this automorphism group is cyclic of order two, but for the remaining graphs in Figures 1 and 2 it is trivial. (Compare 2.10 below.) Each such automorphism gives rise to an automorphism of the spaces

$$H_0^S \subset \mathcal{H}^S \subset \mathcal{C}^S \subset \mathcal{P}^S$$

which is *linear* in the sense that it preserves the affine structure and fixes the base point $f_0$. However, these spaces have a possibly larger group of linear automorphisms, as follows. We first introduce the *full symmetry group* $G(S)$ consisting of all maps $g$ from $|S| \times \mathbf{C}$ to itself which are bijective, commute with $f_0$, and carry each component $v \times \mathbf{C}$ linearly onto some $v' \times \mathbf{C}$.

**Lemma 2.6.** *Conjugation by an element $g \in G(S)$ carries each polynomial map $f \in \mathcal{P}^S$ to a map $g \circ f \circ g^{-1} \in \mathcal{P}^S$. The resulting action of the group $G(S)$ on the space $\mathcal{P}^S$ is linear, and preserves the subsets $H_0^S \subset \mathcal{H}^S \subset \mathcal{C}^S$.*

The proof will be given below.

**Caution.** This action need not be effective: Some elements of $G(S)$ may commute with all maps in $\mathcal{P}^S$, and hence act trivially on the space $\mathcal{P}^S$. We will use the notation $\bar{G}(S)$ for the *effective symmetry group*, that is the quotient group of $G(S)$ which operates effectively on the spaces $\mathcal{H}_0^S \subset \mathcal{H}^S \subset \mathcal{C}^S \subset \mathcal{P}^S$. For details, see 2.9 below.

In order to justify the definition of $\mathcal{P}^S$ and $G(S)$, let us note that any possible dynamical behavior which can be obtained by nowhere linear polynomial maps from a finite union of copies of $\mathbf{C}$ to itself can already be realized by some mapping $f$ which belongs to an appropriate affine space $\mathcal{P}^S$ of monic centered polynomials. (This is an immediate generalization of the statement that every polynomial map $\mathbf{C} \to \mathbf{C}$ of degree $d \geq 2$ is affinely conjugate to a monic centered map.)

**Lemma 2.7.** *Let $M$ be a finite union of disjoint copies of $\mathbf{C}$, and let*

$$\phi : M \to M$$

*be a map whose restriction to each copy of $\mathbf{C}$ is given by a polynomial of degree $\geq 2$. Then there is a mapping schema $S$, unique up to isomorphism, and a bijection $h : M \to |S| \times \mathbf{C}$ which is affine on each copy of $\mathbf{C}$, so that the conjugate map $f = h \circ \phi \circ h^{-1}$ belongs to the space $\mathcal{P}^S$. Furthermore, $h$ is unique up to composition with elements of the finite group $G(S)$ which acts on $|S| \times \mathbf{C}$, and $f$ is unique up to the action of $\bar{G}(S)$ on $\mathcal{P}^S$.*

The proof is easily supplied. $\square$

To describe the group $G(S)$ more explicitly, let $m$ be the number of connected components of $S$, and let $\delta_k$ be the product of the degrees around the unique cycle which is contained in the $k$-th component of $S$. Let $\delta'$ be the product of all of the remaining degrees, which do not lie in cycles.

**Lemma 2.8.** *The group $G(S)$ is finite, and splits canonically as a semi-direct product*

$$1 \to N(S) \to G(S) \leftrightarrows \mathrm{Aut}(S) \to 1\,.$$

*Here $\mathrm{Aut}(S)$ is defined to be the group of automorphisms of $S$ (see 2.5), while $N(S) \subset G(S)$ is a normal abelian subgroup of order $(\delta_1 - 1)\cdots(\delta_m - 1)\,\delta'$. This subgroup can be identified with the group of all functions $\eta$ from $|S|$ to the unit circle which satisfy the identity $\eta(F(v)) = \eta(v)^{d(v)}$.*

**Proof of 2.8 and 2.6.** Let $N(S)$ be the subgroup consisting of those elements of $G(S)$ which carry each copy $v \times \mathbf{C}$ to itself. Then any element $g \in N(S)$ must have the form $g(v, z) = (v\,,\, \eta(v)z)$, where the $\eta(v)$ are non-zero complex numbers. In fact, a brief computation shows that such a map $g$ commutes with $f_0$ if and only if the numbers $\eta(v)$ satisfy the identities $\eta(F(v)) = \eta(v)^{d(v)}$. If $v$ lies in the $k$-th cycle of $S$, then it follows easily that $\eta(v)$ must be a $(\delta_k - 1)$-st root of unity. Evidently, the correspondence $f \mapsto g \circ f \circ g^{-1}$ defines a linear action of the finite group $N(S)$ on the affine space $\mathcal{P}^S$, preserving the subsets $H_0^S$, $\mathcal{H}^S$, and $\mathcal{C}^S$. Further details will be left to the reader. □

**Remark 2.9.** A similar argument shows that the *effective symmetry group* $\bar{G}(S)$ can be identified with the quotient $G(S)/N_0(S)$, where $N_0 \subset N$ is a direct sum of cyclic groups of order two, consisting of all functions from $|S|$ to $\{\pm 1\}$ which are trivial on the image $F(|S|)$, and also trivial on all vertices which have weight $w(v) > 1$. In other words, an automorphism $g \in G(S)$ of the space $|S| \times \mathbf{C}$ commutes with every map $f \in \mathcal{P}^S$ if and only if it belongs to this subgroup $N_0(S)$. *It follows that $G(S) \neq \bar{G}(S)$ if and only if there exists a vertex of weight $w = 1$ which does not belong to the image $F(|S|) \subset |S|$.*

**2.10. Examples.** In the quadratic case $S = (1)$, the symmetry group $G(1)$ is trivial. For the four schemata of weight 2, these groups can be tabulated as follows,

| $S$ | $N(S)$ | G(S) | $\mathrm{Aut}(S)$ | $\bar{G}(S)$ |
|---|---|---|---|---|
| $(2)$ | $\mathbf{Z}/2$ | $\mathbf{Z}/2$ | 0 | $\mathbf{Z}/2$ |
| $(1,1)$ | $\mathbf{Z}/3$ | order 6 | $\mathbf{Z}/2$ | order 6 |
| $(\{1\}1)$ | $\mathbf{Z}/2$ | $\mathbf{Z}/2$ | 0 | 0 |
| $(1)+(1)$ | 0 | $\mathbf{Z}/2$ | $\mathbf{Z}/2$ | $\mathbf{Z}/2$ |

where $G(1,1) = \bar{G}(1,1)$ is the dihedral group of order $6$. Here are four analogous examples, where the total weight $w$ can be arbitrary: For the schema $S = (w)$ with a single vertex, the group $N = G = \bar{G}$ is cyclic of order $w$ and $\mathrm{Aut}(S)$ is trivial. For the cycle $S = (1, 1, \ldots, 1)$ of total weight $w$, the group $\mathrm{Aut}(S)$ is cyclic of order $w$ and $N(S)$ is cyclic of order $2^w - 1$, hence the group $G(S) = \bar{G}(S)$ is non-abelian of order $w\,(2^w - 1)$. For the schema $S = (\{\{\cdots\{\{1\}1\}\cdots 1\}1\}1)$ with vertices

$v_1 \to v_2 \to \cdots \to v_{w-1} \to v_w \circlearrowleft$ , the group $\mathrm{Aut}(S)$ is trivial and $N = G$ is cyclic of order $2^{w-1}$ , while $\bar{G}$ is cyclic of order $2^{w-2}$ . Finally, for the sum $(1) + \cdots + (1)$ of total weight $w$ , the group $G = \bar{G} = \mathrm{Aut}(S)$ is the symmetric group of order $w!$ and $N$ is trivial. The proofs are not difficult.

**Remark 2.11.** An interesting relation between reduced mapping schemata of the same weight can be defined as follows. *Let us say that $S \succ S'$ if and only if the connectedness locus $\mathcal{C}^S$ contains a hyperbolic component with reduced schema isomorphic to $S'$ .* This is clearly a reflexive relation; that is $S \succ S$ in all cases.

As an example, for the four reduced schemata of critical weight $w = 2$ , it is not difficult to check that this partial ordering is transitive, and can be described as follows. We have

$$(2) \succ (1,1) \succ \begin{cases} (\{1\}1) \\ (1)+(1) \end{cases} ,$$

but no other relations, except as implied by reflexivity and transitivity. (There is an analogous partial ordering for the various real forms of these hyperbolic components. Compare 6.2, and Figures 3–5.)

If $S \succ S'$ , where $S' = (|S'|\,,\,F'\,,\,w')$ and $S = (|S|\,,\,F\,,\,w)$ , then evidently there is an associated map $\psi : |S'| \to |S|$ which:

(1) preserves order, in the sense that if $F'(v_1') = v_2'$ then some iterate of $F$ maps $\psi(v_1')$ to $\psi(v_2')$ , and

(2) preserves weight, in the sense that $\sum_{\psi(v')=v} w'(v') = w(v)$ for each $v \in |S|$ .

It is conjectured that $S \succ S'$ if and only if there exists such a map $\psi$ . In particular, it is conjectured that this relation is transitive. In many cases, a hyperbolic component $H_\alpha \subset \mathcal{H}^S$ of type $S'$ seems to be contained in a complete copy of the connectedness locus $\mathcal{C}^{S'}$ , which is homeomorphically embedded into $\mathcal{C}^S$ in such a way that the principal component $H_0^{S'}$ corresponds to $H_\alpha$ . (This is a generalization of the Douady-Hubbard concept of "modulation" or "tuning" for quadratic maps.) Whenever this happens, it follows that any type of hyperbolic component which occurs in $\mathcal{C}^{S'}$ must also occur in $\mathcal{C}^S$ . Thus, in this special case, the relations $S \succ S'$ and $S' \succ S''$ certainly imply that $S \succ S''$ .

**Caution.** It may happen that $S \succ S'$ and $S' \succ S$ , even though $S$ is not isomorphic to $S'$ . For example, this is true for $S = (\{2\}1, \{1\}1, 1)$ and $S' = (\{1\}1, \{2\}1, 1)$ . The proof of this statement is an easy application of the Realization Theorem for "Hubbard forests", as proved by Poirier [P3].

## §3. Blaschke Products and the Model Space $B(S)$ .

Henceforth, all mapping schemata will be reduced. This section will describe a topological model, based on Blaschke products, for hyperbolic components with mapping schema $S$ . First recall some standard facts.

Let $D$ be the open unit disk in $\mathbf{C}$ . For any $a \in D$ , there is one and only one conformal automorphism $\mu_a : \bar{D} \to \bar{D}$ which maps $a$ to zero and fixes the boundary point 1, given by

$$\mu_a(z) \;=\; k\,\frac{z-a}{1-\bar{a}z} \qquad \text{with} \qquad k = \frac{1-\bar{a}}{1-a}\,.$$

It is easy to check that the proper holomorphic maps from $D$ onto itself are precisely the finite products of the form

$$\beta(z) \;=\; c\,\mu_{a_1}(z)\,\cdots\,\mu_{a_d}(z)\,,$$

with $|c| = 1$ . Here $d$ is the degree, $\{a_1\,\ldots,\,a_d\}$ are the pre-images of zero, and $c = \beta(1)$ . Evidently every such map $\beta$ extends uniquely as a rational map from the Riemann sphere $\hat{\mathbf{C}} = \mathbf{C} \cup \infty$ onto itself. In particular, $\beta$ extends continuously over the boundary circle $\partial D$ . Note that the extension of $\beta$ over $\hat{\mathbf{C}}$ commutes with the inversion $z \mapsto 1/\bar{z}$ . In particular, if $z_0$ is a fixed point or a critical point of $\beta$ , then $1/\bar{z}_0$ is also.

> **Lemma 3.1.** *A proper map $\beta$ of degree $d \ge 2$ from the unit disk onto itself induces a $d$-to-one covering map from the circle $\partial D$ onto itself. Such a map $\beta$ has at most one fixed point in the open disk $D$ . If there is an interior fixed point, then the induced map on $\partial D$ is topologically conjugate to the linear map $t \mapsto td$ of the circle $\mathbf{R}/\mathbf{Z}$ . In particular, in this case there are exactly $d-1$ boundary fixed points.*

(On the other hand, if there is no interior fixed point, then there must be $d+1$ boundary fixed points, counted with multiplicity.)

**Proof Outline.** For $|z| = 1$ , a brief computation shows that the logarithmic derivative

$$d\log\beta(z)/d\log(z) \;=\; z\,\beta'/\beta$$

is a sum of $d$ terms, each of which is real and strictly positive. It follows easily that $\beta$ induces a covering map on the boundary, and that any equation of the form $\beta(z) = \text{constant} \in \partial D$ has exactly $d$ distinct solutions. If there is an interior fixed point, then after conjugating by a conformal automorphism, we may assume that the fixed point is $z = 0$ , so that

$$\beta(z) \;=\; c\,z\,\mu_{a_2}(z)\cdots\mu_{a_d}(z)\,.$$

It follows that $|\beta'(0)| = |a_2 \cdots a_d| < 1$ and that $|\beta(z)| < |z|$ for all $z \ne 0$ in the open disk. Thus this fixed point is attracting, and is unique within $D$ . When there is such an interior fixed point, it is not difficult to check that the logarithmic derivative satisfies $z\beta'(z)/\beta(z) > 1$ at all points of $\partial D$ . In other words, the map of $\partial D$ onto itself is strictly expanding. In particular, every boundary fixed point $z_0$ is repelling, with $\beta'(z_0) > 1$ .

Since the algebraic number of fixed points in $\hat{\mathbf{C}}$ is equal to $d+1$ , it then follows that there are exactly $d-1$ distinct fixed points on $\partial D$ .

More explicitly, we can set up a coordinate system on $\partial D$ as follows. Choose one of these boundary fixed points $z_0$ and assign it the coordinate $t(z_0)=0$ . Now assign the $d$ preimages of $z_0$ under $\beta$ the coordinates $t(z)=j/d$ , numbering in counterclockwise order around the circle from $z_0$ . Similarly, assign the $d^2$ preimages under $\beta^{\circ 2}$ the coordinates $j/d^2$ , and so on. Since $\beta$ is expanding on $\partial D$ , the iterated preimages of $z_0$ are everywhere dense on $\partial D$ , so this construction converges to a well defined homeomorphism $t:\partial D \stackrel{\approx}{\to} \mathbf{R}/\mathbf{Z}$ . $\square$

Closely related is the statement that $\beta$ gives rise to a preferred $\beta$-invariant measure $\ell=\ell_\beta$ on $\partial D$ , with the properties that $\ell(\partial D)=1$ and that $\beta$ maps any small interval of measure $\ell(I)=\epsilon$ to an interval of measure $\ell(\beta(I))=\epsilon\, d$ . For example $\ell$ can be defined by the formula $\ell(I)=\lim_{k\to\infty} N(k)/d^k$ , where $N(k)$ is the number of solutions to the equation $\beta^{\circ k}(u)=1$ in the interval $I\subset\partial D$ . Note that this measure is "balanced" in the sense that each of the $d$ components of $\beta^{-1}(I)$ has measure $\ell(I)/d$ . Evidently $\ell$ corresponds to the Lebesgue measure on $\mathbf{R}/\mathbf{Z}$ under the conjugating homeomorphism. (This $\ell$ can also be characterized as the unique invariant measure of maximal entropy.)

**Definition.** Let $\beta: D\to D$ be a proper holomorphic map of degree $d\geq 2$ . We will say that $\beta$ is *boundary-rooted* if its extension over $\partial D$ fixes the point $+1$ , and *fixed point centered* if $\beta(0)=0$ .

**Lemma 3.2.** *If $\phi$ is a proper holomorphic map of degree $d\geq 2$ from $D$ to itself, with a fixed point in the open disk $D$ , then $\phi$ is holomorphically conjugate to a map $h^{-1}\circ\phi\circ h$ which is boundary-rooted and fixed point centered. In fact there are exactly $d-1$ distinct choices for the conjugating Möbius automorphism $h$ . The space $B(d-1)$ consisting of all proper holomorphic $\beta: D\to D$ which are boundary-rooted and fixed point centered of degree $d$ is a topological cell of dimension $2d-2$ .*

**Proof.** Evidently $h$ can be any Möbius automorphism of $\bar{D}$ which maps $0$ to the unique interior fixed point of $\phi$ , and maps $+1$ to one of the $d-1$ boundary fixed points. Let $\mathcal{S}_n(\mathbf{C})$ be the $n$-*fold symmetric product*, consisting of unordered $n$-tuples $\{a_1\,,\,\ldots\,,\,a_n\}$ of complex numbers. This can be identified with the complex affine space consisting of all monic polynomials of degree $n$ , under the correspondence

$$\{a_1\,,\,\ldots\,,\,a_n\}\mapsto(z-a_1)\cdots(z-a_n)=z^n-\sigma_1 z^{n-1}+-\cdots\pm\sigma_n\,,$$

where the $\sigma_j$ are the elementary symmetric functions of $\{a_1\,,\,\ldots\,,\,a_n\}$ . Thus $\mathcal{S}_n(\mathbf{C})$ is homeomorphic to $\mathbf{C}^n\cong\mathbf{R}^{2n}$ . Since $\mathbf{C}$ is homeomorphic to the 2-cell $D$ , it follows that $\mathcal{S}_n(D)$ is also homeomorphic to $\mathbf{R}^{2n}$ . Now consider the space $B(w)$ consisting of all boundary-rooted Blaschke products $\beta$ of degree $d=w+1$ which fix the origin. We can write

$$\beta(z) \;=\; z\,\mu_{a_1}(z)\cdots\mu_{a_w}(z)\,.$$

Evidently this space is homeomorphic to the symmetric product $\mathcal{S}_w(D)$ , and hence is a topological cell, homeomorphic to $\mathbf{R}^{2w}$ . $\square$

We will also need to find a normal form for Blaschke products under one-sided composition with a Möbius transformation. **Definition:** A proper holomorphic map $D \to D$ of degree $d \ge 2$ is *critically centered* if the sum of its $d-1$ (not necessarily distinct) critical points is equal to zero.

**Lemma 3.3.** *If $\phi : D \to D$ is proper and holomorphic of degree $d \ge 2$, then there exist exactly $d$ distinct Möbius automorphisms $h : D \to D$ for which the composition $\beta' = \phi \circ h$ is boundary-rooted and critically centered. The space $B'$ consisting of all boundary-rooted, critically centered proper holomorphic maps of degree $d$ is a topological cell of dimension $2d-2$.*

The proof will be based on the concept of "*conformal barycenter*" for a collection of points in the disk:

**Lemma 3.4.** *Given points $c_1, \ldots, c_n$ in a Riemann surface $W$ isomorphic to $D$, there exists a conformal isomorphism $\eta : W \to D$, unique up to a rotation of $D$, which takes the $c_j$ to points $\eta(c_j)$ with sum $\eta(c_1) + \cdots + \eta(c_n)$ equal to zero.*

It follows that the point $p = \eta^{-1}(0) \in W$ is uniquely defined. **Definition:** This point $p$ will be called the *conformal barycenter* of the $c_j$. A proof of this lemma can easily be constructed from [Douady and Earle, §2]. □

**Proof of 3.3.** Evidently $h$ can be any Möbius automorphism of $D$ which carries zero to the conformal barycenter of the critical points of $\phi$, and which carries $+1$ to one of the $d$ points of $\phi^{-1}(1)$.

To determine the topology of $B'$, we proceed as follows. Note first that the subspace $\mathcal{S}_n(\bar{D}) \subset \mathcal{S}_n(\mathbf{C})$, consisting of $n$-tuples which belong to the closed disk $\bar{D}$, forms a closed $2n$-cell with interior equal to $\mathcal{S}_n(D)$. In fact, for each non-zero $\{a_1, \ldots, a_n\} \in \mathcal{S}_n(\mathbf{C})$ consider the ray consisting of points $\{ta_1, \ldots, ta_n\}$ with $t \ge 0$. Each such ray crosses the boundary of $\mathcal{S}_n(\bar{D})$ exactly once, and the image of each such ray in the space of ordered $n$-tuples $(\sigma_1, \ldots, \sigma_n)$ crosses the unit sphere exactly once. Hence, stretching by an appropriate factor along each such ray, we obtain the required homeomorphism from $\mathcal{S}_n(\bar{D})$ to the unit disk in $\mathbf{C}^n$. Using this construction we see also that the subspace of $\mathcal{S}_n(D)$ consisting of unordered $n$-tuples with sum zero is a topological $(2n-2)$-cell. Thus the set of Blaschke products of the form $\beta(z) = \mu_{a_1}(z) \cdots \mu_{a_d}(z)$ with $a_1 + \cdots + a_d = 0$ is an open topological $(2d-2)$-cell. It follows that the collection $B'$ of boundary-rooted, critically centered Blaschke products $\beta'$ of degree $d$ is also an open topological $(2d-2)$-cell. For given any such $\beta'$, by Lemma 3.4 there is a unique boundary-rooted Möbius automorphism $\eta$ so that $\beta = \beta' \circ \eta$ has the sum of its zeros equal to zero, and similarly given $\beta$ there is a unique $\eta$ so that $\beta' = \beta \circ \eta^{-1}$ is critically centered. □

We will use such Blaschke products to model the map $f^C : W^C \to W^C$ studied in 2.3. Since $W^C$ is isomorphic to a disjoint union of open disks, let us first study proper maps from such a disjoint union to itself.

**Definition.** To any reduced mapping schema $S = (|S|, F, w)$ we associate the *model space* $B(S)$ consisting of all proper holomorphic maps

$$\beta : |S| \times D \rightarrow |S| \times D$$

such that $\beta$ carries each $v \times D$ onto $F(v) \times D$ by a boundary-rooted map of degree $d(v) = w(v) + 1$ which satisfies $\beta(v, 0) = (F(v), 0)$ whenever $v$ is periodic under $F$, and is critically centered whenever $v$ is not periodic under $F$.

> **Lemma 3.5.** *If the schema $S$ has total weight $w$, then the model space $B(S)$ is homeomorphic to an open cell of dimension $2w$.*

**Proof.** This follows easily from 3.2 and 3.3. □

We next show that the various maps in $B(S)$ serve as models for all possible dynamics which can occur within the basins of attracting cycles.

> **Lemma 3.6.** *Let $M$ be a disjoint union of finitely many open disks, and let $\phi : M \rightarrow M$ be any proper holomorphic map, of degree $\geq 2$ on each component of $M$, such that every orbit under $\phi$ converges to an attracting cycle. Then $\phi$ is holomorphically conjugate to a map which belongs to some model space $B(S)$. Here the schema $S$ is uniquely determined up to isomorphism; and the number of distinct conformal isomorphisms from $M$ onto $|S| \times D$ which conjugate $\phi$ to some element of $B(S)$ is equal to the order of the group $G(S)$.*

**Proof.** We may identify the complex manifold $M$ with $\Sigma \times D$, where $\Sigma$ is a finite index set. Note that $\phi$ extends continuously over $\Sigma \times \partial D$. First consider some component $\sigma \times D$ which is mapped to itself by some iterate $\phi^{\circ k}$. Let $d_1 \cdots d_k$ be the degree of $\phi^{\circ k}$ on this component. Then $\phi^{\circ k}$ has a unique fixed point inside $\sigma \times D$. After conjugating by a Möbius automorphism of $\sigma \times D$, we may take this interior fixed point to be the center point $(\sigma, 0)$. Similarly, there are $d_1 \cdots d_k - 1$ fixed points on $\sigma \times \partial D$, and we can rotate $\sigma \times \bar{D}$ so that one of these fixed points is $(\sigma, 1)$. Pushing forward, we find corresponding interior and boundary points for each of the $k$ disks in the cycle. Finally we work outwards, first choosing corresponding preferred points for each of the disks which maps immediately to a disk on this cycle, and then continuing inductively. Details will be left to the reader. □

We can sharpen 3.6 as follows.

> **Lemma 3.7.** *The effective symmetry group $\bar{G}(S)$ of §2 operates smoothly on the model space $B(S)$ in such a way that two maps in $B(S)$ are conformally conjugate to each other if and only if they belong to the same orbit under this action. In fact this action of $\bar{G}(S)$ on $B(S)$ is covered by an action of the full symmetry group $G(S)$ on the product $B(S) \times |S| \times D$ with the following property: Each $g \in G(S)$ carries $(\beta, v, z)$ to a triple of the form $(\beta', h(v, z))$ where $h = h_{g,\beta}$ is a conformal automorphism of $|S| \times D$ depending on $g$ and $\beta$, and where $\beta' = h \circ \beta \circ h^{-1}$.*

**Remark.** We will see in §4 that $B(S)$ is canonically diffeomorphic to the model hyperbolic component $\mathcal{H}_0^S$ . Since $\bar{G}(S)$ acts linearly on $\mathcal{H}_0^S$ , it is hardly surprising that it operates smoothly on $B(S)$ . However, to avoid a circular argument, we must prove this fact from scratch.

**Proof of 3.7.** Since the action of the subgroup $\mathrm{Aut}(S) \subset G(S)$ on $B(S)$ and on $B(S) \times |S| \times D$ is quite easy to describe, let us concentrate on the complementary subgroup $N(S) \subset G(S)$ . In other words, we will only discuss those conformal automorphisms of $|S| \times D$ which carry each component onto itself. If such an automorphism $h$ conjugates some map $\beta \in B(S)$ into another map $\beta' = h^{-1} \circ \beta \circ h$ which also belongs to $B(S)$ , then evidently $h$ must preserve center points, and hence must have the form $h(v\,,\,z) = (v\,,\,\eta_v\,z)$ . (Here $h$ is the inverse of the map considered in the previous paragraph.) Now the condition that $\beta'$ is boundary-rooted takes the form

$$\beta(v\,,\,\eta_v) \;=\; (F(v)\,,\,\eta_{F(v)})\;. \tag{1}$$

Evidently this condition depends on the particular map $\beta$ we have chosen. However, it is not difficult to check that the possible solutions vary continuously as we vary $\beta$ . Since the space $B(S)$ is contractible, this means that if we find a solution for one point $\beta_0 \in B(S)$ , then we obtain corresponding solutions for all points $\beta \in B(S)$ . Define the "center point" $\beta_0 \in B(S)$ to be the mapping defined by

$$\beta_0(v,z) \;=\; f_0(v,z) \;=\; (F(v)\,,\,z^{d(v)})\;. \tag{2}$$

(Compare 2.4 and 3.8.) Then it is true, almost by definition, that the group consisting of all holomorphic conjugacies from $\beta_0$ to itself can be identified with the symmetry group $G(S)$ . Thus, deforming the solutions of (1) continuously, we obtain an action of the group $N(S)$ , and hence of $G(S)$ , on the space $B(S)$ and on $B(S) \times |S| \times D$ .

Next we must ask which elements of the group $N(S) \subset G(S)$ act trivially on $B(S)$ . Note that the automorphism $h(v,z) = (v\,,\,\eta_v\,z)$ of $|S| \times D$ commutes with the map $\beta(v,z) = (F(v)\,,\,\beta_v(z))$ if and only if

$$\beta_v(\eta_v\,z) \;=\; \eta_{F(v)}\beta_v(z)\;. \tag{3}$$

In most cases, we can choose each map $\beta_v$ so that its set of zeros admits no nontrivial rotation. Whenever this is the case, the equation (3) admits only the trivial solution $\eta_v = \eta_{F(v)} = 1$ . However, in the special case where the vertex $v$ has weight $w(v) = 1$ , and is not periodic under $F$ , the map $\beta_v$ must be critically centered of degree two, and hence must have the form

$$\beta_v(z) \;=\; \mu_a(z)\mu_{-a}(z) \;=\; \frac{z^2 - a^2}{1 - \bar{a}^2 z^2}\;.$$

In this special case, the set of zeros of $\beta_v$ necessarily admits an $180°$ rotation. Hence equation (3) admits the solutions $\eta_v = \pm 1\,,\ \eta_{F(v)} = 1$ . In this way, it is not difficult to check that the subgroup $N_0(S) \subset G(S)$ of 2.9, which acts trivially on $\mathcal{P}^S$ , is precisely equal to the subgroup which acts trivially on $B(S)$ . Thus we obtain the required effective action of the quotient group $\bar{G} = G/N_0$ on $B(S)$ . The details of this argument are not difficult, and will be left to the reader. □

Finally, we will need the following result, which is essentially well known.

**Lemma 3.8.** *Each model space $B(S)$ contains one and only one map $\beta_0$ which is post-critically finite. It is given by formula* (2) *above, and is characterized by the properties that each component $v \times D$ contains exactly one critical point, and that $\beta_0$ maps critical points to critical points.*

**Proof.** First consider a proper holomorphic map $\beta : D \to D$, which has a attracting fixed point at zero. If the multiplier $\lambda = \beta'(0)$ is non-zero, then by the Kœnigs linearization theorem we can choose a local coordinate $\zeta$ in a neighborhood of zero so that $\beta$ corresponds to the map $\zeta \mapsto \lambda\zeta$. Let us extend this coordinate system to a conformal isomorphism between an open set $U \subset D$ and an open disk $|\zeta| < r$, with $r$ as large as possible. Note that $r$ cannot be infinite, since no open subset of $D$ is conformally isomorphic to the whole complex line. If $r$ is maximal, then there must be some obstruction to extending further, and this obstruction can only be a critical point lying on the boundary of $U$. Thus, under this hypothesis, there must be a critical point in $D$ whose forward orbit is *not* eventually periodic.

Therefore, in the post-critically finite case, the multiplier $\lambda$ must be zero. Suppose then that $\lambda = 0$. By Böttcher's theorem, we can choose the local coordinate $\zeta$ so that $\beta$ corresponds to the map $\zeta \mapsto \zeta^k$ for some $k \geq 2$. Again we extend to a conformal isomorphism from $U \subset D$ onto the open disk $|\zeta| < r$ with $r$ as large as possible. If the maximal $r$ satisfies $r < 1$, then again there must be a critical point on the boundary of $U$, whose forward orbit is not eventually periodic. Thus, if all critical orbits are eventually periodic, it follows that $r = 1$. But then the boundary of $U$ maps into itself under $\beta$, and it follows easily that $U = D$, so that $\beta$ is conformally conjugate to the map $z \mapsto z^k$. In fact if $\beta(1) = 1$ then $\beta(z) = z^k$. In this case, note that the unique critical point of $\beta$ is the *only* point of $D$ whose forward orbit is eventually periodic.

To complete the proof, we must also consider the case of a critically centered Blaschke product $\beta : D \to D$, with $\beta(1) = 1$. If all of the critical points of $\beta$ map to zero, then we must show again that $\beta(z) = z^d$. But this is clear, since $D$ is exhibited as a branched covering of $D$ with unique branch point at the critical value zero. □

## §4. Hyperbolic Components are Topological Cells.

The object of this section is to prove the following two results.

**Theorem 4.1.** *If $H_\alpha \subset \mathcal{C}^{S_0}$ is any hyperbolic component whose elements $f$ have reduced mapping schema $\bar{S}(f)$ isomorphic to $S$, then $H_\alpha$ is diffeomorphic to the model space $B(S)$. This diffeomorphism is canonically defined, up to composition with an element of the group $\bar{G}(S)$ which acts on $B(S)$.*

The proof will also demonstrate the following.

**Corollary 4.2 (McMullen).** *Each hyperbolic component $H_\alpha$ contains one and only one map $f_\alpha$ which is post-critically finite.*

Equivalently, such a "center" map $f_\alpha$ has the property that each component of the complement of $J(f_\alpha)$ contains one and only one pre-critical point.

To begin the argument, suppose that $f$ is a hyperbolic map belonging to some connectedness locus $\mathcal{C}^{S_0}$, with reduced mapping schema $\bar{S}(f)$ isomorphic to $S$. Then the open set $W^C \subset K(f)$, as defined in 2.3, satisfies the hypothesis of Lemma 3.6. That is, $W^C$ is a union of finitely many components, each conformally isomorphic to $D$, and the first return map $f^C$ from $W^C$ to itself has degree at least two on each component. Hence, by 3.6, there exists a conformal isomorphism $h : W^C \to |S| \times D$ which conjugates $f^C$ to some map $\beta \in B(S)$.

However, we must be careful since $h$ and $\beta$ are not uniquely defined. (Compare 3.7.) To deal with this non-uniqueness, we proceed as follows. Note first that for each component $W_\alpha \subset W^C$ the closure $\bar{W}_\alpha$ is homeomorphic to the closed unit disk $\bar{D}$, in a homeomorphism which is conformal throughout the interior. (See [DH1, pp. 13, 24, 26].)

**Caution.** The closed disks $\bar{W}_v$ are not always disjoint from each other, so the closure of $W^C$ need not be homeomorphic to $|S| \times \bar{D}$. Since boundary points of $W^C$ will play an important role in our discussion, it will be convenient to introduce the following notation. *Let $\hat{W}^C = \coprod \bar{W}_v$ be the disjoint union of the closures of the components of $W^C$, and let $\hat{f}^C : \hat{W}^C \to \hat{W}^C$ be the continuous extension of $f^C$.*

In order to choose some specific conformal isomorphism between $|S| \times D$ and $W^C$, we must first choose some isomorphism $\iota : S \to \bar{S}(f)$. Evidently the number of ways of doing this is equal to the order of the automorphism group $\mathrm{Aut}(S)$. For each component $W_{\iota(v)}$ of $W^C$, we must then choose one boundary point $q(v)$ which will correspond to the boundary base point $(v, 1) \in v \times \bar{D}$. These boundary points are to be chosen as follows.

**Definition 4.3. Boundary markings.** Let $f \in \mathcal{H}^{S_0}$ be a hyperbolic map with reduced schema $\bar{S}(f)$ isomorphic to $S$. By a ***boundary marking*** $q : |S| \to \partial\hat{W}^C$ of $f$, ***covering the isomorphism*** $\iota : S \to \bar{S}(f)$, we will mean a function which assigns to each vertex $v \in |S|$ a boundary point $q(v) \in \partial W_{\iota(v)}$, satisfying the condition that $\hat{f}^C(q(v)) = q(F(v))$.

> **Lemma 4.4.** *Such boundary markings always exist. In fact, the number of distinct boundary markings for $f$ is precisely equal to the order of the symmetry group $G(S)$. Given such a boundary marking $q$, there is one and only one homeomorphism $\hat{q} : |S| \times \bar{D} \xrightarrow{\cong} \hat{W}^C$, conformal throughout the interior, which satisfies $\hat{q}(v, 1) = q(v)$, for which the map $\beta = \hat{q}^{-1} \circ f^C \circ \hat{q} : |S| \times D \to |S| \times D$ belongs to the model space $B(S)$.*

The proof is completely analogous to the proof of 3.6, and will be omitted. □

Note also that as we deform the hyperbolic map $f$ within a small neighborhood, there is a corresponding deformation of any given boundary marking. This follows easily from the theorem of Mañé-Sad-Sullivan and Lyubich, which asserts that the entire Julia set $J(f)$ varies continuously as we vary the hyperbolic map $f$. We can now make a more precise restatement of 4.1:

**Theorem 4.1′.** *If* $H_\alpha \subset \mathcal{C}^{S_0}$ *is any hyperbolic component whose elements* $f \in H_\alpha$ *have mapping schema* $\bar{S}(f)$ *isomorphic to* $S$ *, then:*

(1) *we can choose a boundary marking* $q_f : |S| \to \partial \hat{W}^C(f)$ *which varies continuously as* $f$ *varies over* $H_\alpha$ *,*

(2) *there is an associated extension* $\hat{q}_f : |S| \times \bar{D} \xrightarrow{\cong} \hat{W}^C$ *as in 4.4, and*

(3) *the resulting correspondence* $f \mapsto \hat{q}_f^{-1} \circ \hat{f}^C \circ \hat{q}_f \in B(S)$ *maps the component* $H_\alpha$ *diffeomorphically onto the space* $B(S)$ *of Blaschke products.*

The proof is a generalization of that given by Douady and Hubbard in the quadratic case. However, more care is needed, since it is necessary to make a choice of boundary markings. For example, a priori some component $H_\alpha$ might have a non-trivial fundamental group. If this were the case, then starting with a boundary marking $q_0$ for $f_0$ and deforming it as we follow a loop around $H_\alpha$ we might end up with a different boundary marking for $f_0$ . In order to prove that this cannot happen, We proceed as follows.

Let $\tilde{H}_\alpha$ be the space consisting of all pairs $(f, q)$ where $f$ belongs to $H_\alpha$ and $q$ is a boundary marking for $f$ . This space $\tilde{H}_\alpha$ has a natural topology, and there is a natural projection $(f, q) \mapsto f$ from $\tilde{H}_\alpha$ to $H_\alpha$ . Define a map $\Phi : \tilde{H}_\alpha \to B(S)$ by the construction

$$\Phi : (f, q) \ \mapsto \ \hat{q}^{-1} \circ \hat{f}^C \circ \hat{q} \ ,$$

with $\hat{q}$ as in 4.4. We will prove the following.

**Lemma 4.5.** *The space* $B(S)$ *is evenly covered under this map*

$$\Phi : \tilde{H}_\alpha \to B(S) .$$

Since $B(S)$ is simply connected by 3.5, this implies that there is a section $B(S) \to \tilde{H}_\alpha$ . The composition $B(S) \to \tilde{H}_\alpha \to H_\alpha$ is then the required diffeomorphism from $B(S)$ onto $H_\alpha$ , with inverse as described in 4.1′.

**Proof of 4.5.** Let us start with a map $f_0 \in H_\alpha$ with boundary marking $q_0$ and with associated model map $b_0 = \hat{q}_0^{-1} \circ \hat{f}_0^C \circ \hat{q}_0 \in B(S)$ . Choose two radii $r_1 < r_2 < 1$ so that

(1) every critical point of the map $b_0 : |S| \times D \to |S| \times D$ is contained in the union $|S| \times D_{r_1}$ of disks of radius $r_1$ , and

(2) so that $b_0$ maps $|S| \times \bar{D}_{r_2}$ into $|S| \times D_{r_1}$ .

Let $U \subset B(S)$ be a simply connected neighborhood of $b_0$ which is small enough so that all maps $b \in B(S)$ will satisfy these same conditions. That is, the union $|S| \times D_{r_1}$ must contain all critical points of $b$ , and must contain the image $b(|S| \times \bar{D}_{r_2})$ . Then we will construct a new polynomial map $f_b \in H_\alpha$ by quasi-conformal surgery.

First construct a new function $b'$ from $|S| \times D$ to itself as follows. Let $b'$ coincide with $b$ on $|S| \times \bar{D}_{r_1}$ and with $b_0$ outside of $|S| \times D_{r_2}$ . Now interpolate linearly on each intermediate region $r_1 \le |z| \le r_2$ , setting

$$b'(v, z) \ = \ t\, b_0(v, z) + (1 - t) b(v, z) \ ,$$

where $t = (|z| - r_1)/(r_2 - r_1)$ . We will assume that the neighborhood $U$ is small enough so that this linear interpolation yields a new function $b'$ which is a quasi-conformal local homeomorphism throughout these annuli $r_1 \le |z| \le r_2$ . Note that $b'$ is actually holomorphic outside of these annuli.

We now follow the surgery procedure originated by Douady and Hubbard. (Compare [D1], [DH1], [McM], [Sh1], [Sh2], [Su3].) Identifying $|S| \times D$ with $W^C$ under $\hat{q}_0$ , we obtain a quasi-conformal map $g_b$ from the Riemann sphere to itself by setting $g_b(z) = f(z)$ outside of $W^C$ , with $g_b = \hat{q}_0 \circ b' \circ \hat{q}_0^{-1}$ within $W^C$ . Then $g_b$ is holomorphic except on a collection of annuli, one of which lies in each component of $W^C$ . Since every orbit under $g_b$ passes through these bad annuli at most once, we can pull back the standard conformal structure $\mu^0$ under the iterates of $g_b$ to obtain a new conformal structure $\mu_b$ on $\hat{\mathbf{C}}$ which is invariant under $g_b$ . Using the measurable Riemann mapping theorem, we see that there is a quasi-conformal automorphism $h_b$ of the Riemann sphere which transforms $\mu^0$ to $\mu_b$ . (Compare Ahlfors & Bers or Lehto & Virtanen.) Thus the conjugate mapping $f_b = h_b^{-1} \circ g_b \circ h_b$ preserves the standard structure $\mu^0$ , or in other words is holomorphic. Furthermore, if we choose $h_b$ to be doubly tangent to the identity at infinity, then it is unique, and depends (real) differentiably on the parameter $b$ . Hence the holomorphic map $f_b$ also depends differentiably on $b$ . It is not difficult to check that $f_b$ is a monic centered polynomial with a preferred boundary marking. Thus it belongs to our space $\hat{H}_\alpha$ , and we have constructed a smooth map $b \mapsto f_b = s(b)$ from the open set $U \subset B(S)$ to $\hat{H}_\alpha$ .

We would like to prove that $s$ is a local section of the projection map $\Phi : \hat{H}_\alpha \to B(S)$ . That is, we would like to prove that the composition

$$\Phi(s(b)) \;=\; \Phi(f_b) \in B(S)$$

is equal to $b$ . Examining the construction, we certainly see that if we restrict $b$ to the disks $|S| \times D_{r_1}$ and restrict $\Phi(f_b)$ to a corresponding neighborhood of its critical set, then the two are holomorphically conjugate. It is not too difficult to show that this holomorphic conjugacy extends inductively over the iterated pre-images of $|S| \times D_{r_1}$ , since each of these can be considered as a branched covering of the previous one. Passing to the direct limit, we see that $b$ is holomorphically conjugate to $\Phi(f_b)$ on the entire space $|S| \times D$ .

Thus we have constructed a smooth map $b \mapsto s(b) = f_b$ from an open set $U \subset B(S)$ to $\hat{H}_\alpha$ such that the image $\Phi \circ s(b) = \Phi(f_b) \in B(S)$ is holomorphically conjugate to $b$ . Hence $\Phi \circ s(b) = g_b(b)$ for some group element $g_b \in \bar{G}(S)$ . It follows easily from smoothness that $g_b$ can be chosen as a constant, independent of $b$ . Then $s' = s \circ g_b^{-1}$ is the required local section of the projection $\Phi$ . This completes the proof of 4.5, 4.1$'$ and 4.1. □

**Proof of 4.2.** This follows from the argument above, together with 3.8. □

## §5. Analytic isomorphism between hyperbolic components.

If $H_\alpha \subset \mathcal{C}^{S_1}$ and $H_\beta \subset \mathcal{C}^{S_2}$ are two different hyperbolic components with reduced mapping schema isomorphic to $S$ , then by Theorem 4.1 there are diffeomorphisms

$$H_\alpha \to B(S) \leftarrow H_\beta ,$$

uniquely defined up to a choice among finitely many boundary markings, or equivalently up to the action of the group $\bar{G}(S)$ on $B(S)$ . The composition mapping $H_\alpha$ to $H_\beta$ will be called the *canonical diffeomorphism* between these two sets. We will prove the following.

**Theorem 5.1.** *This canonical diffeomorphism $H_\alpha \to H_\beta$ between open subsets of complex affine spaces is biholomorphic.*

That is, it is holomorphic, with holomorphic inverse. In particular, it follows that the canonical diffeomorphism from $H_\alpha$ to the standard model $H_0^S$ is biholomorphic. Note that this diffeomorphism is unique up to the action of the finite group $\bar{G}(S)$ of linear automorphisms of $H_0^S$ . The proof will be based on the following.

**Definition.** We will say that a map $f \in \mathcal{P}^{S_1}$ satisfies a *critical orbit relation* if either (1) the $w$ critical points of $f$ are not all distinct, or (2) the associated critical orbits are not disjoint from each other, or (3) some critical orbit is eventually periodic. It is not difficult to show that the set of all $f$ which satisfy some critical orbit relation forms a countable union of algebraic varieties in the affine space $\mathcal{P}^{S_1}$ .

**Lemma 5.2.** *If $f_1 \in H_\alpha$ does not satisfy any critical orbit relation, then the canonical diffeomorphism from $H_\alpha$ to $H_\beta$ is biholomorphic throughout a neighborhood of $f_1$ .*

**Proof.** First consider the mapping schema $S = (w)$ , with a single vertex of degree $d = w + 1$ . Then every $f_1 \in H_\alpha$ has a unique attracting periodic orbit. To simplify the notation, let us assume that it is a fixed point $p_1 = p(f_1)$ . We can construct a local holomorphic coordinate system for $H_\alpha$ near $f_1$ as follows. Since there are no critical orbit relations, the multiplier $\lambda_1 = f_1'(p_1) \in D$ cannot be zero. Hence every $f$ close to $f_1$ has a fixed point $p = p(f)$ close to $p_1$ with multiplier $\lambda = \lambda(f) \in D \smallsetminus \{0\}$ . Let us choose a Kœnigs linearizing coordinate $z \mapsto \zeta(z)$ near $p(f)$ so that $\zeta(f(z)) = \lambda\zeta(z)$ . Evidently $\zeta$ extends to a holomorphic map from the basin of attraction $W(f)$ into $\mathbf{C}$ satisfying this same identity $\zeta(f(z)) = \lambda\zeta(z)$ . Since there are no critical orbit relations, the values $\zeta(c_1), \ldots, \zeta(c_w)$ at the critical points of $f$ are all distinct and non-zero. In fact no ratio $\zeta(c_j)/\zeta(c_k)$ can be equal to a power of $\lambda$ . Let us normalize so that $\zeta(c_1) = 1$ . *Then the numbers $\lambda, \zeta(c_2), \ldots, \zeta(c_w)$ form the required local holomorphic coordinates near $f_1$ .*

To see that the correspondence $f \mapsto \zeta(c_j)$ is holomorphic, note that $c_j = c_j(f)$ depends holomorphically on $f$ , and hence that

$$\zeta(c_j) = \lim_{k\to\infty} (f^{\circ k}(c_j) - p)/(f^{\circ k}(c_1) - p)$$

is a uniform limit of holomorphic functions. We must show that these coordinates, in a sufficiently small neighbood of $f_1$ , determine $f$ uniquely. Choose open sets

$U_0 \subset U_1 \subset \cdots \subset \hat{W}(f)$ as follows. Choose some number $0 < r < 1$ which is smaller than all of the $|\zeta(c_j)|$, and let $U_k$ be the component containing the fixed point $p$ in the region $|\lambda^k \zeta(z)| < r$. Then each $U_{k+1}$ can be described as a branched covering of $U_k$, with the restriction of $f$ to $U_{k+1}$ as projection map, and with the critical points of $f$ in $U_{k+1}$ as branch points. Note that $U_0$ can be identified with the open disk $\{\zeta : |\zeta| < r\}$. This entire sequence of Riemann surfaces, together with the holomorphic functions $f$ and $\zeta$ on the $U_k$, and the embedding of $U_k$ into $U_{k+1}$ can be constructed inductively, starting with $U_0$, if we are given the numbers $\lambda$, $\zeta(c_2)$, ..., $\zeta(c_w)$ and $r$ together with the topological data which specifies which covering to take. Evidently this topological data (eg. certain subgroups of finite index in fundamental groups of punctured disks) varies continuously with $f$. Passing to the direct limit of $U_k$ as $k \to \infty$, we have constructed a Riemann surface conformally isomorphic to $W^C(f) \cong D$, together with the map $f^C$ on $W^C(f)$, from the given data. By Theorems 3.6 and 4.1, this determines $f$ uniquely, up to the action of the effective symmetry group.

The proof for an arbitrary connected mapping schema $S$ is similar. If the unique attracting periodic orbit has period $m$, then it is convenient to write the multiplier as $\lambda^m$, and to choose $\zeta : W(f) \to \mathbf{C}$ so that $\zeta(f(v', z) = \lambda\zeta(v', z)$. Proceeding as above, from the data $\lambda$, $\zeta(c_2)$, ..., $\zeta(c_w)$, $r$ and $S$ we can first build up a copy of the component $W_v$ containing one of the periodic points, and then build up the rest of $W^C$ inductively. The case of an $S$ with several connected components now follows easily. Since these same local holomorphic coordinates can be used either in $H_\alpha$ or in $H_\beta$, this proves Lemma 5.2. □

It now follows easily that this diffeomorphism $H_\alpha \to H_\beta$ is holomorphic everywhere. In fact the Cauchy-Riemann equations, which are necessary and sufficient conditions for a $C^1$-smooth map to be holomorphic, are satisfied except on a countable union of proper algebraic subvarieties, intersected with $H_\alpha$. Since such a countable union is nowhere dense in $H_\alpha$, Theorem 5.1 follows by continuity. □

## §6. Real forms.

First consider a real polynomial map $f_{\mathbf{R}} : \mathbf{R} \to \mathbf{R}$ , of degree $d \ge 2$ . We can extend uniquely to a complex polynomial map $f : \mathbf{C} \to \mathbf{C}$ . This $f$ commutes with the complex conjugation operation, which we denote by $\gamma_0(z) = \bar{z}$ . If $f$ is hyperbolic, then as in §2 we form the union $W^C(f)$ of those components of the attractive basin which contain critical points. Evidently $\gamma_0$ is an antiholomorphic mapping which carries this set $W^C(f)$ onto itself. Note that $\gamma_0$ may permute the various components of $W^C(f)$ ; for the critical points of $f$ need not be real and in fact $W^C(f)$ may not contain any real points at all. In order to find an appropriate universal model for such behavior, we consider the following construction.

Let $M$ be a finite union of copies of $\mathbf{C}$ , and let $\gamma'$ be any antiholomorphic involution of $M$ . If $f'$ is a nowhere linear polynomial map from $M$ to itself which commutes with $\gamma'$ , then as in 2.7 conjugation by some affine isomorphism $h : M \xrightarrow{\cong} |S| \times \mathbf{C}$ will carry $f'$ to a map $f = h \circ f' \circ h^{-1}$ from $|S| \times \mathbf{C}$ to itself which is monic and centered on each component $v \times \mathbf{C}$ . Hence $f$ belongs to an appropriate space $\mathcal{P}^S$ of monic centered maps. This same affine conjugation will carry $\gamma'$ to some antiholomorphic involution $\gamma = h \circ \gamma' \circ h^{-1}$ of $|S| \times \mathbf{C}$ which commutes with $f$ . **Definition:** We will say that $f$ belongs to the "real subspace" $\mathcal{P}^S_{\mathbf{R}}(\gamma)$ , consisting of polynomials in $\mathcal{P}^S$ which commute with $\gamma$ .

We can fit this construction into a group theoretic framework as follows. Recall that the group $G(S)$ of 2.5 can be described as the set of all holomorphic automorphisms $g$ of the manifold $|S| \times \mathbf{C}$ which commute with the special map $f_0^S(v, z) = (F(v)\,,\, z^{d(v)})$ . These automorphisms $g$ have the property that for any $f \in \mathcal{P}^S$ the conjugate $g \circ f \circ g^{-1}$ will also belong to $\mathcal{P}^S$ . Let us extend to a larger group $\hat{G}(S)$ by allowing also antiholomorphic automorphisms which commute with $f_0^S$ , or equivalently which conjugate $\mathcal{P}^S$ onto itself. Then it is easy to show that $\hat{G}(S)$ is the split extension of its normal subgroup $G(S)$ by the two element group $\{1, \gamma_0\}$ generated by the standard involution $\gamma_0(v, z) = (v, \bar{z})$ . This involution commutes with the elements of the subgroup $\mathrm{Aut}(S) \subset G(S)$ . However, for $g$ in the normal abelian subgroup $N(S) \subset G(S)$ , conjugation by $\gamma_0$ carries $g$ to $g^{-1}$ . We can also describe $\hat{G}(S)$ as a split extension

$$1 \to N(S) \to \hat{G}(S) \leftrightarrows \big(\{1, \gamma_0\} \times \mathrm{Aut}(S)\big) \to 1 \;.$$

Proofs are easily supplied. Thus we are led to the following.

**Preliminary Definition 6.1.** A *real form* of the mapping schema $S$ is an antiholomorphic involution $\gamma : |S| \times \mathbf{C} \to |S| \times \mathbf{C}$ which commutes with $f_0^S$ , and hence belongs to the group $\hat{G}(S)$ . Two such involutions (ie., two such real forms) are to be considered as *isomorphic* if they belong to the same conjugacy class in $\hat{G}(S)$ . Each real form $\gamma$ gives rise to an anti-linear involution $f \mapsto \gamma \circ f \circ \gamma$ of the complex $w$-dimensional affine space $\mathcal{P}^S$ . The fixed point set of this involution is a real $w$-dimensional affine space $\mathcal{P}^S_{\mathbf{R}}(\gamma)$ , consisting of maps in $\mathcal{P}^S$ which commute with $\gamma$ . We will call $\mathcal{P}^S_{\mathbf{R}}(\gamma)$ the *real form* of $\mathcal{P}^S$ associated with $\gamma$ . If we intersect this real affine space with the connectedness locus $\mathcal{C}^S$ , then we obtain a set $\mathcal{C}^S_{\mathbf{R}}(\gamma)$ which is called a *real connectedness locus*, or a *real form* of $\mathcal{C}^S$ . Similarly, the intersection $\mathcal{H}^S_{\mathbf{R}}(\gamma) = \mathcal{H}^S \cap \mathcal{P}^S_{\mathbf{R}}(\gamma)$ is the associated *real hyperbolic locus*, and its connected components are the associated *real hyperbolic components*. We will see in 6.4 that these real hyperbolic components are just the intersections $H_\alpha \cap \mathcal{P}^S_{\mathbf{R}}(\gamma)$ , where $H_\alpha$ can be any component of $\mathcal{H}^S$ which intersects $\mathcal{P}^S_{\mathbf{R}}(\gamma)$ , or equivalently whose center point $f_\alpha$ commutes with $\gamma$ . As a special case, we can consider the *principal hyperbolic component* $H^S_{0\mathbf{R}}(\gamma) = H^S_0 \cap \mathcal{P}^S_{\mathbf{R}}(\gamma)$ .

The *real symmetry group* $G_{\mathbf{R}}(S, \gamma)$ is defined to be the centralizer of $\gamma$ in $G(S)$ , that is the subgroup consisting of all elements $g \in G(S)$ which commute with $\gamma$ . Evidently each $g \in G(S)$ acts linearly on the spaces $\mathcal{P}^S_{\mathbf{R}}(\gamma) \supset \mathcal{C}^S_{\mathbf{R}}(\gamma) \supset \mathcal{H}^S_{\mathbf{R}}(\gamma) \supset H^S_{0\mathbf{R}}(\gamma)$ by the correspondence $f \mapsto g \circ f \circ g^{-1}$ . Hence an appropriate quotient group $\bar{G}_{\mathbf{R}}(S, \gamma) \subset \bar{G}(S)$ acts effectively on these spaces.

Similarly, if $M$ is a finite union of copies of the unit disk $D$ , and $\gamma'$ is an antiholomorphic involution of $M$ , then we can consider proper holomorphic maps $f' : M \to M$ which commute with $\gamma'$ , and have degree $\ge 2$ everywhere. As in 3.6, we can identify $M$ with some $|S| \times D$ so that $f'$ corresponds to a map $f \in B(S)$ . Furthermore, as in 3.7, the group $\hat{G}(S)$ acts on $B(S) \times |S| \times D$ . Thus $\gamma'$ corresponds to an antiholomorphic involution $\gamma \in \hat{G}(S)$ , and $f$ belongs to the space $B_{\mathbf{R}}(S, \gamma)$ of maps which commute with $\gamma$ . In particular, if $f'$ is any hyperbolic map which belongs to some locus $\mathcal{H}^{S_1}_{\mathbf{R}}(\gamma_1)$ , then the space $M = W^C(f')$ is isomorphic to a finite union of disks, and is mapped into itself by the antiholomorphic involution $\gamma_1$ , so the above discussion applies. If the corresponding map $f : |S| \times D \to |S| \times D$ belongs to the model space $B_{\mathbf{R}}(S, \gamma)$ , then we will say that the real hyperbolic component containing $f'$ has the *type* $(S, \gamma)$ .

In practice, a slightly different construction often seems more natural. Again suppose that $M$ is a finite union of copies of $\mathbf{C}$ , that $f' : M \to M$ is a nowhere linear proper holomorphic map, and that $\gamma'$ is an antiholomorphic involution commuting with $f'$ .

**Lemma 6.2.** *We can choose a conformal isomorphism* $h : M \cong |S| \times \mathbf{C}$ *which conjugates* $\gamma'$ *to a standard involution of the form* $\gamma : (v, z) \mapsto (\bar{v}, \bar{z})$ *, and which conjugates* $f'$ *to a map* $f : |S| \times \mathbf{C} \to |S| \times \mathbf{C}$ *which is centered with leading coefficient* $\pm 1$ *on each component* $v \times \mathbf{C}$ *.*

Here the vertex map $v \leftrightarrow \bar{v}$ can be any element of order $\le 2$ in the group $\mathrm{Aut}(S)$ . Thus we can put the involution $\gamma$ into a convenient standard form, at the cost of allowing both $+1$ or $-1$ as leading coefficients for $f$ . The proof is not difficult. □

With this formulation, the real form of $S$ is described by the involution $v \leftrightarrow \bar{v}$ , together with the leading coefficient function $|S| \to \{\pm 1\}$ . However, one then needs to do a little work to decide when two such real forms are isomorphic to each other.

As an example, consider the space $\mathcal{P}^{(2)}$ of monic centered cubic polynomials. Using the formulation 6.1, we note that $\hat{G}(2)$ is the direct sum of two cyclic groups of order two, and it follows easily that there are two suitable antiholomorphic involutions, namely the standard involution $\gamma_0 : z \mapsto \bar{z}$ and the exotic involution $\gamma_1 : z \mapsto -\bar{z}$ . Thus we consider the two spaces $\mathcal{P}^2_{\mathbf{R}}(\gamma_j)$ consisting of monic centered cubic maps satisfying $\overline{f(z)} = (-1)^j f(\bar{z})$ , for $j = 0, 1$ . Using the formulation 6.2, we consider instead the two spaces $\mathcal{P}^{2\pm}_{\mathbf{R}}$ consisting of centered cubic maps with real coefficients, and with leading coefficient $\pm 1$ . (Compare Figure 3.)

**Figure 3. The spaces $\mathcal{P}_{\mathbf{R}}^{(2)^+}$ and $\mathcal{P}_{\mathbf{R}}^{(2)^-}$ of real cubic maps, intersected with the complex connectedness locus $\mathcal{C}^{(2)}$. These pictures show the $(A, b)$-plane where $x \mapsto \pm x^3 - 3Ax + b$. (Compare [M3].)**

There is a completely analogous discussion for any even $w \geq 2$ . The space $\mathcal{P}^{(w)}$ of polynomials of degree $w+1$ has exactly two real forms, up to isomorphism. In fact $\hat{G}(w)$ is a dihedral group of order $2w$ with two distinct conjugacy classes of anti-holomorphic involutions. It is most convenient to identify the corresponding real forms with the spaces $\mathcal{P}_{\mathbf{R}}^{(w)^+}$ and $\mathcal{P}_{\mathbf{R}}^{(w)^-}$ , consisting of real centered polynomials with leading coefficient $+1$ or $-1$ respectively. The associated groups $G_{\mathbf{R}}((w)^{\pm}) = \bar{G}_{\mathbf{R}}((w)^{\pm})$ are all cyclic of order two, generated by the symmetry $f(z) \mapsto -f(-z)$ . This symmetry is visible as the reflection $(A, b) \mapsto (A, -b)$ in Figure 3. On the other hand, if $w$ is odd, then the dihedral group $\hat{G}(w)$ has just one conjugacy class of antiholomorphic involutions. Hence there is a unique real form, consisting of real monic centered polynomials. In this case, the symmetry group $G_{\mathbf{R}}((w)^+)$ is trivial.

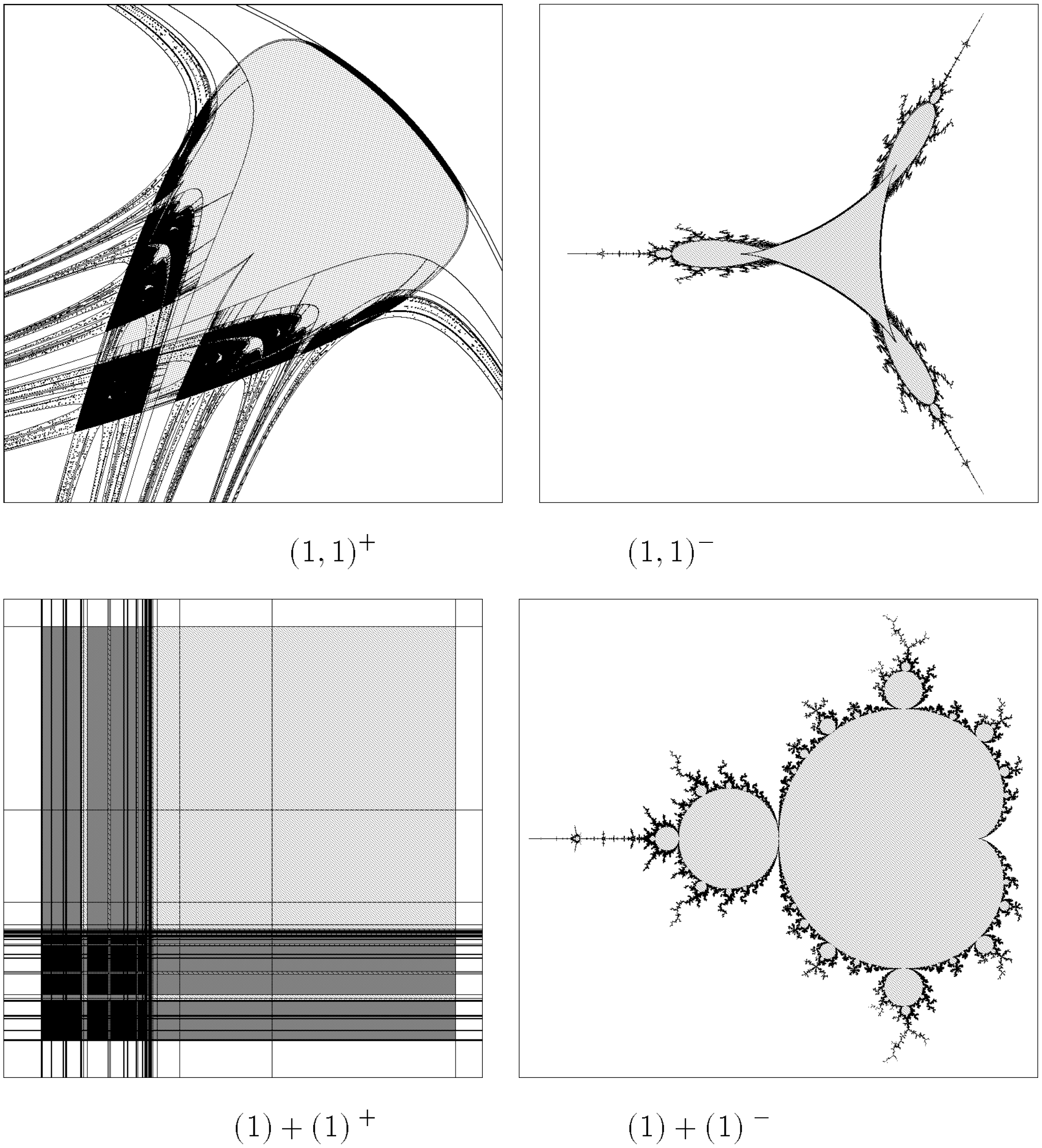


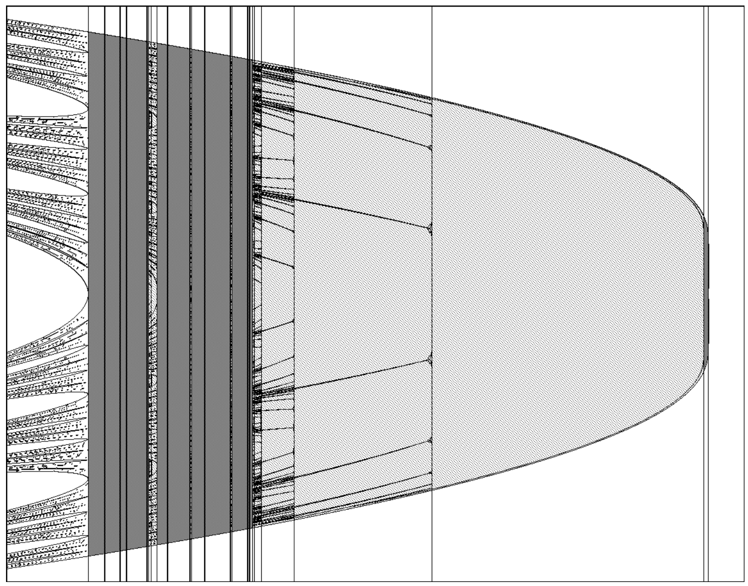

**Figure 4. Connectedness loci for four real forms of weight two.**

For each reduced mapping schema $S$ of weight two we can make a corresponding computation. In each case it turns out that there are exactly two real forms, which it will be convenient to denote by $S^+$ and $S^-$. (Compare Figures 3-5.) Here $S^+$ corresponds to the space made up out of monic real maps. For $S^-$, we need different descriptions in the various cases, as follows:

For $S^- = (1,1)^-$ and $S^- = (1)+(1)^-$ we must consider monic centered maps from $|S|\times\mathbf{C}$ to itself which commute with the antiholomorphic involution $(v\,,\,z)\mapsto(3-v\,,\,\bar{z})$. Such maps have the form

$$(v_1\,,\,z)\mapsto(v_2\,,\,z^2+c)\,,\quad (v_2\,,\,z)\mapsto(v_1\,,\,z^2+\bar{c})\quad \text{for}\quad S=(1,1)$$
$$(v_1\,,\,z)\mapsto(v_1\,,\,z^2+c)\,,\quad (v_2\,,\,z)\mapsto(v_2\,,\,z^2+\bar{c})\quad \text{for}\quad S=(1)+(1)\,.$$

The corresponding connectedness loci are shown in Figure 4. In the first case this locus, known as the "*tricorn*", has a symmetry group $G_{\mathbf{R}}((1,1)^-) \cong G(1,1)$ which is non-abelian of order 6, as is visible in the Figure. (Compare [Wi].) For $S^- = (1)+(1)^-$, the associated connectedness locus is just a copy of the Mandelbrot set, with the involution $c \mapsto \bar{c}$ as unique real automorphism.



$(\{1\}1)^{\pm}$

**Figure 5. Connectedness loci for the remaining two real forms of weight two.**

For $S = (\{1\}1)$, the situation is more confusing, since the connectedness loci for the two real forms $(\{1\}1)^{\pm}$ are indistinguishable from each other. We must consider real maps of the form

$$(v_1\,,\,x) \mapsto (v_1\,,\,x^2 + r_1)\,, \quad (v_2\,,\,x) \mapsto (v_1\,,\,\pm x^2 + r_2)\,,$$

where the choice of sign has very little effect on the dynamics. In this case the group $\bar{G}_{\mathbf{R}}((\{1\}1)^{\pm})$ of dynamic automorphisms is trivial, even though there is an evident set theoretic automorphism $(r_1\,,\,r_2) \mapsto (r_1\,,\,-r_2)$ of the connectedness locus.

**Remark 6.3.** In analogy with 2.9, there is a partial ordering

$$(2)^{\pm} \quad \succ \quad \begin{cases} (1,1)^- & \succ \quad (1)+(1)^- \\ (1,1)^+ & \succ \quad \begin{cases} (1)+(1)^+ \\ (\{1\}1)^{\pm} \end{cases} \end{cases}$$

for the collection of real forms of total weight two.

Our main result in the real case can be stated as follows.

**Theorem 6.4.** *Every hyperbolic component in a real connectedness locus of weight* $w$ *is a topological* $w$*-cell with a unique "center" point, and is real analytically homeomorphic to a uniquely defined principal hyperbolic component* $H^S_{0\mathbf{R}}(\gamma)$ *, or to a suitably defined space* $B_R(S,\gamma)$ *of Blaschke products, under a homeomorphism which is uniquely determined up to the action of the symmetry group* $\bar{G}_{\mathbf{R}}(S,\gamma)$ *.*

The proof involves going through the arguments in previous sections, keeping track of the extra involution, and is not difficult. □

## §7. Polynomials with marked critical points.

By a *critically marked* polynomial map of degree $w+1$ we will mean a polynomial map $f$ together with an ordered list $(c_1, \ldots, c_w)$ of its critical points. Even if $f$ is a real polynomial, this list must include all complex critical points, so that the derivative $f'(z)$ is a constant multiple of $(z-c_1)\cdots(z-c_w)$. Similarly, we can define the concept of a "critically marked" Blaschke product.

Branner and Hubbard have shown the utility of studying critically marked polynomial mappings. All of the principal results of the previous sections extend to the marked case. For any mapping schema $S$ we can define a space $\mathcal{P}^\sharp(S)$ of marked polynomial maps and a space $B^\sharp(S)$ made up out of marked Blaschke products. Then any hyperbolic component of type $S$ in any marked connectedness locus $C^\sharp(S') \subset \mathcal{P}^\sharp(S')$ is canonically homeomorphic to $B^\sharp(S)$. The one step in this program which cause additional difficulty is the analogue of Lemma 3.5, showing that $B^\sharp(S)$ is a topological cell. However, Douady has supplied the author with a proof of the following key result.

**Lemma 7.1.** *The space consisting of all critically marked Blaschke products of degree $w+1$ which fix the points $0$ and $1$ is a topological cell of dimension $2w$.*

**Proof.** We will make use of the standard diffeomorphism $h : D \to \mathbf{C}$, given by $h(z) = w = z/\sqrt{1-|z|^2}$, with inverse $z = w/\sqrt{1+|w|^2}$. Let $\beta : D \to D$ be a proper holomorphic map fixing $0$ and $1$, with (not necessarily distinct) critical points $c_1, \ldots, c_w$. Consider the composition $h \circ \beta : D \to \mathbf{C}$. Pulling back the standard conformal structure of $\mathbf{C}$ under this composition, we obtain a new Riemann surface $D_\beta$, which has underlying space $D$ but is none-the-less conformally isomorphic to $\mathbf{C}$. In fact we can choose the conformal isomorphism $\eta : \mathbf{C} \to D_\beta$ so that $\eta(0) = 0$ and $\lim_{t\to+\infty} \eta(t) = 1$. The composition $h \circ \beta \circ \eta$ will then be a polynomial map of the form

$$z \;\mapsto\; h(\beta(\eta(z))) \;=\; \lambda \int_0 (w+1)(z-\hat{c}_1)\cdots(z-\hat{c}_w)dz$$

with leading coefficient $\lambda > 0$, where $\eta(\hat{c}_k) = c_k$. In fact, after replacing the function $\eta(z)$ by $\eta(z/\lambda^{1/(w+1)})$, we may assume that $\lambda = 1$. Conversely, given such a monic polynomial with marked critical points which fixes the origin, we can reverse this construction, thus obtaining a corresponding Blaschke product with marked critical points. This proves that the required space of Blaschke products is diffeomorphic to the space of $w$-tuples $(\hat{c}_1, \ldots, c_w) \in \mathbf{C}^w$. $\square$

Using 3.4, it follows easily that the space of critically centered Blaschke products with marked critical points and with $\beta(1) = 1$ is also a topological cell. The proof that the model space $B^\sharp(S)$ is a topological cell is now straightforward.

For the case of marked polynomial maps, we must expect a group of symmetries which is much larger than $G(S)$, and many more real forms than in §6. The basic definitions are as follows. For any mapping schema $S$, let $\Pi(S)$ be the cartesian product, over all vertices $v$, of the symmetric group of permutations of the numbers $\{1, 2, \ldots, w(v)\}$ which are used to index the critical points in $W_v$. Then the affine space $\mathcal{P}^\sharp(S)$ of marked centered polynomial maps from $|S| \times \mathbf{C}$ to itself can be defined as a normal branched covering space of $\mathcal{P}^S$, with $\Pi(S)$ acting as the group of deck transformations. Similarly, the spaces $C^\sharp(S) \supset H_0^\sharp(S)$ and $B^\sharp(S)$ are $\Pi(S)$-branched coverings of $\mathcal{C}^S \supset H_0^S$ and of $B(S)$. The full group $G^\sharp(S)$ of symmetries of $\mathcal{P}^\sharp(S)$ splits as a semidirect product

$$1 \to \Pi(S) \to G^\sharp(S) \leftrightarrows G(S) \to 1$$

or $\quad 1 \to N(S) \times \Pi(S) \to G^\sharp(S) \leftrightarrows \mathrm{Aut}(S) \to 1$. (Compare §2.8.)

Rather than working out the full details of this, let me simply describe the simplest case $S = (w)$. If $a_1, \ldots, a_w$ are arbitrary complex numbers with barycenter $a = (a_1 + \cdots + a_w)/w$, then it may be convenient to use the notation

$$z \mapsto f(z) = f(a_1, \ldots, a_w; z)$$

for the unique marked centered monic map of degree $d = w + 1$ with critical points

$$a_1 - a, \; a_2 - a, \; \ldots, \; a_w - a$$

and with $f(0) = a$. Here the symmetric group $\Pi(w)$ of order $w!$ acts by permuting the $a_j$, and the group $N(w)$ of $w$-th roots of unity $\eta$ acts by the rule $f(z) \mapsto \eta f(z/\eta)$, or equivalently

$$f(a_1, \ldots, a_w; z) \quad \mapsto \quad f(\eta a_1, \ldots, \eta a_w; z).$$

The automorphism group $\mathrm{Aut}(w)$ is of course trivial, so that the full symmetry group $G^\sharp(w)$ is just the cartesian product $N(w) \times \Pi(w)$ of order $w\, w!$.

In order to study real forms, we extend $G^\sharp(w)$ by the antiholomorphic involution $\gamma_0$, which commutes with $\Pi(w)$, and look for conjugacy classes of antiholomorphic involutions in the resulting group $\hat{G}^\sharp(w)$. For $d$ even, it turns out that there are $d/2$ distinct real forms, corresponding to polynomial maps with either $1, 3, \ldots$, or $w$ real critical points. For $d$ odd, there are $d+1$ distinct real forms. As an example, for $d = 3$ there are two marked real forms $x \mapsto \pm(x^3 - 3a^2 x) + b$ with real critical points $(a, -a)$ and two marked real forms $x \mapsto \pm(x^3 + 3a^2 x) + b$ with imaginary critical points $(ia, -ia)$. (Compare [Branner and Hubbard].) Further details will be left to the reader.

## Appendix
## Realizing Reduced Schemata.

by Alfredo Poirier

We fix a reduced schema $\bar{S} = (|\bar{S}|, F, w)$. The main purpose of this appendix is to show that there exist a Postcritically Finite Polynomial $f$ of degree $d(\bar{S}) = w(\bar{S}) + 1$, whose associated reduced schema $\bar{S}(f)$ realizes $\bar{S}$:

**Theorem.** *Every reduced schema $\bar{S} = (|\bar{S}|, F, w)$ can be realized.*

We start by constructing a non reduced schema $S$ which has $\bar{S}$ as associated reduced schema (compare Figures A.1 and A.2). Let $V$ be the set consisting of two copies of $|\bar{S}|$. Thus to any $v \in |\bar{S}|$, we associate $v, v' \in V$. We set $w_S(v) = w(v)$, and because we do not want to increase the degree we define $w_S(v') = 0$. Next, we define the function $F_S : V \to V$ as follows. Let $F_S(v) = v'$ and $F_S(v') = F(v)$. We will realize this mapping schema $S = (|S| = V, F_S, w_S)$ by constructing an expanding Hubbard tree (for the definitions and main results in abstract Hubbard Trees we refer to [P1] or [P2]).

In other words we have only inserted between every pair of (critical) vertices a third non-critical one. Clearly when restricted to the set $|\bar{S}|$, $F_S$ satisfies $F_S^{\circ 2} = F$. The next two lemmas follow directly from the above construction.

**Lemma 1.** *The mapping schema $S = (|S|, F_S, w_S)$ has reduced schema $\bar{S}$.*

**Lemma 2.** *Let $v' \in V$ be a non critical vertex. If $F_S(\omega) = v'$ then $v = \omega$.*

In other words every non critical vertex has exactly one preimage.

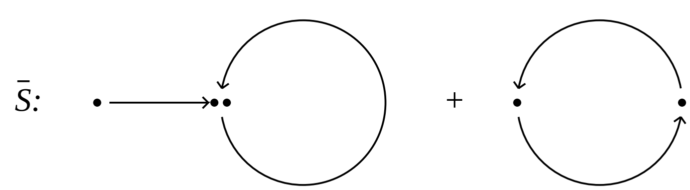



**Figure A.1: A reduced schema.**

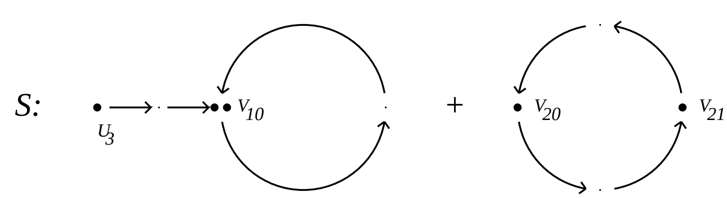


Figure A.2: The associated non-reduced mapping schema.

We begin the proof that the mapping schema $S$ can be realized, with a preliminary construction. Let $\mathcal{C}_i$ be the cycle $v_{i0} \mapsto v'_{i0} \mapsto v_{i1} \mapsto \ldots v_{in} = v_{i0}$. For this cycle we will construct a (star-shaped) expanding Hubbard Tree $H(\mathcal{C}_i)$ as follows. Join the vertices $v_{ij}$ and $v'_{ij}$ to a new vertex $p_i = p_{\mathcal{C}_i}$. We define the angle between consecutive edges at $p_i$ as $1/2n$. The new vertex $p_i$ will be by definition fixed of degree $d(p_i) = 1$. At all other vertices the dynamics and degree is that induced by $S$ (note that by definition $d(v) = w(v) + 1$). As there is only one periodic point which does not belong to a critical cycle, this is trivially an expanding abstract Hubbard tree.

Next, from each critical cycle $\mathcal{C}_i$ we choose a critical vertex $v_i = v_{i0}$ and form the collection $\{v_1, \ldots, v_m\}$; where $m \geq 1$ is the number of components of $S$. Let $u_{m+1}, \ldots, u_{m+r}$ be the critical vertices which do not belong to critical cycles. We construct an expanding abstract Hubbard tree which realizes $S$ as follows. Join the vertices $v_i$ and $u_j$ to a new vertex $q$, by segments $\ell_i$ and $\ell_j$. At $q$ these segments should form angles of non trivial multiples of $1/(r+m)$. Next, for every vertex $u_j$, we join $u_j$ and $u'_j$ by an edge which forms an angle of $1/d(u_j)$ with $\ell_j$. Also for every cycle $\mathcal{C}_i$ we paste the tree $H(\mathcal{C}_i)$ at $v_i$ forming an angle of $1/d(v_i)$ with $\ell_i$. Because of lemma 2 this graph is a topological (angled) tree. If we define $q$ as a fixed vertex of degree $1$, then with the induced dynamics from $S$ and all $H(\mathcal{C}_i)$ this tree is expanding (compare Figure A.3). Now, as every expanding Hubbard Tree can be realized by a Postcritically Finite polynomial (see [P1] or [P2]), the result follows.

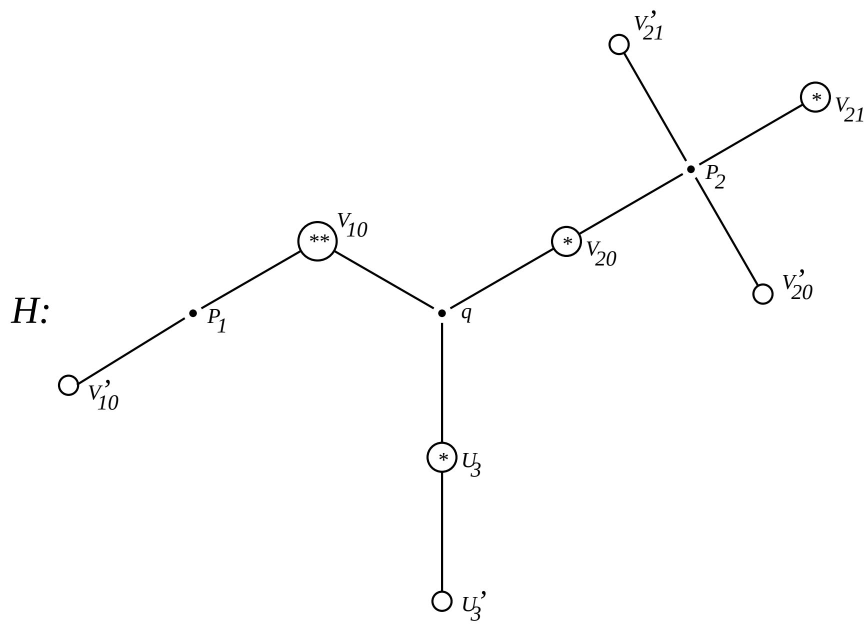



**Figure A.3: An associated abstract Hubbard Tree which realizes this mapping schema. In this figure $*$ represents a critical point ($**$ a double critical point). Dots correspond to points in the Julia set, and circles correspond to centers of Fatou components. As there is no edge whose endpoints correspond to Julia set vertices, this tree is (trivially) by definition expanding.**

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