# Weak Disks of Denjoy Minimal Sets

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**Abstract.** This paper investigates the existence of Denjoy minimal sets and, more generally, strictly ergodic sets in the dynamics of iterated homeomorphisms. It is shown that for the full two-shift, the collection of such invariant sets with the weak topology contains topological balls of all finite dimensions. One implication is an analogous result that holds for diffeomorphisms with transverse homoclinic points. It is also shown that the union of Denjoy minimal sets is dense in the two-shift and that the set of unique probability measures supported on these sets is weakly dense in the set of all shift-invariant, Borel probability measures.

**Section 0: Introduction.** One strategy for understanding a dynamical system is to first isolate invariant sets that are dynamically indecomposable. One then studies the structure of these pieces and how they fit together to give the global dynamics. This idea goes back at least to Birkhoff and has a particularly clear expression in Conley's Morse decompositions.

There are many notions of dynamical indecomposibility in the literature. In this paper we consider a fairly strong one that uses both topology and measure. An invariant set is called *strictly ergodic* if it is both minimal (every orbit is dense) and uniquely ergodic (existence of a unique, invariant Borel probability measure). These properties are preserved under topological conjugacy but *not* measure isomorphism.

The simplest such invariant sets are periodic orbits, and there are many theorems concerning their existence. The next simplest strictly ergodic systems are probably rigid rotations on the circle with irrational rotation number and the closely related Denjoy minimal sets. Elements of these invariant sets are sometimes called (generalized) quasi-periodic points. The models for Denjoy minimal sets are the minimal sets in nontransitive circle homeomorphisms with irrational rotation number. An abstract dynamical system is called a *Denjoy minimal set* if it is topologically conjugate to such a model. One of the questions that motivated this paper is what kind of properties of periodic orbits are also true for more general strictly ergodic invariant sets, in particular, for Denjoy minimal sets?

One way to begin to address this question is to collect these invariant sets into spaces. For a fixed homeomorphism $f$ of a compact metric space $X$, let $\mathcal{S}(X, f)$ denote the set of all strictly ergodic $f$-invariant subsets of $X$. Since different minimal sets are of necessity disjoint, each point in $\mathcal{S}(X, f)$ represents a minimal set that is disjoint from every other minimal set. A strictly ergodic set supports a unique invariant Borel probability measure, so we may use these measures with the weak topology to put a topology on $\mathcal{S}(X, f)$. If $\mathcal{D}(X, f)$ denotes the set of $f$-invariant subsets that are Denjoy minimal sets, then $\mathcal{D}(X, f) \subset \mathcal{S}(X, f)$, so we may use the weak topology on $\mathcal{D}(X, f)$ also.

In ([M]), Mather shows that for a area-preserving monotone twist map of the annulus, $f : A \to A$, the nonexistence of an invariant circle with a given irrational rotation number implies the existence of numerous Denjoy minimal sets with that rotation number. More precisely, using the notation just introduced, $\mathcal{D}(A, f)$ contains topological balls of every finite dimension. From one point of view this is a very surprising result. One has an arbitrarily large dimensional family of minimal sets embedded in a two-dimensional dynamical system. Another question that motivated this paper is how common is this kind of phenomenon in dynamics on finite dimensional manifolds?

It is important to note that even for a smooth system, $\mathcal{S}(M, f)$ can be empty. One example of this is Furstenberg's $C^\omega$-diffeomorphism of the two torus that is minimal but *not* strictly ergodic ([F]). However, for the full shift on two symbols $(\Sigma_2, \sigma)$ one has:

**Theorem 0.1.** *The space $\mathcal{S}(\Sigma_2, \sigma)$ contains a subspace homeomorphic to the Hilbert cube and the space $\mathcal{D}(\Sigma_2, \sigma)$ contains topological balls of dimension $n$ for all natural numbers $n$.*

The basic tool in the proof of this theorem is the main construction. This construction takes a certain type of open set in the circle (a regular one) and produces a compact, invariant set in the full two-shift. The construction uses the open set to produce itineraries with respect to a rigid rotation on the circle by an irrational angle. This process is somewhat analogous to using a Markov partition to produce a symbolic model for a system. Another analogous process is used in the kneading theory of unimodal maps of the interval. The difference here is that the chosen open set, in general, has no relation to the dynamics. The Hilbert cube of strictly ergodic sets is obtained by showing that the invariant sets constructed in the two-shift have unique invariant probability measures that depend continuously on the regular open sets in the appropriate topologies.

The main construction is a generalization of Morse and Hedlund's construction of Sturmian minimal sets as described on page 111 of [G-H]. Such generalizations are a standard tool in topological dynamics. In particular, the main construction is a special case of the almost automorphic minimal extensions of Markley and Paul given in [M-P]. Also of particular relevance are pages 234-241 of [A] and [H-H1].

For any regular open set, the main construction yields a minimal set in the shift. If the open set is a finite union of intervals, it gives a Denjoy minimal set. When the open set is more complicated, the resulting minimal set is more complicated. In particular, it follows from [M-P] that for certain open sets the construction gives minimal sets that have positive topological entropy and are not uniquely ergodic (see Remark 3.4 below).

The full two-shift is frequently embedded in the iterates of a complicated dynamical system. (In fact, this is one definition of a "complicated" dynamical system.) In view of Theorem 0.1 one would therefore expect that that $\mathcal{S}(X, f)$ will frequently contain a Hilbert cube. In the following corollary, the first sentence is a consequence of Theorem 0.1 and the Birkhoff-Smale theorem (a particularly suitable statement of which can be found on page 109 of [Rl]). The second sentence follows from the first and a theorem of Katok ([K]).

**Corollary 0.2.** *If $f : M \to M$ is a diffeomorphism of the compact manifold $M$ that has a transverse homoclinic orbit to a hyperbolic periodic point, then $\mathcal{S}(M, f)$ contains a subspace homeomorphic to the Hilbert cube. In particular, this is the case when $M$ is* *two-dimensional, $f$ is $C^{1+\alpha}$ and has positive topological entropy.*

As was the case with Mather's theorem, one has a large dimensional family of minimal sets (in this case an infinite dimensional family) embedded in finite dimensional dynamics. We shall see in Remark 3.7 below that in many cases this can be viewed as a manifestation of the fact that the Hilbert cube is the continuous, surjective image of the Cantor set. There is an invariant Cantor set $\hat{\Lambda}$ embedded in the dynamics. The orbit closure of each point in $\hat{\Lambda}$ supports a unique invariant probability measure. When the measures are given the weak topology, the map that takes the point to the measure is a continuous surjection of the Cantor set $\hat{\Lambda}$ onto the Hilbert cube.

There are two important examples that illustrate the necessity of the smoothness and dimension in the second sentence of Corollary 0.2. In [R] Rees constructs a homeomorphism of the two torus that is minimal and has positive topological entropy. Herman gives a $C^{\omega}$-difeomorphism of a 4-manifold that is also minimal with positive topological entropy ([Hm]). Neither example is uniquely ergodic, so in these cases $\mathcal{S}(M, f)$ is empty. The second sentence of Corollary 0.2 also raises the question of a converse. Specifically, if $\mathcal{S}(M, f)$ contains a subspace that is homeomorphic to the Hilbert cube, does $f$ have positive topological entropy? Proposition 3.1 shows that this is false on manifolds of dimension bigger than three.

It is an easy exercise to show that periodic orbits are dense in the full two-shift. A somewhat deeper result due to Parthasarathy says that the invariant probability measures supported on period orbits are weakly dense in the set of all shift-invariant probability measures, $\mathcal{M}(\Sigma_2, \sigma)$ ([P]). The next proposition gives the analog of these results for Denjoy minimal sets.

**Proposition 0.3.**

*(a) The set of points that are members of Denjoy minimal sets is dense in $\Sigma_2$.*

*(b) The set of invariant measures supported on Denjoy minimal sets is weakly dense in the set of invariant measures, i.e. $\mathcal{D}(\Sigma_2, \sigma)$ is dense in $\mathcal{M}(\Sigma_2, \sigma)$.*

This paper is organized as follows. Section 1 gives basic definitions, background information and the main construction. Section 2 contains the statement and proof of the main theorem. This theorem describes continuity properties of the main construction and the structure of resulting invariant sets. Section 2 also contains the proof of Theorem 0.3. The proof of Theorem 0.1 is given in Section 3, as is the example that shows that the converse of Corollary 0.2 is false in dimensions three and greater. The last section examines the relationship between the intrinsic rotation number of a Denjoy minimal set and its "extrinsic" rotation number when it is embedded in a map of the annulus. It is also shown that any Denjoy minimal set in the two-shift can be generated from a regular open set in the circle using the main construction.

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**Section 1: Preliminaries.** This section introduces assorted notation and definitions and recalls some basic facts from topology, ergodic theory and topological dynamics. Many of these facts are stated without proof or references. In such cases, the facts are either elementary exercises or can be found in Walters' book [W].

For a set $X, Cl(X), Int(X), X^c$ and $Fr(X)$ denote the closure, interior, complement and frontier of the set, respectively. The operator $\sqcup$ is the disjoint union. Thus $A \sqcup B$ represents the union of the two sets, but conveys the added information that the sets are disjoint. The indicator function of a set $X$ is denoted $\mathbf{I}_X$. Thus $\mathbf{I}_X(x) = 1$ if $x \in X$, and is 0 otherwise. The circle is $S^1 = \mathbf{R}/\mathbf{Z}$ and $R_\eta : S^1 \to S^1$ is rigid rotation by $\eta$, *i.e* $R_\eta(\theta) = \theta + \eta \mod 1$. Haar measure on the circle is denoted by $m$.

A nonempty, proper subset $U \subset S^1$ is called a *regular open* set if $Int(Cl(U)) = U$. The set of all regular open sets is

$$\mathcal{RO} = \{U \subset S^1 : U \text{ is a regular open set}\}.$$

Given an open set $U$, its $*$-*dual* is the interior of its complement and is denoted by $U^* = Int(U^c)$. Note that $U$ is regular open if and only if $S^1$ can be written as the disjoint union of three nonempty sets, $S^1 = U \sqcup F \sqcup U^*$ with $F = Fr(U) = Fr(U^*)$. In consequence, $U \in \mathcal{RO}$ if and only if $U^* \in \mathcal{RO}$.

The set $\mathcal{RO}$ of regular open sets will be topologized using the symmetric difference of sets. For $U, V \in \mathcal{RO}$, their symmetric difference is $U \triangle V = (U \cap V^c) \sqcup (U^c \cap V)$ and the distance between them is $d(U, V) = m(U \triangle V)$. If $U$ and $V$ are regular open, when $U \triangle V$ is nonempty it contains an interval. In particular, $d(U, V) = 0$ if and only if $U = V$. Since $d(U, V) = \int |\mathbf{I}_U - \mathbf{I}_V| dm = \|\mathbf{I}_U - \mathbf{I}_V\|_1$, $\mathcal{RO}$ maybe thought of as a subspace of $L^1(S^1, m)$. This makes it clear that $d$ gives a metric on $\mathcal{RO}$.

If the frontiers of either $U$ or $V$ have positive measure, it could happen that $d(U, V) \neq d(U^*, V^*)$. To avoid this and related situations it is sometimes necessary to restrict attention to the set of regular open sets whose frontiers have measure zero,

$$\mathcal{RO}_0 = \{U \in \mathcal{RO} : m(Fr(U)) = 0\}.$$

A metric that controls both regular open sets and their $*$-duals is given by

$$d_*(U, V) = (d(U, V) + d(U^*, V^*))/2.$$

Unless otherwise noted, the topology on $\mathcal{RO}$ will be that given by the metric $d_*$. Note that when restricted to $\mathcal{RO}_0$, $d$ and $d_*$ give the same metric.

It will also be useful to identify regular open sets that are equal after a rigid rotation of $S^1$. More precisely, say $U \sim V$ if there exists an $\eta \in S^1$ with $V = R_\eta(U)$. Denote the quotient spaces by $\mathcal{RO}' = \mathcal{RO} / \sim$ and $\mathcal{RO}'_0 = \mathcal{RO}_0 / \sim$. Note that the topology generated by the projection $\mathcal{RO} \to \mathcal{RO}'$ can be viewed as being generated by the metric $d'([U], [V]) = \inf\{d_*(U, R_\eta(V)) : \eta \in S^1\}$, where $[U]$ denotes the equivalence class of $U$ under $\sim$.

A related notion is that of a symmetric set. A set $U \in \mathcal{RO}$ is called *symmetric* if there exists an $\eta \neq 0$ with $R_\eta(U) = U$. Because $U$ is open, such an $\eta$ will always be a rational number.

In this paper a *dynamical system* means a pair $(X, h)$ where $X$ is a compact metric space and $h$ is a homeomorphism. Given a point $x \in X$, its orbit is $o(x, h) =$

$\{\ldots, h^{-1}(x), x, h(x), \ldots\}$. A finite piece of the forward orbit is denoted $o(x, h, N) = \{x, h(x), \ldots, h^N(x)\}$. If $(X, h) \to (Y, g)$ is a continuous semiconjugacy, then $(X, h)$ is called an *extension* of $(Y, g)$, and $(Y, g)$ is a *factor* of $(X, h)$. When the semiconjugacy is one to one on a dense $G_\delta$ set, the extension is termed *almost one to one*.

The pair $(X, h)$ is called a *minimal set* if every orbit is dense. The pair is *uniquely ergodic* if there exists a unique invariant Borel probability measure. A useful characterization is: $(X, h)$ is uniquely ergodic if and only if the sequence of functions $(\sum_{i=0}^N f \circ h^i)/(N+1)$ converges uniformly for all $f \in C(X, \mathbf{R})$. A pair that is both minimal and uniquely ergodic is called *strictly ergodic*. Note that the property of being minimal, uniquely ergodic or strictly ergodic is preserved under topological conjugacy. Also, if an extension is strictly ergodic, then so is its factor.

A compact $h$-invariant set $Y \subset X$ is called minimal, uniquely ergodic or strictly ergodic if $h$ restricted to $Y$ has that property. In a slight abuse of notation, this situation is described by saying that $(Y, h)$ is minimal, etc.

Perhaps the simplest nontrivial strictly ergodic system is $(S^1, R_\alpha)$ for an irrational $\alpha$. A homeomorphism $g : S^1 \to S^1$ that has an irrational rotation number and the pair $(S^1, g)$ is *not* minimal is called a *Denjoy example*. Such examples are classified up to topological conjugacy in [My]. The two classifying invariants are the rotation number and the set of orbits that are "blown up" into intervals. A Denjoy example always has a unique minimal set $Y \subset S^1$ with $(Y, g)$ strictly ergodic.

An abstract dynamical system $(X, h)$ is called a *Denjoy minimal set* if it is topologically conjugate to the minimal set in a Denjoy example. Such an $(X, h)$ is always strictly ergodic. Mather points out in [M] that a Denjoy minimal set $(X, h)$ always has a well defined *intrinsic rotation number*, *i.e.* if $(X, h)$ is topologically conjugate to the minimal sets in two Denjoy examples $(S^1, g_1)$ and $(S^1, g_2)$, then either $g_1$ and $g_2$ have the same rotation number or else $g_1$ and $g_2^{-1}$ do. If $(X, h)$ is a Denjoy minimal set with intrinsic rotation number $\alpha$, it is an almost one to one extension of $(S^1, R_\alpha)$.

A general dynamical system $(Z, h)$ can have many invariant subsets that are Denjoy minimal sets or strictly ergodic. These subsets are collected together in the spaces

$$\mathcal{D}(Z, h) = \{Y \subset Z : (Y, h) \text{ is a Denjoy minimal set}\}$$

and

$$\mathcal{S}(Z, h) = \{Y \subset Z : (Y, h) \text{ is strictly ergodic}\}.$$

To topologize these spaces we recall the weak topology on measures. Given a dynamical system $(Z, h)$, the set of all its invariant, Borel probability measures is denoted $\mathcal{M}(Z, h)$. The weak topology on $\mathcal{M}$ can be defined by saying that the measures $\mu_n \to \mu_0$ weakly if and only if $\int f d\mu_n \to \int f \mu_0$ for all $f \in C(Z, \mathbf{R})$. Note that $\mathcal{M}(Z, h)$ with this topology is compact, and when viewed as a subspace of the dual space to $C(Z, \mathbf{R})$, it is convex with extreme points equal to the ergodic measures. Since a strictly ergodic system supports a unique invariant probability measure, there is a natural inclusion $\mathcal{S}(Z, h) \subset \mathcal{M}(Z, h)$. This inclusion induces a topology on $\mathcal{S}(Z, h)$ that will be called the weak topology. The fact that $\mathcal{D}(Z, h) \subset \mathcal{S}(Z, h)$ allows us to use the weak topology on $\mathcal{D}(Z, h)$ also.

In the absence of unique invariant measures we use the Hausdorff metric to measure the distance between compact invariant subsets. Given a compact space $X$, the space consisting of the closed subsets of $X$ with the Hausdorff topology is denoted $\mathcal{H}(X)$. Note that if $X$ is compact metric then so is $\mathcal{H}(X)$. A map $\Phi : E \to \mathcal{H}(X)$ is called *lower semicontinuous* if for all closed subsets $Y \subset X$, the set $\{e \in E : \Phi(e) \subset Y\}$ is closed in $E$. We will need the fact that the following property implies that $\Phi$ is lower semicontinuous: When $e_n \to e$ in $E$ and for some subsequence $\{n_i\}$, $\Phi(e_{n_i}) \to K$ in $\mathcal{H}(X)$ then $\Phi(e) \subset K$. Informally, $\Phi$ is lower semicontinuous if when you perturb $e$, $\Phi(e)$ may get suddenly larger, but never suddenly smaller.

The *full shift on two symbols* is the the pair $(\Sigma_2, \sigma)$ consisting of the sequence space $\Sigma_2 = \{0,1\}^{\mathbf{Z}}$ and the shift map $\sigma$. A *symbol block* $b$ is a finite sequence $b_0, b_1, \ldots, b_{N-1}$ which each $b_i$ equal to 0 or 1. The *length* of the block $b$ is $N$ and the *period* is its period when considered as a cyclic word. A sequence $s \in \Sigma_2$ has *initial block* $b$ if $b_i = s_i$ for $i = 0, \ldots, N-1$. It is notationally convenient to view the topology on $\Sigma_2$ as being generated by a metric $d_\Sigma$ with $d_\Sigma(s,t) < 1/N$ if and only if $s_i = t_i$ for $|i| < N$. A *cylinder set* depends on a block $b$ and an integer $n$ and is a set of the form

$$C_b^n = \{s \in \Sigma_2 : s_{i+n} = b_i, \text{ for } i = 0, \ldots, length(b) - 1\}.$$

If $n = 0$, we write $C_b^0 = C_b$.

Since cylinder sets are both open and closed, their indicator functions are continuous. In fact, the finite linear combinations of such indicator functions form a dense set in $C(\Sigma_2, \mathbf{R})$. This implies that the measures $\mu_n \to \mu_0$ weakly if and only if $\mu_n(C_b^n) \to \mu_0(C_b^n)$ for all cylinder sets $C_b^n$. Since the elements of $\mathcal{M}(\Sigma_2, \sigma)$ are shift invariant measures, any such measure $\mu$ satisfies $\mu(C_b^n) = \mu(C_b)$ for all $n$. Thus the topology on $\mathcal{M}(\Sigma_2, \sigma)$ is in fact generated by the metric

$$d(\mu_1, \mu_2) = \sum |\mu_1(C_{b^{(n)}}) - \mu_2(C_{b^{(n)}})|/2^n$$

where the sum is over some enumeration $b^{(n)}$ of all possible blocks by the natural numbers $n$.

The main construction in this paper takes a regular open set in the circle and produces a compact invariant set in $(\Sigma_2, \sigma)$ along with an invariant measure. As noted in the introduction, it is closely related to the construction given in [M-P]. We are primarily interested here in the dependence of the construction on the open set and a "rotation number". This dependence is encoded in two functions $\lambda : \mathcal{RO}_0 \times S^1 \to \mathcal{M}(\Sigma_2, \sigma)$ and $\Lambda : \mathcal{RO} \times S^1 \to \mathcal{H}(\Sigma_2)$ defined as follows.

Fix $U \in \mathcal{RO}$ and $r \in S^1$ Define $B \subset S^1$ as

$$B = \{x \in S^1 : o(x, R_r) \cap Fr(U) = \emptyset\}.$$

Since $U$ is regular open, $Fr(U)$ is closed and nowhere dense, and thus since $B = \cap_{i \in \mathbf{N}} R_r^i(Fr(U)^c)$, $B$ is dense $G_\delta$. Now define $\phi : B \to \Sigma_2$ so that

$$(\phi(x))_i = \mathbf{I}_U(R_r^i(x)).$$

Thus for any point $x \in B$, the sequence $\phi(x)$ is the "itinerary" of $x$ under $R_r$ with respect to the set $U$, *i.e.* $\phi(x)$ has a 1 in the i$^{th}$ place if $R_r^i(x)$ is in $U$ and 0 if it is in $U^*$. It is easy to see that $\phi$ is continuous.

Now define $\Lambda(U,r) = Cl(\phi(B))$. If $U \in \mathcal{RO}_0$, then $m(U \sqcup U^*) = 1$ and so $m(B) = 1$. Thus we may define a probability measure $\lambda \in \mathcal{M}(\Sigma_2, \sigma)$ by $\lambda = \phi_*(m)$, where as usual this means that $\lambda(X) = m(\phi^{-1}(X))$ for a Borel set $X$.

In this construction, $B$ and $\phi$ depend on the choice of $U$ and $r$. If this dependence needs to be emphasized, we write $B = B_{U,r}$ and $\phi = \phi_{U,r}$. It is clear that for all $\eta \in S^1$ and $U \in \mathcal{RO}$, $\Lambda(R_\eta(U), r) = \Lambda(U,r)$ and $\lambda(R_\eta(U), r) = \lambda(U,r)$. Thus the maps $\Lambda$ and $\lambda$ descend to maps on $\mathcal{RO}' \times S^1$ and $\mathcal{RO}'_0 \times S^1$ that will also be called $\Lambda$ and $\lambda$.

To make the last definition, we need to adopt the notation that $U_0 = U^*$ and $U_1 = U$. For a block of symbols $b$ of length $N+1$, define

$$U_{b,r} = \bigcap_{i=0}^{N} R_r^{-i}(U_{b_i}).$$

The important property of these sets is that for $x \in B$, $x \in U_{b,r}$ if and only if $\phi(x)$ is in the cylinder set $C_b$. As a consequence, for $U \in \mathcal{RO}_0$, $\lambda(U,r)[C_b] = \phi_* m(C_b) = m(U_{b,r})$.

**Lemma 1.** *The following maps are continuous.*
*(a) For fixed $U \in \mathcal{RO}$, the map $S^1 \to \mathbf{R}$ given by $\eta \mapsto d_*(U, R_\eta(U))$.*
*(b) For fixed $U \in \mathcal{RO}$, the map $S^1 \to \mathcal{RO}$ given by $\eta \mapsto R_\eta(U)$.*
*(c) The map $\mathcal{RO} \to \mathbf{R}$ given by $U \mapsto m(U)$.*
*(d) For fixed symbol block $b$, the map $\mathcal{RO} \times S^1 \to \mathcal{RO}$ given by $(U,r) \mapsto U_{b,r}$.*

**Proof of (a) and (b).** We first prove continuity of the map $\eta \mapsto d(U, R_\eta(U))$ at $\eta = 0$. Since $U \in S^1$ is open, we can find a countable set of disjoint intervals $\{I_n\}$ so that $U = \sqcup I_n$. Now given $\epsilon > 0$, pick $M$ so that $\sum_{n>M} m(I_n) < \epsilon/4$ and assume $|\eta| < \epsilon/(4M)$. Now for each $n$, clearly $m(I_n \cap R_\eta(U)^c) < \eta$ and so

$$\begin{aligned} m(U \cap R_\eta(U)^c) &< \sum_{n>M} m(I_n) + \sum_{n \le M} m(I_n \cap R_\eta(U)^c) \\ &< \epsilon/2. \end{aligned}$$

Now since $m(U^c \cap R_\eta(U)) = m(R_{-\eta}(U)^c \cap U)$, we also get $m(U^c \cap R_\eta(U)) < \epsilon/2$ and so $d(U, R_\eta(U)) < \epsilon$.

What we have just shown also implies that $\eta \mapsto d(U^*, R_\eta(U^*))$ is continuous at $\eta = 0$, and thus $\eta \mapsto d_*(U, R_\eta(U))$ is also. Since $d(R_\eta(U), R_{\eta'}(U)) = d(U, R_{\eta-\eta'}(U))$, the continuity of $\eta \mapsto R_\eta(U)$ at all $\eta$ follows. Finally, since $d_*$ is a metric, and therefore a continuous function $\mathcal{RO} \times \mathcal{RO} \to \mathbf{R}$, we get $\eta \mapsto d_*(U, R_\eta(U))$ continuous for all $\eta$.

**Proof of (c).** Given two finite collections of sets $A_i$ and $B_i$ with $i \in \{0, \dots N\}$ using the fact that $d(A,B) = \|\mathbf{I}_A - \mathbf{I}_B\|_1$ and standard integral inequalities it is easy to show that $|m(A) - m(B)| \le d(A,B)$ and $d_*(\cap A_i, \cap B_i) \le \sum d_*(A_i, B_i)$.

The continuity of $U \mapsto m(U)$, follows from the fact that $d_*(U,V) \le \epsilon/2$ implies $\epsilon \ge d(U,V) \ge |m(U) - m(V)|$.

**Proof of (d).** If the length of the fixed block $b$ is $N+1$, then given $\epsilon > 0$ using (a), pick $\delta < \epsilon/(2N+2)$ so that $|\eta| < \delta$ implies $d_*(U, R_\eta(U)) < \epsilon/(2N+2)$.

We therefore have for $(V,s) \in \mathcal{RO} \times S^1$ with $d(U,V) < \delta$ and $|r-s| < \delta/N$,

$$\begin{aligned} d_*(U_{b,r}, V_{b,s}) &= d_*(\cap R_r^{-i}(U_{b_i}), \cap R_s^{-i}(V_{b_i})) \\ &\leq \sum d_*(R_r^{-i}(U_{b_i}), R_s^{-i}(V_{b_i})) \\ &= \sum d_*(R_{i(s-r)}(U_{b_i}), V_{b_i}) \\ &\leq \sum (d_*(R_{i(s-r)}(U_{b_i}), U_{b_i}) + d_*(U_{b_i}, V_{b_i}) \\ &\leq \epsilon. \end{aligned}$$

□

**Section 2: The main theorem.** The main goal of this section is to prove the following theorem. For the reader interested in the quickest route to Theorem 0.1, we note that the lower semicontinuity of $\Lambda$ and the results in part (3) are not needed for that proof.

**Theorem 2.** *Let the maps $\lambda : \mathcal{RO}'_0 \times S^1 \to \mathcal{M}(\Sigma_2, \sigma)$ and $\Lambda : \mathcal{RO}' \times S^1 \to \mathcal{H}(\Sigma_2)$ be as defined in Section 1.*

*(1) The map $\lambda$ is continuous and the map $\Lambda$ is lower semicontinuous.*

*(2) Fix $\alpha \notin \mathbf{Q}$.*

- *(a) For all $U \in \mathcal{RO}'$, $(\Lambda(U,\alpha), \sigma)$ is an almost one to one minimal extension of $(S^1, R_{n\alpha})$ for some natural number $n$.*
- *(b) If $U \in \mathcal{RO}'_0$, then $(\Lambda(U,\alpha), \sigma)$ is uniquely ergodic.*
- *(c) If $Fr(U)$ is a finite set, then $(\Lambda(U,\alpha), \sigma)$ is a Denjoy minimal set with intrinsic rotation number $n\alpha$ for some natural number $n$.*
- *(d) For fixed $\alpha \notin \mathbf{Q}$, when considered as a function of $U$, $\Lambda$ and $\lambda$ are injective.*

*(3) Fix $p/q \in Q$ with $p$ and $q$ relatively prime.*

- *(a) For all $U \in \mathcal{RO}'$, $\Lambda(U, p/q)$ is a finite collection of periodic orbits whose periods divide $q$.*
- *(b) For fixed $p/q \in Q$, when considered as a function of $U$, the image of $\lambda$ is the convex hull of the probability measures supported on the periodic orbits whose periods divide $q$.*

**Proof of (1).** Since $\Lambda(R_\eta(U), r) = \Lambda(U, r)$, it suffices to check the continuity of $\Lambda$ as a map defined on $\mathcal{RO}$. A similar comment holds for $\lambda$.

As noted in the previous section, the weak topology on $\mathcal{M}(\Sigma_2, \sigma)$ is generated by the metric $d(\lambda_1, \lambda_2) = \sum |\lambda_1(C_{b^{(n)}}) - \lambda_2(C_{b^{(n)}})|/2^n$ and $\lambda(U,r)[C_b] = m(U_{b,r})$. Thus to prove the continuity of $\lambda$ it suffices to check that for fixed $b$ the map $U \mapsto m(U_{b,r})$ is continuous. This follows from Lemma 1 (c) and (d).

For the proof of the lower semicontinuity of $\Lambda$, begin by assuming that $(U^{(n)}, r^{(n)}) \to (U^{(0)}, r^{(0)})$. If for some subsequence $\{n_i\}$, $\Lambda(U^{(n_i)}, r^{(n_i)}) \to K$ in the Hausdorff topology, then we will show that $\Lambda(U^{(0)}, r^{(0)}) \subset K$. As noted in the previous section, this implies the desired semicontinuity. Fix an $x_0 \in B^{(0)}$ and integer $N > 0$ and let $b$ be the initial block of length $N+1$ in $\phi(x_0)$. This certainly implies that $U^{(0)}_{b,r^{(0)}}$ is a nonempty open set and therefore has positive measure. Therefore by Lemma 1 (c) and (d) there exists an $M$ so that $n > M$ implies that $m(U^{(n)}_{b,r^{(n)}}) > 0$. In particular, for $n > M$, there exists $x_n \in B^{(n)}$ so that $\phi_n(x_n)$ has its initial block equal to $b$. There therefore exits a sequence $x_j \in B^{(j)}$ with $\phi_j(x_j) \to \phi_0(x_0)$.

Now assuming that for some subsequence $\{n_i\}$, $\Lambda(U^{(n_i)}, r^{(n_i)}) \to K$ in the Hausdorff topology, then if $x_{n_i} \in B^{(n_i)}$ is the appropriate subsequence of the sequence constructed in the previous paragraph, then $\phi_{n_i}(x_{n_i}) \to \phi_0(x_0)$, so certainly $\phi_0(x_0) \in K$. But $x_0 \in B^{(0)}$ was arbitrary, and so $\phi_0(B^{(0)}) \subset K$ and since $\Lambda(U^{(0)}, r^{(0)})$ is the closure of the $\phi_0(B^{(0)})$, we have $\Lambda(U^{(0)}, r^{(0)}) \subset K$, as required.

**Proof of (2).** For the proof of (2), fix an $\alpha \notin \mathbf{Q}$ and for the proof of (2a), (2b) and (2c) a $U \in \mathcal{RO}$. We will suppress the dependence of various objects on $U$ and $\alpha$ and so $\Lambda = \Lambda(U, \alpha)$, etc.

**(2a).** To prove the minimality of $\Lambda$ we use the following characterization of minimality ([O]): If $f : X \to X$ is a homeomorphism of a compact metric space and $x \in X$, then $Cl(o(x, f))$ is a minimal set if and only if given $\epsilon > 0$, there exists an $N$ such that for all $n$, there exists an $i$ with $0 \leq i \leq N$ and $d(f^{n+i}(x), x) < \epsilon$.

To apply this to the case at hand, first note that for $x \in B$, certainly $o(x, R_\alpha)$ is dense in $B$, and so $\Lambda = Cl(o(\phi(x), \sigma))$. Since $(S^1, R_\alpha)$ is minimal, the above property holds for $Cl(o(x, R_\alpha))$. Since $\phi$ is continuous, it also holds for $Cl(o(\phi(x), \sigma)) = \Lambda$, which is therefore minimal.

The proof of the semiconjugacy requires a new definition. Given $U, V \in \mathcal{RO}$, define $\rho(U, V) = \sup\{m(I) : I \text{ is an interval contained in } U \triangle V\}$. Now $\rho$ will not satisfy the triangle inequality but it is easy to see that for fixed $U \in \mathcal{RO}$, the map $\eta \mapsto \rho(U, R_\eta(U))$ is a continuous function $S^1 \to \mathbf{R}$. Also, if $U$ is asymmetric, then $\rho(U, R_\eta(U)) = 0$ if and only if $\eta = 0$.

The first step in the proof of the semiconjugacy is to show that $\phi$ is injective when $U$ is asymmetric. Assume that for $x_1, x_2 \in B$, $\phi(x_0) = \phi(x_1)$, and therefore for all $i$, $\mathbf{I}_U(R^i_\alpha(x_1)) = \mathbf{I}_U(R^i_\alpha(x_2))$. Thus if $x_2 = R_\eta(x_1)$, $\mathbf{I}_U = \mathbf{I}_U \circ R_\eta$ when restricted to the dense set $o(x_1, R_\alpha)$. In particular, $\rho(U, R_\eta(U)) = 0$ and since $U$ is asymmetric, $d(x_1, x_2) = \eta = 0$.

Continuing with the assumption that $U$ is asymmetric, we show that $\phi^{-1}$ is uniformly continuous. Since $\phi(B)$ is certainly dense in $\Lambda$, this implies that we can extend $\phi^{-1}$ to a semiconjugacy from $(\Lambda, \sigma)$ to $(S^1, R_\alpha)$.

Since $S^1$ is compact and $\eta \mapsto \rho(U, R_\eta(U))$ is continuous, given $\epsilon > 0$ there exists a $\delta > 0$ so that $\rho(U, R_\eta(U)) < \delta$ implies $|\eta| < \epsilon$. Pick $N > 0$ so that for every $x \in S^1$, every interval of length $\delta$ contains a point of $o(x, R_\alpha, N)$. Now if $x_1, x_2 \in B$ satisfy $d_\Sigma(\phi(x_1), \phi(x_2)) < 1/N$ and if $x_2 = R_\eta(x_1)$, then $\mathbf{I}_U = \mathbf{I}_U \circ R_\eta$ when restricted to the set $o(x, R_\alpha, N)$. Now if $\rho(U, R_\eta(U)) > \delta$ then $U \triangle R_\eta(U)$ will contain an interval of length $\delta$ and thus a point of $o(x, R_\alpha, N)$, a contradiction. Thus $\rho(U, R_\eta(U)) < \delta$ and so by the choice of $\delta$, $d(x_1, x_2) = |\eta| < \epsilon$, proving the uniform continuity of $\phi^{-1}$. Note that $\phi(B)$ is dense $G_\delta$ in $\Lambda$ so the extension is almost one to one.

Now assume that $U$ is symmetric. The group of numbers $r$ such that $R_r(U) = U$ has a rational generator, say $p/q$, with $0 < p/q < 1$ and $p$ and $q$ relatively prime. If $U' = \pi(U)$ where $\pi : S^1 \to S^1/R_{p/q}$ is the projection, then $\Lambda(U, \alpha)$ has $\Lambda(U', q\alpha)$ as a $q$-fold factor (here we have identified $S^1/R_{p/q}$ with $S^1$). Since $U'$ is asymmetric, $\Lambda(U', q\alpha)$ has $(S^1, R_{q\alpha})$ as a factor, finishing the proof of (2a).

**(2b).** Let $\psi$ denote the extension of $\phi^{-1}$ to a continuous semiconjugacy from $(\Lambda, \sigma)$ to $(S^1, R_{\alpha q})$ and assume that $m(Fr(U)) = 0$. If $\lambda_1$ and $\lambda_2$ are two invariant Borel probability measures supported on $\Lambda$, then since $(S^1, R_{\alpha q})$ is uniquely ergodic, $\psi_*(\lambda_1) = \psi_*(\lambda_2) = m$. If $X \subset \Lambda$ is a Borel set, then since $m(B) = 1$, for $i = 1, 2$, $\lambda_i(X) = \lambda_i(\psi^{-1}(B) \cap X)$. Now since $\psi$ is injective on $B$, this is equal to $\lambda_i(\psi^{-1}(B \cap \psi(X)) = m(B \cap \psi(X))$ and so $\lambda_1 = \lambda_2$.

**(2c).** Now assume $Fr(U)$ is a finite set. In this case, each $x \in B^c$ will have exactly two preimages under $\psi$, namely, the limit of $\phi(x_n)$ as $x_n \to x$ from the right and the limit of $\phi(x_n)$ as $x_n \to x$ from the left. This makes it clear that in this case $\Lambda$ is conjugate to the minimal set in the circle homeomorphism obtained by "blowing up" into intervals points on the orbits of each $x \in Fr(U)$.

**(2d).** When $U \in \mathcal{RO}_0$, $\Lambda(U, \alpha)$ is the support of $\lambda(U, \alpha)$. Thus to prove (2d) it suffices to show that $\Lambda(U, \alpha)$ is an injective function of $U$. Assume that for some $U_1, U_2 \in \mathcal{RO}$, $\Lambda(U_1, \alpha) = \Lambda(U_2, \alpha)$. Using (2a), $\phi(B_1)$ and $\phi_2(B_2)$ are dense $G_\delta$ in the compact metric space $\Lambda(U_1, \alpha) = \Lambda(U_2, \alpha)$. This implies that $\phi(B_1) \cap \phi_2(B_2) \neq \emptyset$, and so there exist $x_1, x_2 \in S^1$ with $\phi_1(x_1) = \phi_2(x_2)$. Thus if $R_\eta(x_1) = x_2$, then $\mathbf{I}_{U_1} = \mathbf{I}_{U_2} \circ R_\eta$ when restricted to the dense set $o(x_1, R_\alpha)$. This implies that $U_1 \triangle R_\eta(U_2)$ contains no intervals. Since the $U_i$ are regular open sets, this means that $U_1 = R_\eta(U_2)$ and so $U_1$ and $U_2$ are in the same equivalence class in $\mathcal{RO}'$, as required.

**Proof of (3).** Fix $p/q \in \mathbf{Q}$ with $p$ and $q$ relatively prime. Since $R^q_{p/q} = Id$, it is clear that any $s \in \Lambda(U, p/q)$ will satisfy $\sigma^q(s) = s$ which implies (3a). Say a symbol block $b$ is *prime* if its length equals its period. For $U \in \mathcal{RO}'_0$, by construction, $\lambda(U, p/q) = \sum m(U_{b,p/q})\mu_b$ where $\mu_b$ is the probability measure supported on the periodic orbit with repeating block $b$ and the sum is over all prime blocks $b$ whose period divides $q$. With this formula in hand it is easy to construct a $U$ so that $\lambda(U, p/q)$ is any desired point in the convex hull given in the statement of (3b). □

**Proof of Proposition 0.3**

(a). A theorem of Parthasarathy says that the measures supported on periodic orbits are dense in $\mathcal{M}(\Sigma_2, \sigma)$ ([P]). Fix one such measure $\mu_0$, and assume it is supported on an orbit of period $q$. Using the formula given in the proof of Theorem 2 (3b), find a regular open set $U$ with $Fr(U)$ a finite set and a $p/q$ with $\lambda(U, p/q) = \mu_0$. Now pick irrationals $\alpha_n \to p/q$. By Theorem 2 (1), $\lambda(U, \alpha_n) \to \mu_0$, and by Theorem 2 (2c), each $\lambda(U, \alpha_n)$ is the unique measure supported on a Denjoy minimal set.

(b). It suffices to show that for any symbol block $b$, there exists an $s \in \Sigma_2$ which has initial block $b$ and $Cl(o(s, \sigma))$ is a Denjoy minimal set. Fix an irrational $\alpha$ and $x_0 \in S^1$. Choose a finite union of intervals $U$ so that $R^i_\alpha(x_0) \in U$ if and only if $b_i = 1$, for $i = 0, \ldots, length(b) - 1$. Further, the open set $U$ should satisfy $o(x_0, R_\alpha) \cap Fr(U) = \emptyset$. If $U$ has these properties, Theorem 2 (2c) shows that $\Lambda(U, \alpha)$ is the desired Denjoy minimal set. □

**Section 3: The Hilbert cube of strictly ergodic sets.** We begin with some definitions in preparation for the proof of Theorem 0.1. A copy of the Hilbert cube is given by the collection of sequences,

$$\mathbf{H} = \{\gamma \in \mathbf{R}^{\mathbf{N}} : 0 \leq \gamma_i \leq \frac{1}{i+2} \text{ for all } i \in \mathbf{N}\}.$$

A subspace of $\mathbf{H}$ that contains topological balls of all dimensions is

$$\mathbf{H}_0 = \{\gamma \in \mathbf{H} : \gamma_i = 0, \text{ for all but finitely many } i\}.$$

For $\gamma \in \mathbf{H}$, define an asymmetric regular open set $U_\gamma$ by

$$U_\gamma = \bigcup_{i \in \mathbf{N}} (\frac{1}{i+2} - \gamma_i^3, \frac{1}{i+2} + \gamma_i^3).$$

Now define a map $\Gamma : \mathbf{H} \to \mathcal{RO}'_0$ via $\Gamma(\gamma) = [U_\gamma]$. It is clear that $\Gamma$ is continuous and injective. Since $\mathbf{H}$ is compact, $\Gamma(\mathbf{H})$ is homeomorphic to $\mathbf{H}$.

**Proof of Theorem 0.1.** Fix an irrational $\alpha$. By Theorem 2 (2ab), the set $\Lambda(\Gamma(\mathbf{H}), \alpha)$ consists of strictly ergodic sets. Since $\Gamma(\mathbf{H})$ is compact, using Theorem 2 (1) and (2d), we have that $\lambda(\Gamma(\mathbf{H}), \alpha)$ is homeomorphic to $\Gamma(\mathbf{H})$ and therefore to $\mathbf{H}$. This proves the first statement in the theorem. To prove the second, note that Theorem 2 (2c) implies that $\lambda(\Gamma(\mathbf{H}_0), \alpha)$ consists of measures supported on Denjoy minimal sets. Since $\lambda(\Gamma(\mathbf{H}_0), \alpha)$ is homeomorphic to $\mathbf{H}_0$, it (and consequently, $\mathcal{D}(\Sigma_2, \sigma)$) contains topological balls of all dimensions. $\square$

**Remarks.**

**(3.1)** In Theorem 0.1 there is an obvious distinction between $\mathcal{S}(\Sigma_2, \sigma)$, which contains a copy of $\mathbf{H}$, and $\mathcal{D}(\Sigma_2, \sigma)$, which contains a copy of $\mathbf{H}_0$. This is because $\Lambda(\Gamma(\mathbf{H}), \alpha)$ contains minimal sets that are not Denjoy. In particular, if $\gamma \in \mathbf{H} - \mathbf{H}_0$ and for some $i \neq 0$, $R_\alpha^i(0) \in Fr(U_\gamma)$, then $\Lambda(U_\gamma, \alpha)$ is not a Denjoy minimal set. In the semiconjugacy from $(\Lambda(U_\gamma, \alpha), \sigma)$ to $(S^1, R_\alpha)$, the inverse image of 0 consists of *three* points.

A Denjoy minimal set is obtained from an irrational rotation on the circle by replacing (or 'blowing up") each element of a collection of orbits by a *pair* of orbits. For all $\gamma$ not of the type just described, $\Lambda(U_\gamma, \alpha)$ is a Denjoy minimal set. When $\gamma \in \mathbf{H}_0$, the number of orbits blown up is the same as the number of distinct orbits containing points of $Fr(U_\gamma)$. For $\gamma \in \mathbf{H} - \mathbf{H}_0$, if for all $i \neq 0$, $R_\alpha^i(0) \notin Fr(U_\gamma)$, then $\Lambda(U_\gamma, \alpha)$ is a Denjoy minimal set with countably many orbits blown up. All the infinite dimensional families we could construct had the property that some minimal set was not Denjoy.

**(3.2)** Morse and Hedlund's construction of Sturmian minimal sets corresponds to the special case $U = (0, \alpha)$. In this case, $\Lambda(U, \alpha)$ is a Denjoy minimal set with a single orbit blown up.

**(3.3)** Theorem 2 (1) states that $\Gamma$ is a lower semicontinuous function whose range is the set of closed subsets of a compact metric space. When such functions have a domain that is a Baire space, they are continuous on a dense, $G_\delta$ set (see page 114 of [C]). It seems unlikely that $\mathcal{RO}$ is a Baire space, but since $\Gamma(\mathbf{H})$ is homeomorphic to the Hilbert cube, we may apply this result to show that the map (for fixed $\alpha$)

$$\Lambda(\cdot\,, \alpha) : \Gamma(\mathbf{H}) \to \mathcal{H}(\Sigma_2)$$

is continuous at a generic point of $\Lambda(\mathbf{H})$. This result can also be obtained directly by showing that the map is, in fact, continuous at all points $\Gamma(\gamma)$ for which all points of $Fr(U_\gamma)$ are on disjoint orbits.

**(3.4)** As is perhaps obvious from Remark (3.1), when $Fr(U)$ is more complicated topologically, so is the structure of $\Lambda(U, \alpha)$ (for irrational $\alpha$). However, Theorem 2 (2b) says that for all $U \in \mathcal{RO}_0$, $\Lambda(U, \alpha)$ is uniquely ergodic. It is in fact measure isomorphic to $(S^1, R_\alpha)$. To get minimal sets with more interesting measure theoretic properties we must have $m(Fr(U)) > 0$. In this case the set $B_{U,\alpha}$ from the main construction is a zero measure, dense $G_\delta$ set in the circle. This leads one to expect that $\Lambda(U, \alpha)$ could support more than one invariant probability measure.

The results of [M-P] show that this is frequently the case. The relevant construction from that paper begins with a Cantor $K$ in the circle. The complement of $K$ is the disjoint union of open intervals. One chooses a set of labels for these open sets with each open set labeled by zero or one. The set of labels is used to construct a minimal set in the two-shift as in the main construction. If $K$ has positive measure, then for most sets of labels (in the appropriate sense) the constructed minimal set is not uniquely ergodic and has positive topological entropy.

However, the constructed minimal set can be uniquely ergodic as the following example suggested by Benjamin Weiss shows. Let $(X, f)$ be a Denjoy minimal set with intrinsic rotation number $\alpha$. Note that $(X, f)$ is both measure isomorphic to and an almost one to one extension of $(S^1, R_\alpha)$. Using results of Jewett and Kreiger we may find a zero-dimensional strictly ergodic system $(Z, h)$ that is mixing and has positive entropy. Let $(Y, g)$ be the product of the two systems. Because $(Z, h)$ and $(X, f)$ are strictly ergodic and $(Z, h)$ is mixing and $(X, f)$ has pure point spectrum, $(Y, g)$ is strictly ergodic.

Now think of Y as an extension of X. The main theorem and the remark following Theorem 4 in [F-W] imply that there is a minimal almost 1-1 extension of X , say $(\tilde{Y}, \tilde{g})$, which maps onto $(X, f)$ in such a way that the invariant measures of $(\tilde{Y}, \tilde{g})$ are in one to one correspondence with the $g$-invariant measures on $Y$. Thus $(\tilde{Y}, \tilde{g})$ is a strictly ergodic, positive entropy, almost 1-1 extension of rotation by alpha. Further, as a consequence of the method of construction in [F-W], since $X$, $Y$, and $Z$ are zero-dimensional, $\tilde{Y}$ is also.

Let $p : \tilde{Y} \to S^1$ denote the given semiconjugacy and let $\tilde{B} \subset \tilde{Y}$ be the dense $G_\delta$ set on which $p$ is injective. Pick two sets, each open and closed, with $V_0 \sqcup V_1 = \tilde{Y}$. Note that $U = (p(V_0))^c$ is a regular open set. Use the partition $\{V_0, V_1\}$ in the usual way to get a symbolic model by defining $k : \tilde{Y} \to \Sigma_2$ so that

$$(k(y))_i = \mathbf{I}_{V_1}(\tilde{g}^i(y)).$$

It is fairly straightforward to show that $\tilde{B} \subset p^{-1}(B_{U,\alpha})$ and thus, $p = \psi \circ k$ where $\psi : \Lambda(U, \alpha) \to S^1$ is the semiconjugacy constructed in the proof of Theorem 2 (2a). This implies that $\Lambda(U, \alpha)$ is a factor of $(\tilde{Y}, \tilde{g})$, and thus is strictly ergodic. Further, we may choose $V_0$ and $V_1$ so that $\Lambda(U, \alpha)$ has positive entropy. To finish, note that $m(Fr(U)) > 0$, for if not, $\Lambda(U, \alpha)$ would be measure isomorphic to the zero entropy system $(S^1, R_\alpha)$.

It would be interesting to have conditions on a regular open set with positive measure frontier that distinguish these two cases. More precisely, give necessary and sufficient conditions for the unique ergodicity of $\Lambda(U, R_\alpha)$. Another interesting question is the structure of the set of its invariant measures in the cases when $\Lambda(U, \alpha)$ is not uniquely ergodic (*cf.* [Wm]).

**(3.5)** Since each point in $\mathcal{S}(X, f)$ represents a disjoint minimal set, the size of $\mathcal{S}(X, f)$ should give some indication of the complexity of the dynamics of $f$. The topological entropy of $(X, f)$, denoted $h(X, f)$, is perhaps the most common way of measuring dynamical complexity. Corollary 0.2 shows that, at least in some cases, when the topological entropy is positive, $\mathcal{S}(X, f)$ is large. If the size of $\mathcal{S}(X, f)$ is to give a measure of dynamical complexity, the converse should be true. The next proposition shows that this is not the case, at least when the "size" of $\mathcal{S}(X, f)$ is measured by the maximal dimension of an embedded ball and $X$ is a manifold of dimension greater than two.

**Proposition 3.1.**

*(a) There exists a compact shift invariant set $\hat{\Lambda} \subset \Sigma_2$ such that $\mathcal{S}(\hat{\Lambda}, \sigma)$ is homeomorphic to the Hilbert cube and $h(\hat{\Lambda}, \sigma) = 0$.*

*(b) On any smooth manifold $M$ with dimension greater than two there exists a $C^\infty$ diffeomorphism $f$ such that $h(f) = 0$ and $\mathcal{S}(M, f)$ contains a subspace homeomorphic to the Hilbert cube.*

**Proof of (a).** Fix an irrational $\alpha$ and let $T = S^1 \times \mathbf{H}$. Define $F : T \to T$ as $F = R_\alpha \times Id$. We will do a construction analogous to the main construction, but now using the space $T$ and the map $F$. To get an open set in $T$ we use the open sets $U_\gamma$ constructed above to define

$$\hat{U} = \bigcup_{\gamma \in \mathbf{H}} U_\gamma \times \{\gamma\}.$$

Next let

$$\hat{B} = \{\beta \in T : o(\beta, F) \cap Fr(\hat{U}) = \emptyset\}$$

and define $\Phi : \hat{B} \to \Sigma_2$ so that

$$(\Phi(\beta))_i = \mathbf{I}_{\hat{U}}(F^i(\beta)).$$

Finally, let $\hat{\Lambda} = Cl(\Phi(\hat{B}))$.

Note that for fixed $\gamma$, $\Phi$ restricted to $(S^1 \times \{\gamma\}) \cap \hat{B}$ is just $\phi_{U_\gamma, \alpha}$ from the main construction and that

$$\hat{\Lambda} = Cl(\bigcup_{\gamma \in \mathbf{H}} \Lambda(U_\gamma, \alpha)).$$

Theorem 2 (2a) and (2d) imply that $\Phi$ is injective. Using an argument similar to one in the proof of Theorem 2 (2a), one gets that $\Phi^{-1}$ is uniformly continuous, and therefore has a continuous extension to a $\Psi : \hat{\Lambda} \to T$ that satisfies $\Psi \circ \sigma = F \circ \Psi$.

The variational principle (see page 190 in [W]) implies that $h(\hat{\Lambda}, \sigma) = 0$ if all ergodic measures for $(\hat{\Lambda}, \sigma)$ have metric entropy zero. If $\eta$ is an ergodic, invariant Borel probability measure for $\hat{\Lambda}$, then $\Psi_*(\eta)$ is such a measure for $(T, F)$ and so $\Psi_*(\eta)$ is Haar measure on $S^1 \times \{\gamma_0\}$ for some $\gamma_0$. This implies that $\eta$ is supported on $\Psi^{-1}(S^1 \times \{\gamma_0\})$. Once again, using an argument virtually identical to one in the proof of Theorem 2 (2b), one obtains $\eta = \lambda(U_{\gamma_0}, \alpha)$. This measure with the shift is measure isomorphic to rotation on the circle by $\alpha$ and therefore has zero metric entropy, as required. Note that the argument just given also shows that $\mathcal{S}(\hat{\Lambda}, \sigma)$ is in fact homeomorphic to the Hilbert cube, $\mathbf{H}$.

**Proof of (b).** We first construct the map on the space $P = D^2 \times [-1, 1]$, where $D^2$ is a closed two-dimensional disk. Let $h : D^2 \to D^2$ be a Smale horseshoe, *i.e.* $h$ is a $C^\infty$-diffeomorphism whose nonwandering set consists of the union of a finite number of fixed points and a set $\Omega$ on which the dynamics are conjugate to the full two-shift. The compact invariant set $\hat{\Lambda}$ constructed in the proof of (a) is embedded in $\Omega$ by the conjugacy. Call this embedded set $\bar{\Lambda}$.

Next, let $h_t$ for $t \in [-1, 1]$ be an isotopy with $h_{-1} = Id$, $h_0 = h$, and $h_1 = Id$. Further, $h_t$ restricted to the boundary of $D^2$ should be the identity for all $t$. Now pick a $C^\infty$-function $w : P \to \mathbf{R}$ with $w \geq 0$ and $w^{-1}(0) = \partial P \sqcup (\bar{\Lambda} \times \{0\})$. Let $g : P \to P$ be the time one map of the flow generated by the vector field $w(u)\frac{\partial}{\partial z}$, where $u = (x, y, z)$ is a point in $P$. Now let $f = g \circ (h_t \times Id)$. By construction, the nonwandering set of $f$ is $\partial P \sqcup (\bar{\Lambda} \times \{0\})$ and thus $h(f) = 0$. Since each point on $\partial P$ is a fixed point for $f$, $\mathcal{S}(P, f)$ is homeomorphic to $\mathcal{S}(\bar{\Lambda}, \sigma) \sqcup \partial P$, which in turn, is homeomorphic to $\mathbf{H} \sqcup \partial P$.

To obtain the result on a general manifold of dimension three or higher, embed a copy of $(P, f)$ in it and extend $f$ by the identity on the rest of the manifold. □

**Remarks**

**(3.6)** This proposition leaves open the possibility of a converse to Corollary 0.2 in dimension 2. In this dimension there are a number of results that show that the existence of certain types of zero entropy invariant sets can imply that a homeomorphism has positive topological entropy. For example, if an orientation-reversing homeomorphism of a compact surface of genus $g$ has periodic orbits with $g + 2$ distinct odd periods, then it has positive entropy ([B-F], [H]). For orientation-preserving homeomorphisms there are restrictions on the periods that occur in zero entropy maps given in [S]. Even a single period orbit can imply positive entropy if the isotopy class on its complement is nontrivial ([Bd]). These results give credence to the conjecture that for a manifold $M$ of dimension 2, if $f : M \to M$ is a homeomorphism and $\mathcal{S}(M, f)$ contains a topological ball of dimension 3, then $h(f) > 0$.

**(3.7)** It was noted in the introduction that the existence of a Hilbert cube of strictly ergodic sets can often be viewed as a manifestation of a standard topological fact, namely, the Hilbert cube is the continuous surjective image of the Cantor set. For concreteness, let $f : M \to M$ be a homeomorphism with an invariant set $\bar{\Lambda}$ with $(\bar{\Lambda}, f)$ conjugate to $(\hat{\Lambda}, \sigma)$, where $\hat{\Lambda}$ is the set constructed in the proof of Proposition 3.1 (b). Using the conjugacy, the proof of Proposition 3.1 (b), and Theorem 2 (2b) one gets that for each $x \in \bar{\Lambda}$, $Cl(o(x, f))$ supports a single invariant probability measure which is $c_*(\lambda(U_{\gamma(x)}, \alpha))$ for the appropriate $\gamma(x)$. Further, the map $x \mapsto c_*(\lambda(U_{\gamma(x)}, \alpha))$ is continuous. (More formally, this map is

$$x \mapsto c_*(\lambda(\Gamma(\pi_2(\Psi(x))), \alpha))$$

where $\pi_2 : S^1 \times \mathbf{H} \to \mathbf{H}$ is the projection). The domain of this map is the invariant Cantor set $\bar{\Lambda}$ and its image is $\lambda(\Gamma(\mathbf{H}), \alpha)$, which is homeomorphic to the Hilbert cube, $\mathbf{H}$.

**(3.8)** The construction in the proof of 3.1 (b) can be used to embed any compact shift invariant subset of $\Sigma_2$ as the only "interesting" dynamics in a three-dimensional diffeomorphism. It is reminiscent of Schweitzer's construction of $C^1$-counterexample to the Seifert conjecture ([Sc]).

**Section 4: Intrinsic and extrinsic rotation numbers.** In the Section 1 it was noted that abstract Denjoy minimal sets have well-defined intrinsic rotation numbers. The next proposition specializes some previous results to the case of fixed intrinsic rotation number.

**Proposition 4.1.** *Fix an irrational $\alpha$ and let $\mathcal{D}_\alpha(\Sigma_2, \sigma)$ denote the set of Denjoy minimal sets in the shift with intrinsic rotation number $\alpha$.*

*(a) When given the weak topology, the space $\mathcal{D}_\alpha(\Sigma_2, \sigma)$ contains topological balls of dimension $n$ for all natural numbers $n$.*

*(b) The set of points that are members of Denjoy minimal sets with intrinsic rotation number $\alpha$ is dense in $\Sigma_2$.*

*(c) If $(D, \sigma)$ is a Denjoy minimal set with intrinsic rotation number $\alpha$, then $D = \Lambda(U, \alpha)$ for some regular open set $U$ with $m(Fr(U)) = 0$. Consequently, $\mathcal{D}_\alpha(\Sigma_2, \sigma) \subset \lambda(\mathcal{RO}_0, \alpha)$.*

**Proof of Proposition 4.1.** When $U$ is asymmetric and $\Lambda(U, \alpha)$ is a Denjoy minimal set, it has intrinsic rotation number $\alpha$. This follows from Theorem 2 (2b) (and its proof). Thus to prove (a) we need only note that the proof of Theorem 0.1 began with a statement, " Fix an irrational $\alpha$". The proof of Proposition 0.3 (b) contains a similar statement, so that proof proves (b).

To prove (c), note that by definition, there exists a conjugacy $c : D \to Y$ where $Y$ is the minimal set in a Denjoy example $g : S^1 \to S^1$ with rotation number $\alpha$. It is a standard fact that there exists a semiconjugacy $h$ of $(S^1, g)$ to $(S^1, R_\alpha)$ with the properties that $h$ is injective on a set that is dense in $Y$ and the lift of $h$ is weakly order preserving, *i.e.* $x < y$ implies $\tilde{h}(x) \leq \tilde{h}(y)$.

Now let $p = h \circ c$ and $U = (p(C_0))^c$. Since $C_0$ is compact in $\Sigma_2$, $U$ is open. Further, the properties given above imply that $U^* = (p(C_1))^c$ and $p(C_0) \cap p(C_1) = Fr(U) = Fr(U^*)$. Thus using a fact from Section 1, $U$ is a regular open set, and by construction, $\Lambda(U, \alpha) = D$. Since $p(C_0) \cap p(C_1)$ is at most countable, $m(Fr(U)) = 0$. $\square$

These results, of course, also hold for homeomorphisms with a full two-shift embedded in their dynamics. In this case, however, one is perhaps more interested in *extrinsic* properties of invariant sets, *i.e.* properties associated with how the sets are embedded in the manifold. Perhaps the simplest such extrinsic property is the extrinsic rotation number, and the simplest case in which this can be defined is for a homeomorphism of the annulus.

If $f : A \to A$ is a homeomorphism of the annulus and $z \in A$, define the rotation number of $z$ under $f$ as

$$\rho(z) = \lim_{n \to \infty} \frac{\pi_1(\tilde{f}^n(\tilde{z})) - \pi_1(\tilde{z})}{n},$$

if the limit exists. Here $\tilde{f} : \mathbf{R} \times [-1, 1] \to \mathbf{R} \times [-1, 1]$ and $\tilde{z}$ are lifts of $f$ and $z$, respectively, and $\pi_1 : \mathbf{R} \times [-1, 1] \to \mathbf{R}$ is the projection. Note that the rotation number is only defined modulo 1 as it depends on the choice of lift.

If $D \subset A$ is a Denjoy minimal set under $f$, then it is uniquely ergodic. Thus for all $z \in D$, $\rho(z) = \int r(z)\, d\mu$, where $\mu$ is the the unique invariant probability measure of $(D, f)$ and $r : S^1 \to \mathbf{R}$ is the map that lifts to $\pi_1 \circ \tilde{f} - \pi_1$. This number will be called the *extrinsic rotation number* of $(D, f)$.

The Denjoy minimal sets constructed by Mather in [M] have monotonicity properties that imply that their extrinsic and intrinsic rotation numbers are rationally related. For Denjoy minimal sets in a general homeomorphism of the annulus this will not be the case. As a specific example, we will consider homeomorphisms $f : A \to A$ that have a *rotary horseshoe* (*cf.* [H-H2]) A picture of the lift of such a map is shown in Figure 1. The dotted vertical lines are the boundaries of fundamental domains.

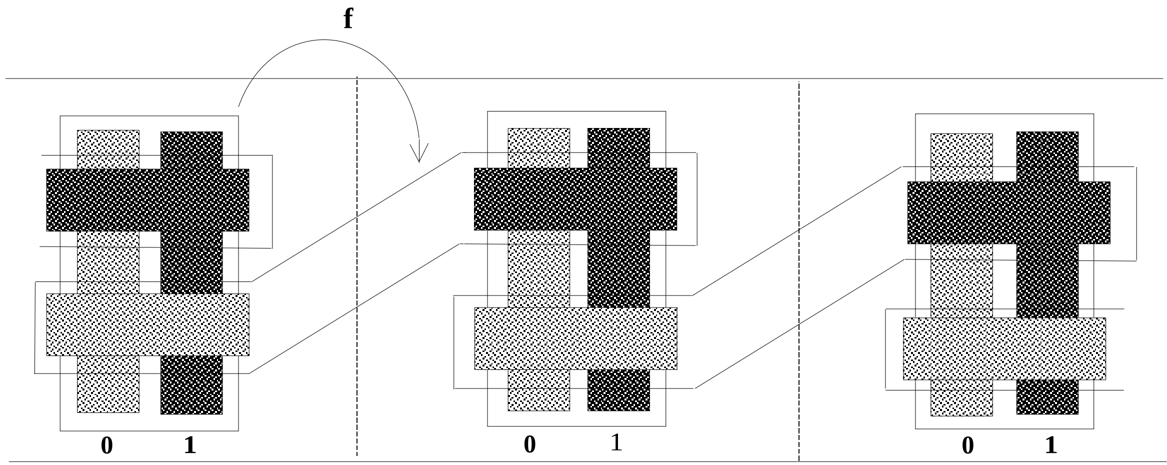


Figure 1: The lift of a rotary horseshoe.

A map contains a rotary horseshoe if it has a compact invariant set $\Omega$ that is conjugate to the full two-shift. The conjugacy $c : \Omega \to \Sigma_2$ is required to have the property that for $z \in \Omega$ the first element in $c(z)$ is 1 if and only if $\tilde{f}$ moves $\tilde{z}$ (approximately) one fundamental domain to the right. More precisely, for $z \in \Omega$ it is required that

$$\rho(z) = \lim_{N\to\infty} \sum_{i=0}^{N} \frac{\mathbf{I}_{C_1}(\sigma^i(c(z)))}{(N+1)}.$$

Thus $\rho(z)$ is the asymptotic average number of ones in the sequence $c(z)$.

We are now almost in a position to state a result about the existence of Denjoy minimal sets with given intrinsic and extrinsic rotation number. For an annulus homeomorphism $f$, let $\mathcal{D}_{\alpha,\beta}(A, f)$ denote the set of all Denjoy minimal sets for $f$ with intrinsic rotation number $\alpha$ and extrinsic rotation number $\beta$.

**Proposition 4.2.** *If a homeomorphism $f : A \to A$ has a rotary horseshoe, then for all irrational $\alpha$, and all $\beta \in S^1$, $\mathcal{D}_{\alpha,\beta}(A, f)$ contains topological balls of dimension $n$ for all natural numbers $n$.*

**Proof of Proposition 4.2.** If for a given $U \in \mathcal{RO}_0$ and irrational $\alpha$, $\Lambda(U, \alpha)$ is a Denjoy minimal set, then the comments above Lemma 1 and unique ergodicity imply that for all $s \in \Lambda(U, \alpha)$,

$$\lim_{N\to\infty} \sum_{i=0}^{N} \frac{\mathbf{I}_{C_1}(\sigma^i(s))}{(N+1)} = \lambda(U, \alpha)[C_1] = m(U).$$

This implies that the corresponding Denjoy minimal set in the annulus has extrinsic rotation number equal to $m(U)$. To finish the proof, one need only imitate the proof of Theorem 0.1 using a family $U_\gamma$ that satisfies $m(U_\gamma) = \beta$, for all $\gamma$. □

Note that the case of rational $\beta$ is included in this result. This means that large dimensional balls of Denjoy minimal sets with a given *rational* extrinsic rotation number are present in the dynamics.

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