Local Connectivity of Julia Sets: Expository Lectures.

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Contents

Introduction

The following notes provide an introduction to recent work of Branner, Hubbard and Yoccoz on the geometry of polynomial Julia sets. They are an expanded version of lectures given in Stony Brook in Spring 1992. I am indebted to help from the audience.

Section 1 describes unpublished work by J.-C. Yoccoz on local connectivity of quadratic Julia sets. It presents only the "easy" part of his work, in the sense that it considers only non-renormalizable polynomials, and makes no effort to describe the much more difficult arguments which are needed to deal with local connectivity in parameter space. It is based on second hand sources, namely Hubbard [Hu1] together with lectures by Branner and Douady. Hence the presentation is surely quite different from that of Yoccoz.

Section 2 describes the analogous arguments used by Branner and Hubbard [BH2] to study higher degree polynomials for which all but one of the critical orbits escape to infinity. In this case, the associated Julia set J is never locally connected. The basic problem is rather to decide when J is totally disconnected. This Branner-Hubbard work came before Yoccoz, and its technical details are not as difficult. However, in these notes their work is presented simply as another application of the same geometric ideas.

Chapter 3 complements the Yoccoz results by describing a family of examples, due to Douady and Hubbard (unpublished), showing that an infinitely renormalizable quadratic polynomial may have non-locally-connected Julia set. An Appendix describes needed tools from complex analysis, including the Grötzsch inequality.

We will assume that the reader is familiar with the basic properties of Julia sets and the Mandelbrot set. (For general background, see for example [Be], [Br2], [D1], [D2], [EL], [L1], as well as the brief outline in §3.) In particular, we will make use of ***external rays*** for a polynomial Julia set $J(f) \subset \mathbf{C}$. (Compare [DH1], [DH2], [M2], [GM].)

§1. Local Connectivity of Quadratic Julia Sets (following Yoccoz).

This section will prove the following.

Theorem 1. *If $f_c(z) = z^2 + c$ is a quadratic polynomial such that:*
(1) the Julia set $J(f_c)$ is connected,
(2) both fixed points are repelling, and
(3) f_c is not renormalizable[1]
then $J(f_c)$ is locally connected.

In terms of the familiar parameter space picture for the family of quadratic maps $f_c(z) = z^2 + c$, Condition (1) says that the parameter value c belongs to the Mandelbrot set M, while (2) says that c does not belong to the closure of the central region bounded by the cardioid, and (3) says that c does not belong to any one of the many small copies of M which are scattered densely around the boundary of M. (Compare Figure 1.)

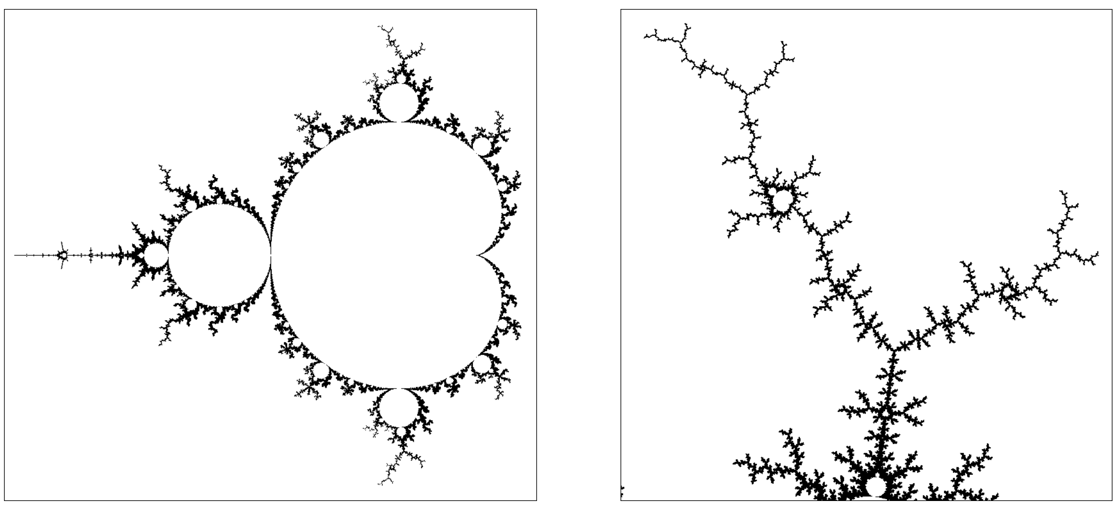

Figure 1. Boundary of the Mandelbrot set M, and a detail of the one-third limb showing several small copies of M.

Remarks: The proof will give much more, since it will effectively describe the Julia set by a new kind of symbolic dynamics. With a little more work, the Yoccoz method can also deal with the finitely renormalizable case. Since the case of a map with attracting or parabolic fixed point had been understood much earlier [DH2], we see that Conditions (2) and (3) can be actually replaced by the following weaker pair of conditions:

($2'$) f has no irrationally indifferent periodic points (Cremer or Siegel points), and

($3'$) f is not *infinitely* renormalizable.

These modified Conditions (1), ($2'$) and ($3'$) are all essential. In fact,

[1] For the definition of renormalizability, see Figure 10 and the associated discussion.

(1) : For $c \notin M$ the Julia set $J(f_c)$ is a Cantor set, which is certainly not locally connected.

(2′) : Sullivan and Douady showed that a polynomial Julia set with a Cremer point is never locally connected. (Compare [Su], [D1], [M2]. For a more explicit description of non local connectivity, see [Sø].) Similarly, Herman has constructed quadratic polynomials with a Siegel disk having no critical point on its boundary. The corresponding Julia set cannot be locally connected. (See [He,§17.1], [D1,II,5], [D4].)

(3′) : In §3, following unpublished work by Douady and Hubbard, we will describe infinitely renormalizable polynomials for which J is not locally connected.

Thus the sharpened version of Theorem 1 comes fairly close to deciding exactly which quadratic polynomial Julia sets are locally connected. Yoccoz has also proved a corresponding result in parameter space: For c in the Mandelbrot set M, if the polynomial $f_c(z) = z^2 + c$ is not infinitely renormalizable, then M is locally connected at c. For varied proofs of this more difficult result, see [Hu2], [HF], [K].

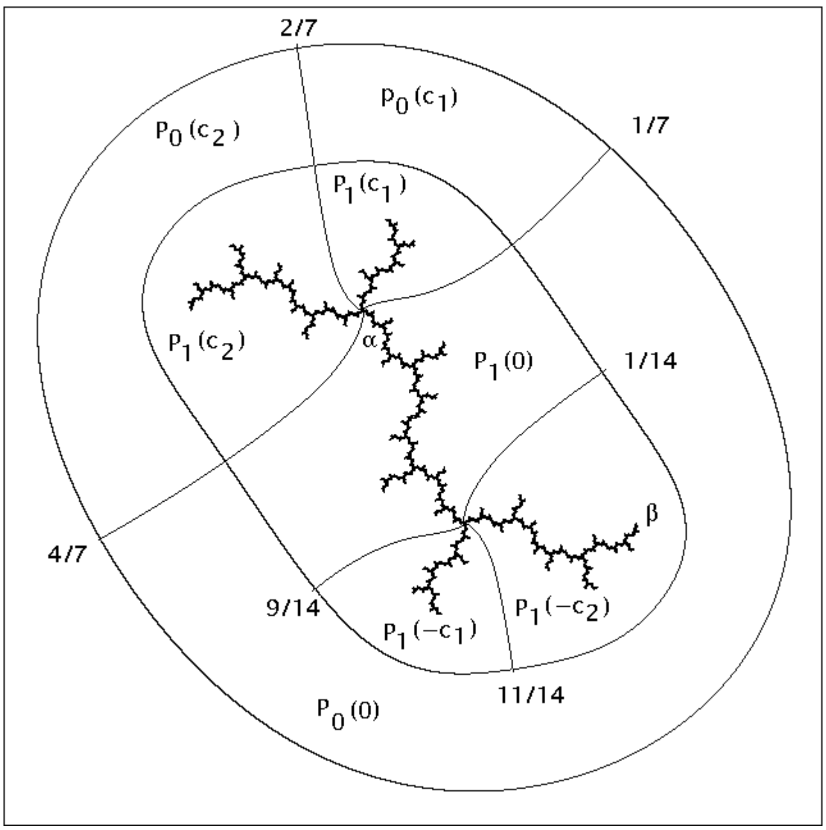

Figure 2. Julia set for $f(z) = z^2 + i$, showing the Yoccoz puzzles of depth zero and one. Here $q = 3$. The $1/7$, $2/7$ and $4/7$ rays land at the fixed point α.

The Yoccoz jigsaw puzzle. The proof of Theorem 1 begins as follows. If a connected quadratic Julia set has two repelling fixed points, then one fixed point (to be called β) is the landing point of the zero external ray, and the other (called α) is the landing point of a cycle of q external rays, where $q \geq 2$. (Compare [M2], [Pe].) Let

$0 = c_0 \mapsto c_1 \mapsto \cdots$ be the critical orbit. The *Yoccoz puzzle* of *depth zero* consists of q pieces $P_0(c_0)$, $P_0(c_1)$, ..., $P_0(c_{q-1})$ which are obtained by cutting the region $G \le 1$ along the q external rays landing at α. Here G is the canonical potential function. The various pieces have been labeled so that each $P_0(c_i)$ contains the post-critical point $c_i = f^{\circ i}(0)$.

Inductive Construction: If $P_d^{(1)}, \ldots, P_d^{(m)}$ are the puzzle pieces of depth d, then the connected components of the sets $f^{-1}(P_d^{(i)})$ are the puzzle pieces $P_{d+1}^{(j)}$ of depth $d+1$. As an example, there are always $2q-1$ pieces of depth one. These consist of q pieces $P_1(c_0)$, $P_1(c_1)$, ..., $P_1(c_{q-1})$ which touch at the fixed point α, together with $q-1$ additional pieces $P_1(-c_1)$, ..., $P_1(-c_{q-1})$ which touch at the pre-image $-\alpha$.

The Main Problem: Let $z \in J(z)$ be any point whose forward orbit never hits[1] the fixed point α, and let $P_d(z)$ be the unique puzzle piece of depth d which contains this point z, so that

$$P_0(z) \supset P_1(z) \supset P_2(z) \supset \cdots .$$

Does the intersection $\bigcap_d P_d(z)$ consist of the single point z?

The associated annuli. Let $P_d(z) \supset P_{d+1}(z)$ be the puzzle pieces of two consecutive depths containing some given $z \in J(f)$. If we are lucky, the smaller puzzle piece $P_{d+1}(z)$ will be contained in the interior of $P_d(z)$. In this case the difference

$$A_d(z) = \text{interior}(P_d(z)) \smallsetminus P_{d+1}(z)$$

is an annulus, whose modulus $\operatorname{mod} A_d(z)$ is a positive real number. (Compare the Appendix.) For example in Figure 2 the annulus $A_0(-c_1)$ has positive modulus. On the other hand, it may happen that $P_{d+1}(z)$ intersects the boundary of $P_d(z)$. In this case we describe $A_d(z)$ as a ***degenerate annulus***, and define its modulus to be zero. For example, in Figure 2 the "annulus" $A_0(0)$ around the critical point is degenerate.

Modified Main Problem (Branner and Hubbard) : Given a point $z \in J(f)$, is the sum $\sum_d \operatorname{mod} A_d(z)$ infinite? If so, using the Grötzsch inequality, it is not difficult to prove that the intersection $\bigcap P_d(z)$ consists of the single point z. (See Appendix.)

Note that f carries any puzzle piece $P_d(z)$ of depth $d > 0$ onto $P_{d-1}(f(z))$. This map $P_d(z) \to P_{d-1}(f(z))$ is either a conformal isomorphism or a two-fold branched covering according as $P_d(z)$ does or does not contain the critical point.

Definition. We describe an annulus as being either *semi-critical* or *critical* or *off-critical* according as the critical point belongs to the annulus itself or to the bounded or the unbounded component of its complement. Thus, in the schematic picture (Figure 3 left), the annulus $A_0(z)$ is critical, while $A_1(z)$ is semi-critical, and $A_2(z)$ is off-critical. Let $A_d(z)$ be the annulus of depth $d > 0$ in the Yoccoz puzzle surrounding a point $z \in J(f)$. If $A_d(z)$ is critical or off-critical, then evidently f maps $A_d(z)$ onto $A_{d-1}(f(z))$:

by an unramified two-fold covering in the critical case,

by a conformal isomorphism in the off-critical case.

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[1] The slight modifications of the argument needed to deal with the case $f^{\circ d}(z) = \alpha$ are described near the end of this section.

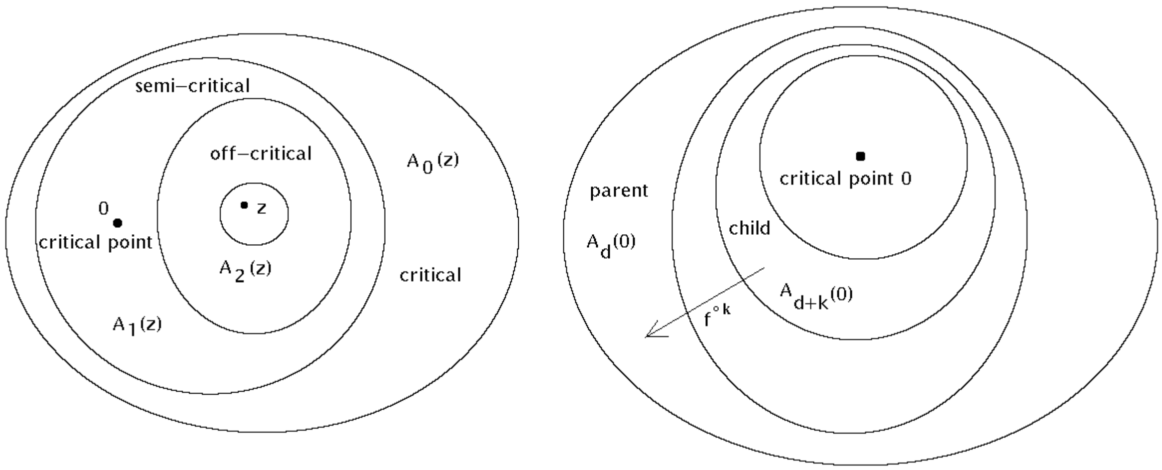

Figure 3. Critical, semi-critical and off-critical annuli (left), and a "child" (right).

In the first case it follows that $\mod A_d(z) = \frac{1}{2} \mod A_{d-1}(f(z))$, while in the second case $\mod A_d(z) = \mod A_{d-1}(f(z))$. In the semi-critical case, the appropriate statement is that $\mod A_d(z) > \frac{1}{2} \mod A_{d-1}(f(z))$. The proof is more complicated. (Compare Problem 1-2 at the end of this section.) Even in the semi-critical case, it is easy to check that: *$A_d(z)$ has positive modulus if and only if $A_{d-1}(f(z))$ has positive modulus.*

Definition (Figure 3 right). The critical annulus $A_{d+k}(0)$ in the Yoccoz puzzle will be called a *child* [1] of the critical annulus $A_d(0)$ if and only if $A_{d+k}(0)$ is an unramified two-fold covering of $A_d(0)$ under the map $f^{\circ k}$.

Note that the modulus of such a child is always exactly half the modulus of the parent. Our strategy for solving the Modified Main Problem can now be summarized as follows:

> *Find a critical annulus of positive modulus, and prove that it has so many descendents, children and grandchildren and so on, that the modulus sum is infinite.*

Definition. The *tableau*[1] associated with an orbit $z_0 \mapsto z_1 \mapsto z_2 \mapsto \cdots$ in $J(f) \smallsetminus \{\alpha\}$ is an array with one column associated with each z_i and one row associated with each depth in the Yoccoz puzzle. We will draw a solid vertical line at depth d in the j-th column to indicate that the annulus $A_d(z_j)$ in the Yoccoz puzzle coincides with the critical annulus $A_d(0)$. A double vertical line will indicate that $A_d(z_j)$ is semi-critical, and no line at all will indicate that it is off-critical. Diagonal arrows (pointing north-east) correspond to iterates of the map f. Thus annuli along the same diagonal line either all have zero modulus or all have non-zero modulus.

This concept of tableau, due to Branner and Hubbard, provides a language which we will use to describe the Yoccoz proof. (It is not the language which Yoccoz himself uses.)

[1] Our terminology is based on [Hu1], but with several modifications. Thus in Hubbard's terminology, the level $d+k$ is called a "legitimate child" of the level d. Similarly, we have replaced Hubbard's "marked grid" by *tableau*, and his "noble" by *excellent*.

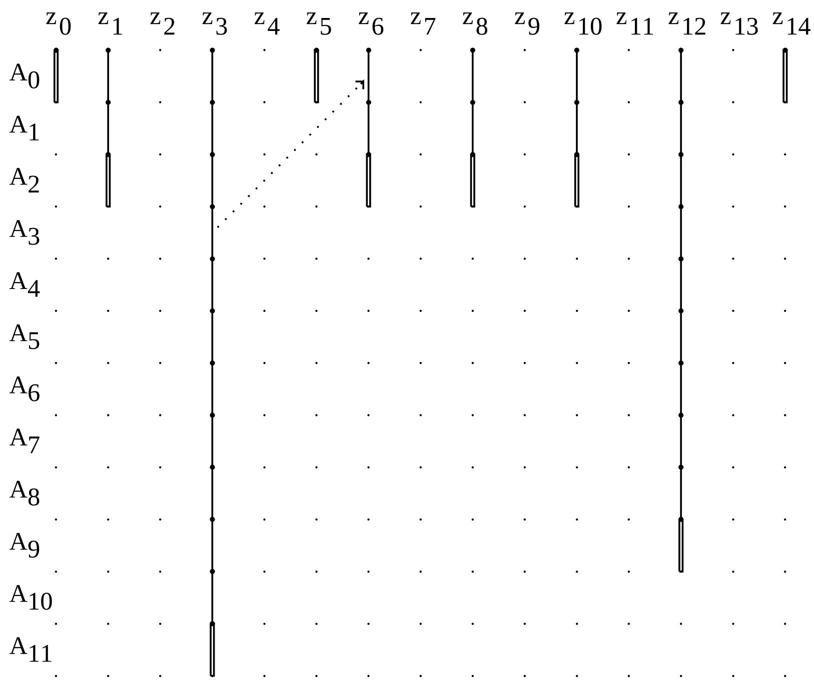

Figure 4. An example: the tableau associated with the orbit of the point $z_0 = 1$ *for the map* $f(z) = z^2 - 1.6$ *. Here we see by following the diagonal arrow that the critical annulus* $A_3(z_3) = A_3(0)$ *is an unramified two-fold covering of the critical annulus* $A_0(z_6) = A_0(0)$ *. In particular, the critical annulus at depth* 3 *is a child of the critical annulus at depth* 0 *. Similarly, the critical annuli at depths* 4 , 6 , 8 , 10 *are children of the critical annulus of depth* 1 *.*

We are particularly interested in the tableau of the critical orbit

$$0 = c_0 \mapsto c_1 \mapsto c_2 \mapsto \cdots ,$$

which has vertical line segments going all the way down in column zero. *To simplify the discussion, we will always assume that the critical orbit is disjoint from the fixed point* α *, so that this critical tableau is well defined.* (If the critical orbit ends at α , then we are in the post-critically finite case, and local connectivity can be established by other methods. Compare [DH2].)

> **First tableau rule:** *Every column of a tableau is either all critical, or all off-critical, or has exactly one semi-critical depth and is critical above and off-critical below.*

(Thus each column, corresponding to a point $z_j \in J(f)$, can be completely described by a single number, the "*semi-critical depth*" $-1 \le \mathrm{scd}(z_j) \le \infty$, defined by the condition that $P_d(z_j) = P_d(0) \iff d \le \mathrm{scd}(z_j)$. A large value of $\mathrm{scd}(z_j)$ means that z_j is very close to the critical point.)

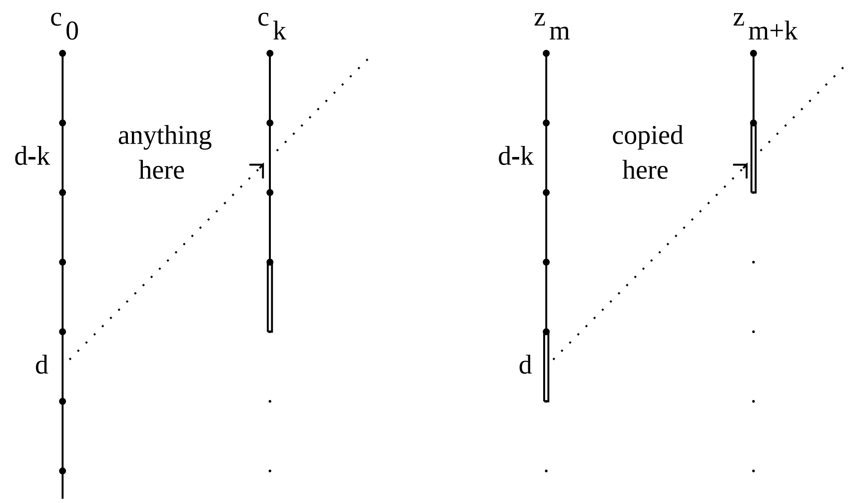

Figure 5. Illustration for the second and third tableau rules, with the critical tableau on the left, and the tableau for $z_0 \mapsto z_1 \mapsto \cdots$ on the right.

Now let us compare the tableau of the critical orbit

$$0 = c_0 \ \mapsto \ c_1 \ \mapsto \ c_2 \ \mapsto \ \cdots$$

with the tableau of some given orbit $z_0 \mapsto z_1 \mapsto \cdots$ in $J(f)$. (The case $z_0 = c_0$ is not excluded.) If the tableau of $\{z_j\}$ is critical or semi-critical at depth d in column m, draw a line "north-east" from this critical or semi-critical annulus, as indicated by the dotted line on the right, and draw a corresponding line north-east from depth d of column zero in the critical tableau.

Second tableau rule: *Everything strictly above the diagonal line on the left must be copied above the diagonal line on the right.*

The proofs of these two rules are easily supplied. □

Now suppose that the critical annulus of depth d is a child of the critical annulus of depth $d-k$, as indicated in the figure, and suppose that the tableau of $\{z_j\}$ is semi-critical at depth d of column m.

Third tableau rule: *Following the diagonal arrow from this semi-critical annulus of depth d in the tableau of $\{z_j\}$, we must reach a semi-critical annulus at depth $d-k$, as illustrated.*

Proof. According to the hypothesis, $f^{\circ k}$ maps $A_d(0)$ onto $A_{d-k}(0)$, where the point z_m is an element of this annulus $A_d(0)$. Therefore $f^{\circ k}(z_m) = z_{m+k}$ must belong to $A_{d-k}(0)$. □

Definition. We will say that a critical annulus $A_d(0)$ is *excellent* if it contains no post-critical points, or equivalently if there is no semi-critical annulus in the d-th row of the critical tableau.

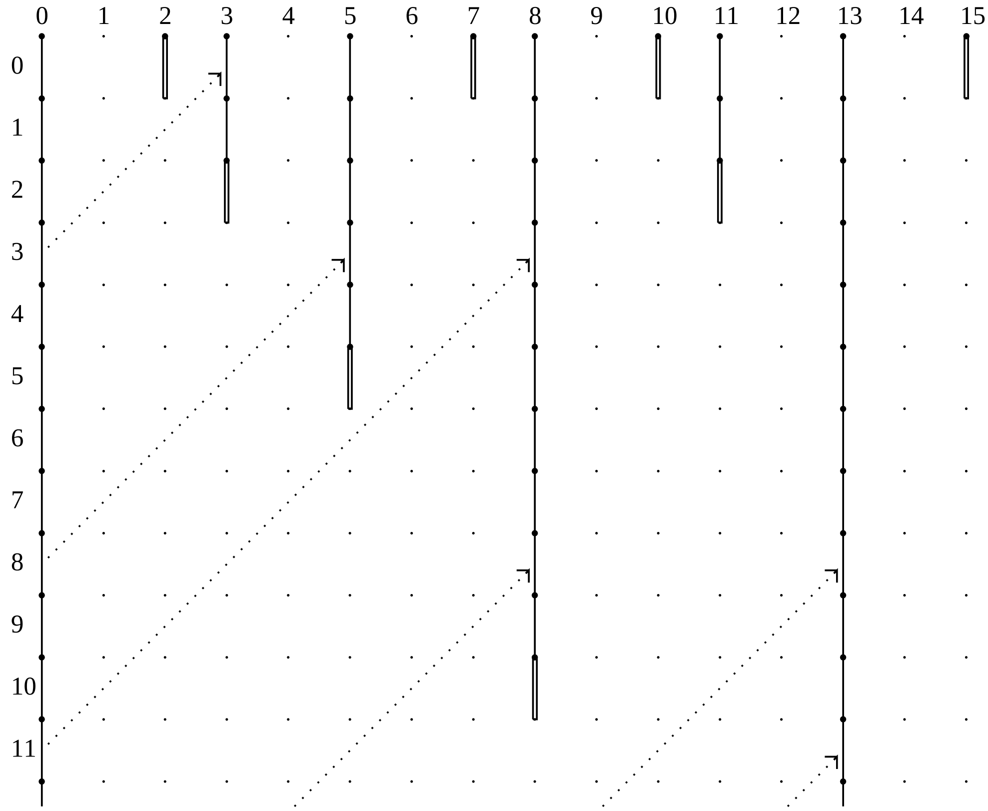

Figure 6. Another example: the Fibonacci tableau. (Compare [BH2].) Here the closest recurrences of the critical orbit come after a Fibonacci number of iterations:

$$1\,,\ 2\,,\ 3\,,\ 5\,,\ 8\,,\ 13\,,\ 21\,,\ 34\,,\ \ 55\,,\ \ldots\ .$$

If u_n is the n-th Fibonacci number, then the semi-critical annulus for column number u_n occurs at depth $u_{n+1}-3$. The diagonal dotted lines illustrate the genealogies

$$\begin{array}{ccccccc} 0 & \leftarrow & 3 & \leftarrow & 8 & \leftarrow & \cdots \\ & & & \nwarrow & & & \\ & & & & 11 & \leftarrow & \cdots \end{array} \qquad (arrow\ from\ child\ to\ parent).$$

In Figure 6, note that the critical annuli at depths 1, 3, 4, 6, 7, 8, 9, ... are excellent. Each one has exactly two children, which are also excellent. On the other hand, 0, 2, 5, 10, 18 (Fibonacci numbers minus three) are not excellent. Each of these has only one child; however this child is excellent.

To see the force of the three tableau rules, suppose that we try to modify this Fibonacci tableau by changing just one column. For example, suppose that we place the semi-critical annulus for column 5 at depth 3 or 4 or at depth ≥ 6, instead of at depth 5, without changing columns 0 through 4. **Exercise:** Show that the tableau rules would then force column 8 to end already at the semi-critical depth 0 or 1 or 2 respectively.

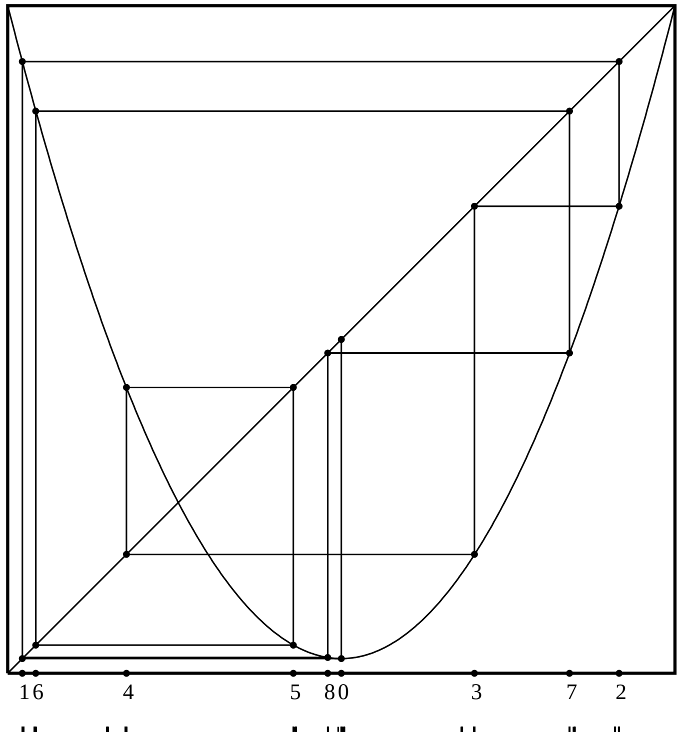

Figure 7. Graph of the unimodal map $x \mapsto x^2 - 1.8705286\cdots$ which realizes the Fibonacci tableau. (Compare [ML].) The first eight points on the critical orbit are marked. The critical orbit closure is a rather thin Cantor set, which is plotted underneath the graph. (Compare Problem 1-7.)

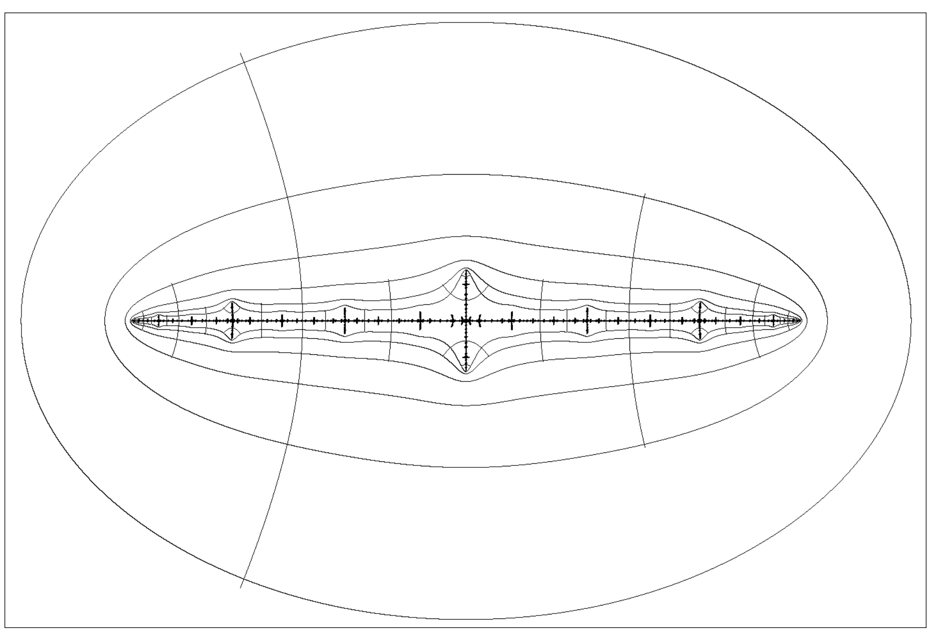

Figure 8. The Yoccoz puzzle at depths zero through five for this quadratic Fibonacci map, drawn to the same scale. Note that the critical pieces at any depth are the biggest ones.

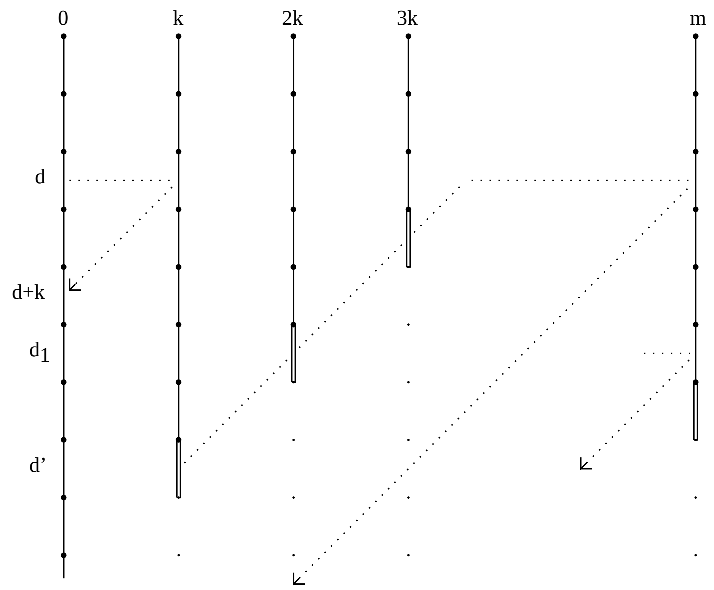

Figure 9. Finding Children.

Recall that a critical annulus $A_d(0)$ is ***excellent*** if the corresponding tableau row contains no semi-critical entries, or equivalently if the annulus $A_d(0)$ contains no post-critical points. **Definition:** We will say that the critical tableau is ***recurrent*** if there are columns with $k > 0$ which go arbitrarily far down; and that it is ***periodic*** if at least one such column goes all the way down, so that $P_d(0) = P_d(c_k)$ for all depths d. (Compare Lemma 2 below.)

Lemma 1: *If the critical tableau is recurrent but not periodic, then:*
- (a) *Every critical annulus has at least one child.*
- (b) *Every excellent critical annulus has at least two children.*
- (c) *Every child of an excellent parent is excellent.*
- (d) *Every only child is excellent.*

Proof of (a). Start on the left at depth d, and march to the right until we first meet another critical annulus (= solid vertical line), say at column k. Now marching diagonally "south-west", we find the first child at depth $d + k$.

Proof of (b). By hypothesis, the k-th column cannot be critical all the way down. Hence it must be semi-critical at some depth $d' > n$. Starting at column k and depth d', proceed diagonally right (north-east). By the tableau rules, column $2k$ must be semi-critical at depth $d' - k$. Similarly column $3k$ must be semi-critical at depth $d' - 3k$, and so on, until we again reach depth d. Furthermore, as we follow this diagonal, we do not meet any other critical or semi-critical annuli. In particular, at the point where we reach depth d, there cannot be a critical or semi-critical annulus. (The hypothesis that d is excellent comes in at this point. Compare Figure 6.) Now let us again march to the right at depth d until we reach a critical annulus, say at column m. Again turning 135° and proceeding diagonally south-west, we cannot meet any critical or semi-critical annulus until we are back at column 0. In this way we prove that the critical annulus of depth $m + d$ is also a child of d.

Proof of (c). This is clear, since if $f(A_{d+k}(0)) = A_d(0)$ where $A_{d+k}(0)$ contains a post-critical point, then $A_d(0)$ does also.

Proof of (d), by contradiction. Consider a child d' which is not excellent, and let $d_1 = n' - k$ be the parent. Then for some $k' \geq k$ the k'-th column has semi-critical annulus at depth d'. (The case $k' = k$ is illustrated.) Following the diagonal up from column k' and depth d', we must meet a semi-critical annulus at depth d_1 by the third tableau rule. (Thus the parent is not excellent.) Now proceed to the right at depth d_1 until we meet a critical annulus, say at column m. Then it follows as above that $d_1 + m$ is a second child; hence d' was not an only child. □

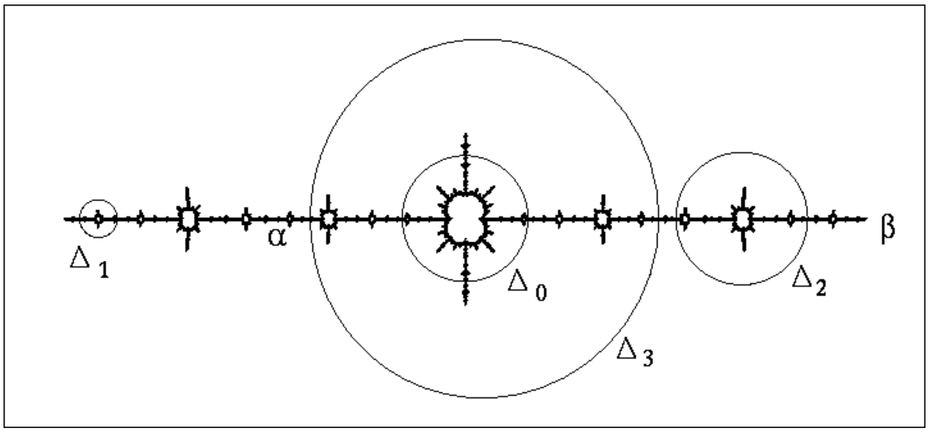

Figure 10. Julia set for $f(z) = z^2 - 1.75$, with renormalization period $p = 3$. (The parameter value $c = -1.75$ is the root point of a small copy of the Mandelbrot set.) The filled Julia set $K(f^{\circ 3}|\Delta_0)$ is a topological disk bounded by a "cauliflower".

Definition. (Compare [DH3].) A quadratic polynomial f is *renormalizable* if there exists an integer $p \geq 2$ and a closed topological disk Δ_0 around the critical point with the following properties:

(1) Δ_0 should be centrally symmetric about the critical point so that its image $\Delta_1 = f(\Delta_0)$ is also a closed topological disk.

(2) f should induce conformal isomorphisms

$$\Delta_1 \xrightarrow{\cong} \Delta_2 \xrightarrow{\cong} \cdots \xrightarrow{\cong} \Delta_p ,$$

where $\Delta_i = f^{\circ i}(\Delta_0)$. In particular, the critical point should be disjoint from $\Delta_1 \cup \cdots \cup \Delta_{p-1}$.

(3) However, the disk Δ_p should contain Δ_0 in its interior.

(4) Finally, the entire orbit of the critical point under the map $f^{\circ p}$ should be contained in the original disk Δ_0.

(For further information, see the discussion of "polynomial-like mappings" in §2, and the discussion of "tuning" in §3.) We will call p the *renormalization period*. If these conditions are satisfied, then the *filled Julia set* $K(f^{\circ p}|\Delta_0)$ can be defined as the compact connected set consisting of points whose orbits remain in Δ_0 under all iterations of $f^{\circ p}$. This is a proper subset of the filled Julia set $K(f)$ of the original map f. We can now state two basic lemmas.

Lemma 2. *If the critical tableau associated with f is periodic, then f is renormalizable. More precisely, if $P_d(c_p) = P_d(0)$ for all depths d, and therefore $P_d(c_{i+p}) = P_d(c_i)$ for all i and d by the Second Tableau Rule, then f is renormalizable of period p.*

The converse is also true, but will not be proved here.

Lemma 3. *If the critical orbit lies completely within the union*

$$P_1(c_0) \cup P_1(c_1) \cup \cdots \cup P_1(c_{q-1})$$

of those puzzle pieces of depth one which touch the fixed point α, then f is renormalizable of period q.

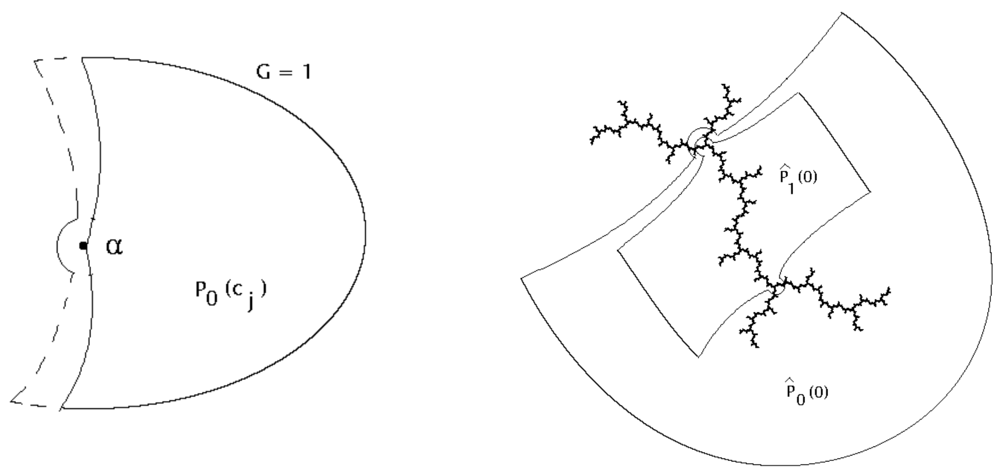

Figure 11. Left: A Puzzle Piece and Thickened Piece of Level 0. Right: Thickened Pieces of Level 0 and 1 for $z \mapsto z^2 + i$.

The following construction will be needed to prove Lemmas 2 and 3. Recall that each of the puzzle pieces $P_0(c_i)$ of depth zero consists of points in some closed subset of the filled Julia set $K(f)$, together with points outside of $K(f)$ which have potential G and external angle t satisfying inequalities of the form

$$0 < G \leq 1, \qquad t_i \leq t \leq t'_i \ .$$

(Here $0 \leq i < q$.) We construct a *thickened puzzle piece* $\widehat{P}_0(c_i) \supset P_0(c_i)$ in two steps, as follows. First choose a small disk $D_\epsilon(\alpha)$ about the fixed point α. Second, choose $\eta > 0$ so small that every external ray whose angle differs from t_i or t'_i by at most η intersects this disk. Now let $\widehat{P}_0(c_i)$ consist of the disk $D_\epsilon(\alpha)$, together with the region bounded by:

(1) the segment $t_i - \eta \leq t \leq t'_i + \eta$ of the equipotential $G = 1$,

(2) the external ray segments of angle $t_i - \eta$ and $t'_i + \eta$ which extend from this equipotential $G = 1$ to their first intersection with $D_\epsilon(\alpha)$, and

(3) an arc of the boundary of $D_\epsilon(\alpha)$.

Evidently this thickened puzzle piece $\widehat{P}_0(c_i)$ contains the original $P_0(c_i)$. We now construct thickened puzzle pieces of greater depth by the usual inductive procedure: If $\widehat{P}_d^{(j)}$ is a thickened puzzle piece of depth d, then each component of $f^{-1}(\widehat{P}_d^{(j)})$ is a thickened puzzle piece of depth $d+1$.

The virtue of these thickened pieces is the following statement, which is easily proved by induction: *If a puzzle piece $P_d^{(j)}$ contains $P_{d+1}^{(k)}$, then the corresponding thickened piece $\widehat{P}_d^{(j)}$ contains $\widehat{P}_{d+1}^{(k)}$ in its interior.* In other words, this construction replaces all of our annuli by non-degenerate annuli.

Caution: For this construction to be useful, we also need the following condition: *A thickened puzzle piece $\widehat{P}_d(z)$ contains the critical point only if $P_d(z)$ already contains the critical point.* However, for a fixed choice of ϵ and η, this condition will only be satisfied for appropriately bounded values of d. For it may well happen that the fixed point α is an accumulation point of the critical orbit. Thus we cannot avoid having points c_d of the critical orbit within $\widehat{P}_0(c_i)\smallsetminus P_0(c_i)$, hence we cannot avoid having thickened puzzle pieces $\widehat{P}_d(z)$ with $0 \in \widehat{P}_d(z)\smallsetminus P_d(z)$, when d is large.

However, we always avoid complication by assuming that the critical orbit does not actually hit the point α. This does not seriously limit the scope of the Theorem, since if $c_i = \alpha$ then f is post-critically finite and one knows already by [DH2] that $J(f)$ is locally connected. With this hypothesis, we can always choose ϵ and η small enough so that the above condition is satisfied for any specified d.

Proof of Lemma 2. Choose $d \ge p$ so large that the critical annulus $A_d(0)$ is a child of $A_{d-p}(0)$, and let $\Delta_0 = \widehat{P}_d(0)$ be the thickened critical puzzle piece of depth d. Then $\Delta_p = f^{\circ p}(\Delta_0)$ is equal to $\widehat{P}_{d-p}(c_p) = \widehat{P}_{d-p}(0)$, which contains Δ_0 in its interior. The hypothesis guarantees that the successive points c_p, c_{2p}, c_{3p}, $\ldots$ all belong to Δ_0. Further details will be left to the reader. □

Proof of Lemma 3 (suggested by M. Lyubich). Let $\Delta_0 = \widehat{P}_1(0)$. Then it is easy to check that the successive images $\Delta_i = f^{\circ i}(\Delta_0)$ are disjoint from the critical point for $0 < i < q$, and that Δ_q contains $\widehat{P}_0(0) \supset \Delta_0$ in its interior. We will prove inductively that $c_{qi} \in P_1(0) \subset \Delta_0$ for every i. It certainly follows from this inductive hypothesis that $c_{qi+1} \in P_0(c_1) \cap K(f) \subset P_1(c_1)$, and similarly that $c_{qi+j} \in K(f) \cap P_0(c_j) \subset P_1(c_j)$ for $0 < j < q$. In particular, c_{qi+q-1} must belong to $P_1(c_{q-1})$, hence $c_{qi+q} \in P_0(c_q) = P_0(0)$. By the hypothesis of Lemma 3, the point c_{qi+q} is known to lie in one of the puzzle pieces $P_1(c_h)$ which lie around α. Evidently it can only lie in $P_1(0)$, as required. □

Corollary. *If f is not renormalizable, then there exists a critical annulus of positive modulus.*

Proof. According to Lemma 3, the critical orbit must visit one of the puzzle pieces $P_1(-c_1)$, $\ldots$, $P_1(-c_{q-1})$ which surround the pre fixed point $-\alpha$. Suppose for example that $c_d \in P_1(-c_i)$. It is easy to check that the corresponding annulus $A_0(-c_i)$ has positive modulus. (Compare Figure 2.) Pulling this annulus back inductively along the critical orbit, it follows that $A_d(0)$ also has positive modulus. □

Here is another application of thickened puzzle pieces. As usual, we assume that the orbit of z_0 does not hit the fixed point α.

Lemma 4. *Suppose that some orbit $z_0 \mapsto z_1 \mapsto \cdots$ in the Julia set never reaches the neighborhood $P_N(0)$ of the critical point. Then the intersection $\bigcap P_d(z_0)$ of the puzzle pieces containing z_0 reduces to the single point z_0.*

Proof. Thickening the puzzle pieces very slightly, we may assume that the orbit of z_0 never reaches $\widehat{P}_N(0)$. Let U_0, U_1, $\ldots U_m$ be the interiors of the various thickened puzzle pieces of depth $N-1$, numbered so that the critical value $c_1 = f(0)$ belongs to U_0. We will make use of the Poincaré metric on the U_i. For $i > 0$, note that there are exactly two branches of f^{-1} on U_i, call them g_1 and g_2. For $i > 0$, each of these maps g_k on U_i, is a holomorphic map which carries U_i into a proper subset of some U_j. Hence it strictly reduces the Poincaré metric. For each puzzle piece of depth N contained in the U_i, it follows that g_k shrinks distances by some definite factor $\lambda < 1$. (We need the thickening at this point, to insure that these puzzle pieces are compactly contained in U_i.) Let C be the maximum of the Poincaré diameters of these puzzle pieces of depth N. Then for a puzzle piece $P_{N+h}(z_0)$, since the successive images $P_{N+h-i}(z_i)$ for $0 \le i \le h$ never meet the critical value region U_0, it follows inductively that the Poincaré diameter of $P_{N+h}(z_0)$ is at most $\lambda^h C$, which tends to zero as $h \to \infty$. □

In order to deal with the possibility that $f^{\circ d}(z_0) = \alpha$, so that the puzzle piece $P_d(z_0)$ is not uniquely defined, we will need the following.

Definition. For any point z in the Julia set let $P_d^*(z)$ be the union of the puzzle pieces of depth d containing z. (In most cases, $P_d^*(z)$ is equal to the unique depth d puzzle piece $P_d(z)$ which contains z. However, if $f^{\circ d}(z) = \alpha$ then $P_d^*(z)$ is a union of q distinct puzzle pieces.)

We can now state and prove the principal result of this section.

Theorem 2. *Suppose as usual that f is quadratic, with connected Julia set, with both fixed points repelling, and not renormalizable. Suppose further that the critical orbit is disjoint from the fixed point α. Then for any $z \in J(f)$ we have $\bigcap_d P_d^*(z) = \{z\}$.*

Proof. Since we assume that f is not renormalizable, we know from the Corollary above that there exists some critical annulus $A_m(0)$ which has positive modulus. We will first prove that $\bigcap P_d(0) = \{0\}$, then prove that $\bigcap P_d(z) = \{z\}$ for any $z \in J(f)$ which is not an iterated pre-image of α, and finally prove the corresponding result when $f^{\circ n}(z) = \alpha$.

Critically Recurrent Case. Suppose that the critical orbit is recurrent, so that Lemma 1 applies. First consider the puzzle pieces $P_d(0)$ around the critical point. If the non-degenerate annulus $A_m(0)$ has at least 2^k descendents in the k-th generation for each k, then each of these contributes exactly $\operatorname{mod} A_m(0)/2^k$ to the sum $\sum_d \operatorname{mod} A_d(0)$. Hence this sum is infinite, as required. On the other hand, if there are fewer descendents in some generation, then one of them must be an only child, hence excellent by Lemma 1d. Using Lemma 1b and 1c, we again see that $\sum_d \mathrm{mod}\, A_d(0)$ is infinite. Therefore the intersection $\bigcap P_d(0)$ reduces to the single point 0.

Now consider a point $z_0 \neq 0$ of the Julia set. We assume that the orbit $z_0 \mapsto z_1 \mapsto \cdots$ is disjoint from α, so that the puzzle pieces $P_d(z_0)$ are well defined. If the orbit of z_0 does not accumulate at zero, then we have $\bigcap P_d(z_0) = \{z_0\}$ by Lemma 4. Suppose, on the other hand, that the origin is an accumulation point of $\{z_n\}$. In other words, suppose that the tableau of the point z_0 has critical annuli reaching down to all depths. For each depth d, let us start at column zero (corresponding to the point z_0 itself) and advance to the right until we first hit a critical annulus at column n, then proceed diagonal back down until we reach column zero at depth $n+d$. It follows from this construction that the annulus $A_{n+d}(z_0)$ is conformally isomorphic to $A_d(0)$. Furthermore, distinct values of d must correspond to distinct values of $n+d$. Thus the sum $\sum \mathrm{mod}\, A_d(z_0)$ is also infinite, hence $\bigcap P_d(z_0) = \{z_0\}$, as required.

Critically Non-recurrent Case. Now suppose that the critical orbit is not recurrent. Then the critical value $f(0)$ satisfies the hypothesis of Lemma 4. Hence $\bigcap P_d(f(0)) = \{f(0)\}$, and it follows easily that $\bigcap P_d(0) = \{0\}$. Next consider a point $z_0 \neq 0$. Again we may assume that the orbit $\{z_n\}$ accumulates at zero, since otherwise the conclusion would follow from Lemma 4. Again the Corollary above tells us that there exists one depth m such that $A_m(0)$ has positive modulus. The corresponding depth m for the tableau of z_0 must have infinitely many columns k which are critical. For each of these, let us proceed diagonally back down in the tableau of z_0, ignoring whatever critical or semi-critical annuli we may meet, until we reach column zero at depth $m+k$. Each time we meet a critical or semi-critical annulus, we lose up to half of the modulus. (Problem 1-2 below.) However, the hypothesis that the critical orbit is non-recurrent guarantees that such losses will only occur a bounded number of times. Thus $\sum \mathrm{mod}\, A_d(z_0)$ has infinitely many summands which are bounded away from zero. Hence this sum is infinite, and we have proved that $\bigcap P_d(z_0) = \{z_0\}$ in this case also.

Iterated Pre-images of α. If some forward image $z_n = f^{\circ n}(z_0)$ is equal to the fixed point α, then the above arguments do not make sense as stated. In this case, there are q distinct puzzle pieces $P_n^{(i)}$ of depth n which have z_0 as common boundary point. Each of these is contained in a unique sequence of nested puzzle pieces $P_n^{(i)} \supset P_{n+1}^{(i)} \supset \cdots$ which have z_0 as common boundary point. **Assertion:** *For each one of these q nested sequences the intersection $\bigcap_d P_d^{(i)}$ reduces to the single point z_0.* In fact, the proof of Lemma 4 applies equally well to this situation. Evidently the statement that $\bigcap_d P_d^*(z_0) = \{z_0\}$ follows immediately.

Thus we have proved Theorem 2: $\bigcap_d P_d^*(z) = \{z\}$ in all cases. Theorem 1, as stated on page 2, is a straightforward consequence. (Compare Problem 1-1 below.) □□

Here are some problems for the reader.

Problem 1-1. Local Connectivity. Prove that the intersection of $J(f)$ with each puzzle piece is connected. Conclude that $J(f)$ is locally connected at z whenever $\bigcap P_d^*(z) = \{z\}$.

Problem 1-2. Semi-Critical Annuli. If $A_d(z)$ is a non-degenerate semi-critical annulus of depth $d > 0$, show that $A_d(z)$ is the union of

(1) a ramified two-fold covering of $A_{d-1}(f(z))$, and

(2) a conformal copy of $P_d(f(z))$.

Using the Grötzsch inequality, prove that $\mod A_d(z) > \frac{1}{2} \mod A_{d-1}(f(z))$.

Problem 1-3. Non-degenerate Annuli. Show that an annulus $A_d(z_m)$ is non-degenerate if and only if the corresponding annulus $A_0(z_{d+m})$ of depth zero is semi-critical.

Problem 1-4. Further Tableau Rules. Let $q \geq 2$ be the number of external rays landing at the fixed point α. Show that the semi-critical depth of a tableau column can never take the values $1, \ldots, q-1$. Show that at most $q-1$ consecutive columns can be completely off-critical (semi-critical depth $\mathrm{scd}(z_i) = -1$), and show that $\mathrm{scd}(z_i) = -1$ for $m < i < m+q$ if and only if the m-th column has semi-critical depth $\mathrm{scd}(z_m) \geq q$.

Problem 1-5. The Critical Orbit is Generically Dense. It is convenient to say that a property of certain points in a compact set is ***generically*** true if it is true throughout a countable intersection of dense open subsets. For example, one can show that for generic c in the boundary of the Mandelbrot set, the map f_c is non-renormalizable[1], with both fixed points repelling. Let $U_d \subset \partial M$ be the set of parameter values c in the boundary of the Mandelbrot set such that every puzzle piece of depth d for f_c contains a post-critical point $c_i = f_c^{\circ i}(0)$ in its interior. Show that U_d contains a dense relatively open subset of ∂M. (To prove density, use the fact that periodic points are dense in $J(f_c)$, and use Montel's Theorem.) *For a generic parameter value $c \in \partial M$, conclude that the closure of the critical orbit is the entire Julia set $J(f_c)$.* Conclude also that no non-degenerate annulus can be excellent in the generic case.

Problem 1-6. The Yoccoz τ-function. For each depth $d > 0$, if there exists an integer $0 \leq \tau < d$ so that $f^{\circ d-\tau}$ maps $P_d(0)$ onto $P_\tau(0)$, then we define $\tau(d)$ to be the largest such integer, and describe the critical puzzle piece $P_{\tau(d)}(0)$ as the "parent" of $P_d(0)$. Otherwise, if no such integer exists, we set $\tau(d) = -1$. Thus $-1 \leq \tau(d) < d$ in all cases. Show that $\tau(d+1) \leq \tau(d) + 1$. Show that the annulus $A_d(0)$ is a child of $A_n(0)$ if and only if $n = \tau(d) = \tau(d+1) - 1 \geq 0$.

Problem 1-7. Persistent Recurrence. The critical orbit is said to be ***persistently recurrent*** if it is non-renormalizable with $\lim_{d\to\infty} \tau(d) = +\infty$. Show that the Fibonacci tableau is persistently recurrent. In the persistently recurrent case, show that the critical orbit is bounded away from the fixed point β. (Otherwise, for any depth d we could choose a post-critical point $c_n \in P_d(\beta)$, then choose the smallest k with $0 \in P_{d+k}(c_{n-k})$, and conclude that $\tau(d+k) = 0$.) Show that the critical orbit is bounded away from every iterated pre-image of β, and hence that its closure is nowhere dense in the Julia set. More generally, show that the critical orbit is bounded away from any periodic point. Using the fact that it is bounded away from α, show that the critical orbit closure is a Cantor set. (For further information, see [L2].)

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[1] Proof outline: With notation as in §3, the set of renormalizable points in the Mandelbrot set (together with associated root points) forms a countable union of compact subsets $H * M$. This union is nowhere dense in the boundary ∂M. In fact the set of c such that the critical orbit of f_c eventually lands on the fixed point β is everywhere dense in ∂M. (Compare [Br2] or [M1, App. A].) Such a map f_c cannot be renormalizable [D3].

§2. Polynomials for which all but one of the critical orbits escape (following Branner and Hubbard).

Let $f : \mathbf{C} \to \mathbf{C}$ be a polynomial of degree $d \ge 2$, with filled Julia set K and with Julia set $J = \partial K$. The object of this section is to study the case where only one critical point has bounded orbit. However, we begin with the simpler case where no critical point has bounded orbit. The following is well known.

Theorem 3. *If all critical orbits of f escape to infinity, then $J = K$ is a Cantor set of measure zero*[1]*. Furthermore, the dynamical system $(J, f|J)$ is homeomorphic to the one-sided shift on d symbols*[2]*.*

Proof. Let $\omega_1, \ldots, \omega_{d-1}$ be the (not necessarily distinct) critical points of f, and let $G : \mathbf{C} \to \mathbf{R}_+$ be the canonical potential function (= Green's function), which satisfies $G(f(z)) = d\,G(z)$ and vanishes precisely on the filled Julia set K. Thus $G(\omega_i) > 0$ by hypothesis. The critical points of G in $\mathbf{C} \smallsetminus K$ are the points ω_i and also all of their preimages under iterates of f. Hence the critical values of G (other than zero) are the numbers of the form $G(\omega_i)/d^k$ with $k \ge 0$.

The *Branner-Hubbard puzzle* of f is constructed as follows. Choose a number G_0, not of the form $G(\omega_i)/d^k$, so that $0 < G_0 < \mathrm{Min}\{G(f(\omega_i))\}$. Then the region $G^{-1}(0, G_0]$ contains no critical values of f, and is bounded by smooth curves since G_0 is not a critical value of G. Similarly, each locus

$$G^{-1}\left[0, \frac{G_0}{d^k}\right] = \left\{z \in \mathbf{C} \;:\; G(z) \le \frac{G_0}{d^k}\right\} \qquad (*_k)$$

is bounded by smooth curves. Note that the complementary region $G^{-1}(G_0/d^k, \infty)$ cannot have any bounded component, since the harmonic function G cannot have a local maximum. It follows that the locus $(*_k)$ is the disjoint union of a finite number of closed topological disks. By definition, each of these closed disks will be called a *puzzle piece* P_k of depth k. Since these puzzle pieces contain no critical value of f, it follows easily that f maps each P_k of depth $k > 0$ by a conformal isomorphism onto some puzzle piece $f(P_k)$ of depth $k-1$.

If $P_k \supset P_{k+1}$ are nested puzzle pieces of depths k and $k+1$, then the set

$$A_k = \mathrm{Interior}(P_k) \smallsetminus P_{k+1}$$

is a well defined annulus of strictly positive modulus. (Figure 12.) We will call such an A_k an *annulus of depth* k in the Branner-Hubbard puzzle. Evidently $f^{\circ k}$ maps each annulus of depth k by a conformal isomorphism onto an annulus of depth zero. *Hence the moduli of all annuli constructed in this way are uniformly bounded away from zero.*

---

[1] Shishikura has shown that the Hausdorff dimension of this Cantor set can be arbitrarily close to two.

[2] Przytycki and Makienko have both announced the sharper result that any rational Julia set which is totally disconnected and contains no critical point must be isomorphic to a one-sided shift.

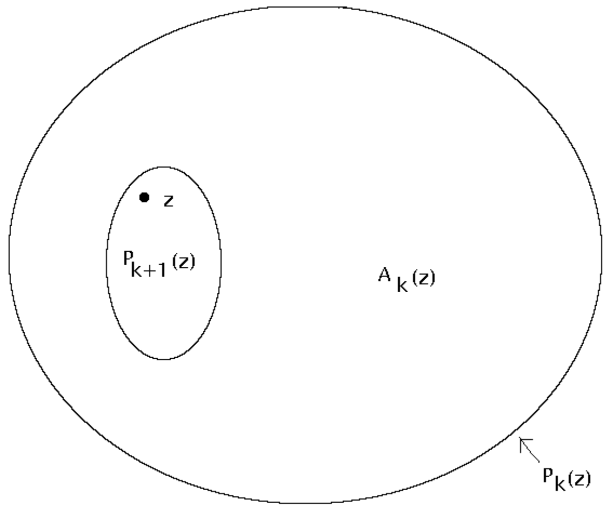


Figure 12. Nested puzzle pieces, and the annulus $A_k(z)$.

For any point z in the filled Julia set we can form the nested sequence of puzzle pieces $P_0(z) \supset P_1(z) \supset \cdots$, all containing z . Since the moduli of the annuli $(\text{Interior}\, P_k(z)) \smallsetminus P_{k+1}(z)$ are bounded away from zero, it follows that the intersection $\bigcap P_k(z)$ reduces to the single point z . Since each boundary circle ∂P_k is disjoint from K , this implies that K is totally disconnected.

The proof that J has measure zero will be based on the *McMullen inequality*

$$\mathbf{area}(P_{k+1}) \;\le\; \frac{\mathbf{area}(P_k)}{1 + 4\pi\, \mathbf{mod}\,(A_k)} \;,$$

with $A_k = \text{Interior}(P_k) \smallsetminus P_{k+1}$ as above. (Compare the Appendix.) However, to apply this inequality in a useful way, we will need to construct annuli somewhat differently.

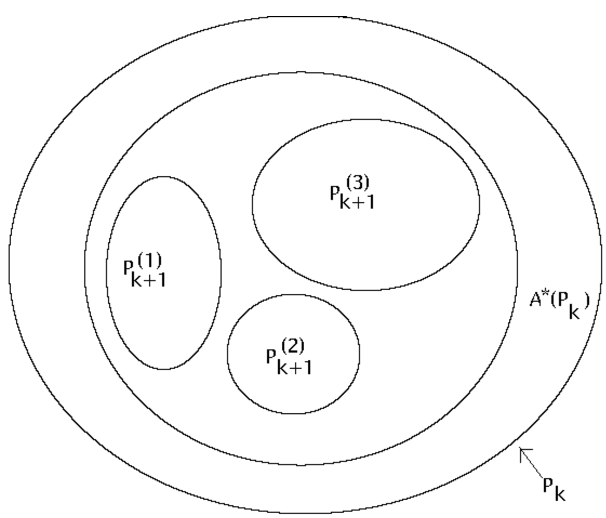


Figure 13. The "thin annulus" $A^*(P_k) \subset A_k \subset P_k$.

Choose a number $\epsilon > 0$ which is small enough so that there are no critical values of G within the closed interval $[G_0 - \epsilon\,,\,G_0]$. Then each connected component of $G^{-1}[(G_0-\epsilon)/d^k\,,\,G_0/d^k]$ is an annulus, and there is one such "thin annulus"

$$A^*(P_k) \quad = \quad P_k \cap G^{-1}\Big[\frac{G_0-\epsilon}{d^k}\,,\,\frac{G_0}{d^k}\Big]$$

within each connected component P_k of $G^{-1}[0\,,\,G_0/d^k]$. The construction is such that:

(a) all of these annuli $A^*(P_k)$ have modulus strictly bounded away from zero, say $\mathbf{mod}\,A^*(P_k) \ge c > 0$, and

(b) every puzzle piece P_{k+1} which is contained in P_k is actually contained in the smaller disk $P_k \smallsetminus A^*(P_k)$ (the bounded component of $\mathbf{C} \smallsetminus A^*(P_k)$).

Thus the McMullen inequality takes the following sharper form. For each fixed puzzle piece P_k ,

$$\sum_{P_{k+1}\subset P_k} \mathbf{area}(P_{k+1}) \quad\le\quad \frac{\mathbf{area}(P_k)}{1+4\pi\,\mathbf{mod}\,(A^*(P_k))} \quad\le\quad \frac{\mathbf{area}(P_k)}{1+4\pi\,c}\;,$$

to be summed over those puzzle pieces of depth $k+1$ which are contained in the given P_k . It now follows inductively that the sum of the areas of *all* puzzle pieces of depth k satisfies

$$\sum \mathbf{area}(P_k) \;\le\; \sum \mathbf{area}(P_0)/(1+4\pi\,c)^k\;.$$

Since this tends to zero as $k \to \infty$, and since $J \subset \bigcup P_k$, it follows that J has area zero.

To prove that $(J\,,\,f|J)$ is isomorphic to the one-sided d-shift, we proceed as follows. We will construct a closed subset $\Gamma \subset \mathbf{C}$, consisting of paths leading out to infinity, such that:

(1) Γ contains all critical values of f ,

(2) $f(\Gamma) \subset \Gamma$, and

(3) the complement $U = \mathbf{C} \smallsetminus \Gamma$ is simply-connected, and contains the Julia set.

In fact, starting from each critical value $f(\omega_i)$, we can follow the gradient directions of G (the path of steepest ascent) until we meet a critical point of G , or all the way out to infinity if we never meet a critical point. At each critical point of G we must make a choice. From a critical point of multiplicity μ , there are $\mu+1$ distinct directions in which we can continue along a path of steepest ascent. Choose one of these $\mu+1$ directions for each critical point of G . However, the choice must be consistent in the following sense: If $f(z) = z'$, where z and z' are both critical points of G , then f must carry the preferred path from z to the preferred path from z' . It is not difficult to make such a consistent choice, simply by ordering the critical points of f so that $G(\omega_1) \le \cdots \le G(\omega_{d-1})$, and making a choice of ascending path first at ω_1 , and its iterated pre-images, then at ω_2 and its iterated pre-images, and so on. Now we can follow the chosen paths from all of the post-critical points $f^{\circ k}(\omega_i)$ out to infinity. The paths from different post-critical points may come together, but once they come together they must stay together, out to infinity.

The union Γ of these preferred paths will be a locally finite union of disjoint topological trees with the required properties.

Since $f(\Gamma) \subset \Gamma$, it follows that $f^{-1}(U) \subset U$. Furthermore, since U is simply connected, and contains no critical values of f, it follows that every one of the d branches of f^{-1} near a point of U extends uniquely to a holomorphic map $g_i : U \to U$. The images

$$g_1(U), \ldots, g_d(U) \subset U$$

are disjoint open sets covering the Julia set. We will show that the intersections

$$J_i = J \cap g_i(U)$$

are disjoint compact sets which form the required *Bernoulli partition* $J = J_1 \cup \cdots \cup J_d$ of the Julia set. That is:

> *For each sequence of integers $i_0, i_1, \ldots$ between 1 and d there exists one and only one orbit $z_0 \mapsto z_1 \mapsto z_2 \mapsto \cdots$ in the Julia set with $z_n \in J_{i_n}$ for every n.*

In fact z_0 can be described as the intersection of the nested sequence of sets $g_{i_0} \circ g_{i_1} \circ \cdots \circ g_{i_n}(J)$. To prove this statement, we use the Poincaré metric on U. Since each g_i restricted to the compact set $J \subset U$ shrinks Poincaré distances by a factor bounded away from one, it follows that each such intersection $\bigcap_n g_{i_0} \circ g_{i_1} \circ \cdots \circ g_{i_n}(J)$ consists of a single point. This proves Theorem 3. □

Maps with exactly one bounded critical orbit.

This will be an exposition of results due to Branner and Hubbard [BH3]. We now suppose that exactly one of the $d-1$ critical points of f has bounded orbit, while the orbits of the remaining $d-2$ critical points, counted with multiplicity, escape to infinity. (Thus we exclude examples such as $z \mapsto z^3 + i$, for which a double critical point has bounded orbit; however, a double critical point escaping to infinity is fine.) Furthermore, we assume that $d \geq 3$, so that at least one critical orbit does escape. Then, according to Fatou and Julia, the Julia set is disconnected, with uncountably many connected components. Let

$$c_0 \mapsto c_1 \mapsto c_2 \mapsto \cdots$$

be the unique bounded critical orbit.

As in the proof of Theorem 3, choose a number $G_0 > 0$ which is not a critical value of G, and so that the region $G^{-1}(0, G_0]$ contains no critical value of f. Again we define the puzzle pieces of depth k to be the connected components P_k of the locus

$$\bigcup P_k = G^{-1}\left[0, \frac{G_0}{d^k}\right].$$

Thus each point $z \in K$ determines a nested sequence $P_0(z) \supset P_1(z) \supset \cdots$, and the central problem is to decide whether or not $\bigcap_k P_k(z) = \{z\}$. Again we look at the intermediate annuli

$$A_k(z) = \text{Interior}\, P_k(z) \smallsetminus P_{k+1}(z).$$

As in the Yoccoz proof, such an annulus is said to be semi-critical, critical, or off-critical according as the critical point c_0 belongs to the annulus itself, or to the bounded or the unbounded component of its complement. (For this purpose, we ignore the other critical points, whose orbits escape to infinity.) This Branner-Hubbard puzzle is easier to deal with than the Yoccoz puzzle for three reasons:

(a) All of the annuli $A_k(z)$ are non-degenerate, with strictly positive modulus.

(b) The various puzzle pieces of depth k are pair-wise disjoint.

(c) For each $z \in K$, the intersection $\bigcap_k P_k(z)$ of the puzzle pieces containing z has an immediate topological description: It is equal to the connected component of the filled Julia set K which contains the given point z. For this intersection is clearly connected, and no larger subset can be connected since each boundary $\partial P_k(z)$ is disjoint from K.

The ***tableau*** of an orbit $z_0 \mapsto z_1 \mapsto \cdots$ can be described as a record of exactly which of the annuli $A_k(z_i)$ are critical or semi-critical or off-critical. First suppose that the tableau of the critical orbit is not periodic. We continue to assume that $d \geq 3$ and that exactly one of the $d-1$ critical points has bounded orbit.

Theorem 4. *If the critical tableau is not periodic, or equivalently if the post-critical points* $c_1, c_2, \ldots$ *are all disjoint from the critical component* $\bigcap_k P_k(c_0)$, *then for every point* z_0 *of the filled Julia set* K *the sum* $\sum_k \mathbf{mod}\, A_k(z_0)$ *is infinite, hence* $\bigcap_k P_k(z_0) = \{z_0\}$. *It follows that* $J = K$ *is a totally disconnected set of area zero.*

Thus J is again homeomorphic to a Cantor set. However, in this case $(J, f|J)$ is not isomorphic to a shift, or even a sub-shift. For there are critical points in J, hence f is not locally one-to-one on J.

The proof of this theorem is quite similar to the proof of the Yoccoz theorem. However there are simplifications, leading to the sharper result which is stated: $\sum \mathbf{mod}\ A_k(z) = \infty$ for *all* $z \in K$. The main difference is the following statement, which is true whether or not the critical tableau is periodic. Consider an orbit $z_0 \mapsto z_1 \mapsto \cdots$.

Lemma 5. *Suppose that the points* $z_1, z_2, \ldots$ *are all disjoint from some neighborhood* $P_N(c_0)$ *of the critical point* c_0. *Then the annuli* $A_k(z_0)$ *have modulus uniformly bounded away from zero, hence* $\sum_k \mathbf{mod}\, A_k(z_0) = \infty$.

Proof. Each annulus $A_k(z_i)$ of depth $k > 0$ has modulus at least half of the modulus of $A_{k-1}(z_{i+1})$. In fact, if $i > 0$ and $k > N$ then these two annuli are conformally isomorphic. Thus

$$\mathbf{mod}\, A_k(z_0) \geq \mathbf{mod}\, A_0(z_k)/2^{N+1},$$

where the right side is bounded away from zero since there are only finitely many annuli of depth zero. □

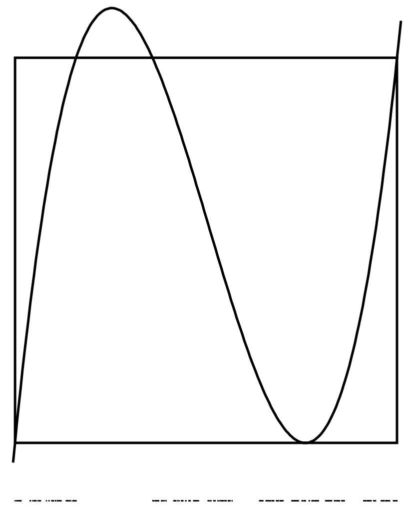


Figure 14. An example: Graph of the function $f(x) = x(x-c_0)^2/(1-c_0)^2$ *on the unit interval* $I=[0,1]$ *, with* $c_0 = .76$ *. There is just one bounded critical orbit* $c_0 \mapsto 0 \mapsto 0 \mapsto \cdots$ *. Since the iterated pre-images of any point of* J *are dense in* J *, and since there are three distinct branches of* f^{-1} *mapping* I *into itself, it follows that the Julia set* J *is completely contained in the real interval* I *. This Julia set is plotted underneath the graph. Since* c_0 *and* 0 *evidently belong to distinct connected components of* J *, it follows from Theorem 4 that* J *is totally disconnected.*

Proof of Theorem 4. If the critical tableau is recurrent, then the proof proceeds exactly as is the Yoccoz argument: Every critical annulus either has at least 2^n descendents in the n-th generation for every n , or else has a descendent with this property. Since all annuli are non-degenerate, it follows that $\sum \textbf{mod}\, A_k(c_0) = \infty$. On the other hand, if the critical tableau is non-recurrent, then it follows from Lemma 5 that $\sum \textbf{mod}\, A_k(c_0) = \infty$. In particular, it follows that the collection of puzzle pieces $\{P_k(c_0)\}$ forms a fundamental system of neighborhoods of the critical point c_0 . The corresponding statements for any iterated pre-image of c_0 follow immediately.

Now consider a point $z_0 \in K$, with orbit $z_0 \mapsto z_1 \mapsto \cdots$ which never meets the critical point c_0 . If this orbit does not accumulate at c_0 , then the statement that $\sum_k \textbf{mod}\, A_k(z_0) = \infty$ follows from Lemma 5. On the other hand, if this orbit does accumulate at c_0 , then since $\sum \textbf{mod}\, A_k(c_0) = \infty$, an easy tableau argument shows that $\sum_k \textbf{mod}\, A_k(z_0) = \infty$ also. Thus $\bigcap_k P_k(z) = \{z\}$ in all cases.

Since each boundary $\partial P_k(z)$ is disjoint from the filled Julia set K , it follows that K is totally disconnected, and hence that $J = K$. To prove that this set has measure zero, we proceed as in the proof of Theorem 3. Choose $\epsilon > 0$ so that the interval $[G_0 - \epsilon\,,\, G_0]$ contains no critical values of the function $G : \mathbf{C} \to \mathbf{R}_+$. Then each puzzle piece $P_k(z)$ contains a unique component $A^*(P_k(z))$ of the set $[G^{-1}[(G_0 - \epsilon)/d^k\,,\, G_0/d^k]$. These annuli $A^*(P_k(z)) \subset A_k(z)$ are also non-degenerate, and the proof above shows equally well that $\sum_k \textbf{mod}\, A^*(P_k(z)) = \infty$. (The proof actually becomes a little easier, since a thin annulus can never contain the critical point c_0 .) Hence, just as in the proof of Theorem 3, for each fixed puzzle piece P_k , the total area of the puzzle pieces P_{k+1} of depth $k+1$ which are contained in P_k satisfies

$$\sum_{P_{k+1}\subset P_k} \mathbf{area}\, P_{k+1} \quad\le\quad \frac{\mathbf{area}\, P_k}{1+4\pi\,\mathbf{mod}\, A^*(P_k)}\ .$$

Define the ratio $\mu(P_k)$ by the formula

$$\mathbf{area}\, P_k \quad=\quad \mu(P_k)\sum_{P_{k+1}\subset P_k}\mathbf{area}\, P_{k+1}$$

so that this McMullen inequality takes the form $1+4\pi\,\mathbf{mod}\, A^*(P_k) \ \le\ \mu(P_k)$. Then, substituting this formula inductively, we can write

$$\sum_{P_0}\mathbf{area}\, P_0 \;=\; \sum_{P_0\supset P_1}\mu(P_0)\,\mathbf{area}(P_1) \;=\;\cdots\;=\; \sum_{P_0\supset\cdots\supset P_k}\mu(P_0)\,\cdots\,\mu(P_{k-1})\,\mathbf{area}(P_k)\ ,$$

where the left hand expression is to be summed over all puzzle pieces of depth zero, the next over all pairs $P_0\supset P_1$, and so on. (If $P_0\supset\cdots\supset P_k$, note that $P_0,\ldots,P_{k-1}$ are uniquely determined by P_k .) Let η_k be the minimum value of the product $\mu(P_0)\cdots\mu(P_{k-1})$ as $P_0,\ldots,P_{k-1}$ varies over all sequences of nested puzzle pieces $P_0\supset P_1\supset\cdots\supset P_{k-1}$. Then we see from this last equality that

$$\sum\mathbf{area}(P_0) \quad\ge\quad \eta_k\sum\mathbf{area}(P_k)\ ,$$

to be summed over all puzzle pieces of depth zero or k respectively. Thus, if we can prove that $\eta_k\to\infty$ as $k\to\infty$, then it will follow that

$$\mathbf{area}(J) \ \le\ \sum\mathbf{area}(P_k) \ \le\ \sum\mathbf{area}(P_0)/\eta_k \ \to 0\ ,$$

hence $\mathbf{area}(J)=0$ as required.

Clearly $1<\eta_1<\eta_2<\cdots$. If these numbers tended to a finite limit $L<\infty$, then for each k we could find puzzle pieces $P_0(k)\supset P_1(k)\supset\cdots\supset P_{k-1}(k)$ so that $\mu(P_0(k))\cdots\mu(P_{k-1}(k))\le L$. Hence we could choose a puzzle piece P_0 which occurs infinity often as $P_0(k)$, then choose $P_1\subset P_0$ which occurs infinitely often as $P_1(k)$, and so on. In this way, we could find a sequence

$$P_0 \ \supset\ P_1 \ \supset\ P_2 \ \supset\ \cdots$$

with $\mu(P_0)\cdots\mu(P_{k-1})\le L<\infty$ for every k . Since $1+4\pi\,\mathbf{mod}\,(A^*(P_i)) \ \le\ \mu(P_i)$, this would imply that

$$1+4\pi\sum\mathbf{mod}\,(A^*(P_i)) \ \le\ L<\infty\ ,$$

contradicting our statement that $\sum\mathbf{mod}\, A_k(z)=\infty$ for all $z\in K$. This completes the proof of Theorem 4. □

On the other hand, if the critical tableau is periodic, then we will prove the following.

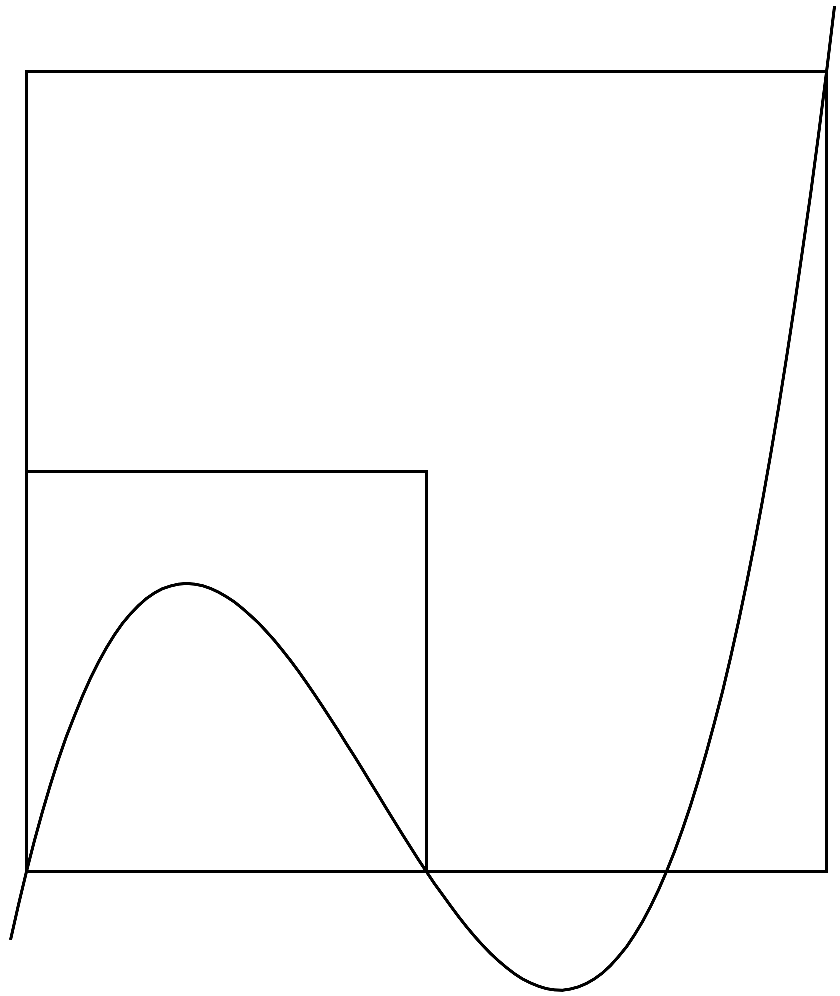


Figure 15. Example for Theorem 5: Graph of the map $f(x)=x(2x-1)(5x-4)$ *on the unit interval. In this case the connected interval* $[0,\frac{1}{2}]$ *is contained in the filled Julia set* K *. The orbit of the critical point* $\frac{2}{3}$ *escapes to* $-\infty$ *.*

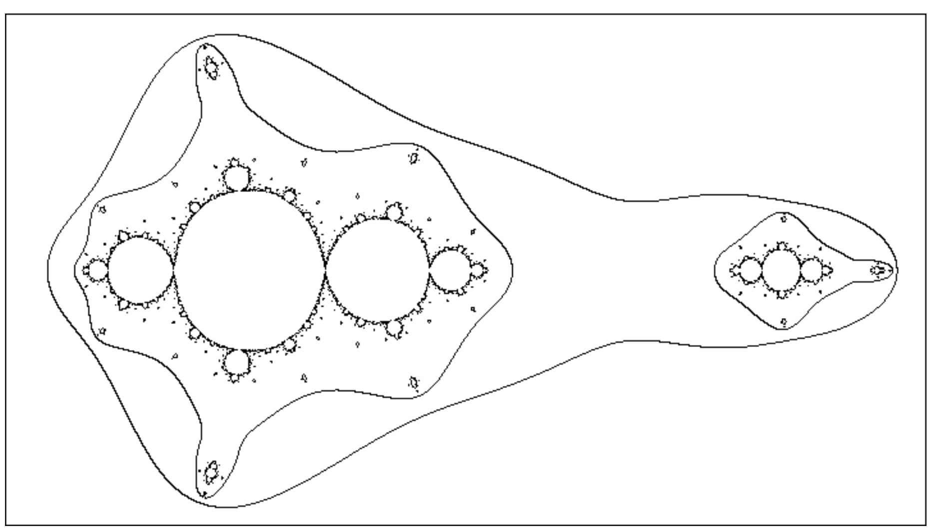


Figure 16. Julia set for this map, drawn to the same scale, showing the puzzle pieces of level zero and one. Each non-trivial component of J *is homeomorphic to a certain quadratic Julia set.*

Theorem 5. *Still assuming that just one critical orbit* $c_0 \mapsto c_1 \mapsto \cdots$ *is bounded, if the critical tableau is periodic of period* $p \geq 1$ *, so that* $P_k(c_0) = P_k(c_p)$ *for all depths* k *, then the connected component of the filled Julia set* $K = K(f)$ *which contains* c_0 *is non-trivial, that is, consists of more than one point. In fact, a component of* K *is non-trivial if and only if it contains some iterated pre-image of* c_0 *.*

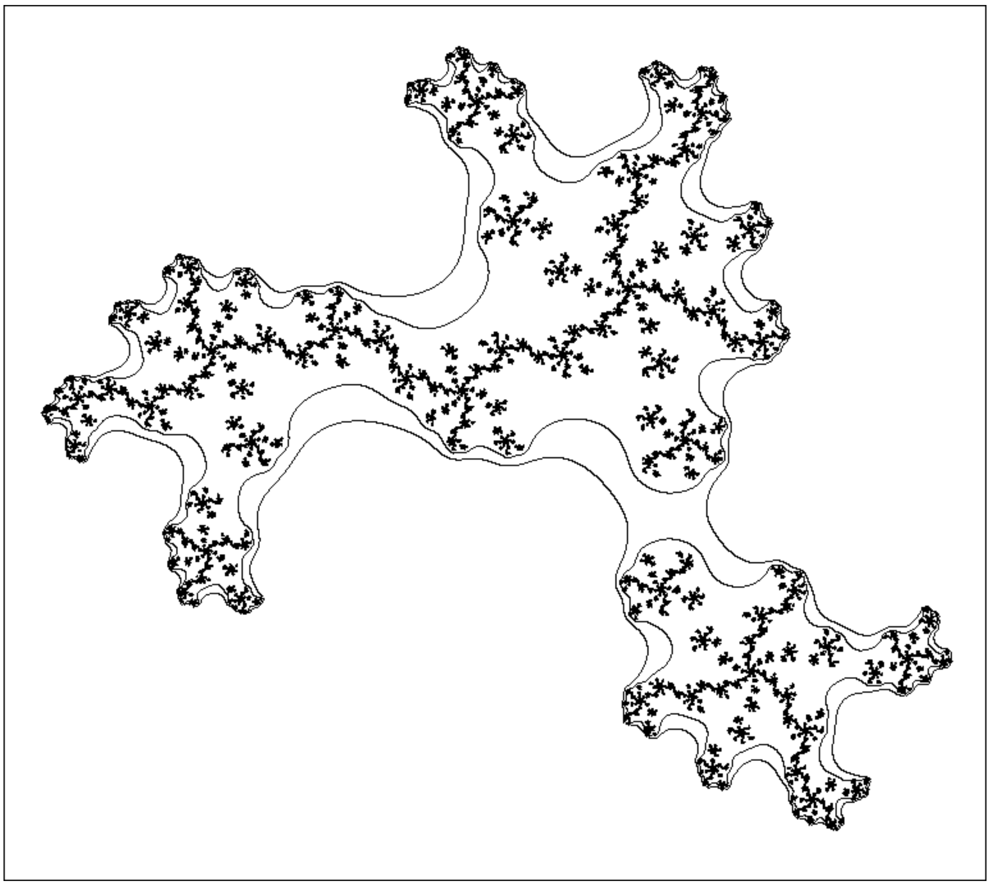


Figure 17. Julia set for $z \mapsto z^3 + a\,z^2 + 1$ *, with* $a = -1.10692 + .63601\,i$ *, showing the puzzle pieces of level zero and one. Each non-trivial component of* J *is homeomorphic to the Julia set for the quadratic map* $z \mapsto z^2 + i$ *.*

Thus there there are countably many non-trivial components of K . These countably many components are everywhere dense in K , since the iterated pre-images of any point of the Julia set are dense in the Julia set. Since a disconnected Julia set necessarily has uncountably many components, it follows that there are uncountably many single points components. (The Julia set of a rational function may have uncountably many non-trivial components. Compare [McM]. However, in the polynomial case no such example is known.)

Proof of Theorem 5. If $P_k(c_0) = P_k(c_k)$ for all k, then it follows from the tableau rules that $P_k(c_i) = P_k(c_{p+i})$ for all i and k. Hence the entire orbit

$$c_0 \mapsto c_p \mapsto c_{2p} \mapsto \cdots$$

of c_0 under $f^{\circ p}$ is contained in the critical component $\bigcap_k P_k(c_0) \subset K(f)$. This intersection certainly has more than one point. For either it contains $c_0 \neq c_p$, or else c_0 is a superattracting point, in which case some entire neighborhood of c_0 belongs to $\bigcap_k P_k(c_0)$.

It follows easily that every pre-critical point in $K(f)$ also belongs to a non-trivial connected component. For if the orbit $z_0 \mapsto z_1 \mapsto \cdots$ intersects the critical component $\bigcap_k P_k(c_0)$, then we can choose the smallest $\ell \geq 0$ for which z_ℓ belongs to this critical component. It follows easily that $f^{\circ \ell}$ maps the component $\bigcap_k P_k(z_0)$ containing z_0 homeomorphically onto this critical component.

Now consider an orbit $z_0 \mapsto z_1 \mapsto \cdots$ in $K(f)$ which is disjoint from this critical component. This means that the tableau of this orbit has no columns which are completely critical. The proof is now divided into two cases:

Case 1. Suppose that there exists a fixed puzzle piece $P_k(c_0)$ which is disjoint from this orbit $\{z_n\}$. Then according to Lemma 5 we have $\sum_k \textbf{mod}\, A_k(z_0) = \infty$, hence $\bigcap_k P_k(z_0) = \{z_0\}$.

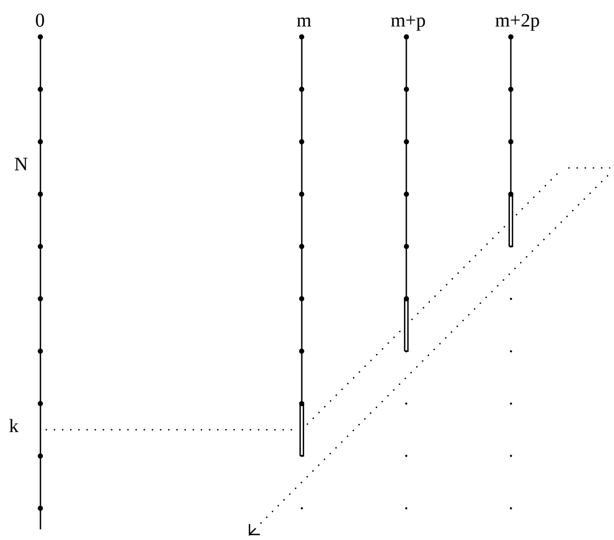


Figure 18. Tableau for an orbit $z_0 \mapsto z_1 \mapsto \cdots$ *which intersects every critical puzzle piece.*

Case 2. If the orbit $\{z_n\}$ intersects *every* critical puzzle piece, then we use a tableau argument as follows. Choose a depth N so that the periodic critical tableau has no semi-critical annuli at depths $\geq N$. By hypothesis, there are infinitely many pairs (k, m), with $k \geq N$, so that the k-th row of the tableau for z_0 is semi-critical in column m and off-critical in earlier columns. (Compare Figure 18.) Using the tableau rules, we can then compute the tableau in column $m + i$ and depth $k - i$ for $0 < i < k - N$. In fact the entries in column $m+jp$ and depth $k-jp$ are semi-critical, and the others are off-critical. It now follows that the annulus $A_{k+m+1}(z_0)$ is conformally isomorphic to an annulus of depth N. Hence this modulus is bounded away from zero. It follows that $\sum_\ell \textbf{mod}\, A_\ell(z_0) = \infty$, which completes the proof of Theorem 5. □

We can understand this proof better by introducing the following concepts, which are due to Douady and Hubbard [DH3].

Definition. By a *polynomial-like map* is meant a pair $(g\,,\,\Delta)$ where $\Delta \subset \mathbf{C}$ is a closed topological disk and g is a continuous mapping, holomorphic on the interior of Δ, which carries Δ onto a closed topological disk $g(\Delta)$ which contains Δ in its interior, such that g maps boundary points of Δ to boundary points of $g(\Delta)$.

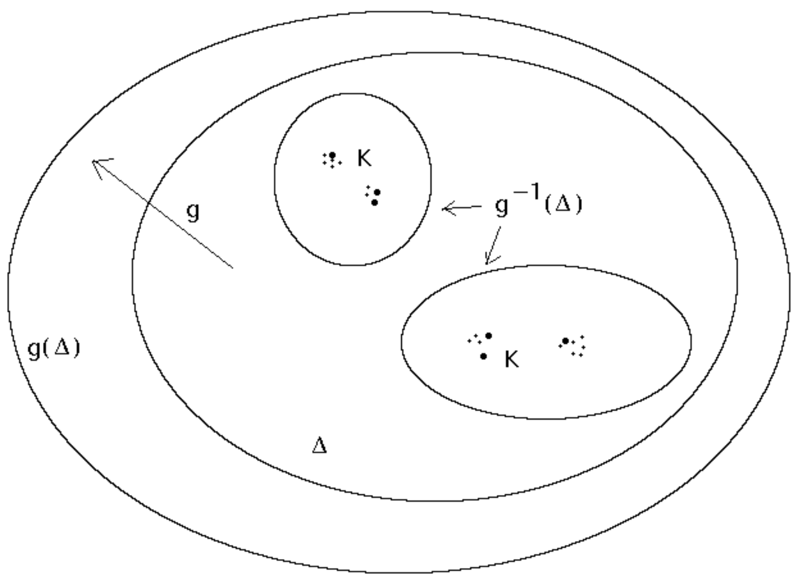


Figure 19. A polynomial-like mapping (g,Δ) *with* $K=K(g,\Delta)$ *totally disconnected.*

The *degree* $d \geq 1$ of such a polynomial-like mapping is a well defined topological invariant. Note that almost every point of $g(\Delta)$ has precisely d pre-images in Δ. The *filled Julia set* $K(g,\Delta)$ is defined to be the compact set consisting of all $z_0 \in \Delta$ such that the entire orbit $z_0 \mapsto z_1 \mapsto \cdots$ of z_0 under g is defined and is contained in Δ.

Lemma 6. *Such a polynomial-like map of degree d has $d-1$ critical points, counted with multiplicity, in the interior of Δ. The filled Julia set $K(g,\Delta)$ is connected if and only if it contains all of these $d-1$ critical points.*

Proof. Consider the nested sequence of compact sets

$$g(\Delta) \supset \Delta \supset g^{-1}(\Delta) \supset g^{-2}(\Delta) \supset \cdots$$

with intersection $K(g,\Delta)$. First suppose that the boundary $\partial\Delta$ contains no post-critical points, that is points $g^{\circ n}(\omega_i)$ with $n>0$ where ω_i is a critical point of g . Then clearly each $g^{-n}(\Delta)$ is a compact set bounded by one or more closed curves. In fact, each $g^{-n}(\Delta)$ is either a closed topological disk or a finite union of closed topological disks, each of which maps onto the entire disk Δ under $g^{\circ n}$. To see this, note that for each component B of $g^{-n}(\Delta)$ the image $g^{\circ n}(B)$ is compact, and that $g^{\circ n}$ maps the boundary ∂B into $\partial\Delta$ and maps the interior of B onto an open subset of the interior of Δ . Since the interior of Δ is connected, this implies that $g^{\circ n}(B)=\Delta$. If some such component B were not simply-connected, then some component B' of $\mathbf{C}\smallsetminus g^{-n}(\Delta)$ would be bounded. A similar argument would then show that $g^{\circ n}$ must map B' onto the complementary disk $\hat{\mathbf{C}}\smallsetminus\text{Interior}(\Delta)$, which is impossible.

Applying the Riemann-Hurwitz formula to the ramified covering $\Delta\to g(\Delta)$, we see that the number of critical points of g in the interior of Δ , counted with multiplicity, is equal to $d\,\chi(g(\Delta))-\chi(\Delta)=d\cdot 1-1$. (Here χ is the Euler characteristic.) Similarly, applying this formula to $g^{-1}(\Delta)\to\Delta$, we see that the number of critical points in $g^{-1}(\Delta)$ is equal to $d\,\chi(g^{-1}(\Delta)-\chi(\Delta)$. Thus all of the $d-1$ critical points are contained in $g^{-1}(\Delta)$ if and only if $\chi(g^{-1}(\Delta))=1$, so that $g^{-1}(\Delta)$ consists of a single topological disk. Similarly, it follows by induction that all of the critical points are contained in $g^{-n}(\Delta)$ if and only if $g^{-n}(\Delta)$ is a topological disk. If every $g^{-n}(\Delta)$ is a disk, then it follows that the set $K=\bigcap g^{-n}(\Delta)$ is connected. On the other hand, if some $g^{-n}(\Delta)$ consists of two or more disks, then each one of these disks must contain a point of K , which is therefore disconnected.

To complete the proof, we must allow for the possibility that $\partial\Delta$ may contain some post-critical point of g . Clearly there can be at most $d-1$ post-critical points in the annular region $g(\Delta)\smallsetminus\Delta$. Hence we can choose a disk $\Delta_1\subset g(\Delta)$ whose boundary avoids these post-critical points. If Δ_1 contains Δ in its interior, and also contains all critical values $g(\omega_i)$ in its interior, then it is easy to check that the pair $(g\,,\,g^{-1}(\Delta_1))$ is a polynomial-like map of the same degree, and with the same filled Julia set, but with no post-critical points in the disk boundary. The proof then proceeds as above. □

Remark. Douady and Hubbard prove much sharper statements: If $(g\,,\,\Delta)$ is polynomial-like of degree $d\geq 2$, with $K(g\,,\,\Delta)$ connected, then there exists a polynomial map ψ of degree d so that ψ on some neighborhood of $K(\psi)$ is quasi-conformally conjugate to g on a neighborhood of $K(g\,,\,\Delta)$. Furthermore, this quasi-conformal conjugacy can be chosen so as to satisfy the Cauchy-Riemann equations (in an appropriate sense) on the compact set $K(\psi)$. The polynomial map ψ is then uniquely determined up to affine conjugacy. In the case $d=2$, one has the further statement that ψ depends continuously on $(g\,,\,\Delta)$.

Now let us return to the situation of Theorem 5.

Lemma 7. *If the critical tableau is periodic of period* $p \geq 1$ *, then for any critical puzzle piece* $P_r(c_0)$ *with* r *sufficiently large, the pair* $(f^{\circ p}, P_r(c_0))$ *is polynomial-like of degree two. Furthermore, the critical orbit*

$$c_0 \mapsto c_p \mapsto c_{2p} \mapsto \cdots$$

under $f^{\circ p}$ *is completely contained in* $P_r(c_0)$ *, so that the filled Julia set* $K(f^{\circ p}, P_r(c_0))$ *is connected. In fact* $K(f^{\circ p}, P_r(c_0))$ *is equal to the intersection of the critical puzzle pieces* $\bigcap_k P_k(c_0)$ *, and hence is precisely equal to the connected component of* $K(f)$ *which contains* c_0 *.*

This is proved by a straightforward tableau argument. Details will be left to the reader. □

In this way, Branner and Hubbard show that each non-trivial component of $K(f)$ is homeomorphic to an appropriate quadratic Julia set. As examples, Figures 16 and 17 illustrate the case $p = k = 1$.

§3. An infinitely renormalizable non locally connected Julia set.

This section describes an unpublished example of Douady and Hubbard. It begins with an outline, with few proofs, of results from Douady and Hubbard [DH1], [DH2], [DH3], [D3].

Background Facts: Julia sets and the Mandelbrot set.

Let $f_c(z) = z^2 + c$. By definition, the ***Mandelbrot set*** M is the compact set consisting of all parameter values $c \in \mathbf{C}$ such that

$$\text{the Julia set } J(f_c) \text{ is connected} \qquad \Longleftrightarrow \qquad 0 \text{ has bounded orbit .}$$

However, it is often convenient to identify M with the corresponding set of polynomials f_c. A map $f_c \in M$ is hyperbolic (on its Julia set) if and only if it has a necessarily unique attracting periodic orbit. The hyperbolic maps in M form an open subset of the plane, and each connected component H in this open set is called a ***hyperbolic component*** in M. Let p be the period of the attracting orbit, and let $\lambda_p = \lambda_p(f_c) \in D$ be its multiplier. The basic facts about hyperbolic components in M are as follows:

(1) *Any two maps in the same hyperbolic component* $H \subset M$ *have attracting orbits of the same period* p. We call $p = p_H$ the ***period*** of H.

(2) *Each hyperbolic component* H *is conformally isomorphic to the open unit disk* D *under the correspondence* $f_c \mapsto \lambda_p(f_c)$. *In fact this correspondence extends uniquely to a homeomorphism between the closure* $\bar{H}$ *and the closed unit disk* $\bar{D}$.

In particular, each H has a unique ***center point*** c_H which maps to $\lambda_p(c_H) = 0$, and each boundary ∂H contains a unique ***root point*** $r_H \in \partial H$ which maps to $\lambda_p(r_H) = 1$. If the map f_c has a superattracting periodic orbit, then evidently c is the center point for one and only one hyperbolic component.

(3) *Similarly, if* f_r *has a parabolic periodic orbit, then* r *is the root point for one and only one hyperbolic component* H. *If the period of this orbit is* p *and the multiplier is* $\lambda_p = \exp(2\pi i m/p')$ *then the period of* H *is* pp'.

By definition, the ***principal hyperbolic component*** $H_\heartsuit$ is the set of f_c having an attracting fixed point. If the multiplier is $\lambda_1 \in D$ then a brief computation shows that $c = \lambda_1(2 - \lambda_1)/4$, hence

$$\lambda_1(c) \;=\; 1 - \sqrt{1 - 4c}\;,$$

taking that branch of the square root which lies in the right half plane. The boundary $\partial H_\heartsuit$ is the ***cardioid***, consisting of all points which have the form $c = e^{i\theta}(2 - e^{i\theta})/4$, so that $\lambda_1(c) = e^{i\theta}$.

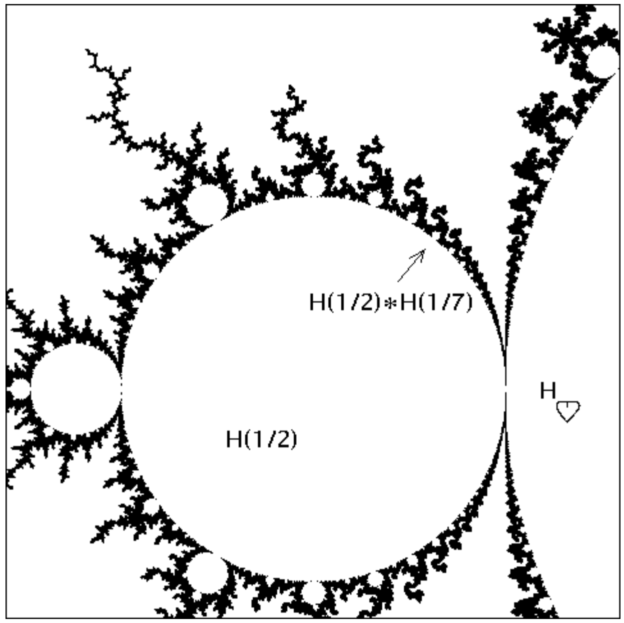


Figure 20. The period two component $H(1/2)$ *in the Mandelbrot set, with an arrow pointing to its satellite* $H(1/2) * H(1/7)$ *.*

Satellites. Given any hyperbolic component H of period $p \geq 1$ and any root of unity $e^{2\pi i m/p'} \neq 1$, it follows from (2) that there is a unique point $r \in \partial H$ with $\lambda_p(r) = e^{2\pi i m/p'}$. According to (3), this r is the root point of a new hyperbolic component H' of period $p\,p'$. We say that H' is a *satellite*, which is *attached* to H at *internal angle* m/p' . In the special case where H is the principal hyperbolic component $H_\heartsuit$, we will use the notation $H' = H(m/p')$ for this satellite, and in general we will use the notation $H' = H * H(m/p')$. (Compare the discussion of "tuning" below.) As an example, taking $m/p' = 1/2$, the point $r = -3/4$ is the root point of the period two component $H(1/2)$, consisting of all c with $|c + 1| < 1/4$.

External rays. We consider not only external rays for the Julia set in the z-plane (= *dynamic plane*), but also external rays for the Mandelbrot set in the c-plane (= *parameter plane*). Let $H \subset M$ be a hyperbolic component of period $p > 1$. Then exactly two external rays in $\mathbf{C} \smallsetminus M$ land at the root point r_H . Let $0 < a < b < 1$ be their angles. These angles have period exactly p under doubling, and hence can be expressed as fractions of the form $n/(2^p - 1)$. As an example, for the satellite component $H(1/p)$ we have $a = 1/(2^p - 1)$ and $b = 2/(2^p - 1)$.

For any $c \in H \cup \{r_H\}$ the corresponding external rays $R_a(c)$ and $R_b(c)$ in $\mathbf{C} \smallsetminus J(f_c)$ land at a periodic point which can be described as the "root point" of the Fatou component containing c . See Figure 21 for a schematic picture of the way these external rays are arranged in the parameter plane and in the dynamic plane. In particular

$$0 < a/2 < b/2 \;\leq\; a < b \;\leq\; 1 + a/2 < (1 + b)/2 < 1 \;,$$

where the $a/2$-ray and the $b/2$-ray always land at distinct points: exactly one of these two rays is periodic.

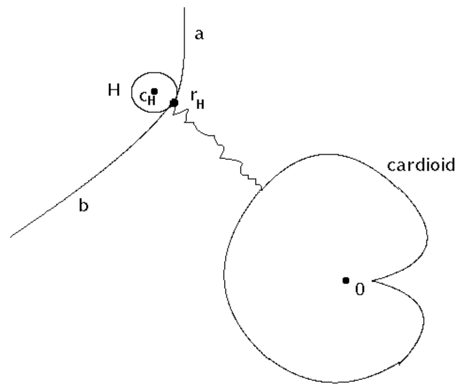
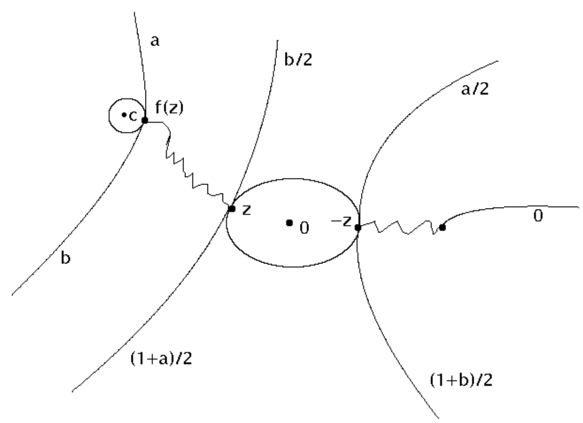


Figure 21. Schematic picture of external rays for a hyperbolic component $H \subset M$ in the c-plane,



and a corresponding picture in the dynamic plane.

Renormalization and Tuning. A map $f_c \in M$ is *renormalizable* if there is an integer $p \geq 2$ called the renormalization period, and a closed topological disk Δ in the z-plane, centrally symmetric about the origin, so that:

(1) $f^{\circ p-1}$ maps the image $f(\Delta)$ by a conformal isomorphism onto a disk $f^{\circ p}(\Delta)$ which contains Δ in its interior, and

(2) the entire orbit of 0 under $f^{\circ p}$ is contained in Δ.

The set of all $f_c \in M$ which are renormalizable of period p consists of a finite number of small copies of M (sometimes with the root point deleted). Each of these small copies contains a unique hyperbolic component of period p. Conversely, each hyperbolic component H of period $p \geq 2$ determines a small copy of M which can be described as the image of an associated mapping $c \mapsto H * c$, which embeds M homeomorphically onto a proper subset $H * M \subset M$. The elements of this form (with the possible exception of the root point r_H), are precisely the elements of M which are renormalizable of period p.

This embedding $c \mapsto H * c$ maps each hyperbolic component H' of period $p' \geq 1$ conformally onto a hyperbolic component $H * H'$ of period pp' in such a way that the

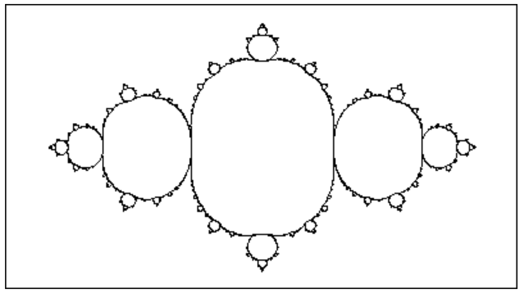
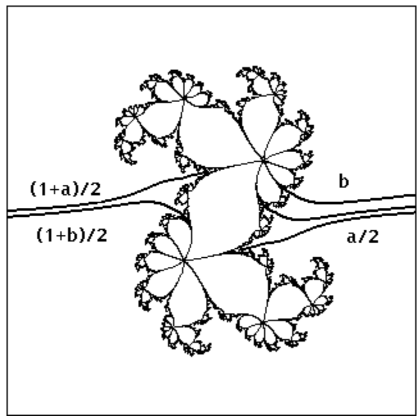
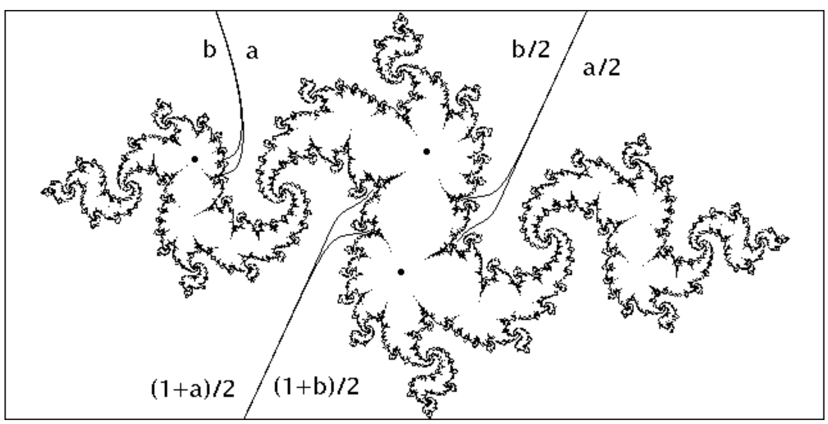


Figure 22. Julia set for the root point of $H(1/2)$ *,*



for the root point of $H(1/7)$ *(here the ray* $a = b/2$ *has not been labeled),*



and for the root point of $H(1/2) * H(1/7)$ *.*

multiplier $\lambda_{pp'}(H * c)$ is equal to $\lambda_{p'}(c)$. In particular, this embedding maps center points to center points and root points to root points. It carries the principal hyperbolic component $H_\heartsuit$ onto H itself, and maps the satellite $H(m/p')$ of $H_\heartsuit$ onto the satellite $H * H(m/p')$ of H . This $*$-product operation between hyperbolic components is associative, and has the principal hyperbolic component $H_\heartsuit$ as two-sided identity element. Thus it makes the collection of hyperbolic components into a monoid (=associative semigroup with identity) which operates as a semigroup of embeddings of M into itself. This monoid is free non-commutative.

Intuitively, the Julia set for $H*c'$ is obtained from the Julia set of a map f_c belonging to the hyperbolic component H by replacing each bounded component of $\mathbf{C}\setminus J(f_c)$ by a copy of $J(f_{c'})$ and defining the dynamics appropriately so that only the copy which is pasted in place of the critical component is mapped non-homeomorphically. The result is described as f_c ***tuned by*** $f_{c'}$.

In terms of external angles, let $a < b$ be the two external angles for the root point of H . These angles have periodic binary expansions of the form $.a_1 \cdots a_p\, a_1 \cdots a_p \cdots$ and $.b_1 \cdots b_p\, b_1 \cdots b_p \cdots$, both with period exactly equal to p . **Assertion:** If the point $c' \in \partial M$ is the landing point of a ray with angle $t \in \mathbf{R}/\mathbf{Z}$, then the image $H * c'$ is the landing point of a ray whose binary expansion can be obtained by inserting the p-tuple $a_1 \cdots a_p$ in place of each zero in the binary expansion of t and the p-tuple $b_1 \cdots b_p$ in place of each one. (See [D3].)

A non locally connected Julia set.

Consider a sequence of integers $1 < p_1 < p_2 < \cdots$.

> **Theorem 6.** *If this sequence diverges to infinity sufficiently rapidly, then the sequence of subsets* $H(1/p_1)*H(1/p_2)*\cdots*H(1/p_k)*M$ *intersects in a single point* $\omega \in M$ *with the property that the Julia set* $J(f_\omega)$ *is not locally connected.*

The proof begins as follows. It will be convenient to use the abbreviation $r(k)$ for the root point $r_{H(1/p_k)}$ on the cardioid. Start with any $p_1 > 1$ and consider the embedding $c \mapsto H(1/p_1)*c$ from M into itself, which carries the period 1 root point $r_{\heartsuit} = 1/4$ to the root point $r(1) = r_{H(1/p_1)}$. Since this embedding is continuous, we can place $H(1/p_1)*c$ as close as we like to $r(1)$ by choosing c close to $1/4$. In particular, if p_2 is sufficiently large, then the entire $1/p_2$-limb will be close to $1/4$ by the Yoccoz inequality, hence every point $c \in H(1/p_1) * H(1/p_2) * M$ will certainly be close to $r(1)$. Under such a small perturbation, note that the parabolic fixed point of $f_{r(1)}$ splits up into a repelling fixed point for f_c , together with a nearby period p_1 orbit. Thus, by choosing p_2 large, we can place this entire period p_1 orbit into an ϵ-neighborhood of the parabolic fixed point of $f_{r(1)}$.

Similarly, choosing p_3 even larger, for any $c \in H(p_1) * H(p_2) * H(p_3) * M$ we can guarantee that the parabolic period p_1 orbit for $H(1/p_1) * r(2)$ is replaced by a nearby period $p_1\, p_2$ orbit. In particular, we can guarantee that this new orbit lies within the $\epsilon + \epsilon/2$ neighborhood of the original parabolic fixed point. Continuing inductively, we can guarantee that all of the new orbits which are constructed lie within the 2ϵ neighborhood of the original fixed point. Furthermore, we can easily guarantee that these successive copies of M have diameter shrinking to zero.

Thus, if ω is the unique point in the intersection, we know that the Julia set $J(f_\omega)$ contains orbits of period 1 , p_1 , $p_1\, p_2$, ... , all lying within the 2ϵ neighborhood of the original parabolic fixed point. These various periodic points can be described as the landing points of external rays of angles $a_k < b_k$ where $0 < a_1 < a_2 < \cdots < b_2 < b_1$. Furthermore, it is easy to check that the difference $b_k - a_k$ tends rapidly to zero, so that $\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k$. On the other hand, as noted in the preceding section, the external rays of angle $a_k/2$ and $b_k/2$ land at different points, which are negatives of each other. Each such landing point is either close to the original parabolic fixed point, or close to its negative. Thus we have a sequence of angles $a_k/2$ and $b_k/2$ tending to a common limit, yet the landing points of the corresponding external rays for $J(f_\omega)$ do not tend to a common limit. According to Caratheodory, this implies that $J(f_\omega)$ is not locally connected. □

More generally, if H_1 , H_2 , ... is any sequence of hyperbolic components which are not centered along the real axis, then it is again true that $\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k$. Let $\omega \in M$ be any element of the intersection $\bigcap_k H_1 * H_2 * \cdots * H_k * M$. Whenever the landing points in $J(f_\omega)$ of the rays of angle $a_k/2$ fail to converge to the critical point, it follows that $J(f_\omega)$ is not locally connected. In particular, this is true whenever the sequence of components H_k converges sufficiently rapidly to the root point $r_\heartsuit = 1/4$.

This seems to be the only known obstruction to local connectivity in the infinitely renormalizable case. Thus whenever the landing points of the $a_k/2$ rays do converge to the critical point, we can ask whether $J(f_\omega)$ is in fact locally connected.

Appendix. Length-Area-Modulus Inequalities.

The most basic length-area inequality is the following. Let $I^2 \subset \mathbf{C}$ be the open unit square consisting of all $z = x + iy$ with $0 < x < 1$ and $0 < y < 1$. By a *conformal metric* on I^2 we mean a metric of the form

$$ds = \rho(z)|dz|$$

where $z \mapsto \rho(z) > 0$ is any strictly positive continuous real valued function on the open square. In terms of such a metric, the *length* of a smooth curve $\gamma : (a, b) \to I^2$ is defined to be the integral

$$L_\rho(\gamma) = \int_a^b \rho(\gamma(t))|d\gamma(t)| ,$$

and the *area* of a region $U \subset I^2$ is defined to be

$$\text{area}_\rho(U) = \int\int_U \rho(x + iy)^2 dx\, dy .$$

In the special case of the Euclidean metric $ds = |dz|$, with $\rho(z)$ identically equal to 1, the subscript ρ will be omitted.

Theorem A.1. *If* $\text{area}_\rho(I^2)$ *(the integral over the entire square) is finite, then for Lebesgue almost every* $y \in (0, 1)$ *the length* $L_\rho(\gamma_y)$ *of the horizontal line* $\gamma_y : t \mapsto (t, y)$ *at height* y *is finite. Furthermore, there exists* y *so that*

$$L_\rho(\gamma_y)^2 \le \text{area}_\rho(I^2) . \tag{1}$$

In fact, the set consisting of all $y \in (0, 1)$ *for which this inequality is satisfied has positive Lebesgue measure.*

Remark 1. Evidently this inequality is best posible. For in the case of the Euclidean metric $ds = |dz|$ we have

$$L(\gamma_y)^2 = \text{area}(I^2) = 1 .$$

Remark 2. It is essential here that we use a square, rather than a rectangle. If we consider instead a rectangle R with base Δx and height Δy, then the corresponding inequality would be

$$L_\rho(\gamma_y)^2 \le \frac{\Delta x}{\Delta y}\text{area}_\rho(R) \tag{2}$$

for a set of y with positive measure.

Proof of A.1. We use the Schwarz inequality

$$\left(\int_a^b f(x)g(x)\,dx\right)^2 \le \left(\int_a^b f(x)^2\,dx\right)\cdot\left(\int_a^b g(x)^2\,dx\right) ,$$

which says (after taking a square root) that the inner product of any two vectors in the Euclidean vector space of square integrable real functions on an interval is less than or equal to the product of their norms. We may as well consider the more general case of a rectangle $R=(0,\,\Delta x)\times(0,\,\Delta y)$. Taking $f(x)\equiv 1$ and $g(x)=\rho(x,y)$ for some fixed y , we obtain

$$\left(\int_0^{\Delta x}\rho(x,y)\,dx\right)^2 \;\le\; \Delta x\int_0^{\Delta x}\rho(x,y)^2\,dx\ ,$$

or in other words

$$L_\rho(\gamma_y)^2 \;\le\; \Delta x\int_0^{\Delta x}\rho(x,y)^2 dx\ ,$$

for each constant height y . Integrating this inequality over the interval $0<y<\Delta y$ and then dividing by Δy , we get

$$\frac{1}{\Delta y}\int_0^{\Delta y}L_\rho(\gamma_y)^2 dy \;\le\; \frac{\Delta x}{\Delta y}\,\mathrm{area}_\rho(A)\ . \qquad (3)$$

In other words, the ***average*** over all y in the interval $(0,\,\Delta y)$ of $L_\rho(\gamma_y)^2$ is less than or equal to $\frac{\Delta x}{\Delta y}\,\mathrm{area}_\rho(A)$. Further details of the proof are straightforward. □

Now let us form a cylinder $\mathcal{C}$ of circumference Δx and height Δy by gluing the left and right edges of our rectangle together. More precisely, let $\mathcal{C}$ by the quotient space which is obtained from the infinitely wide strip $0<y<\Delta y$ in the z-plane by identifying each point $z=x+iy$ with its translate $z+\Delta x$. Define the ***modulus*** $\mathrm{mod}(\mathcal{C})$ of such a cylinder to be the ratio $\Delta y/\Delta x$ of height to circumference. By the ***winding number*** of a closed curve γ in $\mathcal{C}$ we mean the integer

$$w \;=\; \frac{1}{\Delta x}\oint_\gamma dx\ .$$

Theorem A.2 (Length-Area Inequality for Cylinders). *For any conformal metric $\rho(z)|dz|$ on the cylinder $\mathcal{C}$ there exists some simple closed curve γ with winding number $+1$ whose length $L_\rho(\gamma)=\oint_\gamma\rho(z)|dz|$ satisfies the inequality*

$$L_\rho(\gamma)^2 \;\le\; \mathrm{area}_\rho(A)/\mathrm{mod}(A)\ . \qquad (4)$$

Furthermore, this result is best possible: If we use the Euclidean metric $|dz|$ then

$$L(\gamma)^2 \;\ge\; \mathrm{area}(A)/\mathrm{mod}(A) \qquad (5)$$

*for **every** such curve γ .*

Proof. Just as in the proof of A.1, we find a horizontal curve γ_y with

$$L_\rho(\gamma_y)^2 \;\le\; \frac{\Delta x}{\Delta y}\mathrm{area}_\rho(\mathcal{C}) \;=\; \frac{\mathrm{area}_\rho(\mathcal{C})}{\mathrm{mod}(\mathcal{C})}\ .$$

On the other hand in the Euclidean case, for any closed curve γ of winding number one we have

$$L(\gamma) \;=\; \oint_\gamma|dz| \;\ge\; \oint_\gamma dx \;=\; \Delta x\ ,$$

hence $L(\gamma)^2 \;\geq\; (\Delta x)^2 \;=\; \text{area}(\mathcal{C})/\text{mod}(\mathcal{C})$. □

Definitions. A Riemann surface A is said to be an ***annulus*** if it is conformally isomorphic to some cylinder. An embedded annulus $A \subset \mathcal{C}$ is said to be ***essentially embedded*** if it contains a curve which has winding number one around $\mathcal{C}$.
Here is an important consequence of Theorem A.2.

Corollary A.3 (An Area-Modulus Inequality). *Let $A \subset \mathcal{C}$ be an essentially embedded annulus in the cylinder $\mathcal{C}$, and suppose that A is conformally isomorphic to a cylinder $\mathcal{C}'$. Then*

$$\text{mod}(\mathcal{C}') \;\leq\; \frac{\text{area}(A)}{\text{area}(\mathcal{C})}\,\text{mod}(\mathcal{C})\;. \tag{6}$$

In particular:

$$\text{mod}(\mathcal{C}') \;\leq\; \text{mod}(\mathcal{C})\;. \tag{7}$$

Proof. Let $\zeta \mapsto z$ be the embedding of $\mathcal{C}'$ onto $A \subset \mathcal{C}$. The Euclidean metric $|dz|$ on $\mathcal{C}$, restricted to A , pulls back to some conformal metric $\rho(\zeta)|d\zeta|$ on $\mathcal{C}'$, where $\rho(\zeta) = |dz/d\zeta|$. According to A.2, there exists a curve γ' with winding number 1 about $\mathcal{C}'$ whose length satisfies

$$L_\rho(\gamma')^2 \leq \text{area}_\rho(\mathcal{C}')/\text{mod}(\mathcal{C}')\;.$$

This length coincides with the Euclidean length $L(\gamma)$ of the corresponding curve γ in $A \subset \mathcal{C}$, and $\text{area}_\rho(\mathcal{C}')$ is equal to the Euclidean area $\text{area}(A)$, so we can write this inequality as

$$L(\gamma)^2 \leq \text{area}(A)/\text{mod}(\mathcal{C}')\;.$$

But according to (5) we have

$$\text{area}(\mathcal{C})/\text{mod}(\mathcal{C}) \;\leq\; L(\gamma)^2\;.$$

Combining these two inequalities, we obtain

$$\text{area}(\mathcal{C})/\text{mod}(\mathcal{C}) \;\leq\; \text{area}(A)/\text{mod}(\mathcal{C}')\;,$$

which is equivalent to the required inequality (6). □

Corollary A.4. *The modulus of a cylinder is a well defined conformal invariant.*

Proof. If $\mathcal{C}'$ is conformally isomorphic to $\mathcal{C}$ then (7) asserts that $\text{mod}(\mathcal{C}') \leq \text{mod}(\mathcal{C})$, and similarly $\text{mod}(\mathcal{C}) \leq \text{mod}(\mathcal{C}')$. □

It follows that the ***modulus*** of an annulus A can be defined as the modulus of any conformally isomorphic cylinder. Furthermore, if A is essentially embedded in some other annulus A' , then $\text{mod}(A) \leq \text{mod}(A')$.

Corollary A.5 (Grötzsch Inequality). *Suppose that $A' \subset A$ and $A'' \subset A$ are two disjoint annuli, each essentially embedded in A . Then*

$$\text{mod}(A') + \text{mod}(A'') \;\leq\; \text{mod}(A)\;.$$

Proof. We may assume that A is a cylinder $\mathcal{C}$. According to (6) we have

$$\mathrm{mod}(A') \le \frac{\mathrm{area}(A')}{\mathrm{area}(\mathcal{C})}\,\mathrm{mod}(\mathcal{C})\;, \qquad \mathrm{mod}(A'') \le \frac{\mathrm{area}(A'')}{\mathrm{area}(\mathcal{C})}\,\mathrm{mod}(\mathcal{C})\;.$$

where all areas are Euclidean. Using the inequality

$$\mathrm{area}(A') + \mathrm{area}(A'') \le \mathrm{area}(\mathcal{C})\;,$$

the conclusion follows. □

Up to this point, we have only considered cylinders or annuli of finite modulus. If we take an arbitrary Riemann surface with free cyclic fundamental group, then it is always conformally isomorphic to some cylinder, providing that we allow also the possibility of a one-sided infinite or two-sided infinite cylinder (that is, either the upper half-plane or the full complex plane modulo the identification $z \equiv z+1$). By definition, a Riemann surface conformally isomorphic to such an infinite cylinder will be called an ***annulus of infinite modulus.***

Now consider the following situation. Let $U \subset \mathbf{C}$ be a bounded simply connected open set, and let $K \subset U$ be a compact subset such that the difference $A = U \smallsetminus K$ is an annulus (which may have finite or infinite modulus).

Corollary A.6. *Suppose that $K \subset U$ as described above. Then K reduces to a single point if and only if the annulus $A = U \smallsetminus K$ has infinite modulus. Furthermore, the diameter of K is bounded by the inequality*

$$4\,\mathrm{diam}(K)^2 \le \frac{\mathrm{area}(A)}{\mathrm{mod}(A)} \le \frac{\mathrm{area}(U)}{\mathrm{mod}(A)}\;. \tag{8}$$

Proof. According to A.2, there exists a curve with winding number one about A whose length satisfies $L^2 \le \mathrm{area}(A)/\mathrm{mod}(A)$. Since K is enclosed within this curve, it follows easily that $\mathrm{diam}(K) \le L/2$, and the inequality (8) follows. Conversely, if K is a single point then using (7) we see easily that $\mathrm{mod}(A) = \infty$. □

Corollary A.7. (Branner-Hubbard). *Let $K_1 \supset K_2 \supset K_3 \supset \cdots$ be compact subsets of $\mathbf{C}$ with each K_{n+1} contained in the interior of K_n. Suppose further that each interior K_n^o is simply connected, and that each difference $A_n = K_n^o \smallsetminus K_{n+1}$ is an annulus. If $\sum_1^\infty \mathrm{mod}(A_n)$ is infinite, then the intersection $\bigcap K_n$ reduces to a single point.*

Proof. It follows inductively from the Grötzsch inequality that the modulus $\mathrm{mod}(K_1^o \smallsetminus K_n)$ tends to infinity as $n \to \infty$. Hence by (7) and A.7 the intersection of the K_n is a point. □

Corollary A.8 (McMullen Inequality). *Again suppose that $K \subset U \subset \mathbf{C}$, and that $A = U \smallsetminus K$ is an annulus. Then*

$$\mathrm{area}(K) \le \mathrm{area}(U)/e^{4\pi\,\mathrm{mod}(A)}\;. \tag{9}$$

Proof. (Compare [BH, II].) We will first prove the weaker inequality

$$\mathrm{area}(K) \;\le\; \frac{\mathrm{area}(U)}{1+4\pi\,\mathrm{mod}(A)}\;. \tag{10}$$

The *isoperimetric inequality* asserts that the area enclosed by a plane curve of length L is at most $L^2/(4\pi)$, with equality if and only if the curve is a round circle. (See for example [CR].) Combining this with the proof of A.6, we see that

$$\mathrm{area}(K) \;\le\; \frac{L^2}{4\pi} \;\le\; \frac{\mathrm{area}(A)}{4\pi\,\mathrm{mod}(A)}\;.$$

Writing this inequality as $4\pi\,\mathrm{mod}(A) \le \mathrm{area}(A)/\mathrm{area}(K)$ and adding $+1$ to both sides we obtain the completely equivalent inequality $1+4\pi\,\mathrm{mod}(A) \le \mathrm{area}(U)/\mathrm{area}(K)$. This in turn is equivalent to (10).

To sharpen this inequality, we proceed as follows. Cut the annulus A up into n concentric annuli A_i, each of modulus equal to $\mathrm{mod}(A)/n$. Let K_i be the bounded component of the complement of A_i, and assume that these annuli are nested so that $A_i \cup K_i = K^o_{i+1}$ with $K_1 = K$. Let $K^o_{n+1} = A \cup K = U$. Then

$$\mathrm{area}(K_{i+1})/\mathrm{area}(K_i) \;\ge\; 1+4\pi\,\mathrm{mod}(A)/n$$

by (10), hence

$$\mathrm{area}(U)/\mathrm{area}(K) \;\ge\; (\,1\;+\;4\pi\,\mathrm{mod}(A)/n\,)^n\;,$$

where the right hand side converges to $e^{4\pi\,\mathrm{mod}(A)}$ as $n\to\infty$. □

As a final illustration of modulus-area inequalities, consider a *flat torus* $\mathbf{T} = \mathbf{C}/\Lambda$. Here $\Lambda \subset \mathbf{C}$ is to be a 2-dimensional *lattice*, that is an additive subgroup of the complex numbers, spanned by two elements λ_1 and λ_2 where $\lambda_1/\lambda_2 \notin \mathbf{R}$. Let $A \subset \mathbf{T}$ be an embedded annulus.

By the "*winding number*" of A in $\mathbf{T}$ we will mean the lattice element $w \in \Lambda$ which is constructed as follows. Under the universal covering map $\mathbf{C} \to \mathbf{T}$, the central curve of A lifts to a curve segment which joins some point $z_0 \in \mathbf{C}$ to a translate $z_0 + w$ by the required lattice element. We say that $A \subset \mathbf{T}$ is an *essentially embedded annulus* if $w \ne 0$.

Corollary A.9 (Bers Inequality). *If the annulus A is embedded in the flat torus $\mathbf{T} = \mathbf{C}/\Lambda$ with winding number $w \in \Lambda$, then*

$$\mathrm{mod}(A) \;\le\; \frac{\mathrm{area}(T)}{|w|^2}\;. \tag{11}$$

Roughly speaking, if A winds many times around the torus, so that $|w|$ is large, then A must be very skinny. A slightly sharper version of this inequality is given in Problem A-3 below.

Proof. Choose a cylinder C' which is conformally isomorphic to A. The Euclidean metric $|dz|$ on $A \subset \mathbf{T}$ corresponds to some metric $\rho(\zeta)|d\zeta|$ on C', with

$$\mathrm{area}_\rho(C') \;=\; \mathrm{area}(A)\;.$$

By A.2 we can choose a curve γ' of winding number one on C', or a corresponding curve γ on $A \subset \mathbf{T}$, with

$$L(\gamma)^2 \;=\; L_\rho(\gamma')^2 \;\le\; \frac{\mathrm{area}_\rho(C')}{\mathrm{mod}(C')} \;=\; \frac{\mathrm{area}(A)}{\mathrm{mod}(A)} \;\le\; \frac{\mathrm{area}(T)}{\mathrm{mod}(A)}\;.$$

Now if we lift γ to the universal covering space $\mathbf{C}$ then it will join some point z_0 to z_0+w. Hence its Euclidean length $L(\gamma)$ must satisfy $L(\gamma)\ge|w|$. Thus

$$|w|^2 \;\le\; \frac{\mathrm{area}(T)}{\mathrm{mod}(A)}\;,$$

which is equivalent to the required inequality (11). □

Concluding Problems:

Problem A-1. In the situation of Theorem A.1, show that more than half of the horizontal curves γ_y have length $L_\rho(\gamma_y)\le\sqrt{2\,\mathrm{area}_\rho(I^2)}$. (Here "more than half" is to be interpreted in the sense of Lebesgue measure.)

Problem A-2. Show that the converse to A.7 is false: A nest $K_1\supset K_2\supset K_3\supset\cdots$ of compact subsets of $\mathbf{C}$ may intersect in a point even if $\sum_1^\infty \mathrm{mod}(K_n^o\smallsetminus K_{n+1})$ is finite. (For example, do this by showing that a closed disk $\overline{D}'$ of radius $1/2$ can be embedded in the open unit disk D so that the complementary annulus $A=D\smallsetminus\overline{D}'$ has modulus arbitrarily close to zero.)

Problem A-3 (Sharper Bers Inequality). If the flat torus $\mathbf{T}=\mathbf{C}/\Lambda$ contains several disjoint annuli A_i, all with the same "winding number" $w\in\Lambda$, show that

$$\sum \mathrm{mod}(A_i) \;\le\; \mathrm{area}(\mathbf{T})/|w|^2\;.$$

If two essentially embedded annuli are disjoint, show that they necessarily have the same winding number.

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Renormalization and Tuning

ERRATA for

"Local Connectivity of Julia Sets: Expository Lectures"

These notes make the following claim on p. 32:

> *Any map $f_c(z) = z^2 + c$ in the Mandelbrot set M which is renormalizable must belong to one of the small embedded copies of M which are constructed by Douady-Hubbard tuning.*

Curt McMullen points out that this is false; here is a counter-example. Let

$$c = c_1 \approx .41964338 + .60629073\, i\ ,$$

so that the polynomial $f = f_c$ is post-critically finite, with critical orbit

$$0 \;\mapsto\; c_1 \;\mapsto\; c_2 \;\mapsto\; c_3 \;\leftrightarrow\; c_4 = -c_2\ .$$

(See Figure A. The number c can be described as the landing point of the external ray of angle $1/12$ for M. It is nearly the right-most point on the $1/4$-limb of M.) This map f is renormalizable of period 2. In fact the union

$$\Delta_0 \;=\; \widehat{P}_3(0) \cup \widehat{P}_2(c_2) \cup \widehat{P}_2(-c_2)$$

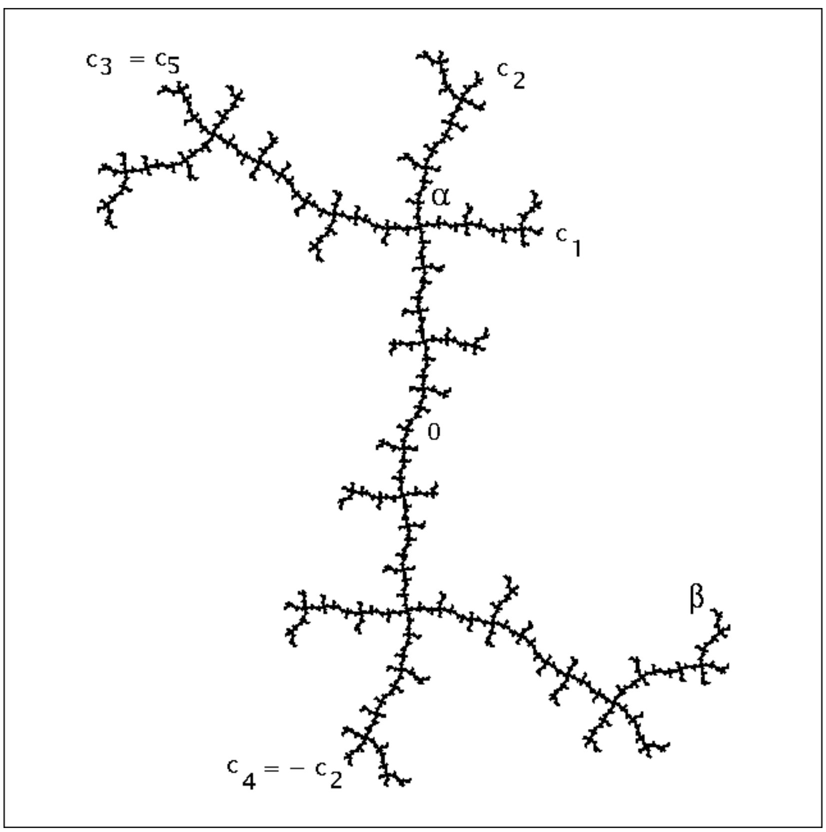

Figure A. Julia set for $f : z \mapsto z^2 + .41964338 + .60629073\, i$.

is a topological disk, centrally symmetric about the origin, and the composition $f^{\circ 2}$ maps Δ_0 by a two-fold branched covering onto a disk

$$\Delta_2 \;=\; \widehat{P}_0(0) \cup \widehat{P}_1(c_2)$$

which compactly contains it. Furthermore, the orbit of 0 under $f^{\circ 2}$ is completely contained in Δ_0. (For definitions and notation, see pp. 11, 12, 32.) The proof is not difficult. In this example, the quadratic-like map $f^{\circ 2}|\Delta_0$ is hybrid-equivalent (in the sense of [DH3]) to the Tchebycheff map $z \mapsto z^2 - 2$. In particular, the filled Julia set $K_0 = K(f^{\circ 2}|\Delta_0)$ is homeomorphic to the filled Julia set for the Tchebycheff map, which is equal to the interval $[-2, 2]$.

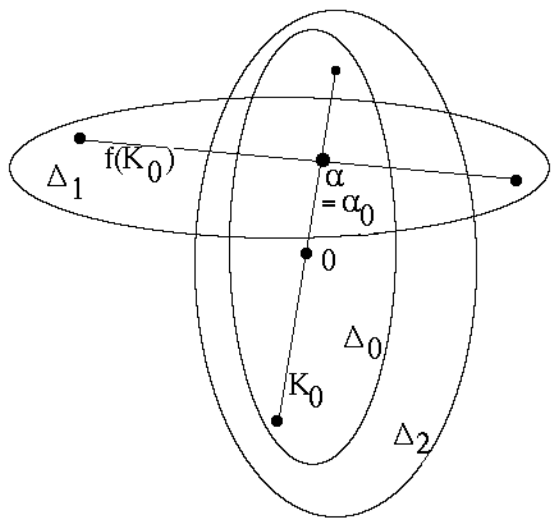

Figure B. Schematic picture of this renormalization.

Note that the critical tableau associated with this map f is not periodic. In fact the orbit of the critical point 0 never returns to the puzzle piece $P_1(0)$. It follows that f cannot be obtained by tuning.

As a different example, let c be any point in a “satellite” hyperbolic component which is attached to the cardioid at internal angle p/q. (See p. 31.) Then McMullen shows that f_c is renormalizable, not only of period q, but also of period n for any divisor n of q.

The curious feature of all of these examples is that the disks Δ_0 and $\Delta_1 = f(\Delta_0)$ cross over each other in a neighborhood of the fixed point α. In fact this fixed point belongs both to the filled Julia set $K_0 = K(f^{\circ 2}|\Delta_0)$ and to its image $f(K_0)$. (See Figure B.) Furthermore, α separates each of these sets.

More generally, let $g = f_{c_1}$ be any renormalizable quadratic map, with $g^{\circ n} : \Delta_0 \to \Delta_n \supset \Delta_0$. The associated filled Julia set $K_0 = K(g^{\circ n}|\Delta_0)$ has two fixed points which we can label as α_0 and β_0. Here β_0 is the non-separating fixed point of $g^{\circ n}|K_0$, corresponding to the end-point of the zero ray for the hybrid-equivalent quadratic map. McMullen shows that a forward image $g^{\circ i}(K_0)$ with $0 < i < n$ can intersect K_0 at most in a single repelling fixed point of $g^{\circ n}$.

Definition: The renormalizable map g is *simply-renormalizable* if the sets K_0 and $g^{\circ i}(K_0)$ intersect at most in the non-separating fixed point β_0 .

This concept will be studied in a forthcoming manuscript by McMullen.

Here is a more detailed list of corrections and comments.

p. 2, Theorem 1: The statement can be sharpened by substituting "simply-renormalizable" in Hypothesis (3). Compare the discussion of Lemmas 2 and 3 below.

p. 2, Hypothesis (3′) : McMullen shows that a map is infinitely renormalizable if and only if it is infinitely simply-renormalizable.

p. 3: For [HF] read [FH].

p. 10. Proof of (b): For $d' > n$ read $d' > d$ and for $d' - 3k$ read $d' - 2k$.

p. 12, Lemma 2: The converse is false as stated. In fact McMullen shows that the critical tableau is periodic if and only if f is simply-renormalizable. (Here we assume as usual that the fixed point α is not parabolic, and also that the critical orbit does not terminate on α so that the critical tableau is well defined.)

p. 12, Lemma 3: Under the hypothesis of this lemma, he shows that f is simply-renormalizable of period q . It follows that the Corollary on p. 13 applies whenever f is not simply-renormalizable.

p. 16, Problem 1-5: The proof as outlined depends on the claim that a generic map in the boundary of the Mandelbrot set is not renormalizable, although the proof sketched in the footnote shows only that it cannot be obtained by tuning. McMullen and also J. Kwapisz point out that the easiest way to prove this statement, as well as Problem 1-5, is by a quite different argument which makes no reference to puzzles. Note first that the family of polynomial maps $c \mapsto f_c^{\circ n}(0)$ fails to be normal in a neighborhood of c precisely for c belonging to the boundary ∂M . Let $U(p, \epsilon)$ be the set of $c \in \partial M$ such that the critical orbit of f_c intersects the ϵ-neighborhood of every cycle of period p . This set is clearly open in ∂M , and using Montel's Theorem one can check that it is dense when $p > 1$. Therefore, a generic f_c in ∂M belongs to the intersection of the $U(p, \epsilon)$. It follows that the critical orbit for a generic f_c is dense in the Julia set $J(f_c)$. As a corollary, it follows that a generic $f_c \in \partial M$ is not renormalizable, since a renormalizable map must have critical orbit bounded away from its β fixed point.

p. 32, Renormalization and Tuning. Replace lines 9–3 from the bottom by:
It seems reasonable to conjecture that a map $f \in M$ is simply-renormalizable of period n if and only if

(1) it belongs to one of the small copies $H * M \subset M$ which are obtained by Douady-Hubbard tuning, where H ranges over the hyperbolic components of period n , and

(2) it is not the root point of a satellite hyperbolic component of period n .

Compare [DH3, p. 332].

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