

Hyperbolicity is Dense in the Real Quadratic Family

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Abstract

It is shown that for non-hyperbolic real quadratic polynomials topological and quasymmetric conjugacy classes are the same.

By quasiconformal rigidity, each class has only one representative in the quadratic family, which proves that hyperbolic maps are dense.

1 Fundamental concepts

1.1 Introduction

Statement of the results. **Dense Hyperbolicity Theorem** *In the real quadratic family*

$$f_a(x) = ax(1 - x), \quad 0 < a \leq 4$$

the mapping f_a has an attracting cycle, and thus is hyperbolic on its Julia set, for an open and dense set of parameters a .

What we actually prove is this:

Main Theorem *Let f and \hat{f} be two real quadratic polynomials with a bounded forward critical orbit and no attracting or indifferent cycles. Then, if they are topologically conjugate, the conjugacy extends to a quasiconformal conjugacy between their analytic continuations to the complex plane.*

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Derivation of the Dense Hyperbolicity Theorem. We show that the Main Theorem implies the Dense Hyperbolicity Theorem. Quasiconformal conjugacy classes of normalized complex quadratic polynomials are known to be either points or open (see [21].) We remind the reader, see [25], that the kneading sequence is aperiodic for a real quadratic polynomial precisely when this polynomial has no attracting or indifferent periodic orbits. Therefore, by the Main Theorem, topological conjugacy classes of real quadratic polynomials with aperiodic kneading sequences are either points or open in the space of real parameters a . On the other hand, it is an elementary observation that the set of polynomials with the same aperiodic kneading sequence in the real quadratic family is also closed. So, for every aperiodic kneading sequence there is at most one polynomial in the real quadratic family with this kneading sequence.

Next, between two parameter values a_1 and a_2 for which different kneading sequences occur, there is a parameter a so that f_a has a periodic kneading sequence. So, the only way the Dense Hyperbolicity Theorem could fail is if there were an interval filled with polynomials without attracting periodic orbits and yet with periodic kneading sequences. Such polynomials would all have to be parabolic (have indifferent periodic orbits). It well-known, however, by the work of [6], that there are only countably many such polynomials. The Dense Hyperbolicity Theorem follows.

Consequences of the theorems. The Dense Hyperbolicity Conjecture had a long history. In a paper from 1920, see [7], Fatou expressed the belief that “general” (generic in today’s language?) rational maps are expanding on the Julia set. Our result may be regarded as progress in the verification of his conjecture. More recently, the fundamental work of Milnor and Thurston, see [25], showed the monotonicity of the kneading invariant in the quadratic family. They also conjectured that the set of parameter values for which attractive periodic orbits exist is dense, which means that the kneading sequence is strictly increasing unless it is periodic. The Dense Hyperbolicity Theorem implies Milnor and Thurston’s conjecture. Otherwise, we would have an interval in the parameter space filled with polynomials with an aperiodic kneading sequence, in a clear violation of the Dense Hyperbolicity Theorem.

The Main Theorem and other results. Yoccoz, [28], proved that a non-hyperbolic quadratic polynomial with a fixed non-periodic kneading sequence is unique up to an affine conjugacy unless it is infinitely renormalizable. Thus, we only need to prove our Main Theorem if the maps are infinitely renormalizable. However, our approach automatically gives a proof for all non-hyperbolic polynomials, so we provide an independent argument. The work of [27] proved the Main Theorem for infinitely renormalizable polynomials of bounded combinatorial type. The paper [18] proved the Main Theorem for some infinitely renormalizable quadratic polynomials not covered by [27]. A different approach to Fatou’s conjecture was taken in a recent paper [22]. That work proves that there is no invariant line field on the Julia set of an infinitely renormalizable real polynomial. This result implies that there is no non-hyperbolic component of the Mandelbrot set containing this real polynomial in its interior, however it is not known if it can also imply the Dense Hyperbolicity Theorem.

Beginning of the proof. Our method is based on the direct construction of a quasiconformal conjugacy and relies on techniques developed in [17] and [18].

The pull-back construction shown in [27] allows one to pass from quasisymmetric conjugacy classes on the real line to quasiconformal conjugacies in the complex plane. Thus, the Main Theorem is reduced to conjugacies on the real line.

We can reduce the proof of the Main Theorem, to the following Reduced Theorem:

Reduced Theorem *Let f and \hat{f} be two real quadratic polynomials with the same aperiodic kneading sequence and bounded forward critical orbits. Normalize them to the form $x \rightarrow ax(1-x)$. Then the conjugacy between f and \hat{f} on the interval $[0, 1]$ is quasisymmetric.*

This paper relies heavily on [17], which describes the inducing process on the real line, and [18] which worked out many ideas and estimates that we use.

Questions remaining. Does the Main Theorem remain true for quadratic S-unimodal maps? I believe that the difficulties here are only technical in nature and the answer should be affirmative. However, in that case the Dense Hyperbolicity Theorem will not follow from the Main Theorem.

Is the Main Theorem true for unimodal polynomials of higher even degree? The present proof uses degree 2 in the proof of Theorem 4. Theorem 4 seems an irreplaceable element of the proof.

Our result implies that the polynomials for which a homoclinic tangency occurs (the critical orbit meets a repelling periodic orbit) are dense in the set of non-hyperbolic polynomials. On the other hand, in view of [15], those correspond to density points in the parameter space of the set of polynomials with an absolutely continuous invariant measure. It seems reasonable to conjecture that the set of non-hyperbolic maps without an absolutely continuous invariant measure has 0 Lebesgue measure. This problem was posed by J. Palis during the workshop in Trieste in June 1992. We do not know of any recent progress in solving this problem.

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1.2 Outline of the paper.

In order to prove the Reduced Theorem we apply the inducing construction, essentially similar to the one used in [18], to f and \hat{f} . We also develop the technique for constructing quasiconformal "branchwise equivalences" in a parallel pull-back construction. The infinitely renormalizable case is treated by constructing a "saturated map" on each stage of renormalization, together with a uniformly quasisymmetric branchwise equivalence, and sewing them to get the quasisymmetric conjugacy. These are the same ideas as used in [18].

The rest of section 1 is devoted to defining and introducing main concepts of the proof. We also reduce the Reduced Theorem to an even simpler Theorem 1. Theorem 1 allows one to eliminate renormalization from the picture and proceed in almost the same way in renormalizable and non-renormalizable cases.

The results of section 2 are summarized in Theorem 2. Theorem 2 represents the beginning stage in the construction of induced mappings and branchwise equivalences. Our main technique of “complex pull-back”, introduced later in section 3, may not immediately apply to high renormalizations of a polynomial, since those are not known to be complex polynomial-like in the sense of [6]. For this reason we are forced to proceed mostly by real methods introduced in [18]. This section also contains an important new lemma about nearly parabolic S-unimodal mappings, i.e. Proposition 2.

In section 3 we introduce our powerful tool for constructing quasiconformal branchwise equivalences. This combines certain ideas of citekus (internal marking) with complex pull-back similar to a construction used in [5]. The main features of the construction are described by Theorem 3. We then proceed to prove Theorem 4. Theorem 4 describes the conformal geometry of our so-called “box case”, which is somewhat similar to the persistently recurrent case studied by [28]. Another proof of Theorem 4 can be found in [10]. However, the proof we give is simpler once we can apply our technique of complex pull-back of branchwise equivalences.

In section 4 we apply the complex pull-back construction to the induced objects obtained by Theorem 2. Estimates are based on Theorems 3 and 4. The results of this section are given by Theorem 5.

Section 5 concludes the proof of Theorem 1 from Theorem 5. The construction of saturated mappings follows the work of [18] quite closely.

The Appendix contains a result related to Theorem 4 and illustrates the technique of separating annuli on which the work of [10], referenced from this paper, is based. The result of the Appendix is not, however, an integral part of the proof of our main theorems.

To help the reader (and the author as well) to cope with the size of the paper, we tried to make all sections, with the exception of section 1, as independent as possible. Cross-section references are mostly limited to the Theorems so that, hopefully, each section can be studied independently.

1.3 Induced mappings

We define a class of unimodal mappings.

Definition 1.1 *For $\eta > 0$, we define the class \mathcal{F}_η to comprise all unimodal mappings of the interval $[0, 1]$ into itself normalized so that 0 is a fixed point which satisfy these conditions:*

- *Any $f \in \mathcal{F}$ can be written as $h(x^2)$ where h is a polynomial defined on a set containing $[0, 1]$ with range $(-1 - \eta, 1 + \eta)$.*
- *The map h has no critical values except on the real line.*
- *The Schwarzian derivative of h is non-positive.*

- The mapping f has no attracting or indifferent periodic cycles.
- The critical orbit is recurrent.

We also define

$$\mathcal{F} := \bigcup_{\eta > 0} \mathcal{F}_\eta .$$

We observe that class \mathcal{F} contains all infinitely renormalizable quadratic polynomials and their renormalizations, up to an affine change of coordinates.

Induced maps The method of inducing was applied to the study of unimodal maps first in [15], then in [13]. In [18] and [17] an elaborate approach was developed to study induced maps, that is, transformations defined to be iterations of the original unimodal map restricted to pieces of the domain. We define a more general and abstract notion in this work, namely:

Definition 1.2 A generalized induced map ϕ on an interval J is assumed to satisfy the following conditions:

- the domain of ϕ , called U , is an open and dense subset of J ,
- ϕ maps into J ,
- restricted to each connected component ϕ is a polynomial with all critical values on the real line and with negative Schwarzian derivative,
- all critical points of ϕ are of order 2 and each connected component of U contains at most one critical point of ϕ .

A restriction of a generalized induced map to a connected component of its domain will be called a *branch* of ϕ . Depending on whether the domain of this branch contains the critical point or not, the branch will be called *folding* or *monotone*. Domains of branches of ϕ will also be referred to as *domains of ϕ* , not to be confused with the domain of ϕ which is U . In most cases generalized induced maps should be thought of as piecewise iterations of a mapping from \mathcal{F} . If they do not arise in this way, we will describe them as *artificial maps*.

The fundamental inducing domain. By the assumption that all periodic orbits are repelling, every $f \in \mathcal{F}$ has a fixed point $q > 0$.

Definition 1.3 If $f \in \mathcal{F}$, we define the fundamental inducing domain of f . Consider the first return time of the critical point to the interval $(-q, q)$. If it is not equal to 3, or it is equal to 3 and there is a periodic point of period 3, then the fundamental inducing domain is $(-q, q)$. Otherwise, there is a periodic point $q' < 0$ of period 2 inside $(-q, q)$. Then, the fundamental inducing domain is $(q', -q')$.

Branchwise equivalences.

Definition 1.4 *Given two generalized induced mappings on J and \hat{J} respectively, a branchwise equivalence between them is an orientation preserving homeomorphism of J onto \hat{J} which maps the domain U of the first map onto the domain \hat{U} of the second map.*

So the notion of a branchwise equivalence is independent of the dynamics, only of domains of the generalized induced mappings.

1.4 Conjugacy between renormalizable maps

The Real K obe Lemma. Consider a diffeomorphism h onto its image (b, c) . Suppose that it has an extension \tilde{h} onto a larger image (a, d) which is still a diffeomorphism. Provided that \tilde{h} has negative Schwarzian derivative, and $\frac{|a-b| \cdot |c-d|}{|c-a| \cdot |d-b|} \geq \epsilon$, we will say that h is ϵ -extendible. The following holds for ϵ -extendible maps:

Fact 1.1 *There is a function C of ϵ only so that $C(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 1$ and for every h defined on an interval I and ϵ -extendible,*

$$|\mathcal{N}h| \cdot |I| \leq C(\epsilon) .$$

Proof:

Apart from the limit behavior as ϵ goes to 1, this fact is proved in [23], Theorem IV.1.2 . The asymptotic behavior can be obtained from Lemma 1 of [11] which says that if \tilde{h} maps the unit interval into itself, then

$$\mathcal{N}h(x) \leq \frac{2h'(x)}{\text{dist}(\{0, 1\}, h(x))} . \tag{1}$$

The normalization condition can be satisfied by pre- and post-composing \tilde{h} with affine maps. This will not change $\mathcal{N}\tilde{h} \cdot |I|$, so we just assume that \tilde{h} is normalized. Since we are interested in ϵ close to 1, the denominator of (1) is large and $h'(x)$ is no more than

$$\exp C\left(\frac{1}{2}\right) \frac{|h(I)|}{|I|} .$$

As $|h(I)|$ goes to 0 with ϵ growing to 1, we are done.

Q.E.D.

Properties of renormalization. A mapping $f \in \mathcal{F}$ will be called *renormalizable* provided that a *restrictive interval* exists for f . An open interval J symmetric with respect to 0 will be called *restrictive* if for some $n > 1$ intervals $J, f(J), f^n(J)$ are disjoint, whereas $f^n(J) \subset J$. These definitions are broadly used in literature and can be traced back at least to [12]. Given an f , the notions of a locally maximal and maximal restrictive interval will be used which are self-explanatory. Observe that if J is locally maximal, then $f^{\partial J} \subset \partial J$.

If f is renormalizable, J is its maximal restrictive interval and n is the first return time from J into itself, we can consider f^n restricted to J which will be called the *first renormalization* of f . If f is in \mathcal{F} , we define its *renormalization sequence* $f_0, f_1, \dots, f_\omega$. Here ω can be finite or infinity meaning that the sequence is infinite. The definition is inductive. f_0 is f . If f_i is renormalizable, then f_{i+1} is the first renormalization of f_i . If f_i is non-renormalizable, the sequence ends. The original mapping f is called infinitely renormalizable if $\omega = \infty$, finitely renormalizable if $0 < \omega < \infty$ and non-renormalizable if $\omega = 0$.

Distortion in renormalization sequences.

Fact 1.2 *1.1 Let $f \in \mathcal{F}_\eta$ and f_i be the renormalization sequence. For every $\eta > 0$ there is a $\tilde{\eta} > 0$ so that for every i f_i belongs to $\mathcal{F}_{\tilde{\eta}}$ after an affine change of coordinates.*

Proof:

A similar estimate appeared in [27]. Our version appears as a step in the proof of Lemma VI.2.1 of [23].

Q.E.D.

Saturated mappings. Let us assume that we have a topologically conjugate pair f and \hat{f} , $f, \hat{f} \in \mathcal{F}$. Let f_i and \hat{f}_i be the corresponding renormalization sequences. As a consequence of f and \hat{f} being conjugate, f_i and \hat{f}_i are conjugate for each i . Also, both renormalization sequences are of the same length.

Definition 1.5 *Let a renormalization sequence f_i be given, and let $i < \omega$. Then, we define the saturated map ϕ_i of f_i as a generalized induced map (Definition 1.2) on the fundamental inducing domain of f_i . The domain of ϕ_i is the backward orbit of the fundamental inducing domain J of f_{i+1} under f_i . Restricted to a connected set of points whose first entry time into J is j , the mapping is f^j .*

Definition 1.6 *If $i < \omega$, a saturated branchwise equivalence v_i is any branchwise equivalence between the saturated maps. If $i = \omega$, the saturated branchwise is also defined equivalence and is just the topological conjugacy on the fundamental inducing domain of f_i .*

In this situation, we have a following fact:

Fact 1.3 *Let f and \hat{f} be a topologically conjugate pair of renormalizable mappings with their renormalization sequences. Assume the existence of a $K > 0$ and a sequence of saturated branchwise equivalences v_i , $i \leq \omega$ which satisfy these estimates.*

- *Every v_i is K -quasisymmetric.*
- *For $i < \omega$ every domain of v_i is adjacent to two other domains, and for any pair of adjacent domains the ratio of their lengths is bounded by K .*
- *For every $i < \omega$ all branches of the corresponding saturated maps are at least $1/K$ -extendible.*

Then, the topological conjugacy between f and \hat{f} is quasimetric with a norm bounded as a function of K only.

Proof:

This is a direct consequence of Theorem 2 of [18].

Q.E.D.

Further reduction of the problem. We will prove the following theorem:

Theorem 1

Suppose that f and \hat{f} are both in \mathcal{F}_η for some $\eta > 0$ and are topologically conjugate. Then, there is a bound K depending only on η so that there is a K -quasimetric saturated branchwise equivalence v_0 . In addition, if $\omega > 0$, the all branches of the saturated maps ϕ_0 and $\hat{\phi}_0$ are $1/K$ -extendible.

Theorem 1 implies the Reduced Theorem. We check the hypotheses of Fact 1.3. To check that v_i is uniformly quasiconformal, we apply Theorem 1 to f_i (after an affine change of coordinates the resulting map is in \mathcal{F}). By Fact 1.1, all these mappings belong to some $\mathcal{F}_{\tilde{\eta}}$ where $\tilde{\eta}$ only depends on η . So, Theorem 1 implies that all saturated mappings are uniformly quasimetric. In the same way we derive the uniform extendibility of saturated maps ϕ_i and $\hat{\phi}_i$. It is a well-known fact (see [12]) that preimages of the fundamental inducing domain of f_{i+1} , or of any neighborhood of 0, by f_i are dense.

We still need to check the condition regarding adjacent domains of ϕ_i and $\hat{\phi}_i$. We only do the check for ϕ_i , since it is the same in the phase space of the other mapping. Inside the domain of f_{i+1} every domain of ϕ_i is adjacent to two others with comparable lengths. For this, see the proof of Proposition 1 in [18] where the preimages of the fundamental inducing domain are explicitly constructed. The computations done there are also applicable in our case since the “distortion norm” used there is bounded in terms of $\tilde{\eta}$ by the Real Kőbe Lemma.

Denote by W the maximal restrictive interval of f_i (so W is the domain of f_{i+1}). Outside of W , consider a connected component of point with the same first entry time j_0 into the enlargement of W with scale $1 + \tilde{\eta}$. By this definition and the Real Kőbe Lemma, the distortion of f_0^j on $f_i^{-j_0}(W)$ is bounded in terms of $\tilde{\eta}$. Now every pair of adjacent domains of ϕ_i can be obtained as the image under some f^{-j_0} in this form of a pair of adjacent domains from the interior of the restrictive interval of f_i . It follows that the condition of Fact 1.3 regarding the ratio of lengths of adjacent domains is satisfied for any pair. Now, the Reduced Theorem follows from Theorem 1.

Theorem 1 for non-renormalizable mappings. Theorem 1 also has content for non-renormalizable mappings. It follows from [28] if f and \hat{f} are in the quadratic family, or renormalizations of quadratic polynomials. Our Theorem 1 is marginally more general.

1.5 Box mappings

Real box mappings.

Definition 1.7 A generalized induced map ϕ on an interval J symmetric with respect to 0 is called a *box mapping*, or a *real box mapping*, if its branches satisfy these conditions. For every branch consider the smallest interval W symmetric with respect to 0 which contains the range of this branch. We will say that W is a *box* of ϕ , and we will also say that the branch ranges through W . For every branch of ϕ , it is assumed that it ranges through an interval which is contained in J , and that the endpoints of this interval are not in the domain U of ϕ . In addition, it is assumed that all branches are monotone except maybe the one whose domain contains 0 and that there is a monotone branch mapping onto J . Also, the central domain of ϕ is always considered a box of ϕ .

Every box mapping has a *box structure* which is simply the collection of all its boxes ordered by inclusion.

Complex box mappings.

Definition 1.8 Let ϕ be a real box mapping. We will say that Φ is a *complex box mapping* and a *complex extension* of ϕ provided that the following holds:

- Φ is defined on an open set V symmetric with respect to the real axis. We assume that connected components of V are topological disks, call them *domains* of Φ , and refer the restriction of Φ to any of its domains as a *branch* of Φ , or simply a *complex branch*.
- If a domain of Φ has a non-empty intersection W with the real line, then W must be a domain of ϕ in the sense of Definition 1.7. Moreover, the complex branch defined on this domain is the analytic continuation of the branch of ϕ defined on W , and this analytic continuation has no critical points in the closure of the domain except on the real line.
- Every domain of ϕ which belongs to a branch that *does not* map onto the entire J is the intersection of the real line with a domain of Φ . If this is true for those domains belonging to long monotone branches as well, we talk of a *complex box mapping with diamonds*.
- If two domains of Φ are analytic continuations of branches of ϕ that range through the same box of ϕ , then the corresponding two branches of Φ have the same image.
- If a domain of Φ is disjoint with the real line, then the branch defined there is univalent and shares its range with some branch of Φ whose domain intersects the real line.
- The boundary of the range of any branch of Φ is disjoint with V .

We see that a complex box mapping also has a box structure defined as the set of the ranges of all its branches. These *complex boxes* are in a one-to-one correspondence with the boxes of the real mapping ϕ . Also, we will freely talk of monotone or folding branches for complex mappings, rather than univalent or degree 2.

Special types of box mappings. The definitions below have the same wording for real and complex box mappings. So we just talk of a “box mapping” with specifying whether it is real or complex.

Definition 1.9 *A box mapping ϕ is said to be of type I if it satisfies these conditions.*

- The box structure contains three boxes, which are denoted $B_0 \supset B' \supset B$.
- B equals the domain of the central branch (i.e. the only domain of ϕ which contains 0.)
- Every monotone branch maps onto B or B_0 . Depending on which possibility occurs, we talk of *long* and *short* monotone branches.
- Every short monotone branch can be continued analytically to a diffeomorphism onto B' and the domain of such a continuation is either compactly contained in B' , or compactly contained in the complement of B' . Restricted to the real line, this continuation has negative Schwarzian derivative.
- The central branch is folding and ranges through B' . It also has an analytic continuation of degree 2 which maps onto a larger set $B'' \subset B_0$ which compactly contains B' . The domain of this continuation is compactly contained in B' . The Schwarzian derivative of this continuation restricted to the real line is negative.
- The closure of the union of domains of all short monotone branches is disjoint with the boundary of B' .
- If the closures of two domains intersect, at least one of these domains is long monotone.

A real box mapping of type I is shown on Figure 1. A complex box mapping of type I is shown on Figure 2.

Definition 1.10 *A box mapping ϕ is said to be of type II if the following is satisfied.*

- The box structure contains three boxes, $B_0 \subset B' \supset B$. B denotes the domain of the central branch.
- All monotone branches of ϕ map either onto B_0 or B' , and are accordingly classified as long or short.
- The central branch is folding, and it has an analytic continuation whose domain is compactly contained in B' , the range is some $B'' \subset B_0$ which compactly contains B' , the degree of this continuation is 2, and the Schwarzian derivative of its restriction to the real line is negative.
- The closure of the union of all domains of short monotone branches is disjoint with the boundary of B' .

- If the closures of two domains intersect, at least one of these domains is long monotone.

Definition 1.11 *A box mapping will be called full if it has only two boxes, $B_0 \supset B$. Also, the central branch must be folding and range through B_0 .*

Hole structures. Given a complex box mapping Φ , the domains which intersect the real line and such that the range of Φ restricted to any of them is less than the largest box will be called *holes*. The central domain is also considered a hole, and described as the central hole. Other domains of Φ which intersects the real line will be called *diamonds*. Boxes, holes and diamonds can then be studied purely geometrically, without any reference to the dynamics. So we consider *hole structures* which are collections of boxes and holes, and *hole structures with diamonds* which also include diamonds. Given a hole structure, the number of its boxes minus one will be called the *rank* of the structure.

Some geometry of hole structures. Although the facts which we will state now have an easy generalization for hole structures of any rank, we formally restrict ourselves to structures of rank 0. We want to specify what bounded geometry is for a hole structure. A positive number K is said to provide the bound for a hole structure if the series of estimates listed below are satisfied. Here we adopt the notation I to mean the box of the structure.

1. All holes are strictly contained in I , moreover, between any hole and I there is an annulus of modulus at least K^{-1} .
2. All holes and I are at least K -quasidisks. The same estimate holds for half-holes and half- I , that is, regions in which a half of the quasidisks was cut off along the real line.
3. Consider a hole and its intersection (a, b) with the real line. Then, there are points $a' \leq a$ and $b' \geq b$ with the property that the hole is enclosed in the diamond of angle $\pi/2 - K^{-1}$ based on a', b' . Furthermore, for different holes the corresponding intervals (a', b') are disjoint.

The “mouth lemma”.

Definition 1.12 *A diamond neighborhood $D(\theta)$ of an interval J given by an angle $0 < \theta < \pi$ is the union of two regions symmetric with respect to the real line and defined as follows. Each region is bounded by a circular arc which intersects the real line in the endpoints of J . We adopt the convention that θ close to 0 given a very thin diamond, while θ close to π gives something called a “butterfly” in [27].*

We now consider an enriched hole structure of rank 0.

Definition 1.13 *Given a rank 0 hole structure, in the part of J which is not covered by the holes we arbitrarily choose a set of disjoint open intervals. For each interval, we define a diamond neighborhood. Each diamond has a neighborhood of modulus K_h^{-1} which is contained in the box. Next, each diamond neighborhood contains two symmetrical arcs. It is assumed*

that each of them joins the endpoints in the interval and except for them is disjoint with the line, and that each arc is K_h -quasiarc. This object will be referred to as a hole structure with diamonds.

A bound for a hole structure with diamonds is the greater of the norm for the hole structure without diamonds and K_h .

For a hole structure with diamonds, consider the “teeth”. i.e. the curve which consists of the upper halves of the boundaries of the holes and the diamonds, and of points of the real line which are outside of all holes and diamonds. Close this curve with the “lip” which means the upper half of the boundary of I . The whole curve can justly be called the “mouth”.

Lemma 1.1 *For a bounded hole structure with diamonds, the mouth is a quasidisk with the norm uniform with respect to the bound of the map.*

Proof:

The proof is based of the three point property of Ahlfors (see [1].) A Jordan curve is said to have this property if for any pair of its points their Euclidean distance is comparable with the smaller of the diameters of the arcs joining them. Moreover, a uniform bound in the three point property implies an estimate on the distortion of the quasicircle. Our conditions of boundedness were set precisely to make it easy to verify the three point property. One has to consider various choices of the two points. If they are both on the same tooth, or both on the lip, the property follows immediately from the fact that teeth are uniform quasicircles. An interesting situation is when one point is on one tooth and the other one on another tooth. We notice that it is enough to check the diameter of a simpler arc which goes along either tooth to the real line, than takes a shortcut along the line, and climbs the other tooth (left to the reader.) Suppose first that both teeth are boundaries of holes. Consider the one which is on the left. We first consider the arc which goes to the b' . By conditions 2 and 3 of the boundedness of hole structures, the diameter of this arc is not only comparable to the distance from the point to b' , but even to the distance from the projection of the point to the line. The estimate follows. The case when one of the teeth is a diamond is left to the reader. Another interesting case is when one point is on the lip and another one on a tooth. If the tooth is a diamond, the estimate follows right away from the choice of angles. If the tooth is a hole, we have to use condition 1. We use the bounded modulus to construct a uniformly bounded quasiconformal map straightens the lip and the tooth to round circles without changing the modulus too much. The estimate also follows.

Q.E.D.

Extension lemma. We are ready to state our main result which in the future will be instrumental in extending real branchwise equivalences to the complex domain. Given two hole structures with diamonds, a homeomorphism h is said to establish the equivalence between them if it is order preserving, and for any hole, box, or diamond of one structure, it transforms its intersection with the real axis onto the intersection between a hole, box or diamond respectively of the other structure with the real axis.

Lemma 1.2 *For a uniform constant K , let two K -bounded hole structures with diamonds be with an equivalence establishing K -quasisymmetric homeomorphism h . We want to extend h to the complex plane with prescribed behavior on the boundary of each tooth and on the lip. This prescribed behavior is restricted by the following condition. For each tooth or the lip consider the closed curve whose base is the interval on the real line, and the rest is the boundary of the tooth or the lip, respectively. The map is already predefined on this curve. We demand that it maps onto the corresponding object of the other hole structure, and that it extends to a K -quasiconformal homeomorphism. Then, h has a global quasiconformal extension with the prescribed behavior and whose quasiconformal norm is uniformly bounded in terms of K .*

Proof:

We first fill the teeth with a uniformly quasiconformal map which is something we assumed was possible. Next, we extend to the lower half-plane by [3]. Then, we want an extension to the region above the lip and above the real line. This is certainly possible, since the boundary of this region is a uniform quasicircle (because the lip was such, use the three point property to check the rest). But the map can be defined below this curve by filling the lip and using [3] in the lower half-plane. Then one uses reflection (see [1].) Now, the map is already defined outside of the mouth, one uses Lemma 1.1 and quasiconformal reflection again to finish the proof.

Q.E.D.

1.6 Introduction to inducing

Standard extendibility of real box mappings. For monotone branches we have the notion of extendibility which is compatible with our statement of the Real Kőbe Lemma (Fact 1.1.) That is, a monotone branch with the range equal to (b, c) will be deemed ϵ -*extendible* if it has an extension as a diffeomorphism with negative Schwarzian derivative onto a larger interval (a, d) and so that

$$\frac{|a - b| \cdot |d - c|}{|c - a| \cdot |d - b|} \geq \epsilon .$$

This amounts to saying that the length of (b, c) in the Poincaré metric of (a, d) is no more than $-\log \epsilon$.¹ The interval (a, d) will be called the *margin of extendibility* while the domain of the extension will be referred to as the *collar of extendibility*.

We can also talk of extendibility for the central folding branch. To this end, we represent the folding branch as $h(x^2)$. Suppose that the central branch ranges through a box (b, c) . Then the folding branch is considered ϵ -extendible provided that h has an extension as a diffeomorphism with negative Schwarzian derivative onto a larger interval (a, d) and the Poincaré length of (b, c) inside (a, d) is at most $-\log \epsilon$. Again, (a, d) is described as the margin of extendibility and its preimage by the extension of the folding branch is the collar of extendibility.

¹See [23] for a discussion of the Poincaré metric on the interval.

Definition 1.14 Consider a box mapping ϕ on J . Standard ϵ -extendibility for ϕ means that the following properties hold.

- There is an interval $I \supset J$ so that the Poincaré length of J in I is no more than $-\log \epsilon$. Every branch of ϕ which ranges through J is extendible with the margin of extendibility equal to I . Furthermore, for each such branch the collar of extendibility also has Poincaré length not exceeding ϵ in I ; moreover if the domain of the branch is compactly contained in J , so is the the collar of extendibility.
- For every other box B in the box structure of ϕ , there is an interval $I_B \supset \overline{B}$ which serves as the margin of extendibility for all branches that range through B . I_B must be contained in any box larger than B . If the domain of a branch is contained in B , the collar of extendibility of this branch must be contained in I_B , moreover, if this domain is compactly contained in B , the collar is contained in B .

We will call a choice of I and I_B the *extendibility structure* of ϕ .

Refinement at the boundary. Consider a generalized induced map ϕ on J . If we look at an endpoint of J , clearly one of two possibilities occurs. Either the domains of ϕ accumulate at the endpoint, or there is one branch whose domain has the endpoint on its boundary. In the first case, we will say that ϕ is *infinitely refined* at its (left, right) boundary. In the second case the branch adjacent to the the endpoint will be called the (left, right) *external branch*.

Assume now that an external branch ranges through J , and that its common endpoint with J is repelling fixed point of the branch. Call the external branch ζ . We will describe the construction of *refinement at a boundary*. Namely, we can define ϕ^1 as ϕ outside of the domain of ζ , and as $\phi \circ \zeta$ on the domain of ζ . Inductively, ϕ^{i+1} can be defined as ϕ outside of the domain of ζ , and $\phi^i \circ \zeta$ on the domain of ζ . The endpoint is repelling. Thus, the external branches of ϕ^i adjacent to the boundary point will shrink at an exponential rate. We can either stop at some moment and get a version of ϕ *finitely refined at the boundary* or proceed to an L^∞ limit of this construction to obtain a mapping infinitely refined at this boundary. There is an analogous process for the other external branch, ζ' (it is exists). Namely, define $(\phi^i)'$ as ϕ outside the domain of ζ' and $\phi^{i-1} \circ \zeta'$ on the domain of ζ' .

We call the mapping obtained in this process a *version of ϕ refined at the boundary* (finitely or infinitely.) Observe that the refinement at the boundary does not change either the box structure of the map or its extendibility structure.

Operations of inducing. Loosely speaking, inducing means that some branches of a box mapping are being replaced by compositions with other branches. Below we give precise definitions of five procedures of inducing.

Simultaneous monotone pull-back. Suppose that a box mapping ϕ on J is given with a set of branches of ϕ which range through J . We denote $\phi' = \phi_b$ where ϕ_b is a version of ϕ refined at the boundary. Then on each branch ζ from this set replace ϕ with $\phi_b^t \zeta$ where ϕ_b^t is ϕ_b with the central branch branch replaced by the identity. Observe that this operation

does not change the box or extendibility structures. This is a tautology except for the one which is just the restriction of ζ to the preimage of the central domain. This is extendible with margin I , so the standard extendibility can be satisfied with the same ϵ .

Filling-in. To perform this operation, we need a box mapping ϕ and ϕ' with the same box and extendibility structures as ϕ and with a choice of boxes B_1 smaller than $B_0 := J$ but bigger than the central domain B . We denote $\phi_0 := \phi'$ and proceed inductively by considering all monotone branches of ϕ which map onto B_1 . On the domain of each such branch ζ , we replace ζ with $\phi_i^t \zeta$. Here, ϕ_i^t denotes ϕ_i whose central branch was replaced by the identity. The resulting map is ϕ_{i+1} . In this way we proceed to the L^∞ limit, and this limit ϕ_∞ is the outcome of the filling-in. Note that B_1 drops from the box structure of ϕ_∞ , but otherwise the new map has the the same boxes with the same margins of extendibility as ϕ . Thus, if ϕ satisfied standard ϵ -extendibility, so does ϕ_∞ .

Simple critical pull-back. For this, we need a box mapping ϕ whose central branch is folding, and another box mapping ϕ' about which we assume that it has the same box structure as ϕ and is not refined at either boundary. Let ϕ^t denote ϕ' whose central branch was replaced by the identity, and ψ be the central branch of ϕ . Then the outcome is equal to ϕ modified on the central domain by substituting ψ with $\phi^t \circ \psi$.

Almost parabolic critical pull-back. The arguments of this operation are the same as for the simple critical pull-back. In addition, we assume that the critical value of ψ is in the central domain of ψ (and ψ'), but 0 is not in the range of ψ from the real line. Also, it is assumed that $\psi^l(0) \notin B$ so some $l > 0$, and let l denote the smallest integer with this property. Then, for each point $x \in B$ we define the *exit time* $e(x)$ as the smallest j so that $\psi^j(x) \notin B$. Clearly, $e(x) \leq l$ for every $x \in B$. Then the outcome of this procedure is ψ modified on the subset of points with exit times less than l by replacing ψ with

$$x \rightarrow \psi' \circ \psi^{e(x)}(x) .$$

On the set of point with exit time equal to l , which form a neighborhood of 0, the map is unchanged.

Critical pull-back with filling-in. This operation takes a box mapping ϕ which must be of type I or full, and another mapping ϕ' with the same box structure and not refined at either boundary. Let ψ denote the central branch of ϕ , and let χ be the generic notation for a short monotone branch of ϕ . We define inductively two sequences, ϕ_i and ϕ_i^r . Let $\phi_0 := \phi$ and ϕ_0^r be ϕ' with the central branch replaced by the identity. Then ϕ_{i+1} is obtained from ϕ by replacing the central branch ψ with $\phi_i^r \psi$ and every short monotone branch χ of ϕ with $\phi_i^r \circ \psi \circ \chi$. Then ϕ_{i+1}^r is obtained in a similar way, but only those short monotone and folding branches of ϕ are replaced whose domains are contained in the range of ψ . Others are left unchanged. At the end, the central branch of the map obtained in this way is replaced by the identity. The outcome is the L^∞ limit of the sequence ϕ_i . This is the most complicated procedure and is described in [18], and called filling-in, in the case when ϕ is full. We note

here that ϕ_∞ is either of type I, full or Markov depending on the position of the critical value of ψ . If the critical value was in a long monotone domain of ϕ' , then the outcome is a full mapping. If it was in a short monotone domain of ϕ' or in the central domain, the result is a type I map. If the critical value was not in the domain of ϕ' , the result is a Markov map, that is a mapping whose all branches are monotone and onto J .

Boundary refinement. The five basic operations described above obviously fall in two classes. The first two involve composing with monotone branches and do not affect standard extendibility. The last three involve composing with folding branches and can affect standard extendibility. Each of these last three operations can be preceded by a process called *boundary refinement*. To do the boundary refinement, assume that a box mapping ϕ is given whose central branch ψ is folding. Then look at the set of “bad” branches of ϕ . For the steps of simple critical pull-back and critical pull-back with filling-in, a branch of ϕ is considered “bad” if it ranges through J , its domain is in the range of ψ and its collar of standard extendibility contains the critical value. For almost parabolic critical pull-back the last condition is weakened so that the branches which contain $\psi^i(0)$, $0 < i \leq l$ are also considered bad. Then boundary refinement is defined as the simultaneous monotone pull-back on the set of bad branches with ϕ' that may vary with the branch. There are two possibilities here. If we do *infinite* boundary refinement, as ϕ' we use the version of ϕ infinitely refined at both endpoints. If we do *minimal* boundary refinement, we use the version of ϕ refined at the endpoint of the side of the critical value only, and refined to the minimal depth which will enforce standard extendibility.

This gives the mapping ϕ' which then enters the corresponding procedure of critical pull-back.

Lemma 1.3 *Let ϕ be a type I or full mapping with standard ϵ -extendibility. Apply critical pull-back with filling-in preceded by boundary refinement. Then, the resulting map ϕ_∞ also has standard ϵ -extendibility.*

Proof:

Boundary refinement assures us that for every i the branches of ϕ_i or ϕ_i^r which range through J are extendible with the same margin as in the original ϕ . This ends the proof if ϕ_∞ is full. Otherwise, let B denote the central domain of ϕ_∞ (the same as the central domain of ϕ_i , any $i > 0$), and B' the central domain of ϕ . Observe that all short branches of ϕ_∞ are extendible with margin B' . Thus, one can set $I_{B'}$ inherited from the extendibility structure of ϕ , and $I_B := B'$. One easily checks that the conditions of standard extendibility hold.

Q.E.D.

2 Initial inducing

2.1 Real branchwise equivalences

Statement of the result. Our definitions follow [18].

Definition 2.1 Let v_1 be a quasisymmetric and order-preserving homeomorphism of the line onto itself. Let v_2 be a quasisymmetric order-preserving homeomorphism from an interval J onto another interval, say J' . We will say that v_2 replaces v_1 on an interval (a, b) with distortion L if the following mapping v is an L -quasisymmetric order preserving homeomorphism of the line onto itself:

- outside of (a, b) , the mapping v is the same as v_1 ,
- inside (a, b) , v has the form

$$v = A' \circ v_2 \circ A^{-1}$$

where A and A' are affine and map J onto $[a, b]$ and J' onto $v_1([a, b])$ respectively.

Definition 2.2 We will say that a branchwise equivalence v between box mappings on J satisfies the standard replacement condition with distortion K provided that

- v restricted to any domain of the box mapping replaces v on J ,
- v restricted to any box of this mapping replaces v on J .

Definition 2.3 A box map ϕ is said to be α -fine if for every domain D and box B so that $D \subset B$ but $\overline{D} \not\subset B$

$$\frac{|D|}{\text{dist}(D, \partial B)} \leq \alpha .$$

Definition 2.4 A branch is called external in the box B provided that the domain D of the branch is contained in B , but the closure of D is not contained in D . We will say that the map is not refined at the boundary of B provided that an external branch exists at each endpoint of B .

Proposition 1 Suppose that box mappings ϕ and ϕ' are given which will enter one of the five inducing operations defined in section 1.6. Assume further that $\hat{\phi}$ is topologically conjugate to ϕ , and $\hat{\phi}'$ is topologically conjugate to ϕ' , with the same topological conjugacy. Let v_0 and v'_0 be the branchwise equivalences between ϕ and $\hat{\phi}$ as well as ϕ' and $\hat{\phi}'$ respectively. Suppose that all these branchwise equivalences coincide outside of J . Assume in addition that

- all branches of $\phi, \phi', \hat{\phi}, \hat{\phi}'$ are ϵ -extendible,
- all four box mappings are α -fine,
- v_0 and v'_0 satisfy the standard replacement condition with distortion K and are Q -quasisymmetric,

- ϕ' and $\hat{\phi}'$ are not refined at the boundary of any box smaller than B_0 ,
- if D is the domain of an external branch of in a box B (for ϕ' or $\hat{\phi}'$), then $|D|/|B| \geq \epsilon_1$.

Then for all ϵ , K , Q , α and ϵ_1 there are bounds L_1 and L_2 so that the following holds. Perform one of the five inducing operations on ϕ and ϕ' as well as $\hat{\phi}$ and $\hat{\phi}'$. Call the outcomes ϕ_∞ and $\hat{\phi}_\infty$, or $\phi_{\infty,b}$ and $\hat{\phi}_{\infty,b}$ for their versions refined at the boundary. Then there are branchwise equivalences v_∞ and $v_{\infty,b}$ between ϕ_∞ and $\hat{\phi}_\infty$, or $\phi_{\infty,b}$ and $\hat{\phi}_{\infty,b}$ respectively which satisfy

- v_∞ and $v_{\infty,b}$ are the same as v outside of J ,
- v_∞ and $v_{\infty,b}$ satisfy the standard replacement condition with distortion L_1 ,
- v_∞ and $v_{\infty,b}$ are both L_2 -quasisymmetric.

Some technical material. The proof of Proposition 1 splits naturally in five cases depending on the kind of inducing operation involved. Among these the almost parabolic critical pull-back stands out as the hardest. Proposition 1 in all remaining cases follows rather straightforwardly from the work of [18]. We begin by recalling the technical tools of [18].

Definition 2.5 *Given an interval I on the real line, its diamond neighborhood is a set symmetrical with respect to the real axis constructed as follows. In the upper half-plane, the diamond neighborhood is the intersection of a round disk with the upper half-plane, while on the intersection of the same disk with the real line is I . For a diamond neighborhood, its height refers to twice the Hausdorff distance between the neighborhood and I divided by the length of I .*

This is slightly different from the definition used in [18], but this is not an essential difference in any argument. The nice property of diamond neighborhoods is the way they are pull-back by polynomial diffeomorphisms.

Fact 2.1 *Let h be a polynomial which is a diffeomorphism from an interval I onto J . Assume that h preserves the real line and that all its critical values are on the real line. Suppose also that h is still a diffeomorphism from a larger interval $I' \supset I$ onto a larger interval $J' \supset J$ so that the Poincaré length of J inside J' does not exceed $-\log \epsilon$. Then, there are constants $K_1 > 0$ and K_2 depending on ϵ only so that the distortion of h (measured as $|h''/(h')^2|$) on the diamond neighborhood of I with height K_1 is bounded by a K_2 .*

Proof:

Observe that the inverse branch of h which maps J' onto I' is well defined in the entire slit plane $\mathbf{C} \setminus J'$. Since h is a local diffeomorphism on I , the inverse branch can be defined on a diamond neighborhood of J' with sufficiently small height. As we gradually increase the height of this neighborhood we see that the inverse branch can be defined until the boundary of the neighborhood hits a critical value of h . That will never occur, though, since all critical

values are on the real line by assumption. Now, the diamond neighborhood of J with height 1 is contained in $\mathbf{C} \setminus J'$ with modulus bounded away from 0 in terms of ϵ . So, by Kőbe's distortion lemma, the preimage of this diamond neighborhood by the inverse branch of h contains a diamond neighborhood of I of definite height, and the distortion of h there is bounded.

Q.E.D.

Pull-back of branchwise equivalences.

Definition 2.6 *A pull-back ensemble is the conglomerate of the following objects:*

- two equivalent induced mappings, one in the phase space of f , the other in the phase space of \hat{f} , with a branchwise equivalence \mathcal{Y} and a pair of distinguished branches Δ and $\hat{\Delta}$ whose domains, D and \hat{D} respectively, correspond by \mathcal{Y} ,
- a branchwise equivalence Υ which must map the critical value of Δ to the critical value of $\hat{\Delta}$ in case if Δ is folding.

The following objects and notations will always be associated with a pull-back ensemble. If Δ is monotone, let $(-Q, Q')$, $\frac{6}{5}Q > Q' \geq Q$, be its range. If Δ is folding, make $Q' = Q$ and define $(-Q, Q')$ to be the smallest interval symmetrical with respect to 0 which contains the range of Δ . Analogously, we define (\hat{Q}, \hat{Q}') to be the range of $\hat{\Delta}$ or the maximal interval symmetric with respect to 0 containing the range of $\hat{\Delta}$, and we also assume that $\frac{6}{5}\hat{Q} > \hat{Q}' \geq \hat{Q}$.

In addition, the following assumptions are a part of the definition:

- $\Upsilon((-Q, Q')) = (-\hat{Q}, \hat{Q}')$,
- $\Upsilon(\bar{z}) = \overline{\Upsilon(z)}$ and $\Upsilon(-\bar{z}) = -\overline{\Upsilon(z)}$ for every $z \in \mathbf{C}$,
- outside of the disc $B(0, \frac{4}{3}Q)$ the map Υ has the form

$$z \rightarrow \frac{\hat{Q}' - \hat{Q}}{Q' - Q} z,$$

- the mapping Υ restricted to the set

$$\mathbf{R} \setminus (-Q, Q')$$

has a global λ -quasiconformal extension,

- the branches Δ and $\hat{\Delta}$ are ϵ -extendible,
- if Δ is folding, then its range is smaller than $(-Q, Q')$, but the length of the range is at least $\epsilon_1 Q$; likewise, if $\hat{\Delta}$ is folding, then its range does not fill the interval $(-\hat{Q}, \hat{Q}')$, but the length of the range is at least $\epsilon_1 \hat{Q}$.

The numbers ϵ , ϵ_1 and λ will be called *parameters* of the pull-back ensemble.

Vertical squeezing.

Definition 2.7 *Suppose that two parameters $s_1 \geq 2s_2 > 0$ are given. Consider a differentiable monotonic function $v : \mathbf{R} \rightarrow \mathbf{R}$ with the following properties:*

- $v(-x) = -x$ for every x ,
- if $0 < x < s_2$, then $v(x) = x$,
- if $s_1 < x$, then $v(x) = x - s_1 + 2s_2$.

Each v defines a homeomorphism of the plane \mathcal{V} defined by

$$\mathcal{V}(x + iy) = x + iv(y) .$$

We want to think of \mathcal{V} as being quasiconformal end depending on s_1 and s_2 only. To this end, for a given s_1 and s_2 pick some v which minimizes the maximal conformal distortion of \mathcal{V} . We will call this \mathcal{V} the vertical squeezing map for parameters s_1 and s_2 .

The Sewing Lemma.

Fact 2.2 *Consider a pull-back ensemble with $Q = Q'$. Let \mathcal{Y} be a branchwise equivalence with D as a domain, and suppose that restricted to D it replaces Υ on $(-Q, Q)$ with distortion M . Suppose that for some $R > 0$ the map Υ transforms the diamond neighborhood with height R of $(-Q, Q)$ exactly on the diamond neighborhood with height R of $(-\hat{Q}, \hat{Q})$. Choose $r > 0$ and $C \leq 1$ arbitrary and assume that the diamond neighborhood with height r of D is mapped by \mathcal{Y} onto the diamond neighborhood with height Cr of \hat{D} . Assuming that $R \leq R_0$ where R_0 only depends on the parameter ϵ of the pull-back ensemble, for every such choice of C, M, r, R , and a set of parameters of the pull-back ensemble, numbers K and L , parameters $s_1, s_2, \hat{s}_1, \hat{s}_2$, a mapping $\tilde{\Upsilon}$ as well as a branchwise equivalence $\tilde{\mathcal{Y}}$ exist so that:*

- if \mathcal{V} is the vertical squeezing with parameters s_1, s_2 , and $\hat{\mathcal{V}}$ is the vertical squeezing map with parameters \hat{s}_1 and \hat{s}_2 , then $\tilde{\mathcal{Y}}$ has the form

$$\tilde{\mathcal{Y}} = \hat{\mathcal{V}} \circ \tilde{\Upsilon} \circ \mathcal{V}^{-1}$$

on the image of the domain of $\tilde{\Upsilon}$ by \mathcal{V} ,

- the domain and range of $\tilde{\Upsilon}$ are contained in diamond neighborhoods with height $1/10$ of D and \hat{D} respectively,
- on the diamond neighborhood with height K of D

$$\tilde{\Upsilon} = \Delta^{-1} \circ \Upsilon \circ \Delta$$

which means the lift to branched covers, order-preserving on the real line in the case of Δ folding,

- outside of this diamond neighborhood the conformal distortion of $\tilde{\Upsilon}$ is bounded by the sum of L and the conformal distortion of Υ outside of the image of the diamond neighborhood of D with height K by Δ ,
- $\tilde{\Upsilon}$ coincides with \mathcal{Y} outside of the diamond neighborhood with height r of D ,
- on the set-theoretical difference between the diamond neighborhoods with heights r and $r/2$, the map $\tilde{\Upsilon}$ is quasiconformal and its conformal distortion is bounded as the sum of L and the conformal distortion of \mathcal{Y} on the diamond neighborhood of D with height r ,
- outside of the diamond neighborhood with height $r/2$ of D , the mapping $\tilde{\Upsilon}$ is independent of Υ ,
- on the set-theoretical difference between the diamond neighborhood of D with height $r/2$ and U the map $\tilde{\Upsilon}$ has conformal distortion bounded almost everywhere by L .

Proof:

This fact is a consequence of Lemmas 4.3 and 4.4 of [18]. The context of [18] is slightly different from our situation, since that paper works in the category of S-unimodal mappings and uses their “tangent extensions” instead of analytic continuation. However, proofs given in [18] work in our situation with only semantic modifications, so we do not repeat them here.

Q.E.D.

As a consequence of the Sewing Lemma, we get

Fact 2.3 *Suppose that a pull-back ensemble is given so that Υ transforms the diamond neighborhood with height R of $(-Q, Q)$ exactly to the homothetic diamond neighborhood of $(-\hat{Q}, \hat{Q})$. Assume also that $R \leq R_0$ as required by the hypothesis of the Sewing Lemma, and that \mathcal{Y} restricted to D replaces Υ on $(-Q, Q)$ with distortion M . Suppose finally that both Υ and \mathcal{Y} are Q -quasiconformal. Then there is a map $\tilde{\Upsilon}$ which differs from \mathcal{Y} only on the diamond neighborhood of D with height $1/2$, is equal to*

$$\hat{\Delta}^{-1} \circ \Upsilon \circ \Delta$$

on D , and is L -quasiconformal. The number L only depends on M , Q and the parameters of the pull-back ensemble.

Proof:

This follows directly from Fact 2.2 when one chooses $r = 1/2$ and observes that C is bounded as a function of Q .

Q.E.D.

Beginning the proof of Proposition 1. In order to be able to use Fact 2.2 we need a lemma that will allow us to build pull-back ensembles.

Lemma 2.1 *Suppose that a branchwise equivalence v between topologically conjugate box mappings ϕ and $\hat{\phi}$ is given on $(-Q, Q)$. Assume that this branchwise equivalence is Q -quasisymmetric and satisfies the standard replacement condition with distortion K . Suppose also that a pair of branches Δ and $\hat{\Delta}$ are given so that Δ and $\hat{\Delta}$ range through boxes that correspond by v . Moreover, assume that if Δ one is folding, the other one is, too, and the critical values correspond by the topological conjugacy. Suppose also that ϕ and $\hat{\phi}$ are both α -fine and all of their branches are ϵ -extendible. If Δ and $\hat{\Delta}$ are also ϵ extendible, then there is a mapping v' which coincides with v on $(-\frac{6}{5}Q, \frac{6}{5}Q)$ except on the domain which contains the critical value of Δ , and an L -quasiconformal extension of v' , called Υ , which together with Δ and $\hat{\Delta}$ gives a pull-back ensemble with parameters ϵ , ϵ_1 and λ . Numbers L and λ depend on Q and K only. The mapping Υ transforms the diamond neighborhood with height R_0 of $(-Q, Q)$ exactly onto the diamond neighborhood of $(-\hat{Q}, \hat{Q})$ with the same height, where R_0 is determined in terms of ϵ by Fact 2.2.*

Proof:

We have two tasks to perform: one is to build v' so that it maps the critical value of Δ to the critical value of $\hat{\Delta}$, and second is to construct the proper quasiconformal extension. Let us address the second problem first. We use this fact.

Fact 2.4 *Let v be a Q -quasisymmetric mapping of the line into itself, and let $(-\hat{Q}, \hat{Q}) = v((-\frac{6}{5}Q, \frac{6}{5}Q))$. For every $R < 1$ there an L -quasiconformal homeomorphism Υ of the plane with these properties:*

- Υ maps the diamond neighborhood of $(-Q, Q)$ with height R exactly onto the diamond neighborhood of $(-\hat{Q}, \hat{Q})$ with the same height,
- Υ restricted to $(-\frac{6}{5}Q, \frac{6}{5}Q)$ equals v ,
- Υ maps $B(0, \frac{4}{3}Q)$ exactly onto $B(0, \frac{4}{3}\hat{Q})$ and is affine outside of this ball.

The bound L depends on R (continuously) and Q only.

Fact 2.4 follows from the construction done in [18] in the proof of Lemma 4.6. From Fact 2.4 we see that once v' is constructed with the desired properties, Lemma 2.1 follows.

To get v' , we first construct take any point c and construct v'' which maps c to $H(c)$, where H is the topological conjugacy, and $v'' = v$ outside of the box that contains c . To construct v'' , we consider two cases. If the c is not in an external domain, one can compose v'' with a diffeomorphism of bounded distortion which moves $v(c)$ to $H(c)$ and is the identity outside of the box. If c is an external domain, map it by the external branch into the phase space of a version of ϕ' refined at the boundary. By the previous argument, this image requires only a push by a diffeomorphism of bounded distortion. This way we get some branchwise equivalence v_1 . Using Fact 2.4, we build the pull-back ensemble which consists of the external branch, its counterpart in the phase space of $\hat{\phi}'$, and the appropriate extension Υ_1 of v_1 . To this pull-back ensemble we can apply Fact 2.3 and get v'' on the real line. By

the way, in this case v'' can immediately be taken as v' . Generally, in order to obtain v' from v'' we apply a similar procedure. Take the branch which contains the critical value of Δ and use the image of the critical value by this branch as the c to build v'' . Then construct the pull-back ensemble by Fact 2.4 and pull v'' back to get v' . Again, v' is quasymmetric from Fact 2.3

Q.E.D.

We will next prove that Lemma 2.1 and the Sewing Lemma allow us to construct v_∞ and $v_{\infty,b}$ which are quasymmetric as needed. The construction of $v_{\infty,b}$ is quite the same as for v_∞ , only we start with ϕ_b . So we only focus on the construction of v_∞ .

Bounded cases of inducing. Suppose that we are in the situation of Proposition 1. In the case of a simple critical pull-back and simultaneous monotone pull-back, the quasymmetric estimate for v_∞ follows directly. By Lemma 2.1 for each branch being refined we construct a pull-back ensemble, and get $\tilde{\mathcal{Y}}$ which is quasiconformal with an appropriate bound by Fact 2.3. Note that in the case of the simultaneous pull-back the operations on various branches do not interfere, since each modifies \mathcal{Y} only in the diamond neighborhood with height $1/2$.

The case of filling-in. We begin by constructing the extension Υ of v'_0 using Lemma 2.1. For this, choose R equal to R_0 which is given by the Sewing Lemma depending on ϵ , and $(-Q, Q)$ equal to the box B_1 . Denote $\Upsilon_0 = \Upsilon$. Also, take as \mathcal{Y} any quasiconformal extension of v_0 with a norm bounded as a function of Q . We will proceed inductively and obtain Υ_{i+1} by applying the Sewing Lemma always with the same \mathcal{Y} , Δ ranging over the set of all branches which map onto B_1 , and Υ_i .

Let δ be the generic notation for the domains of branches of ϕ mapping onto B_1 , and let δ^{-m} denote similar domains of ϕ_m^t . Let us choose r so small that the diamond neighborhoods of domains δ of ϕ are contained in the diamond neighborhood with height R of $(-Q, Q)$. In our inductive construction, this will assure that Υ_i for any i is the same as Υ on the boundary of the diamond neighborhood of $(-Q, Q)$ with height R , so that this diamond is always mapped on the homothetic diamond. In particular, parameters ϵ , λ and C remain fixed since only the Υ component changes in pull-back ensembles. Note that C is determined by the conformal distortion of \mathcal{Y} , i.e. by the parameter Q . Also, since Υ_{i-1} coincides with Υ on the real line outside of $(-Q, Q)$ the condition that Υ restricted to a domain replaces Υ_{i-1} on $(-Q, Q)$ is satisfied with the same distortion M . Thus, for all i the pull-back ensemble used to construct Υ_i satisfies the hypotheses of the Sewing Lemma with the same parameters. So we will regard estimates claimed by these Lemmas as constants.

Next, look at the neighborhood of D where Υ_i has the form

$$\Upsilon_i = \hat{\mathcal{V}} \circ \hat{\Delta}^{-1} \circ \Upsilon_{i-1} \circ \Delta \circ \mathcal{V}^{-1} .$$

By Fact 2.1, this contains a diamond neighborhood with height K which is mapped by Δ with bounded distortion. In particular, by choosing r possibly even smaller, but still controlled by K , we can make sure that the diamond neighborhoods with height r of all domains δ^{-0}

are inside the image of this region (called the *inner pull-back region*) by Δ . Now, address the issue of where Υ_{i-1} and Υ_i differ. If $i = 1$, they differ only on the union of diamond neighborhoods with height r of domains δ^{-0} . For $i > 1$, Υ_i and Υ_{i-1} differ on the preimage of the set where Υ_{i-2} and Υ_{i-1} differ by maps

$$\Delta \circ \mathcal{V}^{-1}$$

with Δ ranging over the set of all branches mapping onto B_1 .

Next, we see inductively that the set on which Υ_i and Υ_{i-1} differ is contained in the union of diamond neighborhoods with fixed height of domains δ^{-i+1} of ϕ_{i-1} . This is clearly so for $i = 1$. In the general case, we need to consider the preimage of the set where Υ_{i-1} differs from Υ_{i-2} by $\Delta \circ \mathcal{V}^{-1}$. Suppose that Υ_{i-2} and Υ_{i-1} differ only on the union of diamond neighborhoods with height r_{i-2} of domains δ^{-i+2} . Pick some domain δ^{-i+2} and observe that Δ extends as a diffeomorphism onto B_1 . It is easily seen that sizes of domains δ^{-m} decrease with m at a uniform exponential rate. Therefore that Δ restricted to the preimage of δ is x -extendible with $|\log x|$ going up exponentially fast with i . So, Δ restricted to $\Delta^{-1}(\delta)$ is almost affine with distortion going down exponentially fast with i (which follows from Kőbe's distortion lemma.) Next, provided that $r_{i-2} < 1$, the height of the diamond neighborhood of δ with height r_{i-2} with respect to δ_0 goes down exponentially with i . Since the distortion goes down exponentially fast, the preimage of this diamond neighborhood by Δ with fit inside the diamond neighborhood of $\delta^{-i+1} = \Delta^{-1}(\delta^{-i+2})$ with height $r_{i-2}(1 + b(i))$ where $b(i)$ decreases exponentially fast with i . Thus, by choosing r small enough, we can ensure that $r_i < 1$ for all i . The same reasoning can be conducted for the phase space of \hat{f} to prove that the images of these diamonds by Υ_i are contained in the diamond neighborhoods with height 1 of the corresponding domains δ^{-i+1} of $\hat{\phi}$.

For any branch Δ of φ , consider the region W defined as the intersection of the inner pull-back region with the set on which \mathcal{V} is the identity (i.e. the horizontal strip of width $2s_2$.) This contains a diamond neighborhood with fixed height of the domain of Δ . So, for $i > i_0$ (i_0 depends on how fast the sizes of δ^{-m} decrease with m and can be bounded through ϵ and α), the set on which Υ_{i-2} and Υ_{i-1} differ is contained in the image of W by Δ , and its image by Υ_{i-1} is contained in the image of \hat{W} by $\hat{\Delta}$. By Fact 2.3 applied i_0 times, Υ_{i_0} is still quasiconformal because each step adds only a bounded amount of distortion. For $i \geq i_0$, Υ_i is obtained on this region by replacing Υ_{i-1} with

$$\hat{\Delta}^{-1} \circ \Upsilon_{i-1} \circ \Delta.$$

This means that for $i \geq i_0$ the conformal distortion of Υ_i is the same as the conformal distortion of Υ_{i_0} . Thus, the limit of Υ_∞ exists and is quasiconformal with the same bounded norm.

Critical pull-back with filling-in. This is very similar to the filling-in case and once we have built the initial pull-back ensemble, the proof goes like in Lemma 4.5 of [18]. We leave the details out.

Almost parabolic critical pull-back. This case requires a different set of tools. Let us first set up the core problem in abstract terms.

Definition 2.8 We define an almost parabolic map φ be the following properties.

- The map φ is defined on and interval $[0, a)$ with $a < 1$.
- $\varphi(x) = g(x^2)$ where g is an orientation-preserving diffeomorphism onto the image of φ , has negative Schwarzian, and is ϵ -extendible, $\epsilon > 0$.
- $\varphi(a) = 1$ and $\varphi(x) > x$ for every x .
- On the set (c, a) , $S\varphi \geq -K$.

The numbers a, ϵ, K will referred to as parameters of φ .

Proposition 2 Suppose that two almost parabolic maps φ and φ' are given. Suppose that the parameters a are both bounded from below by $\alpha > 0$ and from above by $\beta < 1$, likewise the parameter ϵ is equal for both mappings, and the parameters K are both bounded from above by K_0 . Consider the finite sequence a_i defined by $a_0 = a$ and $\varphi(a_i) = a_{i-1}$ for $i > 0$ or the sequence ends when such a_i cannot be found. Define a'_i in the same manner using φ' and assume that the lengths of both sequences are equal to the same l . Define a homeomorphism u from (a_l, a) onto (a'_l, a') by the requirement that $u(a_i) = a'_i$ for every i and that u is affine on each segment (a_i, a_{i-1}) . Then u is a Q -quasi-isometry with Q only depending on α, β, K_0 and ϵ .

A comment about Proposition 2. What really matters is that u is quasimetric, which follows from its being a quasi-isometry. Proposition 2 is easily accepted by specialists in the field, perhaps because it is an easy fact when φ is known to be complex polynomial-like. However, I was unable to find a fair reference in literature regarding the negative Schwarzian setting, though I am aware of an unpublished work of J.-C. Yoccoz in which a similar problem was encountered and solved in the study of critical circle mappings of unbounded type. The approach we use here owes to the work [9].

Easy bounds. We now assume that φ is a mapping that satisfies the hypotheses of Proposition 2. Throughout the proof we will refer to bounds that only depend on α, β, ϵ and K_0 as *constants*. And so we observe that the derivative of g is bounded from both sides by positive constants. This means that the derivative of φ is similarly bounded from above. Next, c is bounded from below by a positive constant, or it would be impossible to maintain $\varphi(x) > x$. The second derivative of φ is also bounded in absolute value by a constant, from the Real Kőbe's Lemma. Next, there is exactly one point z where $\varphi(x) - x$ attains a minimum. That is since because of the negative Schwarzian there are at most two points where the derivative of φ is one. Also observe that if l is large enough, then the distances from z to a_i and a_0 are bounded from below by constants. That is because the shortest of intervals (a_i, a_{i-1}) is the one containing z , or the adjacent one. If l is large this becomes much smaller than the constants bounding from below the lengths of (a_j, a_{j-1}) or (a_{l-j}, a_{l-j-1}) for $j < 10$. A somewhat deeper fact is this.

Lemma 2.2 There are positive constants l_0, K_1 and K_2 so that if $\varphi(x) - x < K_1$ and $l \geq l_0$, then $\varphi''(x)/\varphi'(x) \geq K_2$.

Proof:

The proof almost copies the argument used in the demonstration of Proposition 2 in [9]. We use $\mathcal{N}g$ to mean g''/g' . Take into account the differential equation

$$D\mathcal{N}g = Sg + 1/2(\mathcal{N}g)^2 . \quad (2)$$

which is satisfied by every C^3 diffeomorphism g (a direct check, also see [23] page 56.) Then consider an abstract class of diffeomorphisms $\mathcal{G}(w, L)$ defined as the set of functions defined in a neighborhood of 0 and having the following properties:

1. their Schwarzian derivatives are negative and bounded away from 0 by some $-\beta$.
2. for any $g \in \mathcal{G}$, $g(0) = w$,
3. there is no $x \in (-L, L)$ where g is defined and $g(x) \leq x$

Observe that the function

$$g_0(y) = \varphi(y + x)$$

belongs to $\mathcal{G}(w, L)$ with

$$w := \varphi(x) - x$$

and L a constant equal to the minimum of distances from x to c and from x to a . This is a constant provided l_0 is large enough and K_1 is sufficiently small. So, it suffices to show that for every $L > 0$ there is a constant $K_1 > 0$ so that $w < K_1$ implies $\mathcal{N}g(0) > K_2 > 0$.

We observe that every function from $g \in \mathcal{G}(w, L)$ is uniquely determined by three parameters: a continuous function $\psi = Sg$ and two numbers ν and μ equal to $\mathcal{N}g(0)$ and $g'(0)$ respectively. Indeed, given ψ and ν , $\mathcal{N}g$ is uniquely determined by the differential equation (2), this together with μ determines g' , and finally g is also defined by w . Observe that with μ fixed, g is an increasing function of ψ and ν . Indeed, a look at the equation (2) reveals that if $\psi_1 \geq \psi_2$ with the same ν , then the solution $\mathcal{N}g_1(x) \geq \mathcal{N}g_2(x)$ for $x \geq 0$ while $\mathcal{N}g_1 \leq \mathcal{N}g_2$ for $x \leq 0$. This is immediate if $\psi_1 > \psi_2$ since we see that at every point where the solutions cross $\mathcal{N}g_1$ is bigger on a right neighborhood and less on a left neighborhood. Then we treat $\psi_1 \geq \psi_2$ by studying $\psi' = \psi_1 + c$ where c is a positive parameter and using continuous dependence on parameters. As g' is clearly an increasing function of $\mathcal{N}g$ and ν , the monotonicity with respect to ψ and ν follows. So, if we can show that for some $\tilde{\psi}$ and $\tilde{\nu}$ and every μ the condition $g(x) \geq x$ is violated on $(-L, L)$, it follows that there exists $\delta > 0$ so that for every $\psi \leq \tilde{\psi}$, we must have $\nu > \tilde{\nu} + \epsilon$ if the function is in $\mathcal{G}(w, L)$.

Pick $\tilde{\psi} = -\beta$ and $\nu = 0$. The problem becomes quite explicit. From another well-known differential formula $u'' = Sg \cdot u$ satisfied by $u = 1/\sqrt{g'}$ we find

$$g'(x) = \frac{\mu}{\sqrt{\cosh \beta x}};$$

Let $w_n \rightarrow 0$ and pick μ_n so that the corresponding g satisfies $g(x) \geq x$, or escapes to $+\infty$, on $(-L, L)$. Observe that μ_n must be a bounded sequence, since we have $g(x) \leq \frac{\mu}{C(L, \beta)}x + w$ for $x < 0$ where $C(L, \beta)$ is the upper bound of $\cosh \beta x$ on $[-L, 0]$. Thus if such a sequence existed, we could take a limit parameter μ_∞ which would preserve $g(x) \geq x$ even for $w = 0$, and this cannot be.

Q.E.D.

Approximation by a flow. For l large, the condition $\varphi(x) - x < K_1$ of Lemma 2.2 is satisfied on a neighborhood W of z . On this neighborhood, we can bound

$$B(x - z)^2 + \delta \leq \varphi(x) - x \leq A(x - z)^2 + \delta$$

where $\delta = \varphi(z) - z$. The numbers A and B are constants, and while the upper bound is easy and satisfied in the entire domain of φ , the lower one follows from Lemma 2.2 is guaranteed to hold only on W . Note that the number of points a_i outside of W is bounded by a constant. So we can change the definition of W a bit so that it is preserved by the map u between two almost parabolic maps. That is, we can make points a_m and a_{l-m} the endpoints of W , and if m is big enough, the condition $\varphi'(x) - x$ will also hold on $u(W)$. Incidentally, for l large, this means that $z' \in W'$. The map u is clearly a quasi-isometry outside of W , so we have reduced the problem to considering it on W . Let $T = l - 2m$, that is the number of iterations an orbit needs to travel through W , and let T_0 be chosen so that $[a_{T_0}, a_{T_0+1})$ contains z . If analogous times are considered for φ' , note that $T' = T$, but T_0 and T'_0 have no reason to be the same.

Lemma 2.3 *Let $x_1, x_2 \in W$ with $\varphi^t(x_1) = x_2$. Then, if $z - x_1 < K$, K constant*

$$\begin{aligned} & \sqrt{\frac{1}{4A\delta}} (\tan^{-1}(\sqrt{\frac{A}{\delta}}(x_2 - z)) - \tan^{-1}(\sqrt{\frac{A}{\delta}}(x_1 - z))) - 1 \leq t \\ & \leq \sqrt{\frac{4}{B\delta}} (\tan^{-1}(\sqrt{\frac{B}{\delta}}(x_2 - z)) - \tan^{-1}(\sqrt{\frac{B}{\delta}}(x_1 - z))) + 1 . \end{aligned}$$

Proof:

To get the lower estimate, compare φ with the time one map of the flow

$$\frac{dx}{dt} = 2A(x - z)^2 + \delta .$$

We claim that iterations of the time one map of this flow overtake iterations of the map $x \rightarrow x + A(x - z)^2$ provided that δ is small enough, i.e. if l is sufficiently large. Let $x \in (x_1, x_2)$. We want

$$2A(x + A(x - z)^2 - z)^2 \geq A(x - z)^2 .$$

Observe that there is nothing to prove if $x > z$. Choose K so that $KA < 1/2$ to conclude the proof.

Then the lower estimate follows by integrating the flow. The upper estimate can be demonstrated in analogous way.

Q.E.D.

We will assume that the assumption $z - x_1 < K$ is always satisfied, perhaps by making W even smaller. As a corollary to Lemma 2.3, we note that for l large, $\delta \sim T^{-2}$ where the \sim sign means that the ratio of the two quantities is bounded from both sides by positive constants. Indeed, taking $x_1 - a_m$ and $x_2 - a_{l-m}$ we see that the arguments of the arctan terms are very large for large l , meaning that they values are close to $\pi/2$, or $-\pi/2$ respectively. In particular, $\delta \sim \delta'$ for two different almost parabolic maps.

Lemma 2.4 For every $b > 0$ there are constants l_0 and $\beta > 0$ with the property that if $|a_p - z| \leq b\sqrt{\delta}$, then

$$\frac{p - m}{T} > \beta \text{ and } \frac{l - m - p}{T} \beta .$$

Proof:

Note that z is separated from the boundary of W by a distance which is at least a positive constant. This is because the “step” of the orbit of a is still quite large at the boundary of W . When δ is sufficiently small (depending on b) this means that the entire neighborhood $(z - b\sqrt{\delta}, z + b\sqrt{\delta})$ is in a positive distance from the boundary of W . This means that for $x_1 = a_m$ and $x_2 = a_p$ the time of passage is $\sim \sqrt{1/\delta}$. The same reasoning applies to the passage from a_p to the other end of W . The claim follows from Lemma 2.3.

Q.E.D.

The approximation formula given by Lemma 2.3 works well for points which are within a distance of the order of $\sqrt{\delta}$ from 0. We need a different formula for points further away.

Lemma 2.5 Choose a point a_p so that the distance from a_p to z is bigger than $b\sqrt{\delta}$. Let t denote $p - m$ if $a_p < z$ or $l - m - p$ otherwise. There are constants $l_0, b_0, C > 0$ and $c > 0$ so that if $l \geq l_0$ and $b \geq b_0$, then

$$\frac{c}{|z - x|} \leq t \leq \frac{C}{|z - x|} .$$

Proof:

We begin by comparing the iterations of φ with flows like in the proof of Lemma 2.3. Given a flow

$$\frac{dx}{dt} = \delta + A(x - z)^2$$

we compare it with the flow

$$\frac{dx}{dt} = A(x - z)^2 .$$

The discrepancy between time $t < T$ maps of these flows is less than $T\delta \sim \sqrt{\delta}$. So, for b_0 large, approximating by the simpler flow modifies $|z - x|$ only by a bounded factor. Integration gives the desired estimate.

Q.E.D.

Observe that Lemma 2.5 implies a “converse” of Lemma 2.4. Namely,

Lemma 2.6 Under the hypotheses and using notations of Lemma 2.5, if $t/T > \beta > 0$, then there is a number q depending on β so that

$$|a_p - z| \leq q\sqrt{\delta} .$$

Proof:

From Lemma 2.5, we get

$$x \sim t^{-1} \sim \beta^{-1}T^{-1} \sim \beta^{-1}\sqrt{\delta} .$$

Q.E.D.

Proof of Proposition 2. Let us conclude the proof. If $|a_p - z| \leq b\sqrt{\delta}$, then by Lemmas 2.6 and 2.4 we get $|a'_p - z'| \leq b'\sqrt{\delta}$ where b' depends on b . Then on the interval (a_p, a_{p+1}) the function u indeed has a slope bounded from both sides by positive numbers depending on b . This follows from the approximation given by Lemma 2.3 used with $x_1 := a_p$ and $x_2 := a_{p+1}$ or $x_1 := a'_p$ and $x_2 := a'_{p+1}$ respectively. Since the arguments of the arctan functions are bounded in all cases, the estimate follows. Because of the symmetry of the problem, one can also bound the slope of u on (a_p, a_{p+1}) assuming that $|a'_p - z'| \leq b'\sqrt{\delta'}$. The case which remains is when both $|a_p - z|$ and $|a'_p - z'|$ are large compared with δ . But then Lemma 2.5 is applicable showing that $|a_p - z| \sim |a'_p - z'|$ and since

$$|a_p - a_{p+1}| \sim (a_p - z)^2 \sim (a'_p - z')^2 \sim |a'_p - a'_{p+1}|$$

the slope is bounded as needed.

We finally remark that all was done assuming l sufficiently large. In the case of bounded l Proposition 2 is easy. So, the proof has been finished.

Construction of the branchwise equivalence. Faced with a case of almost parabolic critical pull-back, we first “mark” the branchwise equivalence so that upon its first exit from the central domain the critical value of ϕ is mapped onto the critical value of $\hat{\phi}$. This will add only bounded conformal distortion by Lemma 2.1. Next, we build a quasimetric map u which transforms B (the box through which the central branch ranges) onto \hat{B} , and the backward orbit of the endpoint a of the central domain by the central branch onto the corresponding orbit in the phase space of $\hat{\phi}$. The hard part of the proof is that this u is uniformly quasimetric, and that follows from Proposition 2. Then we change u between the boundary of B and the central domain by v_0 , and between a_i and a_{i+1} , $i \leq l - 2$ by

$$\hat{\phi}^{-i-1} \circ v_0 \circ \phi^{i+1} .$$

By Lemma 3.14 of [18], this will give a uniformly quasimetric map on $J \setminus (a_{l-2}, -a_{l-2})$ provided that it is quasimetric on any interval (a_{i-1}, a_{i+1}) . This is clear since we are pulling-back by bounded diffeomorphisms, with the exception of the last interval (a_{l-2}, a_{l-2}) where this is a quadratic map composed with a diffeomorphism. Still, it remains quasimetric.

Conclusion of the proof of Proposition 1. To finish the proof of Proposition 1, we still need to check that the standard replacement condition is satisfied with uniform norm. The fact that v_∞ restricted to each individual domain replaces v_∞ on J is implied by the fact that on each newly created domain v_∞ is a lift by diffeomorphism or folding branches, always ϵ -extendible, of v' from another branch. The technical details of this fact are provided in [18], Lemma 4.6. Then the fact that v_∞ restricted to each box also replaces v_∞ from J follows automatically. We just notice that boxes of ϕ_∞ , with the exception of the central domain, are the same as the boxes of ϕ . The construction of the Sewing Lemma can be conducted for \mathcal{Y} extending v_0 from one box only, so it will not affect the replacement condition. This concludes the proof of Proposition 1.

Theorem about initial inducing.

Definition 2.9 *A real box map is called suitable if the critical value of the central branch is in the central domain and stays there forever under iterations of the central branch.*

The inducing construction described earlier in this section hangs up on a suitable map. It is easy to see that if a map is suitable, there must be an interval symmetric with respect to 0 which is mapped inside itself by the central branch. In fact, this interval must be a restrictive interval of the original f .

Theorem 2 *Suppose that f and \hat{f} belong to \mathcal{F}_η for some $\eta > 0$ and are topologically conjugate. Then, for every $\alpha > 0$, we claim the existence of conjugate box mappings ϕ_s and $\hat{\phi}_s$ on the respective fundamental inducing domains J and \hat{J} , and a branchwise equivalence v_s between them so that a number of properties are satisfied.*

- ϕ_s and $\hat{\phi}_s$ are either full, or of type I (both of the same type.)
- ϕ_s and $\hat{\phi}_s$ both possess standard ϵ -extendibility in the sense given by Definition 1.14.
- v_s satisfies the standard replacement condition with distortion K_2 in the sense of Definition 2.2.
- v_s is K_1 -quasisymmetric.

In addition, ϕ_s and $\hat{\phi}_s$ either are both suitable or this set of conditions is satisfied:

- for any domain D of ϕ_s or $\hat{\phi}_s$ which does not belong to monotone branch which ranges through the fundamental inducing domain,

$$\frac{|D|}{\text{dist}(D, \partial J)} \leq \alpha,$$

- if the mappings are of type I, then

$$\frac{|B'|}{\text{dist}(B, \partial J)} \leq \alpha$$

and the analogous condition holds for $\hat{\phi}_s$,

- they both have extensions as complex box mappings with hole structures satisfying the geometric bound K_3 .
- on the boundary of each hole, the mapping onto the box is K_4 -quasisymmetric.

The bounds ϵ and K_i depend only on η and α .

An outline of the proof of Theorem 2. Theorem 2 is going to be the main result of this section. The strategy will be to obtain ϕ_s and its counterpart in a bounded (in terms of η and α) number of basic inducing operations. Then ϵ -extendibility will be satisfied by construction. The main difficulty will be to obtain the bounded hole structure. Next, the branchwise equivalence will be constructed based on Proposition 1, so not much extra technical work will be required other than checking the assumptions on this Proposition.

2.2 Inducing process

The starting point.

Fact 2.5 *Under the hypotheses of Theorem 2, conjugate induced mappings ϕ_0 and $\hat{\phi}_0$ exist with a branchwise equivalence v_0 . Also, versions of ϕ_0 and $\hat{\phi}_0$ exist which are refined at the boundary with appropriate branchwise equivalences. In particular, $v_{0,b}$ is infinitely refined at both endpoints. The following conditions are fulfilled for ϕ_0 , $\hat{\phi}_0$ and all their versions refined at the boundary. For simplicity, we state them for ϕ_0 only.*

- ϕ_0 has standard ϵ -extendibility,
- every domain except for two at the endpoints of J is adjacent to two other domains and the ratio of lengths of any two adjacent domains is bounded by K_1 ,
- v_0 restricted to any domain replaces $v_{0,b}$ with distortion K_2 and coincides with $v_{0,b}$ outside of J ,
- v_0 is K_3 -quasisymmetric,
- there are two monotone branches which range through J with domains adjacent to the boundary of J , and the ratio of length of any of them to the length of J is at least ϵ_1 .

The estimates ϵ , ϵ_1 and K_i depend only on η .

Proof:

This fact is a restatement of Proposition 1 and Lemma 4.1 of [18].

Q.E.D.

The general inducing step. Suppose that a real box mapping ϕ is given which is either full, of type I, or of type II and not suitable. Let us also assign a *rank* to this mapping which is 0 if ϕ is full. By the assumption that the critical value is recurrent, the critical value $\phi(0)$ is in the domain of ϕ . There are three main distinctions we make.

Close and non-close returns. If the critical value is in the central domain, the case is classified as a *close return*. Otherwise, it is described as *non-close*. For a close return, we look at the number of iterations of the central branch needed to push the critical value out of the central domain. We call it the *depth* of the close return.

Box and basic returns. The situation is described as *basic* provided that upon its first exit from the central domain the critical value falls into the domain of a monotone branch which ranges through J . Otherwise, we call it a *box case*.

High and low returns. We say that a return is *high* if 0 is the range of the central branch, otherwise the case is classified as *low*. Clearly, all eight combinations can occur, which together with three possibilities for ϕ (full, type I, type II) gives us twenty-four cases. In the description of procedures, we use the notation of Definition 1.9 for boxes of ϕ .

In all cases, we begin with minimal boundary-refinement.

A low, close and basic return. In this case the almost parabolic pull-back is executed. The resulting box mapping is usually not of any classified type. This is followed by the filling-in of all branches ranging through B' or B . This gives a type I mapping. Finally, the critical pull-back is applied to give us a full map.

A low, close and box return. Again, we begin by applying almost parabolic pull-back. Directly afterwards, we apply simple critical pull-back. The result will always be a type II mapping.

A high return, ϕ full or type I. Critical pull-back with filling-in is applied. The outcome is of type I in the box case, and full in the basic case.

A high return, ϕ of type II. Apply filling-in to obtain a type I map, then follow the previous case.

A low return, non-close and basic, ϕ not of type II. Apply critical pull-back with filling-in. The outcome is a full map.

A low return, non-close and basic, ϕ of type II. Use filling-in to pass to type I, then follow the preceding step.

A low and box return. Apply simple critical pull-back. The outcome is a type II map.

This is a well-defined algorithm to follow. The rank of the resulting mapping is incremented by 1 in the box case, and reset to 0 in the basic case.

Notation. A box mapping is of rank n , we will often denote $B_n := B$ and $B_{n'} := B'$.

Features of general inducing.

Lemma 2.7 *If ϕ is a type I or II induced box mapping of rank n derived from some $f \in \mathcal{F}$ by a sequence of generalized inducing steps, then*

$$\frac{|B_n|}{|B_{n'}|} \leq 1 - \epsilon$$

where ϵ is a constant depending on η only.

Proof:

If ϕ is full, the ratio is indeed bounded away from 1 since a fixed proportion of B_0 is occupied by the domains of two branches with return time 2. In a sequence of box mappings this ratio remains bounded away from 1 by Fact 4.1 of [10].

Q.E.D.

Lemma 2.8 *Let ϕ_n denote the mapping obtained from ϕ_0 given by Fact 2.5 in a sequence of n general inducing steps. There are sequences of positive estimates $\epsilon(n)$ and $\epsilon_1(n)$ depending otherwise only on η so that the following holds:*

- ϕ_n has standard $\epsilon(n)$ -extendibility,
- the margin of extendibility of branches ranging through J remains unchanged in the construction,
- all branches are $\epsilon(n)$ -extendible,
- for every box B smaller than J , both external domains exist, belong to monotone branches mapping onto J , and the ratios of their lengths to the length of B are bounded from below by $\epsilon_1(n)$.

Proof:

This lemma follows directly from analyzing the multiple cases of the general inducing step. The second claim follows directly since we use minimal boundary refinement. For the first claim, in most cases one uses Lemma 1.3. The more difficult situation is encountered if a simple critical pull-back occurs. In this situation, the extendibility of short monotone branches may not be preserved. However, one sees by induction that a short monotone domain is always adjacent to two long monotone domains. The ratio of their lengths can be bounded in terms of η and n . So, if the extendibility of this short monotone branch is drastically reduced, then the critical value must be in this adjacent long monotone domain. But then a basic return has occurred in which the general inducing step always uses critical pull-back with filling-in, which preserves extendibility.

Q.E.D.

Main proposition

Proposition 3 *For every $\alpha > 0$, there are a fixed integer N and $\beta_0 > 0$, both independent of the dynamics, for which the following holds. Given a mapping ϕ_0 obtained from Fact 2.5 there is a sequence of no more than N general inducing steps followed by no more than N steps of simultaneous monotone pull-back which give an induced map ϕ that satisfies at least one of these conditions:*

- ϕ is of type I and suitable,
- ϕ is full and α -fine,
- ϕ is of type I, has a hole structure of a complex box mapping with geometrical norm less than β_0 , and for every domain of a short monotone branch or the central domain, the ratio of its length by the distance from the boundary of I is less than α .

Proposition 3 is a major step in proving Theorem 2. Observe that the first and third possibilities are already acceptable as ϕ_s in Theorem 2.

Taking care of the basic case. We construct an induced map $\phi(\alpha)$ according by applying the general inducing step until the central domain and all short monotone domains D satisfy $|D|/\text{dist}(D, \partial J) \leq \alpha$. This will take a bounded number of general inducing steps because the sizes of central domains shrink at a uniform exponential rate (see 2.7.) Of course, we may be prevented from getting to this stage by encountering a suitable map before; in that case Proposition 3 is already proven. From the stage of $\varphi(\delta)$ on, whenever we hit a basic return we are in the second case of Proposition 3. Indeed, the ratio of length of the central domain or any short monotone domain to the distance from the boundary of J is small. What is left is only to shorten long monotone, non-external domains. This is done in a bounded number of simultaneous monotone pull-back steps. So we can assume that box returns exclusively occur beginning from $\varphi(\alpha)$.

2.3 Finding a hole structure

We consider the sequence (φ_k) of consecutive box mappings obtained from $\varphi_0 := \varphi(\alpha)$ in a sequence of general inducing steps. The objective of this section is to show that for some k which is bounded independently of everything else in the construction, a uniformly bounded hole structure exists which extends φ_k as a complex box mapping in the sense of Definition 1.8.

The case of multiple type II maps. We will prove the following lemma:

Lemma 2.9 *Consider some φ_m of rank n . There is a function $k(n)$ such that if the mappings $\varphi_{m+1}, \dots, \varphi_{m+k(n)}$ are all of type II, then $\varphi_{m+k(n)}$ has a hole structure which makes it a complex box mapping. The geometric norm of this hole structure is bounded depending solely on n and η .*

Proof:

If a sequence of type II mappings occurs, that means that the image of the central branch consistently fails to cover the critical point. The rank of all branches is fixed and equal to n . The central domain shrinks at least exponentially fast with steps of the construction at a uniform rate by Lemma 2.7. Thus, the ratio of the length of the central domain of φ_{m+k} to the length of B_n is bounded by a function of k which also depends on n , and for a fixed n goes to 0 as k goes to infinity.

This means that we will be done if we show that a small enough value of this ratio ensures the existence of a bounded hole structure. We choose two symmetrical circular arcs which intersect the line in the endpoints of B_n at angles $\pi/4$ to be the boundary of the box. (α will be chosen in a moment, right now assume $\alpha < \pi/2$). We take the preimages of the box by the monotone branches of rank n . They are contained in similar circular sectors circumscribed on their domains by Poincaré metric considerations given in [27]. Now, the range of the central branch is not too short compared to the length of B_n since it at least covers one branch adjacent to the boundary of the box (otherwise we would be in the basic case). We claim that the domain of that external branch constitutes a proportion of the box bounded depending on the return time of the central branch. Indeed, one first notices that the return time of the external branch is less than the return time of the central branch. This follows

inductively from the construction. But the range of the external domain is always the whole fundamental inducing domain, so the domain of this branch cannot be too short. On the other hand, the size of the box is uniformly bounded away from 0. To obtain the preimage of the box by the complex continuation of the central branch, we write the central branch as $h(z - 1/2)^2$. The preimage by h is easy to handle, since it will be contained in a similar circular sector circumscribed on the real preimage. Since the distortion of h is again bounded in terms of n , the real range of $(x - 1/2)^2$ on the central branch will cover a proportion of the entire $h^{-1}(B_n)$ which is bounded away from 0 uniformly in terms of n . Thus, the preimage of the complex box by the central branch will be contained in a star-shaped region which meets the real line at the angle of $\pi/4$ and is contained in a rectangle built on the central domain of modulus bounded in terms of n .

It follows that this preimage will be contained below in the complex box if the middle domain is sufficiently small. As far as the geometric norm is concerned, elementary geometrical considerations show that it is bounded in terms of n .

Q.E.D.

Formation of type I mappings. We consider then same sequence φ_k and we now analyze the cases when the image of the central branch covers the critical point. Those are exactly the situations which lead to type I maps in general inducing with filling-in.

A tool for constructing complex box mappings. The reader is warned that the notations used in this technical fragment are “local” and should not be confused with symbols having fixed meaning in the rest of the paper. The construction of complex box mappings (choice of a bounded hole structure) in the remaining cases will be based on the technical work of [20]. We begin with a lemma which appears there without proof.

Lemma 2.10 *Consider a quadratic polynomial ψ normalized so that $\psi(1) = \psi(-1) = -1$, $\psi'(0) = 0$ and $\psi(0) = a \in (-1, 1)$. The claim is that if $a < \frac{1}{2}$, then $\psi^{-1}(D(0, 1))$ is strictly convex.*

Proof:

This is an elementary, but somewhat complicated computation. We will use an analytic approach by proving that the image of the tangent line to $\partial\psi^{-1}(D(0, 1))$ at any point is locally strictly outside of $D(0, 1)$ except for the point of tangency. We represent points in $D(0, 1)$ in polar coordinates (r, ϕ) centered at a so that $\phi(1) = 0$, while for points in preimage we will use similar polar coordinates r', ϕ' . By school geometry we find that the boundary of $D(0, 1)$ is given by

$$(r + a \cos \phi)^2 + a^2 \sin^2 \phi = 1 .$$

By symmetry, we restrict our considerations to $\phi \in [0, \pi]$. We can then change the parameter to $t = -a \cos \phi$, which allows us to express r as a function of t for boundary points, namely

$$r(t) = \sqrt{1 - a^2 + t^2} + t . \tag{3}$$

Now consider the tangent line at the preimage of $(r(t), t)$ by ψ . By conformality, it is perpendicular to the radius joining to 0, so it can be represented as the set of points (r', ϕ')

$$r' = \frac{\sqrt{r(t)}}{\cos^2(\theta/2)}, \quad \phi' = \frac{\pi}{2} - \frac{\phi}{2} + \frac{\theta}{2}$$

where θ ranges from $-\pi$ to π . The image of this line is given by $(\hat{r}(\theta), \phi - \theta)$ where

$$\hat{r}(\theta) = \frac{2r(t)}{1 + \cos \theta}.$$

Let us introduce a new variable $t(\theta) := -a \cos(\phi - \theta)$ so that $t = t(0)$. Our task is to prove that

$$r(t(\theta)) < \hat{r}(\theta) \tag{4}$$

for values of θ in some punctured neighborhood of 0. This will be achieved by comparing the second derivatives with respect to θ at $\theta = 0$. By the formula [3]

$$r(t(\theta)) = \sqrt{1 - a^2 + t(\theta)^2} + t(\theta).$$

The second derivative at $\theta = 0$ is

$$\begin{aligned} & \frac{1 - a^2}{(\sqrt{1 - a^2 + t(\theta)^2})^3} (a^2 - t^2) - t - \frac{t^2}{\sqrt{1 - a^2 + t(\theta)^2}} = \\ & = -\sqrt{1 - a^2 + t(\theta)^2} - t + \frac{1 - a^2}{(\sqrt{1 - a^2 + t(\theta)^2})^3}. \end{aligned}$$

The second derivative of the right-hand side of the desirable inequality [4] is more easily computed as

$$\frac{\sqrt{1 - a^2 + t(\theta)^2} + t}{2}.$$

Thus, the proof of the estimate [4], as well as the entire lemma, requires showing that

$$\frac{3}{2}(\sqrt{1 - a^2 + t(\theta)^2} + t) - \frac{1 - a^2}{(\sqrt{1 - a^2 + t(\theta)^2})^3} > 0$$

for $|a| < 1/2$ and $|t| \leq |a|$. For a fixed a , the value of this expression increases with t . So, we only check $t = -a$ which reduces to

$$\frac{3}{2}(1 - a) - (1 - a^2) > 0$$

which indeed is positive except when $a \in [1/2, 1]$.

Q.E.D.

The main lemma. Now we make preparations to prove another lemma, which is essentially Lemma 8.2 of [20]. Consider three nested intervals $I_1 \subset I_0 \subset I_{-1}$ with the common midpoint at $1/2$. Suppose that a map ψ is defined on I_1 which has the form $h(x - 1/2)^2$ where h is a polynomial diffeomorphism onto I_{-1} with non-positive Schwarzian derivative. We can think ψ as the central branch of a generalized box mapping. We denote

$$\alpha := \frac{|I_1|}{|I_0|}$$

. Next, if $0 < \theta \leq \pi/2$ we define $D(\theta)$ to be the union of two regions symmetrical with respect to the real axis. The upper region is defined as the intersection of the upper half plane with the disk centered in the lower $\Re = 1/2$ axis so that its boundary crosses the real line at the endpoints of I_0 making angles θ with the line. So, $D(\pi/2)$ is the disk having I_0 as diameter.

Lemma 2.11 *In notations introduced above, if the following conditions are satisfied:*

- ψ maps the boundary of I_{-1} into the boundary of I_0 ,
- the image of the central branch contains the critical point,
- the critical value inside I_0 , but not inside I_1 ,
- the distance from the critical value to the boundary of I_0 is no more than the (Hausdorff) distance between I_{-1} and I_0 ,

then $\psi^{-1}(D(\theta))$ is contained in $D(\pi/2)$ and the vertical strip based on I_1 . Furthermore, for every $\alpha < 1$ there is a choice of $0 < \theta(\alpha) < \pi/2$ so that

$$\psi^{-1}(D(\theta(\alpha))) \subset D(\theta(\alpha))$$

with a modulus at least $K(\alpha)$, and $\psi^{-1}(D(\theta(\alpha)))$ is contained in the intersection of two convex angles with vertices at the endpoints of I_1 both with measures less than $\pi - K(\alpha)$. Here, $K(\alpha)$ is a continuous positive function.

Proof:

By symmetry, we can assume that the critical value, denoted here by c , is on the left of $1/2$. Then t denotes the right endpoint of I_0 , and t' is the other endpoint of I_0 . Furthermore, x means the right endpoint of B_{n-1} . By assumption, h extends to the range (t', x) . To get the information about the preimages of points t, t', c, x one considers their cross-ratio

$$C = \frac{(x - t)(c - t')}{(x - c)(t - t')} \geq \frac{1 + \alpha}{4}$$

where we used the assumption about the position of the critical value relative I_0 and I_{-1} . The cross ratio will not be decreased by h^{-1} . In addition, one knows that h^{-1} will map the disk of diameter I_0 inside the disk of diameter $h^{-1}(B_n)$ by the Poincaré metric argument of [27]. As a consequence of the non-contracting property of the cross-ratio, we get

$$\frac{h^{-1}(c) - h^{-1}(t')}{h^{-1}(t) - h^{-1}(t')} < \frac{1 + \alpha}{4}. \quad (5)$$

When we pull back the disk based on $h^{-1}(I_0)$, we will get a figure which intersects the real axis along I_1 . Notice that by the estimate [5] and Lemma 2.10, the preimage will be convex, thus necessarily contained in the vertical strip based on I_1 . Its height in the imaginary direction is

$$\frac{|I_1|}{2} \sqrt{\frac{h^{-1}(t) - h^{-1}(c)}{h^{-1}(c) - h^{-1}(t')}} < \frac{|I_1|}{2} \sqrt{\frac{3 - \alpha}{1 + \alpha}}, \quad (6)$$

where we used the estimate [5] in the last inequality. Clearly,

$$\psi^{-1}(D(\pi/2))$$

is contained in the disk of this radius centered at $1/2$. To prove that

$$\psi^{-1}(D(\pi/2)) \subset D(\pi/2),$$

in view of the relation [6] we need

$$\alpha \sqrt{\frac{3 - \alpha}{1 + \alpha}} < 1 \quad (7)$$

By calculus one readily checks that this indeed is the case when $\alpha < 1$. To prove the uniformity statements, we first observe that

$$\psi^{-1}(D(\theta)) \subset \psi^{-1}(D(\pi/2))$$

for every $\theta < \pi/2$. Since [7] is a sharp inequality, for every $\alpha < 1$ there is some range of values of θ below $\pi/2$ for which $\psi^{-1}(D(\theta)) \subset D(\theta)$ with some space in between. We only need to check the existence of the angular sectors. For the intersection of $\psi^{-1}(D(\theta))$ with a narrow strip around the real axis, such sectors will exist, since the boundary intersects the real line at angles θ and is uniformly smooth. Outside of this narrow strip, even $\psi^{-1}(D(\pi/2))$ is contained in some angular sector by its strict convexity.

Q.E.D.

The assumption of extendibility to the next larger box is always satisfied in our construction.

The case when there is no close return. We now return to our construction and usual notations. We consider a map φ_k , type II and of rank n , whose central branch covers the critical point, but without a close return. Then:

Lemma 2.12 *Either the Hausdorff distance from B_n to B_{n-1} exceeds the Hausdorff distance from B_{n-1} to B_{n-2} , or φ_k has a hole structure uniformly bounded in terms of n .*

Proof:

Suppose the condition on the Hausdorff distances fails. We choose the box around B_n and the hole around B_{n+1} by Lemma 2.11. Observe that the quantity α which plays a role in that Lemma is bounded away from 1 by Lemma 2.7. The box is then pulled back by these monotone branches and its preimages are inside similar figures built on the domains of branches by the usual Poincaré metric argument of [27]. For those monotone branches, the desired bounds follow immediately.

Q.E.D.

Proof of Proposition 3. This is just a summary of the work done in this section. We claim that we have proved that either a map with a box structure can be obtained from $\varphi(\delta)$ in a uniformly bounded number of steps of general inducing, or the Proposition 3 holds anyway. Since Lemmas 2.9 and 2.12 provide uniform bounds for the hole structures in terms of k or the rank which is bounded in terms of k , it follows that the hole structure is bounded or the starting condition holds anyway. If the inducing fails within this bounded number of steps because of a suitable map being reached, then the stopping time on the central branch of the suitable map is bounded, hence Proposition 3 again follows.

So, we need to prove that claim. If the claim fails, then by Lemma 2.9 the situations in which the image of the central branch covers the critical point have to occur with definite frequency. That is, we can pick a function $m(k)$ independent of other elements of the construction which goes to infinity with k such that among $\varphi_1, \dots, \varphi_k$ the situation in which the critical point is covered by the image of the central branch occurs at least $m(k)$ times. But each time that happens, we are able to conclude by Lemma 2.12 that the Hausdorff distance between more deeply nested boxes is more than between shallower boxes. Initially, for φ_0 whose rank was n , the B_n distance between and B_{n-1} was a fixed proportion of the diameter of B_n . So only a bounded number of boxes can be nested inside B_{n-1} with fixed space between any two of them. So we have a bound on the value of $m(k)$, thus on k . This proof of the claim is a generalization of the reasoning used in [20]. The claim concludes the proof of Proposition 3.

2.4 Proof of Theorem 2

Immediate cases of Theorem 2. Given two conjugate mappings ϕ_0 and $\hat{\phi}_0$ obtained by Fact 2.5, we apply the generalized inducing process to both. By Proposition 3 after a bounded number of generalized inducing steps we get to one of the three possibilities listed there: a suitable map, a full map, or a type I complex box map with a bounded hole structure. In the first and third cases all we need in order to conclude the proof is the existence of a branchwise equivalence v_s with needed properties. In the second case extra work will be required.

We will show that generally after n general inducing steps stair from a pair of conjugate ϕ_0 and $\hat{\phi}_0$ we get a branchwise equivalence v which satisfies

- v coincides with $v_{0,b}$ outside of J ,
- v satisfies the standard replacement condition with distortion $K_2(n)$ in the sense of Definition 2.2.
- v_s is $K_1(n)$ -quasisymmetric.

The bounds $K_1(n)$ and $K_2(n)$ depend only on η and n .

This follows by induction with respect to n from Proposition 1 using Lemma 2.8. This means Theorem 2 has been proved except in the second case of Proposition 3. In the remaining case, we still have the branchwise equivalence with all needed properties.

The case of ϕ full. We need to construct a hole structure. Observe that we can assume that the range of the central branch of ϕ is not contained in an external branch. In that case we could compose the central branch with this external branch until the critical value leaves the external domain. This would not change the branchwise equivalence, box or extendibility structures of the map. Also, instead of ϕ rather consider the version ϕ_b refined at the endpoint not in the range of the central domain enough times to make all domains inside this external domain of ϕ shorter than some α' .

A full map with two basic returns. Let us first assume that ϕ_b shows a basic return. Carry out a general inducing step. The resulting full mapping ϕ_1 has short monotone branches which are ϵ' -extendible and ϵ' goes to 1 as α and α' go to 0. Indeed, these short branches extend with the margin equal to the central domain of ϕ , and if α is small this is much larger than the central domain of ϕ_1 . Let δ denote the distance from the critical value of ϕ_1 to the boundary of J . We claim that for very $\delta_0 > 0$ there is an $\alpha > 0$, otherwise only depending on η , so that if ϕ was α -fine, then a bounded hole structure exists. Indeed, take the diamond neighborhood with height β of J as the complex box. Its preimage by the central branch is quasidisc with norm depending of δ , β and the extendibility (thus ultimately on η .) If β is very small, the preimage is close to the “cross” which is the preimage of J by the central branch is the complex plane. In particular, for β small enough it fits inside a rombe symmetrical with respect to the real axis with the central domain as a diagonal. The diameter of this rombe is bounded in terms of δ . If the extendibility of short monotone branch is sufficiently good, then the preimages of this rombe by the short monotone branches will be contained in similar rombes around short monotone domains (see Fact 2.1 and use K obe’s distortion lemma.) This gives us a bounded hole structure.

Finally, we show how to modify ϕ_1 so that its critical value is in a definite distance from the boundary of J . If the range of the central branch is very small, this is very easy. Just compose the central branch with the external branch a number of times to repel the critical value from the endpoint, but so that it is still inside the external domain, thus giving a basic return. This may require an unbounded number of compositions, but they will not change the branchwise equivalence. If the range of the central branch is almost the entire J , choose a version of ϕ_1 which is refined to the appropriate depth at the endpoint of J not in the range of the central branch. The appropriate depth should be chosen so that the image of the critical value by the external branch of the refined version is still in an external domain, but already in a definite distance from the boundary. The possibility of doing this follows since we can bound the eigenvalue of the repelling periodic point in the boundary of J from both sides depending on η (see Fact 2.3 in [18].) Then apply the general inducing step to this version of ϕ_1 , and the construct the hole structure as indicated above for the resulting map.

Since in this process we only use a bounded number of inducing operations, or operations that do not change the branchwise equivalence, the branchwise equivalence between the maps for which we constructed hole structures will satisfy the requirements of Theorem 2. So, in the case of a double basic return we finished the proof.

A box return. In this case, we apply the general inducing step once to get a type II mapping ϕ_1 . Observe that the central domain of ϕ_1 is very short compared to the size of the box (the ratio goes to 0 with α , independently of α' .) Again, if the critical value is in a definite distance from the boundary of the box, compared with the size of the box, then we can repeat the argument of Lemma 2.9 to build a hole structure. Otherwise, the critical value is in a long monotone branch external in the box. So one more application of the general inducing step will give us a mapping which is twice basic, so we apply the previous step. Again, we see that the branchwise equivalence satisfies the requirements of Theorem 2. This means that Theorem 2 has been proved in all cases.

3 Complex pull-back

3.1 Introduction

We will introduce a powerful tool for constructing branchwise equivalences while preserving their quasiconformal norm. Since the work is done in terms of complex box mappings, we have to begin by defining an inducing process on complex box mappings.

Complex inducing.

A simple complex inducing step. Suppose that ϕ is a complex box mapping which is either full or of type I. We will define an inducing step for ϕ which is the same as the general inducing used in the previous section on the real, with the only difference that infinite boundary-refinement is used. First, perform the infinite boundary refinement. This has an obvious meaning for complex box mappings. Namely, one finds bad long monotone branches and composes their analytic continuations with ϕ_r . Call the resulting map ϕ' . Next, replace the central branch of ϕ' with the identity to get ϕ^r . Next, construct $\tilde{\phi}$ which is the same as ϕ outside of the central domain, and put

$$\tilde{\phi} := \phi^r \circ \phi$$

on the central domain. Finally, fill in all short univalent branches of $\tilde{\phi}$ to get a type I or full mapping. Observe that on the real line this is just boundary refinement followed by critical pull-back with filling-in, so standard extendibility is preserved.

We also distinguish *complex inducing without boundary-refinement*. This is the same as the procedure described above only without boundary refinement.

A complete complex inducing step. A complete complex inducing step, which will also be called a *complex inducing step* is the same as the simple inducing step just defined provided that ϕ does not show a close return, that is, the critical value of ϕ is not in the central domain. If a close return occurs, a complete inducing step is a sequence of simple inducing steps until a mapping is obtained which shows a non-close return. Then the simple inducing step is done once again and the whole procedure gives a complete inducing step. If simple complex inducing steps are used without boundary refinement, we talk about a *complex inducing step* without boundary refinement.

External marking. Consider two equivalent conjugate complex box mappings, φ and $\hat{\varphi}$, see Definition 1.8 of complex box mappings.

Definition 3.1 *A quasiconformal homeomorphism Υ is called an externally marked branchwise equivalence if it satisfies this list of conditions:*

- restricted to the real line, Υ is a branchwise equivalence in the sense of Definition 1.4,
- Υ maps each box of φ onto the corresponding box of $\hat{\varphi}$,
- Υ maps each complex domain of φ onto a complex domain of $\hat{\varphi}$ so that these domains range through corresponding boxes,
- On the union of boundaries of all holes complex domains of φ , the functional equation

$$\hat{\varphi} \circ \Upsilon = \Upsilon \circ \varphi$$

holds.

The last condition of this definition will be referred to as the *external marking* condition by analogy to *internal marking* which will be introduced next.

Internal marking.

Definition 3.2 *Let v be a branchwise equivalence. An internal marking condition is defined to a choice of a set S so that S is contained in the union of monotone rank 0 domains of v and each such domain contains no more than one point of S . The branchwise equivalence v will be said to satisfy the internal marking condition if v coincides with the conjugacy on S .*

Definition 3.3 *We will say that a branchwise equivalence Υ is completely internally marked if for each internal marking condition Υ can be modified without changing it on the boundaries of holes and diamonds so that the marking condition is satisfied. We will say that an estimate, for example a bound on the quasiconformal norm, is satisfied for a fully internally marked Υ , if all those modifications can be constructed so as to satisfy this estimate.*

Definition 3.4 *Two generalized induced mappings ϕ and $\hat{\phi}$ are called equivalent provided that a branchwise equivalence v exists between them such that for every domain D of ϕ , we have*

$$\hat{\phi}(v(D)) = v(\phi(D)) .$$

Equivalence of induced mappings is weaker than their topological conjugacy and it merely means that the branchwise equivalence respects the box structures, that corresponding branch range through corresponding boxes, and that the critical values are in corresponding domains.

We are ready to state out main result.

Theorem 3 *Suppose that ϕ and $\hat{\phi}$ are conjugate complex box mappings with diamonds of type I or full. Suppose that Υ is Q -quasiconformal externally marked and completely internally marked branchwise equivalence. Also, a branchwise equivalence Υ_b is given between the*

versions of ϕ and $\hat{\phi}$ infinitely refined at the boundary. The map Υ_b is also Q -quasiconformal, externally marked and completely internally marked and coincides with Υ on the boundaries of all boxes. Suppose also that Υ or Υ_b restricted to the domain of any branch that ranges through J replaces Υ_b on J with distortion K' and suppose ϕ and $\hat{\phi}$ satisfy standard ϵ -extendibility. Suppose that ϕ_1 and $\hat{\phi}_1$ are obtained from ϕ and $\hat{\phi}$ respectively after several complex inducing steps, some perhaps without boundary refinement (but always the same procedure is used on both conjugate mappings.) Then, there is an externally marked and completely internally quasiconformal branchwise equivalence Υ_1 between ϕ_1 and $\hat{\phi}_1$. Furthermore, if ϕ_1 is full, then Υ_1 is Q -quasiconformal. Otherwise, Υ is Q -quasiconformal on the complement of the central domain and the union of short univalent domains. In boundary refinement is used at each step, then ϕ_1 and $\hat{\phi}_1$ still have standard ϵ -extendibility and Υ_1 restricted to any domain of a branch that ranges through J replaces Υ_1 on J with distortion K . The number K only depends on ϵ and K' . If the construction without boundary refinement is used and only the box case occurs, the theorem remains valid if ϕ and $\hat{\phi}$ are complex box mappings without diamonds.

3.2 Proof of Theorem 3.

Complex pull-back.

Historical remarks. The line of “complex pull-back arguments” initiated by [6] and used by numerous authors ever since. We only mention the works which directly preceded and inspired our construction. In [5] the idea of complex pull-back was applied to the situation with multiple domains and images, all resulting from a single complex dynamical system. According to [14], a similar approach was subsequently used in [28] in the proof of the uniqueness theorem for non-renormalizable quadratic polynomials. Also, [20] used a complex pull-back construction to study metric properties of the so called “Fibonacci unimodal map”.

Description. Given ϕ and $\hat{\phi}$ full or of type I with an externally marked branchwise equivalence Υ , we will show how to build a branchwise equivalence between ϕ_1 and $\hat{\phi}_1$ obtained in just one complex inducing step. The first stage is boundary refinement. Correspondingly, on the domain of each complex branch ζ being refined, we replace Υ with

$$\hat{\zeta}^{-1} \circ \Upsilon_b \circ \zeta$$

where Υ_b is the branchwise equivalence between the versions refined at the boundary which coincides with Υ on the boundaries of boxes. The map obtained in this way is called Υ' .

Next comes the “critical pull-back” stage. We replace Υ on the central domain with the lift to branched covers

$$\hat{\psi}^{-1} \circ \Upsilon' \circ \psi$$

where ψ and $\hat{\psi}$ are central branches. For this to be well-defined, we need $\Upsilon(\psi(0)) = \hat{\psi}(0)$. If this is a basic return, this condition may be assumed to be satisfied because of internal marking. In the box case, we have to modify Υ inside the complex branch which contains

the critical value. We can do this modification in any way which gives a quasiconformal mapping and leaves Υ unchanged outside of this complex domain. We also choose the lift which is orientation-preserving on the real line. Call the resulting map $\tilde{\Upsilon}$.

Finally, there is the infinite filling-in. This is realized as a limit process. Denote $\Upsilon^0 := \tilde{\Upsilon}$. Then Υ^{i+1} is obtained from $\tilde{\Upsilon}$ by replacing it on the domain of each short univalent branch ζ with $\hat{\zeta}^{-1} \circ \Upsilon^i \circ \zeta$. Then one proceeds the limit almost everywhere in the sense of measure.

Note that the branchwise equivalence $\Upsilon_{1,b}$ between versions of ϕ_1 and $\hat{\phi}_1$ infinitely refined at the boundary can be obtained in the same way using ϕ_b and $\hat{\phi}_b$ instead of ϕ and $\hat{\phi}$ respectively. Also observe that this procedure automatically gives an externally marked branchwise equivalence, and Υ_1 and $\Upsilon_{1,b}$ are equal on the boundaries of all boxes.

Complex pull-back on fully internally marked maps. We observe that the construction of complex pull-back we just described is well defined not only on individual branchwise equivalences with holes, but also on families of fully internally marked branchwise equivalences. This follows from the recursive nature of the construction. Suppose that ζ , the dynamics inside a hole or a diamond is used to pull back a branchwise equivalence Υ . We will show that any marking condition on newly created long monotone domains can be satisfied by choosing Υ appropriately marked. Indeed, long monotone branches of the induced map which arises in this pull-back step are preimages of long monotone branches of the map underlying Υ . Thus, if $s \in S$, one should impose the condition $\zeta(s)$ in the image. This can only lead to some ambiguity if ζ is 2-to-1. In this case ζ is an univalent map H followed by a quadratic polynomial. One prepares two versions of marking which are identical on the part of the real line not in the real image of ζ , and pulls back both of them by H . When they are finally pull back by the quadratic map, they will be the same on the vertical line through the critical point of ζ as a consequence of their being equal on the part of real axis not in the real image. So one can then match the two versions along this vertical line.

Induction. Now Theorem 3 is proved by induction with respect to the number of complex inducing steps. For one step, we just proved it. In the general step of induction, the only this which is not obvious is why a Q -quasiconformal branchwise equivalence is regained after a basic return even though the branchwise equivalence on the previous stage was not Q -quasiconformal on short monotone domains and the central domain. To explain this point, use the same notation as in the description of complex pull-back. The mapping $\tilde{\Upsilon}$ is Q -quasiconformal except on the union of domains of *short monotone branches*. The key point is to notice that it is Q -quasiconformal on the central domain, because this is a preimage of a long monotone domain (we assume a basic return!). Then for Υ^i the unbounded conformal distortion is supported on the set of points which stay inside short monotone branches for at least i iterations. The intersection of these sets has measure 0, which is very easy to see since each short monotone branch is an expanding in the Poincaré metric of the box. So, Υ^i form a sequence of quasiconformal mappings which converge to a quasiconformal limit and their conformal distortions converge almost everywhere. This limit almost everywhere is bounded by Q . By classical theorems about convergence of quasiconformal mappings, see [19], the limit of conformal conformal distortions equal to the conformal distortion of the limit mappings. Thus, the limit is indeed Q -quasiconformal.

To finish the proof of Theorem 3 we have to check extendibility and the replacement condition in the case when boundary refinement is being used. Extendibility follows directly from Lemma 1.3. Then, Υ_1 restricted to any domain whose branch ranges through J is a pull-back by an extendible monotone or folding branch of Υ (or Υ_b) from another such domain. The replacement condition is then satisfied, which follows from Lemma 4.6 of [18].

3.3 The box case

Box inducing. Suppose that a complex box mapping ϕ is given which is either full or of type I and shows a box return. We then follow the complex inducing step without boundary refinement. This will be referred to as *box inducing* and is the same as the box inducing used in [10]. This certainly works on complex box mappings without diamonds.

Complex moduli. Given a complex box mapping ϕ of type I or full, consider the annulus between the boundary of B' and the boundary of B . Denote its modulus $v(\phi)$. By our definition of box mappings, $v(\phi)$ is always positive and finite.

Theorem 4 *Let ϕ be complex box mapping of type II. Suppose that ϕ has a hole structure with a geometric bound not exceeding K . Let ϕ_0 denote the type I mapping obtained from ϕ by filling-in. Assume that box inducing can be applied to ϕ_0 n times, giving a sequence maps ϕ_i , $i = 0, \dots, n$. Then there is number $C > 0$ only depending on the bound of the hole structure so that*

$$v(\phi_i) \geq Ci$$

for every i .

Related results. We state the related results which will be used in the proof. This does not exhaust the list of related results in earlier papers, see “historical comments” below. The first one will be called “the starting condition”.

Definition 3.5 *We say that a type I or type II box mapping of rank n satisfies the starting condition with norm δ provided that $|B_n|/|B_{n'}| < \delta$ and if D is a short monotone domain of ϕ , then also $|D|/\text{dist}(D, \partial B_{n'}) < \delta$.*

Fact 3.1 *Let ϕ_0 be a real box mapping, either of type I of rank n or full. Pick $0 < \tau < 1$ and assume that the central branch is τ -extendible. For every τ , there is a positive number $\delta(\tau)$ with the following property. Suppose that ϕ_i , $i \geq 0$ is sequence of type I real box mappings such that ϕ_{j+1} arises from ϕ_j by a box inducing step. Let ϕ_0 satisfy the starting condition with norm $\delta(\tau)$. Then,*

$$|B_{n+i}|/|B_{(n+i)'}| \leq C^i$$

where C is an absolute constant less than 1.

Proof:

This is Fact 2.2 of [10]. A very similar statement, but for a slightly different inducing process is Proposition 1 of [17].

Q.E.D.

Fact 3.2 *Let ϕ_0 be a complex box mapping. Let $\phi_0, \phi_1, \dots, \phi_n$ be the sequence in which the next map is derived from the preceding one by a box inducing step. Let B_i denote the central domain of ϕ_i on the real line, and let $B_{n'}$ denote the box on the real line through which the central branch ranges. Suppose that for some $C < 1$ and every i*

$$\frac{|B_n|}{|B_{n'}|} \leq C^i$$

and suppose that the hole structure of ϕ_0 satisfies a geometric bound β . For every $C < 1$ and β there is a positive \bar{C} so that

$$v(\phi_i) \leq \bar{C}i$$

for every i .

Proof:

This is Theorem D of [10].

Q.E.D.

Historical comments. Theorem 4, in the way we state it, follows from Theorems B and D of [10]. However, we give a different proof based on Theorem D, but not on Theorem B. Weaker results saying that $|B_i|/|B_{i'}|$ decrease exponentially fast were proved in various cases in [17], [20] and later papers, including an early preprint of this work. In the Appendix we show a result similar to Fact 3.2. Even though this result is not a necessary step in the proof of the Main Theorem, we think that it may be of independent interest and also its proof shows the main idea of the proof of Fact 3.2

Artificial maps.

Introduction. Artificial maps were introduced in [20]. In our language, [20] showed how to prove that the starting condition must be satisfied at some stage of the box construction for a concrete “Fibonacci” unimodal map.² The idea was to use an artificial map “conjugated” to the box map obtained as the result of inducing. Clearly, an artificial map can be set up so as to satisfy the starting condition. Next, one shows that the induced map and its artificial counterpart are quasisymmetrically conjugate. Since for the artificial map the starting condition is satisfied with progressively better norms, after a few box steps it will be forced upon the induced map by the quasisymmetric conjugacy.

This strategy has a much wider range of applicability than the Fibonacci polynomial and it is at the core of our proof of Theorem 4.

²The Fibonacci map is persistently recurrent in the sense of [28], or of infinite box type in the sense of [17].

Topological conjugacy.

Lemma 3.1 *Given a box mapping induced by some f from \mathcal{F} and any homeomorphism of the line into itself, a box map (artificial) can be constructed that is topologically conjugate to the original one and whose branches are all either affine or quadratic and folding. The homeomorphism then becomes a branchwise equivalence.*

Proof:

The domains and boxes of the artificial map that we want to construct are given by the images in the homeomorphism. Make all monotone branches affine and the central branch quadratic. We show that by manipulating the critical value we can make these box maps equivalent. We prove by induction that if two maps are similar, by just changing the critical value of one of them we can make them equivalent. The induction proceeds with respect to the number of steps needed to achieve the suitable map. The initial step is clearly true by the continuity of the kneading sequence in the C^1 topology, and the the induction step consists in the remark that by manipulating the critical value in similar maps we can ensure that they remain similar after the next inducing and the critical value in the next induced map can be placed arbitrarily.

Q.E.D.

The construction of an original branchwise equivalence. We will prove the following lemma:

Lemma 3.2 *Under the assumptions of Theorem 4 and for every $\delta > 0$ there is an artificial map Φ with all branches either affine or quadratic and folding which is conjugate to ϕ . Next, Φ also has a hole structure which makes it a complex box mapping of type II. Also, the type I mapping obtained from Φ by filling-in satisfies the starting condition with norm δ . Also, we claim that an externally marked branchwise equivalence exists between ϕ and φ which is Q quasiconformal. The number Q only depends on δ and the geometric norm of the hole structure for ϕ (K in Theorem 4).*

Proof:

First construct the artificial map Φ . To this end, we choose a diffeomorphism h which is the identity outside B' and squeezes the central branch so that the after filling-in the type I map satisfies the starting condition. We observe that the “nonlinearity” h''/h' can be made bounded in terms of the bound on the hole structure. Then by Lemma 3.1 we can adjust the critical value of this artificial map, without moving domains of branches around, so that the map is equivalent to ϕ . This defines Φ .

Since Φ has affine or quadratic branches, the existence of a bounded hole structure equivalent to the structure already existing is clear. One simply repeats the arguments of Lemmas 2.12 and 2.9 with “infinite extendibility” which makes the problem trivial.

The final step is to construct the branchwise equivalence by Lemma 1.2. First, we decide that on the real line the branchwise equivalence is h . We next define Υ inside the complex box around B' . On the boundary of the box, we just take a bounded quasiconformal map. Again by K obe’s lemma this propagates to the holes with bounded distortion. Since h is a

diffeomorphism of bounded nonlinearity inside each hole, it can be filled with a uniformly quasiconformal mapping. Lemma 1.2 works to build Υ inside the complex box belonging to B' with desired uniformity. Then the map is extended outside of the box. This can also be achieved by Lemma 1.2 regarding B' as a tooth, holes outside of the box as other teeth, and choosing a big mouth.

Q.E.D.

3.4 Marking in the box case

In the box case Theorem 3 does not imply that quasiconformal norms stay bounded. On the other hand, internal marking does not work either in the box case. We should a special procedure of achieving an internal marking condition. We remind the reader that $v(\phi)$ denotes the modulus between B' and B when ϕ is of type I, or between B_0 and B when ϕ is full.

Proposition 4 *Let $\tilde{\phi}$ and $\tilde{\Phi}$ be conjugate complex box mappings either full or of type I, not suitable and showing a box return. Suppose that ϕ and Φ are derived from $\tilde{\phi}$ and $\tilde{\Phi}$ respectively, and not suitable and show a box return. Let Υ be a Q -quasiconformal branchwise equivalence acting into the phase space of Φ . Let v denote the minimum of $v(\tilde{\Phi})$ and $v(\Phi)$. Then there is a branchwise equivalence Υ' with the following properties:*

- Υ' equals Υ except on the complex domain of ϕ which contains the critical value of ϕ ,
- $\Upsilon(\phi(0)) = \Phi(0)$,
- the quasiconformal norm of Υ' is bounded by

$$Q + K_1 \exp(-K_2 v)$$

where K_1 and K_2 are positive constants.

Note that the estimate of the quasiconformal norm is independent of the geometry of ϕ .

An auxiliary lemma.

Lemma 3.3 *Let Φ and $\tilde{\Phi}$ and v be as in the statement of Proposition 4. Let $B_0 \supset B' \supset B$ be the box structure of Φ . Choose a complex domain $b \in B'$ of Φ which is either short monotone. There exists an annulus A and a constant $0 < C < 1$ with the following properties:*

- A surrounds b and is contained in B' .
- the modulus of A is at least Cv and the modulus of the annulus separating A from the boundary of B' is at least Cv ,
- the boundary of A intersects the real line at four points, and these points are topologically determined, meaning that if A' is constructed for a mapping Φ' conjugated to Φ , the conjugacy will map these four points onto the four points of intersection between A' and the real line.

Proof:

We split the proof in two cases depending on whether $\tilde{\Phi}$ showed a close return or not. In the case of a non-close return, the outer boundary of A is chosen as the boundary of the domain of the analytic extension of the branch from b onto the range B' . This means that the annulus of A is equal to $v(\Phi)$. On the other hand, the domain of this extension is contained in the preimage by the central branch of $\tilde{\Phi}$ of some domain of $\tilde{\Phi}$. A lower bound by $v(\tilde{\Phi})/2$ is evident. Also, the topological character of this construction is clear.

In the case of a close return, however, this construction would not work because the annulus between the extension domain and the boundary of B' may become arbitrarily small. So, let us consider the central domain of Φ . Recall that the box inducing step in the case of a close return is a sequence of simple box inducing steps with close returns ended by a simple box inducing step with a non-close return. Let Φ_1 mean the mapping obtained for $\tilde{\Phi}$ by this sequence of simple box inducing steps with close returns. The central branch of Φ is the composition of a restriction of ψ_1 (the central branch of Φ_1) with a short univalent branch b_1 of Φ_1 . By construction, this short univalent branch has an analytic continuation whose domain is contained in the range of ψ_1 and whose range is the range of the central branch of $\tilde{\Phi}$. Inside this extension domain, there a smaller domain δ mapped onto the central domain of Φ only. Consider the annulus W between B and the boundary of $\psi^{-1}(\delta)$. The modulus of W is a half of the modulus of the annulus between b_1 and the boundary of δ , which is $v(\tilde{\Phi}) \cdot 2^{-l}$ where l is the number of subsequent iterations of ψ which keep the critical value inside the central domain. It follows that the modulus of W is at least $v(\tilde{\Phi})/4$. Next, look at the annulus W' separating W from the boundary of B' . This is at least a half of the annulus separating δ from the larger extension domain (onto the range of ψ .) This last annulus has modulus $v(\tilde{\Phi})$. So W' has modulus at least $v(\tilde{\Phi})/2$. Now, to pick A take the extension of the branch from b onto B' and define A to be the preimage of W by this extension. Since A is separated from the boundary of the extension domain, let alone from the boundary of B' , by the preimage of W' , the estimate claimed by this Lemma follows. Also, the topological character of the intersection of A with the real line is clear.

Q.E.D.

Proof of Proposition 4. Suppose that the first exit time of the critical value from the central domain under iteration by the central branch is l . Call $C = \Phi^l(0)$ and $c = \phi^l(0)$. Suppose C belongs to a short monotone domain b .

The point $\Upsilon(c)$ must also be in b . Take the annulus A found for b from Lemma 3.3. Also, call A' the annulus separating A from the boundary of B' . Perturb Υ to Υ_1 so that $\Upsilon_1(c) = C$ and $\Upsilon = \Upsilon_1$ outside of $\Upsilon^{-1}(A)$. We accept as obvious that this can be done by composing Υ with a mapping whose conformal distortion is bounded by $k_1 \exp(-k_2 \text{mod } A)$. By Lemma 3.3, note that this is a correction allowed by Proposition 4. Then, we apply complex pull-back by the central branch ψ to Υ_1 l -times. Call the resulting branchwise equivalence Υ_2 . If $l = 1$ it is clear that the conformal distortion of Υ_2 is the same as for Υ_1 . If $l > 1$ it less clear since we will have to adjust the branchwise equivalence $l - 1$ times inside shrinking preimages of B to make c and C correspond. So until just before the last simple box inducing step the conformal distortion is not well-controlled. However, we show as in the proof of Theorem 3 that in this last simple box inducing step, the region where the

distortion was unbounded has preimage of measure 0 because of filling-in, so that ultimately the distortion of Υ_2 is the same as for Υ_1 almost everywhere.

Next, construct Υ_3 which is the same as Υ outside of $\Upsilon^{-1}(b)$ and equals $\Phi^{-1} \circ \Upsilon_2 \circ \phi$ on b . Both mappings match continuously because of external marking. Inside b , now look at $A_3 = \Phi^{-l-1}(A)$ and $A'_3 = \Phi^{-l-1}(A')$. By construction, the mapping Φ^{l+1} in the region encompassed by these annuli is a branched cover of degree 2, so $\text{mod } A_3 = \frac{1}{2} \text{mod } A$ and $\text{mod } A'_3 = \frac{1}{2} \text{mod } A'$. We consider two cases. If C belongs to the region encompassed by the outer boundary of A , then $\Upsilon_3(c)$ also belongs to this region. This is because the points of intersection of A , and therefore of A_3 , with the real line are topologically determined by Lemma 3.3. So, we perturb Υ_3 to leave it unchanged outside of $\Upsilon^{-1}(b)$ and to make the critical values correspond. Like in the previous paragraph, we claim that this adjustment will only add $k_1 \exp(-k_2 \text{mod } A'_3)$ to the conformal distortion. In this case, we are done with the proof of Proposition 4. Otherwise, C belong to the preimage of some short univalent domain b' of Φ by Φ^{l+1} . Since b' is nested inside B' with a modulus at least $v(\Phi)$, then this preimage is nested inside b with a modulus at least half that large. Also, $\Upsilon_3(c)$ must belong to the same preimage of b' . Again, we adjust Υ_3 to make the critical values correspond and are done with the proof.

Proof of Theorem 4. Given ϕ from Theorem 4, construct a conjugate artificial mapping Φ from Lemma 4.2 choosing δ from Fact 3.1 for a large $\tau = 1001$ (the artificial map has arbitrary extendibility.) This will guarantee by Fact 3.2 that if the sequence Φ_i is derived from $\Phi_0 := \Phi$ by box inducing, then the moduli $v(\Phi_i)$ grow at least at a uniform linear rate. Lemma 4.2 also gives us an externally marked and uniformly quasiconformal branchwise equivalence Υ_0 from the phase space of ϕ to the phase space of Φ .

Now proceed by complex pull-back as defined in the proof of Theorem 3, however do the marking corrections by Proposition 4. We get a sequence of uniformly quasiconformal branchwise equivalences between ϕ_i and Φ_1 . Since quasiconformal mappings preserve complex moduli up to constants, $v(\phi_i) \geq K v(\Phi_i)$ with K only depending on the conformal distortion of Υ_0 , thus ultimately (Lemma 4.2) only on the geometric bound of the hole structure of ϕ . Theorem 4 follows.

4 Construction of quasisymmetric conjugacies

Here is the main result of this section.

Theorem 5 *Suppose that f and \hat{f} are topologically conjugate and belong to some \mathcal{F}_η . Assume also that if f is renormalizable, then the first return time of its maximal restrictive interval to itself is greater than 2. Then for every $\delta > 0$ there exist conjugate real box mappings, ϕ^ϵ and $\hat{\phi}^\epsilon$, either full or of type I and infinitely refined at the boundary, with a branchwise equivalence v between them so that the following list of properties holds:*

- ϕ and $\hat{\phi}$ are either suitable or δ -fine,
- ϕ and $\hat{\phi}$ both have standard ϵ -extendibility,
- v is Q -quasisymmetric,

- v restricted to any long monotone domain replaces v on the fundamental inducing domain with distortion K ,
- v restricted to any short monotone domain replaces v on B with norm K .

The numbers $\epsilon > 0$, Q and K depend on η only.

4.1 Towards final mappings

Technical details of the construction. The next lemma tells that given mappings ϕ_s , $\hat{\phi}_s$ and v_s obtained from Theorem 2, we can modify and extend v_s to an externally marked and fully internally marked quasiconformal branchwise equivalence.

Lemma 4.1 *Let ϕ and $\hat{\phi}$ be topologically conjugate complex box mappings, full or of type I, both infinitely refined at the boundary. Suppose that*

- both have hole structures geometrically bounded by K' ,
- on the boundary of each hole the mapping is K'' quasisymmetric,
- for the domain D of any branch of ϕ , $|D|/\text{dist}(D, \partial J) \leq \alpha$ holds with some fixed α ; the same holds for every domain of $\hat{\phi}$,
- if the mappings are of type I, then $|B'|/\text{dist}(B', \partial J) \leq \alpha$ and the same holds for $\hat{\phi}$,
- all long monotone branches of ϕ and $\hat{\phi}$ are ϵ -extendible, $\epsilon > 0$,
- v exists which is a completely internally marked branchwise equivalence between ϕ and $\hat{\phi}$,
- v is Q -quasisymmetric,
- v restricted to any domain of ϕ replaces v on J with distortion K .

We claim that there a bound α_0 depending on K' only so that if $\alpha < \alpha_0$, the following holds:

- ϕ and $\hat{\phi}$ can be extended to complex box mappings with diamonds, call them ϕ^d and $\hat{\phi}^d$,
- an externally and completely internally marked branchwise equivalence Υ_0 exists between ϕ^d and $\hat{\phi}^d$,
- Υ_0 is L -quasiconformal,
- bounds L and α_0 only depend on ϵ , K , K' , K'' and Q .

Proof:

Let us first pick α_0 . Recall Fact 2.1 and choose κ as K_1 picked for ϵ by this Fact. Make $\kappa \leq 1/2$ as well. Consider the diamond neighborhood with height κ of J . The bound α_0 should be picked so as to guarantee that all holes of ϕ , as well as the box B' in case ϕ is of type I, sit inside this diamond neighborhood, moreover, that they have annular “collars” of definite modulus (say 1) which are also contained in this diamond neighborhood. All holes quasidisks bounded in terms of K' . So, there is a bounded ratio between how far they extend in the imaginary direction and the length of the real domain they belong to. Now it is evident that α_0 small will imply this, and α_0 depends only on K' . Also, by making α_0 even smaller, we can ensure that diamond neighborhoods with height κ of all domains inside B' are contained inside the complex box corresponding to B' , also with annular margins 1.

Next, we choose the diamonds. We will take the diamond neighborhood with height κ of J as the complex box B_0 . The diamonds will simply be preimages of this B_0 by all monotone branches. They will be contained in diamond neighborhoods with height κ of corresponding domains of branches by the Poincaré metric argument of [27]. Also, the diamonds will be preimages of B_0 with bounded distortion (Fact 2.1 again.) We can do the same thing for $\hat{\phi}$. This gives us ϕ^d and $\hat{\phi}^d$.

The final step is to construct the branchwise equivalence by Lemma 1.2. First, we decide that on the real line the branchwise equivalence is v . On the box around B_0 extend it in any way that maps B_0 onto \hat{B}_0 and gives a uniformly quasiasymmetric (in terms of Q and κ) mapping on the union of J and the upper (lower) half of B_0 . Then Υ_0 on the boundaries of diamonds is determined by pull-back. Observe, however, that on the curve consisting of the upper (lower) half of the diamond on the domain on the real line, the map is quasiasymmetric, and that is because of the replacement condition. It follows that diamonds can be filled with uniformly quasiconformal mappings as Lemma 1.2 demands.

We first define Υ_0 inside the complex box around B' . On the boundary of the box, we just take a map transforming it onto the boundary of \hat{B}' and quasiconformal with a bounded norm (in terms of K' and Q) on the union of the upper (lower) half of the boundary of the complex box and the real box B' . This propagates to the holes with bounded deterioration of the quasiasymmetric norm (in terms of K'' .) Thus, each hole can be filled with a uniformly quasiconformal mapping. Lemma 1.2 works to build Υ_0 inside the complex box corresponding to B' with desired uniformity. This having been achieved, Lemma 1.2 is again used inside the entire complex box around B_0 . Here, the complex box around B' is formally regarded as a tooth. Finally, Υ_0 is extended to the plane by quasiconformal reflection.

Q.E.D.

The initial branchwise equivalence. Suppose now that we are in the situation of Theorem 2 with mappings ϕ_s and $\hat{\phi}_s$ not suitable. We want to build an externally marked and completely internally marked branchwise equivalence Υ_s between them. For that, we will use Lemma 4.1 with $\phi := \phi_{s,b}$ and $\hat{\phi} := \hat{\phi}_{s,b}$ (the additional subscript b denotes versions infinitely refined at the boundary.) Comparing the assumptions of Lemma 4.1 with claims of Theorem 2, we see that two conditions that are missing are the complete internal marking, and $|D|/\text{dist}(D, \partial J)$ when D is a long monotone domain. Let us first show that the second property can be had by doing more inducing on long monotone domains. On the level of

inducing, we simply compose long monotone domains whose domains are too large with ϕ_s ($\hat{\phi}_s$ respectively) until we reduce their sizes sufficiently. This can take many inducing steps on any given domain. Note that the hole structure can simply be pull-back. The geometric bound of the hole structure will be worsened only in a bounded fashion provided that α is small enough. This follows from Fact 2.1. Also, the replacement condition will not suffer too much because we are pulling back by maps of bounded distortion (or one can formally use Proposition 1). The only hard point is the quasimetric norm. This does not directly follow from Proposition 1, since we may have to do a large number of simultaneous monotone pull-backs. However, this is easily seen if we proceed by complex pull-back (like in the proof of Theorem 3). To this end, we pick the diamond neighborhood of J with height $1/2$ as B_0 , and a homothetic neighborhood of \hat{J} as \hat{B}_0 . The diamonds are the preimages of B_0 (\hat{B}_0 resp.) by long monotone branches. We do not have any holes. Then the argument used in the proof of Lemma 4.1 applies and allows us to build a branchwise equivalence Υ' “externally marked” on the boundaries of all diamonds (but not on the boundaries of holes.) We can then perform the complex pull-back on long monotone branches any number of times without increasing the quasiconformal norm.

Next, we need to show that the complete internal marking can be realized with a bounded worsening of the bounds. This follows directly from the proof of Lemma 2.1. This Lemma shows only how to implement the marking condition at the critical value, but the argument works the same way for any marking condition.

The final maps. Now we can apply Lemma 4.1 to these modified mappings, and get $\phi^d, \hat{\phi}^d$ and an externally marked completely internally marked branchwise equivalence Υ^d between them. Now, they satisfy the hypotheses of Theorem 3. Proceed by complex inducing with boundary refinement starting from ϕ^d and $\hat{\phi}^d$. It might be that full mappings occur infinitely many times in this sequence. Otherwise, we can define *final* mappings $\phi^f, \hat{\phi}^f$ with their branchwise equivalence Υ^f as either the initial triple ϕ^d , etc., if no full mapping occurs in the sequence derived by complex inducing, or the last triple in this sequence with ϕ^f and $\hat{\phi}^f$ full. Observe that Theorem 3 is applicable with $\Upsilon = \Upsilon_b = \Upsilon^d$. So, Υ^f has all properties postulated by Theorem 3 for Υ_1 .

4.2 Proof of Theorem 5

Getting rid of boundary refinement.

Lemma 4.2 *Let ϕ be a box mapping, either full or of type I, infinitely refined at the boundary, which undergoes k steps of box inducing. Suppose that ϕ has standard ϵ -extendibility. Then there is a mapping ϕ^r obtained from ϕ by a finite number of simultaneous monotone pull-back steps using $\phi^l := \phi$ so that after k box inducing steps starting from ϕ^r the resulting map has standard ϵ -extendibility.*

Proof:

The point is that box inducing skips boundary refinement. However, we show that this can be offset by doing enough “boundary refinement” before entering the box construction. Observe first the following thing. Under the hypotheses of the Lemma, suppose that after k

box inducing steps we get a mapping ϕ_k and then do a simultaneous monotone pull-back on all long monotone branches of ϕ_k using $\phi' := \phi_k$. Then the same mapping can be obtained by doing a simultaneous monotone pull-back on all long monotone branches of the original ϕ . The proof of this remark proceeds by induction. For $k = 1$ this is rather obvious. For the induction step from $k - 1$ to k consider $\phi := \phi_1$ and use the hypothesis of induction. It follows that we need to perform inducing on all long branches of ϕ_1 , and for that use the fact again with $k = 1$. Now the lemma follows immediately, since each time one needs boundary refinement in a general inducing step, the appropriately refined mapping can be obtained by simultaneous monotone pull-back on some or all long monotone branches of ϕ .

Q.E.D.

Main estimate.

Lemma 4.3 *Let ϕ and $\hat{\phi}$ be a pair of topologically conjugate complex box mapping with diamonds of type I or full which undergo k steps of complex box inducing resulting in mappings ϕ_k and $\hat{\phi}_k$. Suppose that both hole structures can be assigned the separation index K . Suppose that box are infinitely refined at the boundary and have standard ϵ extendibility. Also, suppose that a branchwise equivalence Υ exists which is Q -quasiconformal, externally marked and completely internally marked, and satisfies restricted to any long monotone domain on the real line replaces Υ on the fundamental inducing domain with distortion K' . Then there are numbers L_1 which only depends on K and L_2 depending on K' and ϵ , with complex box mappings Φ and $\hat{\Phi}$ of type I and a branchwise equivalence Υ' between them so that the following conditions are satisfied:*

- Υ' is $Q + L_1$ -quasiconformal,
- Υ' is externally marked and completely internally marked,
- Υ' restricted to any long monotone domain replaces Υ' on the fundamental inducing domain with distortion L_2 ,
- Φ has the same box structure as ϕ_k , while $\hat{\Phi}$ has the same box structure as $\hat{\phi}_k$,
- Φ and $\hat{\Phi}$ both have standard ϵ -extendibility.

Proof:

The mapping Φ as obtained as ϕ' for ϕ from Lemma 4.2. $\hat{\Phi}$ is obtained in the same way for $\hat{\phi}$. They are topologically conjugate. By Lemma 4.2 this means that we should obtain some φ_0 by a series of simultaneous monotone pull-backs on long monotone branches of ϕ , and $\hat{\varphi}_0$ is obtained in an analogous way for $\hat{\phi}$. Then we perform box inducing on φ_0 , to get a sequence φ_i with $\varphi_k = \Phi$ and the same is done for $\hat{\varphi}_0$ which gives $\hat{\Phi} = \hat{\varphi}_k$. The branchwise equivalence is obtained by complex pull-back.

Among the claims of Lemma 4.3 the extendibility is clear and the replacement condition follows in the usual way based on Lemma 4.6 of [18]. The hard thing is the quasiconformal estimate for Υ' . The procedure used in the proof of Theorem 3 does not give a uniform

estimate for mappings which are not full. However, by Proposition 4 and Theorem 4 modifications required to obtain the marking in the box case can be done with distortions which diminish exponentially fast at a uniform rate.

Q.E.D.

Conclusion. For the proof of Theorem 5, we begin by Theorem 2 which tells us that either we hit a suitable map first, or we can build induced mappings ϕ^s , $\hat{\phi}^s$ and ν^s . If we encounter the suitable map first, then the conditions of Theorem 5 follow directly from Theorem 2. Note that the assumption about the return time of the restrictive interval into itself is needed to make sure that the suitable mapping has monotone branches, and thus can be infinitely refined at the boundary.

Otherwise, we proceed to obtain final maps with the branchwise equivalence between them. To this end, we build the complex branchwise equivalence, by Lemma 4.1, and proceed by Theorem 3 to obtain the branchwise equivalence between final maps. If final maps do not exist, it means that infinitely many times in the course of the construction we obtain full mappings. By Theorem 3, we get them with uniformly quasiconformal branchwise equivalences. In this sequence of full mappings the sizes of domains other than long monotone ones go to 0. So, having been given a δ we proceed far enough in the construction, and then get the δ -fine mapping by applying simultaneous monotone pull-back on long monotone domains. Theorem 5 follows in this case as well.

So we are only left with the case when final maps exist. Then we pick up the construction by Lemma 4.3. Observe that the assumption about the separation index is satisfied for the following reason. The final map is either the same as ϕ^s , in which case the bound follows directly from Theorem 2, or is full and its holes are inside the holes of ϕ^s constructed by Theorem 2, so the separation index is even better. If f was renormalizable, we choose k in Lemma 4.3 equal to the number of box inducing steps needed to get the suitable map. Then the conditions of Theorem 5 follow directly from Lemma 4.3. The replacement condition on short monotone domains is a consequence of the fact that by construction the branchwise equivalence on short monotone domains is the pull-back of the branchwise equivalence from B , and short monotone branches are extendible by Theorem 4. When f is non-renormalizable we choose a large k depending on δ and follow up with a simultaneous monotone pull-back on all long domains. Theorem 5 likewise follows.

5 Proof of Theorem 1

5.1 The non-renormalizable case

Theorem 1 in the non renormalizable case follows directly from Theorem 5. Choose a sequence δ_n tending to 0. The corresponding branchwise equivalences obtained by Theorem 5 for δ :

δ_n will tend to the topological conjugacy in the C^0 norm. Since they are all uniformly quasisymmetric in terms of η , so is the limit.

5.2 Construction of the saturated map

In the renormalizable case, the only missing piece is the construction of saturated maps with quasimetric branchwise equivalences between them. Also, we need to make sure that the branches of the saturated map are uniformly extendible. The case when the return time of the maximal restrictive interval into itself is 2 is not covered by Theorem 5. In this case, we simply state that Theorem 1 is obvious and proceed under the assumption that the return time is bigger than 2. So, Theorem 1 follows from this proposition:

Proposition 5 *Suppose that conjugate suitable real box mappings, φ and $\hat{\varphi}$ are given, both either full or of type I and infinitely refined at the boundary, with a branchwise equivalence Υ between them so that the following list of properties holds:*

- ϕ and $\hat{\phi}$ both have standard ϵ -extendibility,
- Υ is Q -quasimetric,
- Υ restricted to any long monotone domain replaces v on the fundamental inducing domain with distortion K ,
- Υ restricted to any short monotone domain replaces v on B with norm K .

Then, their saturated mappings φ^s and $\hat{\varphi}^s$ are ϵ -extendible. Also a Q' -quasimetric saturated branchwise equivalence Υ' exists. Q' depends on Q , K , and ϵ only.

The proof of Proposition 5 is basically quoted from [18] with only minor adjustments.

An outline of the construction. Let ψ mean the central branch. Let I denote the restrictive interval. First, we want to pull the branches φ into the domain of ψ . We notice that each point of the line which is outside of the restrictive interval will be mapped outside of the domain of ψ under some number of iterates of ψ . We can consider sets of points for which the number of iterates required to escape from the domain of ψ is fixed. Each such set clearly consists of two intervals symmetric with respect to the critical point. The endpoints of these sets form two symmetric sequences accumulating at the endpoints of the restrictive interval, which will be called *outer staircases*. Consequently, the connected components of these sets will be called steps.

This allows us to construct an induced map from the complement of the restrictive interval in the domain of ψ to the outside of the domain ψ with branches defined on the steps of the outer staircases. That means, we can pull-back Υ to the inside of the domain of ψ .

Next, we construct the *inner staircases*. We notice that every point inside the restrictive interval but outside of the fundamental inducing domain inside it is mapped into the fundamental inducing domain eventually. Again, we can consider the sets on which the time required to get to the fundamental inducing domain is fixed, and so we get the steps of a pair of symmetric inner staircases.

So far, we have obtained an induced map which besides branches inherited from φ has uniformly extendible monotone branches mapping onto I . Denote it with φ^1 . We now proceed by filling-in to get rid of short monotone branches. We conclude with refinement

of remaining long monotone branches. Thus, we will be left with branches mappings onto J only, so this is a saturated map. Its extendibility follows from the standard argument of inducing. The same inducing construction is used for $\hat{\varphi}$. Because the extendibility is obvious, we only need to worry the branchwise equivalence Υ' .

Outer staircases. Suppose that the domain of ψ is very short compared with the length of the the domain of φ . This means that the domain of ψ is extremely large compared with the restrictive interval. This unbounded situation leads to certain difficulties and is dealt with in our next lemma.

Lemma 5.1 *One can construct a branchwise equivalence Υ^1 which is a pull-back of Υ with quasiconformal norm bounded as a uniform function of the norm of Υ . Furthermore, an integer i can be chosen so that the following conditions are satisfied:*

- *The functional equation*

$$\Upsilon \circ \psi^j = \hat{\psi}^j \Upsilon^1$$

holds for any $0 \leq j \leq i$ whenever the left-hand side is defined.

- *The length of the interval which consists of points whose i consecutive images by ψ remain in the domain of ψ forms a uniformly bounded ratio with the length of the restrictive interval.*

Proof:

We rescale affinely so that the restrictive intervals become $[-1, 1]$ in both maps. Denote the domains of ψ and $\hat{\psi}$ with P and \hat{P} respectively. Then, ψ can be represented as $h(x^2)$ where h''/h' is very small provided that $|P|$ is large. We can assume that $|P|$ is large, since otherwise we can take $\Upsilon^1 := \Upsilon$ to satisfy the claim of our lemma. We consider the round disk B (“box”) whose diameter is the box of φ ranged through by ψ . Because ψ is extendible, and its domain was assumed to be small compared to P , the preimage of B by ϕ , called B_1 sits inside B with a large annulus between them. Analogous objects are constructed for $\hat{\varphi}$. It is easy to build a quasiconformal extension v of Υ which satisfies

$$\hat{\varphi} \circ v = v \circ \varphi$$

on B_1 . With that, we are able to perform complex pull-back by ψ and $\hat{\psi}$.

Also, assuming that $|P|$ is large enough, we can find a uniform r so that the preimages of $B(0, r)$ by ψ , $\hat{\psi}$ and $z \rightarrow z^2$ are all inside $B(0, r/2)$. Also, we can have $B(0, r)$ contained in B_1 as well as \hat{B}_1 . Next, we choose the largest i so that $[-r, r] \subset \psi^{-i}(P)$. Then, we change ψ and $\hat{\psi}$. We will only describe what is done to ψ . Outside of $B(0, r)$, ψ is left unchanged. Inside the preimage of $B(0, r)$ by $z \rightarrow z^2$ it is $z \rightarrow z^2$. In between, it can be interpolated by a smooth degree 2 cover with bounded distortion. The modified extension will be denoted with ψ' .

Next, we pull-back Υ by ψ' and $\hat{\psi}'$ exactly i times. That is, if Υ_0 is taken equal to Υ , then Υ_{j+1} is Υ refined by pulling-back Υ_j onto the domain of ψ . Now we need to check whether Υ^1 has all the properties claimed in the Lemma. To see the functional equation condition,

we note that all branches of any Υ_j are in the region where ψ coincides with ψ' . The last condition easily follows from the fact that r can be chosen in a uniform fashion. Also, the quasiconformal norm of Υ_i grows only by a constant compared with Υ , since points pass through the region of non-conformality only once.

Q.E.D.

The staircase construction. We take Υ^1 obtained in Lemma 5.1 and confine our attention to its restriction to the real line, denoted with v_1 . We rely on the fact that v_1 is a quasisymmetric map and its qs norm is uniformly bounded in terms of the quasiconformal norm of Υ^1 .

Completion of outer staircases. We will construct a induced maps φ_2 and $\hat{\varphi}_2$ with a branchwise equivalence v_2 with following properties:

- The map v_2 coincides with v_1 outside of the domain of ψ . Also, it satisfies

$$v_2 \circ \psi^j = \hat{\psi}^j \circ v_2$$

on the complement of the restrictive interval provided that ψ^j is defined.

- Inside the restrictive interval, it is the “inner staircase equivalence”, that is, all endpoints of the inner staircase steps are mapped onto the corresponding points.
- Its qs norm is uniformly bounded as a function of the qc norm of Υ .

Outer staircases constructed in Lemma 5.1 connect the boundary points of the domain of ψ to the i -th steps which are in the close neighborhood of the restrictive interval. Also, the i -th steps are the corresponding fundamental domains for the inverses of ψ in the proximity of the boundary of the restrictive interval.

By bounded geometry of renormalization, see [27], the derivative of ψ at the boundary of the restrictive interval is uniformly bounded away from one. Then, it is straightforward to see that the equivariant correspondence between infinite outer staircases which uniquely extends v_1 from the i -th steps is uniformly quasisymmetric (see a more detailed argument in the last section of citekus).

Inside the restrictive interval, the map is already determined on the endpoints of steps, and can be extended in an equivariant way onto each step of the inner staircase.

Rebuilding a complex map. Due to the irregular behavior of the branchwise equivalence in the domains of branches of φ_2 that map onto the central domain, they cannot be filled-in by critical pull-backs used in [18]. Instead, we will construct an externally marked branchwise equivalence and apply complex filling-in. The external marked will be achieved by Lemma 1.2. As the lip, we choose the circular arc which intersects the real line at the endpoints of the central domain and makes angles of $\pi/4$ with the line. The teeth will be the preimages of the lip by short branches inside the central domain. We check that the norm of this mouth is bounded. By the geodesic property (see [27]), the teeth are bounded by

corresponding circular arc of the same angle. The only property that needs to check is the existence of a bounded modulus between the lip and any tooth. This will follow if we prove that the intersection of a tooth with the real has a definite neighborhood (in terms of the cross-ratio) which is still inside the central domain. Since short branches inside the central domain are preimages of short monotone branches from the outside by a negative Schwarzian map, it is enough to see the analogous property for the domains of short branches in the box. If the number of box steps leading to the suitable map was bounded, this follows from the estimates of [17], as the long monotone branches adjacent to the boundary of the box continue to have a uniformly large size. Otherwise, one uses Theorem 3.

Now we construct the branchwise equivalence on the lip. To this end, we take the straight down projection from the lip to the central domain, and lift the branchwise equivalence from the line. The resulting map on the lip is quasisymmetric with the norm bounded in terms of the quasisymmetric norm of φ_2 . Now we pull-back this map to the teeth by dynamics. By the complex Kőbe lemma, the maps that we use to pull-back are diffeomorphisms of bounded nonlinearity, so they will preserve quasisymmetric properties. Since we assumed the replacement condition for short monotone domains, we can fill each tooth with a uniformly quasiconformal map. This leaves us in a position to apply Lemma 1.2 to fill the mouth with a uniformly quasiconformal branchwise equivalence. Call the mouth W .

Finally, we have to extend the branchwise equivalence to the whole plane. To this end, we choose a half-disk with the fundamental inducing domain normalized to $[-1, 1]$ as the diameter, and make the branchwise equivalence identity there. Next, we regard W and all its preimages by short monotone branches as teeth. This time, it is quite clear that the norm is bounded. The branchwise equivalence of the teeth is pulled back from W by dynamics. The same argument as we made in the preceding paragraph show that Lemma 1.2 can be used to construct a complete branchwise equivalence on the plane.

Construction of the saturated map. We now apply filling-in to all branches which map onto the central domain W . On the level of inducing, the only branches still left are those with the range equal to the fundamental inducing domain of the renormalized map, and long monotone branches onto the whole previous fundamental inducing domain.

Final refinement. We end with a simultaneous monotone pull-back on all long monotone branches. The limit exists in L^∞ and is the saturated map. The branchwise equivalence we get is quasiconformal as well. This can be seen by choosing diamonds for all long monotone branches, marking it externally, and using complex pull-back. The paper [18] offers an alternative way which does not require external marking and instead relies on a version of the Sewing Lemma (Fact 2.2 in our paper.) This closes the proof of Proposition 5, and therefore of all our remaining theorems.

Appendix

5.3 Estimates for hole structures

Separation symbols for complex box mappings.

Definition of the symbols. Now, let φ be a type I complex box mapping of rank n . An ordered quadruple of real non-negative numbers:

$$s(B) := (s_1(B), \dots, s_4(B))$$

will be said to give a separation symbol for B if certain annuli exist as described below. The annuli are either open or degenerate to curves. Figure 3 shows a choice of separating annuli for domain B , which is the same as domain B from Figure 2.

We first assume that there are annuli $A_1(B)$ and $A_2(B)$. Both annuli are contained in $B_{n'}$. The annulus $A_2(B)$ surrounds B_n separating it from the domain of the analytic extension of B with range $B_{n'}$. Then $A_1(B)$ separates $A_2(B)$ from the boundary of $B_{n'}$. We must have

$$s_2(B) \leq \text{mod } A_2(B) \text{ and}$$

$$s_1(B) \leq \text{mod } A_2(B) + \text{mod } A_1(B) .$$

Next, three annuli are selected around B which will give the meaning of the two remaining components of the symbol. First, the annulus $A'(B)$ is chosen exactly equal to the difference between the domain of the canonical extension of the branch defined on B and the domain of B . Then, the existence of $A_3(B)$ is postulated which surrounds $A'(B)$ separating it from B_n and from the boundary of $B_{n'}$. Finally, $A_4(B)$ separates $A_3(B)$ from the boundary of $B_{n'}$. Then

$$s_3(B) \leq \text{mod } A'(B) + \text{mod } A_3(B) \text{ and}$$

$$s_4(B) \leq \text{mod } A'(B) + \text{mod } A_3(B) + \text{mod } A_4(B) .$$

The dependence on B will often be suppressed, in our subsequent notations.

Normalized symbols. We will now arbitrarily impose certain algebraic relations among various components of a separation symbol. Choose a number β , and $\alpha := \beta/2$, together with λ_1 and λ_2 . Assume $\alpha \geq \lambda_1$, $\lambda_2 \geq -\alpha$ and $\lambda_1 + \lambda_2 \geq 0$. If these quantities are connected with a separation symbol $s(B)$ as follows

$$s_1(B) = \alpha + \lambda_1 ,$$

$$s_2(B) = \alpha - \lambda_2 ,$$

$$s_3(B) = \beta - \lambda_1 ,$$

$$s_4(B) = \beta + \lambda_2 .$$

we will say that $s(B)$ is normalized with norm β and corrections λ_1 and λ_2 .

Separation index of a box mapping. For a type I complex box mapping ϕ a positive number β is called its *separation index* provided that valid normalized separation symbols with norm β exist for all univalent branches.

Monotonicity of separation indexes. The nice property of separation indexes is that they do not decrease in the box inducing process. In fact, one could show that they increase at a uniform rate and this will be the final conclusion to be drawn from Theorem 4. For now, we prove

Lemma 5.2 *Let ϕ_i , i between 0 and m be a sequence of complex box mappings of type I with the property that the next one arises from the previous one in a simple box inducing step. If β_0 is a separation index of ϕ_0 , then β_0 is also a separation index of ϕ_i for any $i < m$. In addition, if ϕ_i arises after a non-close return, then $v(\phi_i) \geq \beta_0/4$.*

Proof:

The proof of Lemma has to be split into a number of cases. As analytic tools, we will use the behavior of moduli of annuli under complex analytic mappings. Univalent maps transport the annuli without a change of modulus, analytic branched covers of degree 2 will at worst halve them, and for a sequence of nesting annuli their moduli are super-additive (see [19], Ch. I, for proofs, or [5] for an application to complex dynamics.) To facilitate the discussion, we will also need a classification of branches depending on how they arise in a simple box inducing step.

Some terminology. Consider an abstract setting in which one has a bunch of univalent branches with common range B' and fills them in to get branches mapping onto some $B \subset B'$. The original branches mapping onto B' will be called *parent branches* of the filling-in process. Clearly, every branch after the filling-in has a dynamical extension with range B' . For two branches, the domains of these respective extensions may be disjoint or contain one another. In the first case we say that the original branches were *independent*. Otherwise, the one mapped with a smaller extension domain is called *subordinate* to the other one. Note that if b' is subordinate to b , then the extension of b maps b' onto another short univalent domain.

We then distinguish the set of "maximal" branches subordinate to none. They are mapped by their parent branches directly onto the central domain. Therefore, the domains of extensions of maximal branches mapping onto B' are disjoint. They also cover domains of all branches. The extensions of maximal branches are exactly *parent branches* of the filling-in process. These extensions with range B' will be called *canonical extensions*.

Now, in a simple box inducing step, the parent branches are the short monotone branches of $\tilde{\phi}$. Among these we distinguish at most two *immediate branches* which are restrictions of the central branch of ϕ to the preimage of the central domain. All non-immediate parent branches are compositions of the central branch of ϕ with short monotone branches of ϕ . For example, in the non-close return the first filling gives a set of parent branches, two of which may be immediate, which later get filled in. In the close return filling-in is done twice, so we will be more careful in speaking about parent branches. Figure 2 shows examples of independent and subordinate domains.

We will sometimes talk of branches meaning their domains, for example saying that a branch is contained in its parent branch. We assume that ϕ has rank n , so $B = B_n$ and $B' = B_{n'}$. Let ψ be the central branch of ϕ . Let B_{n+1} denote the central domain of the newly created map ϕ^1 . Observe that $B_{(n+1)'} = B_n$. Suppose that β is a separation index of ϕ . We will now proceed to build separation symbols with norm β for all short univalent domains of ϕ^1 . Let g be a short univalent branch of ϕ^1 and p denote the parent branch g . The parent branch necessarily has the form $P' \circ \psi$. Let P be the branch of ϕ whose domain contains the critical value. Objects (separation annuli, components of separation symbols) referring to ϕ^1 will be marked with bars.

Reduction to maximal branches. Note that it is sufficient show that symbols with norm β exist for maximal branches. Indeed, suppose that a separation symbol exists for a maximal branch b and let b' be subordinate to b . We can take $A_1(b') = A_1(b)$ and $A_2(b') = A_2(b)$. Likewise, we can certainly adopt $A_4(b') = A_4(b)$, and $A_3(b')$ can be chosen to contain $A_3(b)$. The annulus $A'(b)$ is the preimage of the annulus $B_{n'} \setminus B_n$ by the parent branch of b . The annulus $A'(b')$ is the preimage of the same annulus by the canonical extension of b , so it has the same modulus. Since the domain of the canonical extension of b' is contained in the parent domain (equal to the domain of the canonical extension of b), the assertion follows.

Non-close returns. Let us assume that ϕ makes a non-close return, that is $P' \neq \psi$.

The case of p immediate. Let b denote the maximal branch in p . The new central hole B_{n+1} is separated from the boundary of B_n by an annulus of modulus at least $(\beta + \lambda_2(B))/2$. The annulus $\bar{A}_2(b)$ around B_{n+1} will be the preimage by the central branch of the region contained in and between $A_3(P)$ and $A'(P)$. Then, $\bar{A}_1(b)$ is the preimage of A_4 . It follows that we can take

$$\begin{aligned}\bar{s}_1 &= \frac{\beta + \lambda_2(P)}{2} \text{ and} \\ \bar{s}_2 &= \frac{\beta - \lambda_1(P)}{2} .\end{aligned}$$

Of course, since components of the symbol are only lower estimates, we are always allowed to decrease them if needed. The annulus \bar{A}' is naturally given as the preimage of the annulus between B_{n+1} and the boundary of B_n by the central branch, likewise \bar{A}_3 is the preimage of $A_2(P)$, and \bar{A}_4 is the preimage of $A_1(P)$. Since the first two preimages are taken in an univalent fashion, we get

$$\begin{aligned}\bar{s}_3 &= \frac{\beta + \lambda_2(B)}{2} + \alpha - \lambda_2(B) \text{ and} \\ \bar{s}_4 &= \bar{s}_3 + \frac{\lambda_1(B) + \lambda_2(B)}{2} = \frac{\beta}{2} + \alpha + \frac{\lambda_1(B)}{2} .\end{aligned}$$

Thus, if we put

$$\bar{\lambda}_1 = \frac{\lambda_2(B)}{2}, \quad \bar{\lambda}_2 = \frac{\lambda_1(B)}{2}$$

we get a valid separation symbol with norm β . In the remaining non-immediate cases, the branch P' is defined.

P' and P non-immediate and independent. To pick $\overline{A}_2(b)$, we take the preimage by ψ of the annulus separating P from the boundary of the domain of its canonical extension with range $B_{n'}$, i.e. $A'(P)$. We claim that its modulus in all cases is estimated from below by $\alpha + \delta$ where δ is chosen as the supremum of $-\lambda_2(b')$ over all univalent domains b' of ϕ . Indeed, P is carried onto B_n by the extended branch, and the estimate is α plus the maximum of $\lambda_1(b')$ with b' ranging over the set of all short univalent domains of ϕ . and $\lambda_1(P')$ which is at least $\alpha + \delta$. The assertion follows since $\lambda_1(b') + \lambda_2(b') \geq 0$ for any b' . To pick $\overline{A}_1(b)$, consider the annulus separating $A'(P)$ from the boundary of $B_{n'}$, i.e. the region in and between $A_3(P)$ and $A_4(P)$. Pull this region back by the central branch to get $\overline{A}_1(b)$. By the hypothesis of the induction, the estimates are

$$\overline{s}_1 = \frac{\beta + \lambda_2(P)}{2} \text{ and}$$

$$\overline{s}_2 = \frac{\alpha + \delta}{2} .$$

Since b is maximal $\overline{A}'(b)$ is determined with modulus at least \overline{s}_1 . The annulus $\overline{A}_3(b)$ will be obtained as the preimage by the central branch of $A'(P')$. This has modulus at least $\alpha + \delta$ in all cases as argued above. The annulus $\overline{A}_4(b)$ is the preimage of the region in and between $A_3(P')$ and $A_4(P')$. By induction,

$$\overline{s}_3 = \frac{\beta + \lambda_2(P)}{2} + \alpha + \delta \text{ and}$$

$$\overline{s}_4 = \overline{s}_3 + \frac{\beta + \lambda_2(P') - \alpha - \delta}{2} .$$

We put $\overline{\lambda}_1 = \frac{\lambda_2(P)}{2}$ and $\overline{\lambda}_2 = \frac{\alpha - \delta}{2}$. We check that

$$\overline{s}_3 + \overline{\lambda}_1 = \frac{\beta}{2} + \alpha + \lambda_2(P) + \delta \geq \beta - \lambda_2(P) + \lambda_2(P) \geq \beta .$$

In a similar way one verifies that

$$\overline{s}_4 - \overline{\lambda}_2 \geq \beta .$$

Also, the required inequalities between corrections $\overline{\lambda}_i$ follow directly.

P' subordinate to P . This means that some univalent mapping onto $B_{n'}$ transforms P onto B_n and P' onto some P'' . Consider $A_2(P'')$ which separates B_n from P'' , and a larger annulus $A_1(P'')$. Their preimages first by the extended branch and then by the central branch give us $\overline{A}_2(b)$ and $\overline{A}_1(b)$ respectively. The estimates are

$$\overline{s}_2 = \frac{\alpha - \lambda_2(P'')}{2} \text{ and}$$

$$\overline{s}_1 = \frac{\alpha + \lambda_1(P'')}{2} .$$

The annulus $\overline{A}'(b)$ is uniquely determined with modulus \overline{s}_1 , and $\overline{A}_3(b)$ will be the preimage of the annulus separating P'' from B_n . Finally, $\overline{A}_4(b)$ will separate the image of $\overline{A}_3(b)$ from $B_{n'}$. The estimates are

$$\begin{aligned}\overline{s}_3 &= \frac{\alpha + \lambda_1(P'')}{2} + \beta - \lambda_1(P'') = \beta + \frac{\alpha - \lambda_1(P'')}{2} \text{ and} \\ \overline{s}_4 &= \overline{s}_3 + \frac{\lambda_1(P'') + \lambda_2(P'')}{2} = \beta + \frac{\alpha + \lambda_2(P'')}{2}.\end{aligned}$$

Set

$$\begin{aligned}\overline{\lambda}_1 &= \frac{-\alpha + \lambda_1(P'')}{2} \text{ and} \\ \overline{\lambda}_2 &= \frac{\alpha + \lambda_2(P'')}{2}.\end{aligned}$$

The requirements of a normalized symbol are clearly satisfied.

P subordinate to P' . This situation is analogous to the situation of immediate parent branch considered at the beginning. Indeed, by mapping P' to B_n and composing with the central branch one can get a folding branch with range $B_{n'}$ defined on P' . We now see that the situation inside the domain of the canonical extension of P is analogous to the case of immediate parent branches, except that the folding branch maps onto a larger set $B_{n'}$. So the estimates can only improve.

A close return. In this case there are no immediate parent branches and we really have only one case to consider. Fix some short univalent branch b of ϕ^1 , let p be its parent branch, and denote $p = P' \circ \psi$. Consider $A_2(P')$ and $A_1(P')$. Their preimages by the central branch give us $\overline{A}_2(b)$ and $\overline{A}_1(b)$ respectively. The estimates are

$$\begin{aligned}\overline{s}_2 &= \frac{\alpha - \lambda_2(P')}{2} \text{ and} \\ \overline{s}_1 &= \frac{\alpha + \lambda_1(P')}{2}.\end{aligned}$$

The annulus $\overline{A}'(b)$ is uniquely determined with modulus \overline{s}_1 , and $\overline{A}_3(b)$ will be the preimage of the annulus separating P' from B_n i.e. the annulus containing $A_3(P')$ and $A'(P')$ together with the region between them. Finally, $\overline{A}_4(b)$ will be the preimage of $A_4(P')$ by ψ . The estimates are

$$\begin{aligned}\overline{s}_3 &= \frac{\alpha + \lambda_1(P')}{2} + \beta - \lambda_1(P') = \beta + \frac{\alpha - \lambda_1(P')}{2} \text{ and} \\ \overline{s}_4 &= \overline{s}_3 + \frac{\lambda_1(P') + \lambda_2(P')}{2} = \beta + \frac{\alpha + \lambda_2(P')}{2}.\end{aligned}$$

Set

$$\begin{aligned}\overline{\lambda}_1 &= \frac{-\alpha + \lambda_1(P')}{2} \text{ and} \\ \overline{\lambda}_2 &= \frac{\alpha + \lambda_2(P')}{2}.\end{aligned}$$

The requirements of a normalized symbol are clearly satisfied. Not quite surprisingly, these are the same estimates we got in the non-close case with P' subordinate to P .

Conclusion. We already proved by induction that β_0 remains a separation index for all ϕ_i . It remains to obtain the estimate $v(\phi_{i+1}) \geq \beta_0/4$ under the assumption that ϕ_i makes a non-close return. This is quite obvious from considering the separation symbol for the branch P which contains the critical value. Since $s_4(P) = \beta_0 + \lambda_2(P) \geq \beta_0/2$ and because of superadditivity of conformal moduli, there is an annulus with modulus at least $\beta_0/2$ separating P from B' , and its pull-back by ψ gives as an annulus with desired modulus. This all we need to finish the proof of Lemma 5.2. Note, however, that we cannot automatically claim $\bar{s}_1 \geq \beta_0/4$ even if the preceding return was non-close.

Q.E.D.

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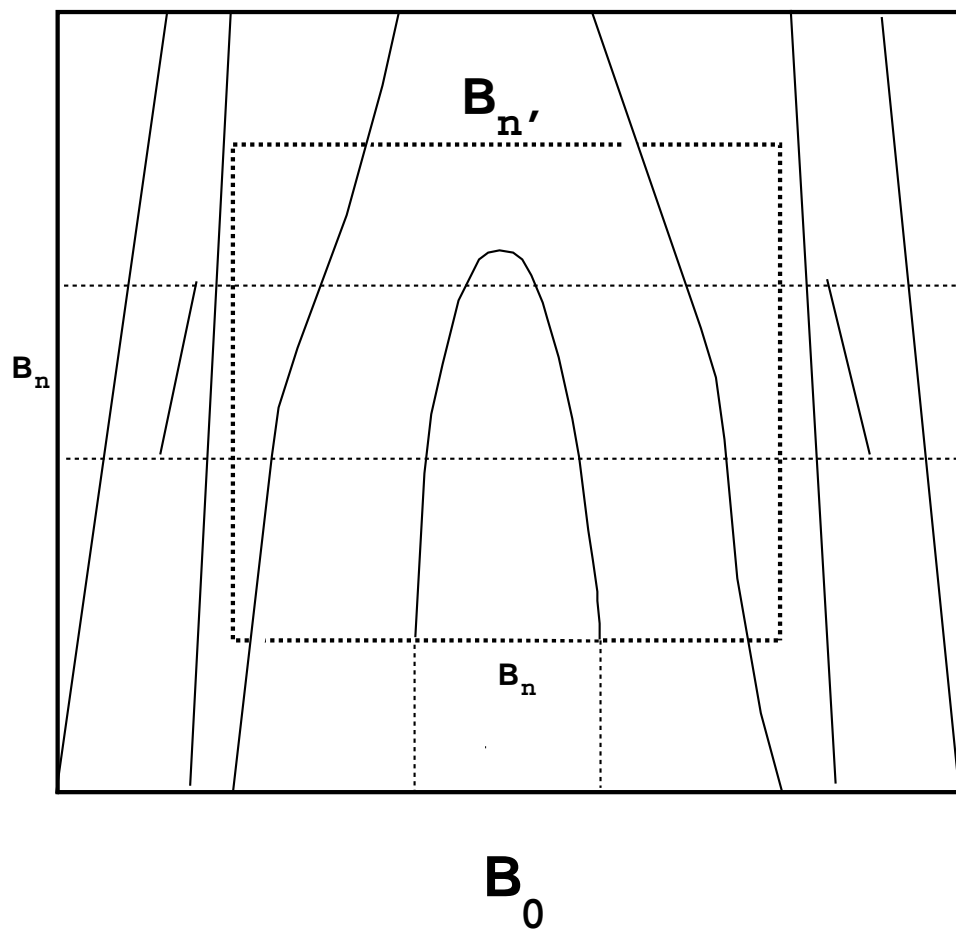


Figure 1: A sample graph of a type I real box mapping. All three permissible types of branches: central folding, long and short monotone are shown. Be aware that typically one has infinitely many branches.

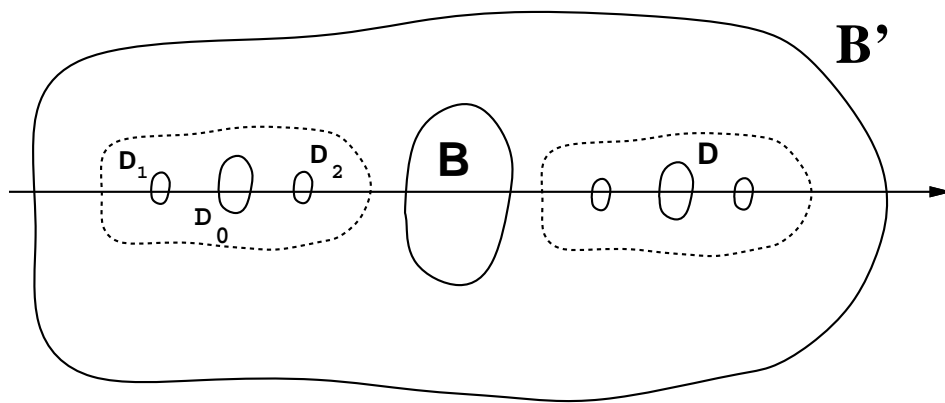


Figure 2: A type I complex box mapping. Dotted lines show domains of canonical extensions. Domains D_0 and D look like they are maximal. Then D_1 and D_2 are subordinate to D_0 , but apparently independent from one another as well as from D . D_0 and D are also independent. There may be univalent domains outside of B' , not shown here.

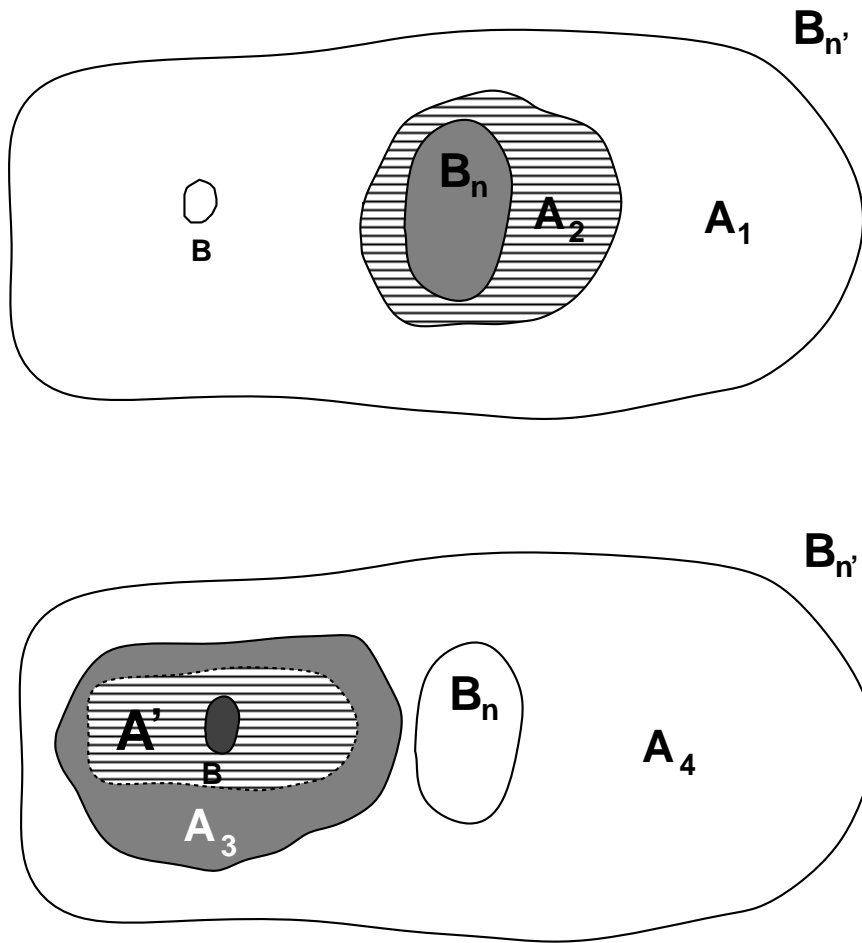


Figure 3: A choice of separating annuli for B . Note that the outermost annuli A_1 and A_4 are filled in white.