The Teichmüller space of an Anosov diffeomorphism of T^2

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July 17, 1991

1 Introduction

In this paper we consider the space of smooth conjugacy classes of an Anosov diffeomorphism of the two-torus. A diffeomorphism f of a manifold M is Anosov if there is a continuous invariant splitting of the tangent bundle $TM = E^s \bigoplus E^u$, where the subbundle E^s is contracted by f, and E^u is expanded. More precisely, if $\| \cdot \|$ is a Riemannian metric on M, there are constants $c > 0$ and $\lambda < 1$ such that

$$\begin{aligned} \|Df^n \cdot v\| &\leq c\lambda^n \|v\| \text{ for v in } E^s \\ \|Df^{-n} \cdot v\| &\leq c\lambda^n \|v\| \text{ for v in } E^u \end{aligned}$$

for all positive integers n. If f is $C^{1+\alpha}$ for $0 < \alpha < 1$, it can be shown that the splitting is in fact Hölder continuous.

The only 2-manifold that supports an Anosov diffeomorphism is the 2-torus [F]. Franks and Manning showed that every Anosov diffeomorphism of $\mathbf{T}^2$ is topologically conjugate to a linear example; that is, to an automorphism defined by a hyperbolic element of $GL(2, \mathbf{Z})$ whose determinant has absolute value one [F], [M]. Consider f and g which are topologically conjugate, so there is a homeomorphism taking the orbits of f to the orbits of g. If the conjugacy is in fact smooth, then f and g must have the same expanding and contracting eigenvalues at corresponding periodic points. De la Llave, Marco, and Moriyon have shown that the eigenvalues at periodic points are a *complete* smooth invariant: if the eigenvalues of f are the same as the eigenvalues of g at periodic points that correspond under a topological conjugacy, then the conjugacy is smooth [Ll],[MM1], [MM2].

The question arises: what sets of eigenvalues occur as the Anosov diffeomorphism ranges over a topological conjugacy class? The information in the set of expanding eigenvalues of f is recorded by a Hölder cyclic cohomology class associated to f. The real-valued function

$$x \mapsto \phi_u(x) = -\log \parallel Df(x) \parallel_u ,$$

where $\parallel Df \parallel_u$ is the Jacobian of f along the unstable bundle E^u, can be described as a "cocycle" over f. The cohomology class of this cocycle, that is, its residue class modulo the space of coboundaries $x \mapsto u(f(x)) - u(x)$, is independent of the choice of Riemannian metric on M. Moreover, the cohomology class of a Hölder cocycle defined over an Anosov diffeomorphism is determined by the sums of values of the cocycle over the various periodic orbits, which for ϕ_u is simply minus the logarithm of the expanding eigenvalue at the periodic point. Similarly, the information in the set of contracting eigenvalues is recorded by the cohomology class of the cocycle defined by $\phi_s = \log \parallel Df \parallel_s$ where $\parallel Df \parallel_s$ is the Jacobian of f along the stable bundle E^s. The sign convention that makes these cocycles negative is chosen for consistency with the notation of Bowen, Ruelle, and Sinai in the theory of Gibbs and equilibrium measures for Anosov systems.

The question asked above can be reformulated: what pairs of cohomology classes (one determined by the expanding eigenvalues, and one by the contracting eigenvalues) occur as the diffeomorphism ranges over a topological conjugacy class? The cohomology is defined over the entire conjugacy class by pulling the Jacobian cocycles back to a fixed representative of the conjugacy class. The purpose of this paper is to answer this question: *all* pairs of Hölder reduced cohomology classes occur. (The reduced cohomology is the cyclic cohomology divided out by the constant cocycles. The pair of reduced cohomology classes is still sufficient information to determine the smooth conjugacy class.)

The Teichmüller space $T(f)$ of an Anosov diffeomorphism f is defined to be the set of smooth structures preserved by the topological dynamics determined by f. This is the smooth category version of the Teichmüller space of a rational map, which was studied by McMullen and Sullivan [MS]. We show that for an Anosov diffeomorphism of $\mathbf{T}^2$, there is a natural bijection

from the Teichmüller space to the product $G(f) \times G(f^{-1})$, where $G(f)$ denotes the real-valued Hölder reduced cyclic cohomology over f.
The main theorem is:

Theorem 1 *Let $f : \mathbf{T}^2 \to \mathbf{T}^2$ be an Anosov diffeomorphism. Then there is a natural bijection $T^{1+H}(f) \leftrightarrow G(f) \times G(f^{-1})$.*

An easy corollary will be

Theorem 2 *Let $f : \mathbf{T}^2 \to \mathbf{T}^2$ be a volume preserving Anosov diffeomorphism. Then there is a natural bijection $T^{1+H}_{vol}(f) \leftrightarrow G(f)$.*

Here $G(f)$ is the Hölder reduced cyclic cohomology over f, where the Hölder exponent is allowed to vary between 0 and 1. More precisely: let C^α denote the space of α-Hölder functions on T^2. Let $C^H = \cup_{\alpha \in (0,1)} C^\alpha$. Then $G(f)$ is the quotient of C^H by the subspace of "almost coboundaries"

$$x \mapsto u(f(x)) - u(x) + K \ ,$$

where u is a function on T^2 and $K \in \mathbf{R}$. (It follows in this setting that $u \in C^H$). $T^{1+H}(f)$ is the Teichmüller space of C^{1+H} invariant smooth structures. An Anosov diffeomorphism is called *volume preserving* if it admits an invariant measure that is absolutely continuous with repect to Lebesgue. $T^{1+H}_{vol}(f)$ is the restriction of this Teichmüller space to the volume preserving elements. See the appropriate sections for precise definitions.

The main theorem can be restated as follows:

Theorem 1′ *Let L be a hyperbolic automorphism of the torus $\mathbf{T}^2$. Given two Hölder functions ϕ_u and ϕ_s from $\mathbf{T}^2$ to $\mathbf{R}$, there exist uniquely defined constants P_u and P_s, and a unique C^{1+H} smooth structure on $\mathbf{T}^2$ which is preserved by L, and determines the cohomology classes $< \phi_u - P_u >$ and $< \phi_s - P_s >$. Moreover, L is Anosov in this smooth structure.*

Remark. The Hölder exponent of the new smooth structure depends on the Hölder norms and exponents of the pair of functions, and on a dynamically defined norm (the *Bowen*, or *variation* norm) of the cohomology classes. [B3]

Remark. The topological conjugacy between C^{1+H} Anosov diffeomorphisms is in fact Hölder continuous [Mn]. Therefore this description is independent of the choice of "base point" as the linear mapping.

Corollary 1 *Let λ_u and λ_s denote the unstable and stable eigenvalues at a periodic point of period n. The numbers $|\lambda_u|^{1/n}$ and $|\lambda_s|^{1/n}$ can be prescribed arbitrarily on any finite set of periodic points, up to (non-unique) constant factors $\exp(-P_u)$ and $\exp(P_s)$, respectively.*

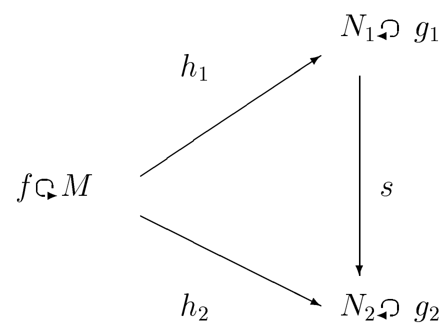
Remark. The correction factors in the corollary are asymptotically unique along any sequence of periodic point sets P_i with the property that the normalized dirac mass on P_i converges to Haar measure (which is also the measure of maximal entropy) on $\mathbf{T}^2$.

We give a sketch of the proof of Theorem 1. The main step is to show that a cohomology class over an Anosov diffeomorphism f determines a canonical invariant transverse measure class to the stable foliation. This is constructed using the Gibbs measure class defined by the cohomology class. When the stable foliation is co-dimension 1, the transverse measure class can be interpreted as a transverse smooth structure. In dimension 2, when both foliations are co-dimension 1, the two transverse structures (determined by the two given cohomology classes) define a product smooth structure, which is invariant by f.

Ruelle and Sullivan [RS], and Sinai [Si], gave a transverse interpretation of a particular Gibbs measure, namely that associated to the constant cocycle. In this case, and only this case, one has a transverse measure, as opposed to a transverse measure class. Under the isomorphism of Theorem 1, this corresponds to the linear member of the topological conjugacy class. The present paper shows how to extend the decomposition of a Gibbs measure into transverse stable and transverse unstable part, as described in [RS] for the constant cocycle measure, to the general case.

The organization of the paper is as follows. In section 2 we define the Teichmüller space $T(f)$. In section 3 we recall facts about cyclic cohomology over a $\mathbf{Z}$ action, over a foliation or an equivalence relation, and describe the Jacobian and Radon-Nykodym cocycles. In Section 4, the map which gives the isomorphism in Theorem 1 is described. Section 5 describes the cocycle properties of Gibbs measures, and collects some necessary results. Section 6 gives the statements of the transverse structure realization results, and proves Theorem 1 assuming these. The main body of the proof is in Section 7, where the tranverse measure class is constructed. Section 8 gives a simple description of the smooth structure defined by a pair of cohomology classes in terms of explicit coordinates on rectangles in a Markov partition.

Figure 1: Teichmüller space of $f : M \to M$.



Acknowledgements. It is a pleasure to thank Dennis Sullivan for many hours of conversation, for inspiration and encouragement. Jack Milnor made many helpful comments on the manuscript. I also thank IHES and the Institute for Mathematical Sciences at Stony Brook for their hospitality while this work was being completed.

2 The Teichmüller space of an Anosov diffeomorphism

Let $f : M \to M$ be a C^r Anosov diffeomorphism, $0 < r \le \omega$. The C^r Teichmüller space $T^r(f)$ is defined as follows. Consider triples (h, N, g) where $g : N \to N$ is a C^r Anosov diffeomorphism, and $h : M \to N$ is a homeomorphism satisfying $g \circ h = h \circ f$. We call such a triple a *marked Anosov diffeomorphism modeled on* $f : M \to M$. Two such triples (h_1, N_1, g_1) and (h_2, N_2, g_2) are *equivalent* if the homeomorphism $s : N_1 \to N_2$ defined by $s \circ h_1 = h_2$ is in fact a C^1 diffeomorphism. The Teichmüller space $T^r(f)$ is the space of equivalence classes of triples.

We also define the C^{1+H} Teichmüller space $T^{1+H}(f)$. Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism for some $0 < \alpha < 1$. Consider all marked

Anosov diffeomorphisms $g : N \to N$ modeled on f where g is $C^{1+\alpha'}$ for some $0 < \alpha' < 1$ (that is, the Hölder exponent of g is not necessarily the same as that of f). Then T^{1+H} is the space of equivalence classes, where equivalence is defined just as it was in the C^r category.

3 Cyclic cohomology.

3.1 Group actions and the Jacobian and Radon-Nykodym cocycles.

We describe here the notion of cocycle over a group action, and the associated notions of coboundary and cohomology (see [Z] and [K].) We work in the topological category since all cocycles we are interested in have at least this degree of regularity. We consider Γ, a locally compact, second countable group, and a continuous right action of Γ on a topological space M. We will be especially interested in the case of a $\mathbf{Z}$ action defined by a diffeomorphism of a manifold M.

A real-valued (additive) cocycle over the action of Γ is a continuous map

$$\Phi : M \times \Gamma \to \mathbf{R}$$

satisfying:

$$\Phi(x, \gamma_1 \cdot \gamma_2) = \Phi(x, \gamma_1) + \Phi(x \cdot \gamma_1, \gamma_2)$$

Here $x \to x \cdot \gamma$ denotes the action of γ on the point x. A *coboundary* is a cocycle of the form $\Phi(x, \gamma) = u(x \cdot \gamma) - u(x)$ where $u : M \to \mathbf{R}$ is a continous function. The function u is called the *transfer function* of the coboundary Φ. Two cocycles are *equivalent* or *cohomologous* if their difference is a coboundary. The cohomology over the action of Γ is the space of equivalence classes of cocycles. A cocycle of the form $\Phi(x, \gamma) = u(x \cdot \gamma) - u(x) + K(\gamma)$, where $K : \Gamma \to \mathbf{R}$ is a homomorphism, is called an *almost coboundary.* The space of cocycles modulo almost coboundaries is the *reduced cohomology* over the action. If Φ is a cocycle, we denote its cohomology class and reduced cohomology class $< \Phi >$ and $< \Phi >_*$, respectively. When the space M has additional structure, we can consider e.g. Hölder, Lipshitz, or smooth cocycles (and coboundaries and cohomology).

The cohomology equivalence relation on cocycles has the following meaning. Let $F(M, \mathbf{R})$ be the space of continuous maps from M to $\mathbf{R}$. Then a cocycle Φ defines an action of Γ on $F(M, \mathbf{R})$ as follows. If $T : M \to \mathbf{R}$, then $(\gamma \cdot T)(x) = \Phi(x, \gamma) + T(x \cdot \gamma)$. Note that if Φ is the identically 0 cocycle, then this action is just the usual pull-back of functions by the group action. When the two cocycles Φ and Ψ are cohomologous, the actions they define are equivalent. If u is the transfer function of the coboundary that relates Φ to Ψ, then the map $U : F(M, \mathbf{R}) \to F(M, \mathbf{R})$ defined by $T \to T + u$ is an isomorphism which conjugates the action defined by Φ to the action defined by Ψ.

Example 1: The Jacobian cocycle. Suppose that Γ acts by diffeomorphisms on a Riemannian manifold M. We define the (additive) Jacobian cocycle $J : M \times \Gamma \to \mathbf{R}$ by $J(x, \gamma) = \log \parallel D\gamma(x) \parallel$ where $\parallel \cdot \parallel$ is the Riemannian metric. The chain rule for differentiation is precisely the cocycle condition. If we choose a new Riemannian metric $\parallel \cdot \parallel_1$ on M, then the new Jacobian cocycle is cohomologous to the original one. The transfer function is simply the logarithm of the ratio of volume elements with repect to the two metrics. Hence there is a cohomology class, the Jacobian class $< J >$, naturally associated to a smooth group action on a smooth manifold. The apriori coarser invariant, the reduced Jacobian class $< J >_*$, is also defined.

Example 2: The Radon-Nykodym cocycle. Suppose Γ acts on the measure space (M, μ), quasi-preserving the measure μ. The (additive) Radon-Nykodym cocycle $R : M \times \Gamma \to \mathbf{R}$ is defined by $R(x, \gamma) = \log \frac{d\mu(\gamma(x))}{d\mu(x)}$. (Since we have retricted to the topological category, we assume that the Radon-Nykodym is continuous.) Again the cocycle condition is the chain rule. If ν is a measure that is equivalent to μ, with Radon-Nykodym derivative $r = \frac{d\nu}{d\mu}$, then the corresponding cocycles are cohomologous, via the transfer function $u = \log(\mathrm{r})$. So there is a cohomology class, the Radon-Nykodym class $< R >$, and a reduced cohomology class $< R >_*$, naturally associated to an action that preserves a measure class.

We now focus on the case $\Gamma = \mathbf{Z}$. A cocycle over a $\mathbf{Z}$ action is determined by its values on the generator: $\phi(x) =: \Phi(x, 1)$. The cocycle condition implies that $\Phi(x, n) = \sum_{k=0}^{n-1} \phi(x \cdot k)$ where $x \cdot k$ denotes the action of k on the point x. If $f : M \to M$ is the action of the generator, then we write $\Phi(x, n) = \sum_{k=0}^{n-1} \phi \circ f^k(x)$. Hence the cocycles over a $\mathbf{Z}$ action can be identifed with continuous functions on M. A coboundary is a function of the form $u \circ f - u$. An almost coboundary is a function of the form $u \circ f - u + K$, where K is a constant.

Example 3: the BRS classes of an Anosov diffeomorphism. Suppose $f : M \to M$ is an Anosov diffeomorphism, with $TM = E^s \bigoplus E^u$. Let $\| \cdot \|$ be a Riemannian metric on M. Since the subbundle E^u is preserved by f, we can define the *unstable Jacobian cocycle* to be the cocycle over the action of f determined by the function $\phi_u(x) = -\log \| Df(x) \|_u$. Here $\| Df(x) \|_u$ denotes the Jacobian in the unstable direction. (The minus sign is a convention in the theory of Bowen, Ruelle, and Sinai.) Similarly we can define a stable Jacobian cocycle $\phi_s(x) = \log \| Df(x) \|_s$. (With this sign convention, the stable Jacobian cocycle of f is the same as the unstable Jacobian cocycle of f^{-1}.) If g is an Anosov diffeomorphism which is smoothly conjugate to f, say $g \circ h = h \circ f$, then the corresponding Jacobian cocycles are cohomologous. Namely, $\phi_u^g \circ h = \phi_u^f + u \circ f - u$ where $u = -\log \| Dh \|_u$, and $\phi_s^g \circ h = \phi_s^f + v \circ f - v$ where $v = \log \| Dh \|_s$. In other words, the smooth conjugacy class of f naturally determines a pair of cohomology classes, $(< \phi_u >, < \phi_s >)$, which we will refer to as the unstable and stable *BRS classes* (for Bowen, Ruelle, and Sinai) of f. The corresponding reduced classes will be referred to as the reduced BRS classes of f.

3.2 Cocycles over a foliation.

It will be useful in what follows to have the notion of a cocycle over a foliation $\mathcal{F}$ of a manifold M. The definition depends on the *graph*, or holonomy groupoid, of the foliation, which was constructed by Winkelnkemper [Co],[Wi]. An element γ of the graph $GR(\mathcal{F})$ is a pair of points of M, $x = s(\gamma)$ and $y = r(\gamma)$, together with an equivalence class of smooth paths $\gamma(t)$ from x to y, tangent to the foliation. Two paths $\gamma_1(t)$, $\gamma_2(t)$ from x to y are equivalent if the holonomy of the path $\gamma_2 \circ \gamma_1^{-1}$ from x to x is the identity. There is a natural composition law on GR, defined when the endpoint of one pair is the first point of another pair. A cocycle over the foliation $\mathcal{F}$ is a function

$$\Phi : GR(\mathcal{F}) \to \mathbf{R}$$

such that

$$\Phi(\gamma_1 \circ \gamma_2) = \Phi(\gamma_1) + \Phi(\gamma_2)$$

whenever the composition $\gamma_1 \circ \gamma_2$ is defined. A coboundary is a cocycle of the form $\Phi(\gamma) = u(r(\gamma)) - u(s(\gamma))$ where u is a function on M. If the leaves of $\mathcal{F}$ have trivial holonomy, then the point $\gamma \in GR(\mathcal{F})$ depends only on the pair $x = r(\gamma)$ and $y = s(\gamma)$. In this case we will use the notation $\Phi(x \to y)$ for the cocycle.

3.3 Cocycles over an equivalence relation

Let $\mathcal{F} \subset M \times M$ be defined by an equivalence relation $\sim$ on M. Then a cocycle over $\mathcal{F}$ is a function

$$\Phi : \mathcal{F} \to \mathbf{R}$$

satisfying

$$\Phi(x, z) = \Phi(x, y) + \Phi(y, z)$$

whenever y and z belong to the equivalence class of x. See [HK] A coboundary is a cocycle of the form $\Phi(x, y) = u(y) - u(x)$ where $u : M \to \mathbf{R}$.

4 The Bowen-Ruelle-Sinai map

Let $f : \mathbf{T}^2 \to \mathbf{T}^2$ be a $C^{1+\alpha}$ Anosov diffeomorphism. Let $G(f)$ denote the Hölder reduced cohomology over f, where the Hölder exponent is allowed to vary. The Bowen-Ruelle-Sinai map,

$$BRS : T^{1+H}(f) \to G(f) \times G(f^{-1})$$

is defined as follows. Let (h, N, g) be a representative of a point in the Teichmüller space $T^{1+H}(f)$. Let $\phi_u(g)$ and $\phi_s(g) = \phi_u(g^{-1})$ be the unstable and stable Jacobian cocycles of g. We map the Teichmüller point to the pair of reduced cohomology classes $(< \phi_u(g) \circ h >_*, < \phi_u(g^{-1}) \circ h >_*)$. Because the topological conjugacy between two $C^{1+\alpha}$ Anosov diffeomorphisms is always Hölder, these are Hölder cohomology classes. The image point is independent of the choice of representative, because a smooth conjugacy changes the Jacobian cocycles by a coboundary.

Theorem 3 (de la Llave, Marco, Moriyon) *The map BRS is injective.*

Remark. Two points need to be added to the theorem proved by de la Llave, Marco, and Moriyon ([Ll],[MM1],[MM2]) to give the stated result. They consider C^r diffeomorphisms where $2 \leq r \leq \omega$. They prove that the pair of BRS cohomology classes is a complete C^r conjugacy invariant. The first point is that if ϕ and ψ are BRS cocycles over an Anosov diffeomorphism whose difference is an almost coboundary, then in fact the difference is a coboundary. (See the remarks about the *pressure* in the next section.) The second point is that the C^{1+H} smoothness case is simpler than the higher smoothness case, because the foliations have C^{1+H} transverse smoothness. An easy modification of the ideas of de la Llave, Marco, and Moriyon proves that the BRS cohomology classes are a complete C^{1+H} conjugacy invariant.

To prove Theorem 1, we need to show that the map BRS is *surjective*. That is, given a pair of reduced cohomology classes $(<\phi_u>_*, <\phi_s>_*)$ over f, we need to construct a marked Anosov diffeomorphism (h, N, g) modeled on f, whose pair of unstable and stable reduced BRS cohomology classes, pulled back to $f: M \to M$ is this pair.

This problem can be restated as follows. We consider the smooth torus M to be defined by a $C^{1+\alpha}$ system of charts (U_β, η_β) on a topological torus $\mathbf{T}^2$. The mapping f is defined on the topological torus, and is assumed to define a $C^{1+\alpha}$ Anosov diffeomorphism when viewed in the smooth charts (U_β, η_β). Given a pair of reduced cohomology classes $(<\phi_u>_*, <\phi_s>_*)$ over f, the problem is to construct a new smooth system of charts (U'_β, η'_β) on the topological torus $\mathbf{T}^2$, with the property that in these charts the mapping f is $C^{1+\alpha'}$, for some $0 < \alpha' < 1$, and the reduced BRS cohomology classes in this smooth structure are $<\phi_u>_*$ and $<\phi_s>_*$. In other words, we identify the underlying point sets of the smooth tori M and N via the homeomorphism h, and vary the point in Teichmüller space by varying the smooth structure on this point set.

Proof of Theorem 2 from Theorem 1. An Anosov diffeomorphism is volume preserving if and only if the forward and backward BRS cohomology classes coincide (under the canonical identification of the forward and backward cohomology) [B2]. Hence the BRS map restricted to the volume preserving diffeomorphisms maps onto the "diagonal" in $G(f) \times G(f^{-1})$, which is naturally identified with $G(f)$.

5 Gibbs measures and associated cocycles

In this section we recall some properties of Gibbs measures. We emphasize the dynamically defined *equivalence relations* and associated cocycles that define a Gibbs measure. The transverse measure class constructed in this paper can be viewed as a "transverse Gibbs measure." It is obtained by focusing on a different dynamically defined equivalence relation and its associated cocycle.

Gibbs measures are defined for dynamical systems with a local product structure. See Baladi's thesis [Ba] for a nice exposition and proofs of some basic results for the general case. Here we only need consider the two classes: Anosov diffeomorphisms and subshifts of finite type. The latter arise because an Anosov diffeomorphism has a presentation as a quotient of a subshift of finite type.

Let A be an $r \times r$ matrix of 0's and 1's. Consider the setof bi-infinite sequences $\Sigma_A = \{\mathbf{x}\}$, where $\mathbf{x} = \ldots x_{-k}x_{-k+1}\ldots x_{-1}x_0x_1\ldots x_lx_{l+1}\ldots$ is in Σ_A if and only if $x_i \in \{1,\ldots,r\}$ and $A_{x_ix_{i+l}} = 1$ for all i. The shift map $\sigma : \Sigma_A \to \Sigma_A$ is defined by $\sigma(\mathbf{x}) = \mathbf{y}$ where $y_i = x_{i+1}$. We give Σ_A the topology defined by the metric $d(\mathbf{x},\mathbf{y}) = \Sigma_{x_i\neq y_i}2^{-|i|}$. The metric space Σ_A together with the shift map is called the *subshift of finite type* defined by A.

If $\mathbf{x} \in \Sigma_A$, we define the *stable set* $W^s(\mathbf{x})$ to be the set of all $\mathbf{y} \in \Sigma_A$ such that $d(\sigma^n\mathbf{x}, \sigma^n\mathbf{y}) \to 0$ as $n \to \infty$. Similarly we define the *unstable set* $W^u(\mathbf{x})$ to be the set of $\mathbf{y} \in \Sigma_A$ such that $d(\sigma^{-n}\mathbf{x}, \sigma^{-n}\mathbf{y}) \to 0$ as $n \to \infty$. A *local stable set* $W^s_\epsilon(\mathbf{x})$ is the set of points $\mathbf{y}$ such that $d(\sigma^n(\mathbf{x}), \sigma^n(\mathbf{y})) < \epsilon$ for $n \geq 0$. Local unstable sets are defined similarly. A stable set has an intrinsic topology defined by the metric $d^-(\mathbf{x},\mathbf{y}) = \sum_{x_i\neq y_i} 2^i$. Similarly, an unstable set has an intrinsic topology defined by the metric $d^+(\mathbf{x},\mathbf{y}) = \sum_{x_i\neq y_i} 2^{-i}$. We will refer to the collection of stable (unstable) sets as the stable (unstable) foliation of Σ_A.

If $f : M \to M$ is an Anosov diffeomorphism, the stable foliation $\mathcal{W}^s$ and unstable foliation $\mathcal{W}^u$ are defined similarly, and have tangent distributions E^s and E^u, respectively.

Anosov diffeomorphisms and subshifts of finite type have a *local product structure.* For every point x in M (or Σ_A) there is a neighborhood U of x, and a homeomorphism

$$u : W^s_\epsilon(x) \times W^u_\epsilon(x) \to U$$

that takes verticals $\{w\} \times W^u_\epsilon(x)$ onto local unstable sets, and horizontals $W^s_\epsilon(x) \times \{z\}$ onto local stable sets.

There is an equivalence relation on M, and Σ_A, defined by the pair of foliations $\mathcal{W}^s$ and $\mathcal{W}^u$. Namely,

$$x \sim y \Leftrightarrow x \in W^s(y) \cap W^u(y).$$

If $x \sim y$, then there are neighborhoods U_x of x and U_y of y, and a homeomorphism

$$\theta : U_x \to U_y$$

such that $z \sim \theta(z)$ for all $z \in U_x$. These are called *conjugating homeomorphims.* The pseudogroup of conjugating homeomorphisms generates this equivalence relation (referred to in the sequel as the Gibbs equivalence relation).

The following definition is due to Capocaccia [Ca].

Definiton 1 *Let $f : M \to M$ be Anosov. Let $\phi : M \to \mathbf{R}$ be continuous. A measure μ on M is a* Gibbs measure for ϕ *if*

$$\log\left(\frac{d\mu(\theta(x))}{d\mu(x)}\right) = \sum_{k=-\infty}^{\infty} (\phi \circ f^k(\theta(x)) - \phi \circ f^k(x))$$

for every conjugating homeomorphism θ.

It is implicit in the definition that both sides of the equation are well-defined. The left hand side of the equation is the logarithmic Radon-Nykodym cocycle associated to the Gibbs equivalence relation and the measure μ. The function ϕ should be regarded as a cocycle over f. If ϕ is changed by an almost coboundary $u \circ f - u + K$ the expression on the right hand side does not change. Hence a Gibbs measure is associated to a reduced cohomology class over f, and is defined by an associated cocycle (referred to as the Gibbs cocycle) over the Gibbs equivalence relation. The definition of Gibbs measure in the subshift of finite type case is exactly analagous.

Now we describe the cocycle properties of the transverse measure class we are going to construct. Instead of the Gibbs equivalence relation, we consider only the stable foliation, and the corresponding holonomy pseudogroup. If $\phi : M \to \mathbf{R}$ is continuous, and if

$$\Phi(x \to y) =: \sum_{k=0}^{\infty} (\phi \circ f^k(y) - \phi \circ f^k(x)) \tag{1}$$

is finite whenever $x \in W^s(y)$, and this expression defines a continuous function of x on a small transversal to $\mathcal{W}^s$, then Φ is a cocycle over W^s, which we will refer to as the transverse Gibbs cocycle. Moreover, if ϕ is changed by an almost coboundary $u \circ f - u + K$, then Φ changes by the coboundary $U(x \to y) = -u(y) + u(x)$. The transverse Gibbs measure class to be constructed will have the following properties. We associate to a reduced cohomology class $< \phi >_*$ over the mapping the cohomology class $< \Phi >$ over the stable foliation defined by equation 5.1. If μ is a representative measure on a small transversal τ, then the logarithmic Radon-Nykodym cocycle of μ over the holonomy hol : $\tau \to \tau$ will be of the form $\Phi(x \to y) + u(y) - u(x)$, where $u : \tau \to \mathbf{R}$ is a continuous function. (In fact u will be Hölder but we postpone the discussion of this until later.)

The following result of Bowen leads us to consider Hölder cocycles over f.

Proposition 1 (Bowen) *[B2] If $\phi : M \to \mathbf{R}$ is Hölder, then the associated Gibbs cocycle and transverse Gibbs cocycle exist, and are Hölder.*

That is, the Gibbs and transverse Gibbs cocycles are Hölder on the domains of the homeomorphisms in the associated pseudogroups. (In fact they are Hölder cocycles over the relevant *metric* equivalence relations, but we will not need this.) The proposition is true for Hölder cocycles over a subshift of finite type, as well.

Remark. There is a larger class of cocycles for which the Gibbs cocycles are finite. In fact there is a natural norm identified by Bowen (the *variation norm*) which defines a Banach space of cocycles with well-defined associated Gibbs cocycles. [B3] One easily carries out the construction of a transverse *continuous* measure class for these cocycles. An open problem is to determine a regularity description of these transverse measure classes.

The Hölder cohomology over an Anosov diffemorphism is naturally associated to the topological conjugacy class of the map. This is because the conjugacy between two C^{1+H} Anosov diffeomorphisms is always Hölder continuous. The Hölder coboundaries over an Anosov diffemorphism form a *closed* subspace of the Hölder cocycles. This is a consequence of the Livshitz theorem, which states that the cohomology class of a Hölder cocycle ϕ over an Anosov diffeomorphism is determined by the values of the cocycle over periodic orbits, i.e. by the sums $\sum_{k=0}^{n-1} \phi \circ f^k(p)$ where $f^n(p) = p$ [Li]. Since these sums are 0 for a coboundary, and this is a closed condition, the coboundaries are a closed subspace.

We now collect various results which will be needed in the sequel. The *one-sided subshift of finite type* Σ_A^+ associated to a $0-1$ matrix A is defined just as in the subshift of finite type case, but one considers only one-sided sequences. The shift map $\sigma : \Sigma_A^+ \to \Sigma_A^+$ is then an expanding endomorphism. The following lemma is due to Sinai and Bowen [B2]:

Lemma 1 *Let $\phi : \Sigma_A \to \mathbf{R}$ be a Hölder function. Then ϕ is cohomologous (via a Hölder transfer function) to a Hölder function ϕ^+ with the property that $\phi^+(\mathbf{x})$ depends only on the forward part of the sequence, i.e. on $x_0, x_1, x_2, \ldots$.*

So the Hölder cyclic cohomologies over (Σ_A, σ) and over (Σ_A^+, σ) are isomorphic.

We give a brief description of Bowen's proof, as we will need to know the form of ϕ^+. Let $[i]$ denote the "rectangle" consisting of those sequences $\mathbf{x} \in \Sigma_A$ with $x_0 = i$. Let $W^u(\mathbf{x}, i)$ be the local unstable set through $\mathbf{x}$ intersected with $[i]$. For each i, choose $\mathbf{x}^i \in [i]$. Define $r : \Sigma_A \to \Sigma_A$ by projecting along local stable sets in the rectangle $[i]$, onto $W^u(\mathbf{x}^i, i)$. Let $u : \Sigma_A \to \mathbf{R}$ be defined by

$$u(\mathbf{y}) = \sum_{k=0}^{\infty} (\phi \circ \sigma^k(\mathbf{y}) - \phi \circ \sigma^k(r(\mathbf{y})))$$

The function u is Hölder along local unstable sets, by Proposition 1 above in the subshift of finite type setting. It can be checked that

$$\phi^+ = \phi + u \circ \sigma - u$$

depends only on the forward part of a sequence.

□

In the general setting of a homeomorphism $f : M \to M$ of a compact metric space, the *pressure* function $P : C(M) \to \mathbf{R}$ is defined on the space $C(M)$ of continuous functions on M. See [W3] or [B2]. In fact, the pressure is defined on the set of cohomology classes: if ϕ is cohomologous to ψ, then $P(\phi) = P(\psi)$. In addition, $P(\phi + K) = P(\phi) + K$. For the purposes of the present paper, the pressure can be viewed as defining an imbedding of the reduced cohomology into the (unreduced) cohomology. Namely, the reduced class $<\phi>_*$ is mapped to the unreduced class $<\phi - P(\phi)>$. The image is precisely the set of cohomology classes with pressure zero. We note that if f is an Anosov diffeomorphism, and $<\phi>$ is the unstable BRS class, then $P(<\phi>) = 0$ [B2].

Theorem 4 (Bowen-Ruelle-Sinai Theorem) *Let Σ_A^+ be a transitive one-sided subshift of finite type (transitive means that there is a dense orbit). Let $\phi : \Sigma_A^+ \to \mathbf{R}$ be a Hölder function with $P(\phi) = P$. Then there is a unique probability measure μ on Σ_A^+ such that $\log \frac{d\mu(\sigma(\mathbf{x}))}{d\mu(\mathbf{x})} = -\phi(\mathbf{x}) + P$.*

See [B2],[W2]. The measure μ is positive on open sets.

The measure μ has the following "one-sided Gibbs" property. Suppose that for some $n > 0$, we have $\sigma^n(\mathbf{x}) = \sigma^n(\mathbf{y})$. Then there is a homeomorphism $T_{\mathbf{xy}}$ from a neighborhood of $\mathbf{x}$ to a neighborhood of $\mathbf{y}$, defined by the property $\sigma^n(T_{\mathbf{xy}}(\mathbf{z})) = \sigma^n(\mathbf{z})$. For μ as in the BRS Theorem, we have

$$\log(\frac{d\mu(T_{\mathbf{xy}}(\mathbf{z}))}{d\mu(\mathbf{z})}) = \sum_{k=0}^{n-1} (\phi(\sigma^k(T_{\mathbf{xy}}(\mathbf{z}))) - \phi(\sigma^k(\mathbf{z}))).$$

We will need the following

Corollary 2 (Local Uniqueness Corollary) *Let Σ_A^+ be a transitive one-sided subshift of finite type. Let $\phi : \Sigma_A^+ \to \mathbf{R}$ be a Hölder function. Let $P = P(\phi)$. Let ν be a finite measure, supported on an open set $V \subset \Sigma_A^+$. Suppose that ν satisfies the property: $\log(\frac{d\nu(\sigma^n(\mathbf{x}))}{d\nu(\mathbf{x})}) = -\sum_{k=0}^{n-1} \phi \circ \sigma^k(\mathbf{x}) + nP$, whenever $\mathbf{x} \in V$ and $\sigma^n(\mathbf{x}) \in V$. Then ν coincides, up to a constant factor, with the measure μ associated to ϕ by the Bowen-Ruelle-Sinai Theorem, restricted to V.*

Proof of Corollary 1. There is an open subset $U \subset V$ and an n such that σ^n is injective on U and $\sigma^n(U) = \Sigma_A^+$. Define a measure $\tilde{\nu}$ on all of Σ_A^+ by

$$\frac{d\tilde{\nu}(\sigma^n(\mathbf{x}))}{d\nu(\mathbf{x})} = \exp(-\sum_{k=0}^{n-1} \phi \circ \sigma^k(\mathbf{x}) + nP)$$

where $\mathbf{x} \in U$. Then $\tilde{\nu}$ agrees with ν on V, by the derivative hypothesis on ν, and $\log \frac{d\tilde{\nu}(\sigma(\mathbf{x}))}{d\tilde{\nu}(\mathbf{x}} = -\phi(\mathbf{x}) + P$. Now apply the uniqueness part of the Bowen-Ruelle-Sinai Theorem.

□

Proposition 2 (The invariant measure.) *Let Σ_A^+ be a transitive subshift of finite type, and let $\phi : \Sigma_A^+ \to \mathbf{R}$ be Hölder. Then there exists a Hölder function h with the following property. Let $\phi' = \phi + h - h \circ \sigma - P(\phi)$. Then $\phi' < 0$ and $\mu_{\phi'}$ (the measure associated to ϕ' by the BRS theorem) is invariant under σ.*

See [L],[W1]. Let μ_ϕ and $\mu_{\phi'}$ be the measures associated to ϕ and ϕ', respectively, by the BRS theorem. Then the construction yields

$$\frac{d\mu_{\phi'}}{d\mu_\phi} = \log(h).$$

Since $P(\phi') = 0$, the Radon-Nykodym derivative of $\mu_{\phi'}$ under the shift map is $\exp(-\phi') \geq c > 1$. Hence the invariant measure is expanded by the shift map.

We now address an important subtlety of the smooth structure construction. Ultimately we obtain a smooth structure by integrating a representative measure on a transversal. A different representative measure differs by a Hölder Radon-Nykodym derivative, that is *Hölder with respect to the underlying metric of, say, the linear toral diffeomorphism.* We need to know that the Radon-Nykodym derivative is Hölder with respect to the *new* smooth coordinate, namely the measure itself. This will follow from the following proposition.

Suppose we have a mixing one-sided subshift of finite type Σ_A^+. A cylinder set C_n of length n associated to a finite word $x_0 x_1 \ldots x_{n-1}$ is the set of all sequences in Σ_A^+ that begin with this word. Suppose we have a Hölder map $\pi : \Sigma_A^+ \to I$ onto an interval $I \subset \mathbf{R}$ satisfying the following properties. First, π is such that $\pi(\mathbf{x}) = \pi(\mathbf{y})$ implies that $\pi(\sigma(\mathbf{x})) = \pi(\sigma(\mathbf{y}))$. Second, we assume that the image by π of a cylinder set is an interval. Let ϕ' be a Hölder function on I. Let μ be the measure on Σ_A^+, associated to the pull-back of ϕ by π, constructed in the Bowen-Ruelle-Sinai theorem. Let μ_0 be the measure corresponding to the constant function. Assume that π is injective on a set of full measure with respect to both μ_0 and μ. There are two metrics d_0 and d_ϕ defined on I by the push-forward by π of μ_0 and μ, respectively.

Proposition 3 *The identity map $\iota : (I, d_0) \to (I, d_\phi)$ is quasisymmetric.*

See [Ja] and [Ji] for the proof of quasisymmetry. We give the outline of the proof in the appendix.

Corollary 3 *A function on I is Hölder in the d_0 metric if and only if it is Hölder in the d_ϕ metric.*

Proof. A quasisymmetric homeomorphism is Hölder [A].

6 Realizing cohomology classes as transverse structures

6.1 Transverse measure class to a foliation

Let M be a smooth n-dimensional manifold, and let $\mathcal{F}$ be a k-dimensional foliation of M. That is, $M = \cup_{F \in \mathcal{F}} F$ where each $F \in \mathcal{F}$ is a smooth submanifold of M. M is covered by *flow-boxes* $D^k \times D^{n-k}$ with the property that each leaf $F \in \mathcal{F}$ meets a flow-box in a collection of disks of the form $D^k \times \{y\}$. A *transversal* τ to the foliation is a smooth $(n-k)$-dimensional submanifold that meets each leaf F transversely.

A *transverse measure* $\mu_{\mathcal{F}}$ assigns to each small transversal a measure, with finite total mass, with the property that the measure is invariant under the holonomy pseudogroup [Co],[RS].

By relaxing the condition on invariance under holonomy, we arrive at the notion of a *transverse measure class*, $\boldsymbol{\mu}_{\mathcal{F}}$. This object assigns to each small transversal a measure *class*, with finite total mass, which is invariant under the holonomy pseudogroup. If the foliation has sufficient transverse regularity so that it preserves a smoothness class Λ, e.g. where Λ denotes Hölder, Lipshitz, or C^r regularity, then we can define a *transverse* Λ *measure class* by requiring that the representative measures on a transversal are equivalent with Radon-Nykodym derivatives in the class Λ.

Example. Suppose $\mathcal{F}$ is a foliation of M with transverse regularity C^{1+H}, i.e. the holonomy maps on transversals are C^{1+H}. Then Lebesgue measure on transversals defines a transverse Hölder measure class.

Let $f : M \to M$ be a diffeomorphism, preserving the foliation $\mathcal{F}$. Then we say that f preserves the transverse Λ measure class $\boldsymbol{\mu}$ if for every small transversal τ, and measurable subset $E \subset \tau$, E has positive $\boldsymbol{\mu}$-measure if and only if $f(E) \subset f(\tau)$ has positive $\boldsymbol{\mu}$-measure, and moreover the Radon-Nykodyn derivative has regularity Λ.

There is a Λ cohomology class over f naturally associated to an f-invariant transverse Λ measure class $\boldsymbol{\mu}_{\mathcal{F}}$. It can be defined as follows. Pick a finite covering of M by flow-boxes $B_i = D_i^k \times D_i^{n-k}$. Choose representative measures μ_i on transversals $\tau_i = \{x_i\} \times D_i^{n-k}$ in each flow-box. Let $\{\alpha_i\}$ be a smooth partition of unity subordinate to the covering by flow-boxes. Define

$$\phi(x) = -\log \frac{d\mu(f(x))}{d\mu(x)}$$

where $\mu = \sum_i \alpha_i(x)\mu_i(x)$, regarded as a measure on the local quotient space obtained by projecting along the leaf factors $D_i^k \times \{y\}$ in the flow boxes containing x.

The Λ cohomology class of ϕ is independent of the choice of covering by flow-boxes, the representative measures, and the partition of unity. We will call this the Radon-Nykodym class of f acting on the transverse measure class $\boldsymbol{\mu}_{\mathcal{F}}$.

In a similar way, we can define the notion of a *transverse smooth structure.* This is an assignment of a smooth structure to each small transversal, with the property that the holonomy pseudogroup acts smoothly with respect to this smooth structure. Note that the assigned smooth structure on a transversal in general will have nothing to do with that induced on a transversal by the ambient smooth structure of the manifold M. If, as above, f is a diffeomorphism preserving the foliation $\mathcal{F}$, then an *f-invariant transverse smooth structure* is one in which the action of f on transversals is smooth (with respect to the assigned structure). There is a natural cohomology class associated to the action of f, defined as in the transverse measure class case, but with Radon-Nykodym derivative replaced by the Jacobian. We will refer to this as the transverse Jacobian class of the action of f on the transverse smooth structure.

6.2 Radon-Nykodym realization

We are now ready to describe the main step in the construction of an invariant smooth structure from a pair of cohomology classes over an Anosov diffeomorphism f.

Theorem 5 (Radon-Nykodym realization.) *Let $f : M \to M$ be an Anosov diffeomorphism. Let $< \phi >_*$ be a Hölder reduced cohomology class over f. Then there is an f-invariant transverse Hölder measure class $\boldsymbol{\mu}$ to the stable foliation W^s, with the property that the reduced Radon-Nykodym class of f acting on $\boldsymbol{\mu}$ is $< \phi >_*$.*

The transverse measure class $\boldsymbol{\mu}$ in the theorem has the additional property that the measure class on a transversal is positive on open subsets of the transversal. When $M = T^2$, transversals to W^s are one-dimensional. In this case, an $\alpha-$ Hölder transverse measure class that is positive on open sets is equivalent to a $C^{1+\alpha}$ transverse smooth structure.

Theorem 6 (Jacobian realization in dimension 2) *Let $f : T^2 \to T^2$ be Anosov. Let $< \phi >_*$ be a Hölder reduced cohomology class over f. Then there is an f-invariant transverse $C^{1+\alpha}$ smooth structure to the stable foliation W^s, with the property that the reduced Jacobian class of f acting on this transverse smooth structure is $< \phi >_*$.*

The Radon-Nykodym realization theorem is proved in section 7.

6.3 Complementary transverse smooth structures

Let $\mathcal{F}$ and $\mathcal{G}$ be foliations of complementary dimension. We say that the foliations *intersect transversely* if the leaves of $\mathcal{F}$ and $\mathcal{G}$ meet transversely. We assume that there is a system of simultaneous flow-boxes of the form

$$D^k \times D^l$$

where each leaf of $\mathcal{F}$ meets a flow-box in a collection of disks of the form $D^k \times \{y\}$, and each leaf of $\mathcal{G}$ meets a flow-box in a collection of disks of the form $\{x\} \times D^l$.

Let $\mathcal{F}$ and $\mathcal{G}$ be foliations of complementary dimension, intersecting transversely. Then a pair of transverse smooth structures, one for each foliation, determines a canonical smooth structure on M as follows. Consider a simultaneous flow-box $D^k \times D^l$. Pick a point (x, y) in the flow-box. Then the disk $\{x\} \times D^l$ is a transversal to the foliation $\mathcal{G}$. Similarly the disk $D^k \times \{y\}$ is a transversal to the foliation $\mathcal{F}$. So there is a product smooth structure on the flow-box determined by the transverse smooth structures on these disks.

The overlap maps for these charts are block diagonal, with the blocks being the derivative of the holonomy for each foliation. So the product structure has the same degree of smoothness as the transverse structures.

Proof of Theorem 1 We show that the BRS map is surjective, assuming the Jacobian realization theorem. Let $< \phi_u >_*$ and $< \phi_s >_*$ be the reduced cohomology classes which we want to realize. We have complementary, transverse foliations W^s and W^u, which by the Jacobian realization lemma can be equipped with a transverse smooth structures, with associated reduced Jacobian classes equal to $< \phi_u >_*$ and $< \phi_s >_*$. We define a product structure as just described. In this smooth structure, the unstable reduced Jacobian class of f is simply the tranverse reduced Jacobian class of f acting on the transverse smooth structure, i.e. $< \phi_u >_*$. Similarly, the reduced stable Jacobian class is $< \phi_s >_*$. It remains to see that f is Anosov in the new smooth structure. This follows from Proposition 2 of Section 5, and the construction of the transverse measure from the Gibbs measure associated to ϕ. We postpone this simple argument to Section 8, where explicit charts in the smooth structure are decribed.

□

7 Proof of Radon-Nykodym realization

An Anosov diffeomorphism has a presentation as the quotient of the shift map on a subshift of finite type [Si]. The subshift of finite type is defined by the transition properties of rectangles in a Markov partition under the action of the diffeomorphism. We will show that a Hölder cocycle over the shift map defines a "transverse Hölder measure class" to the stable sets in the subshift of finite type. The transverse measure class pushes down to a transverse measure class to the stable foliation on M provided the cocycle passes down to a cocycle on M, i.e. when it is generated by a function on the subshift which is constant on fibers of the quotient to M.

This section is organized as follows. In subsection 1, we recall the definition of a Markov partition for an Anosov diffeomorphism, and construct the quotient from the shift to the diffeomorphism. We define the notion of transverse measure class for subshifts of finite type in subsection 2, and in subsection 3 show that a Hölder cocycle determines a Hölder transverse measure class. In subsection 4 we show that the transverse measure class pushes forward to the quotient space when the cocycle does.

7.1 Markov partitions

The simultaneous flow-boxes for the stable and unstable foliations of an Anosov diffeomorphism $f : M \to M$ define a *local product structure* $D^s \times D^u$ on the manifold M. A *rectangle* is a closed set of small diameter which is a product in the local product structure: $R = A^s \times A^u$. A rectangle is *proper* if it is the closure of its interior. We define the *stable* and *unstable boundary of* R respectively: $\partial^s R = A^s \times \partial A^u$; $\partial^u R = \partial A^s \times A^u$.

A *Markov partition* for f is a set $\mathcal{C} = \{R_1, \ldots, R_r\}$ of small proper rectangles whose union is M, and satisfying:

i. each R_i is connected

ii. $\mathrm{int}(R_i) \cap \mathrm{int}(R_j) = \emptyset$ for $i \neq j$

iii. $f(\partial^s \mathcal{C}) \subset \partial^s \mathcal{C}$ where $\partial^s \mathcal{C} = \cup_{i=1}^{i=r} \partial^s R$

iv. $f^{-1}(\partial^u \mathcal{C}) \subset \partial^u \mathcal{C}$ where $\partial^u \mathcal{C} = \cup_{i=1}^{i=r} \partial^u R$

Sinai proved that Anosov diffeomorphisms have Markov partitions of arbitrarily small diameter, where the diameter of the partition is defined to be the largest diameter of a rectangle in the partition [Si].

A Markov partition defines a 0-1 matrix A where $A_{ij} = 1$ if $f(\mathrm{int} R_i) \cap \mathrm{int} R_j \neq \emptyset$ and is 0 otherwise. The properties of the Markov partition guarantee that if $\mathbf{x} \in \Sigma_A$, then the intersection $\cap_{i=-\infty}^{\infty} f^{-i} R_{x_i}$ consists of a single point. This defines a map $\pi : \Sigma_A \to M$ which semi-conjugates the shift map to the mapping f.

7.2 Transverse structures on a subshift of finite type.

We recall that a subshift of finite type Σ_A has a *local product structure* defined by the stable and unstable foliations. That is, there is a homeomorphism defined on a neighborhood U of a point $\mathbf{x}$,

$$u : U \to W^s_\epsilon(\mathbf{x}) \times W^u_\epsilon(\mathbf{x})$$

with the property that local stable sets map to sets of the form $W_\epsilon^s(\mathbf{x}) \times \{\mathbf{z}\}$, and local unstable sets map to sets of the form $\{\mathbf{w}\} \times W_\epsilon^u(\mathbf{x})$.

A small *transversal* to the stable foliation is a set which in the local product structure is represented as the graph of a continuous function $\tau : W_\epsilon^u(\mathbf{x}) \to W_\epsilon^s(\mathbf{x})$.

We assume that Σ_A is a transitive subshift of finite type, i.e. there is a dense orbit. Suppose $\mathbf{x} \in W^s(\mathbf{y})$. Then there is $\epsilon > 0$ and a canonical homeomorphism $h : W_\epsilon^u(\mathbf{x}) \to W_\epsilon^u(\mathbf{y})$, such that for every $\mathbf{z} \in W_\epsilon^u(\mathbf{x})$, $h(\mathbf{z}) \in W^s(\mathbf{z})$. These homeomorphisms are called (unstable) conjugating homeomorphisms. The *holonomy pseudogroup of the stable foliation* is defined to be the pseudogroup of homeomorphisms between transversals generated by projections onto the unstable factor in local product charts, and unstable conjugating homeomorphisms between local unstable sets.

A *transverse measure class* to the stable foliation is an assignment of a measure class to each small transversal, with the property that the holonomy transformations preserve the measure class. A Hölder tranverse measure class is one in which the representative measures on a transversal are required to be equivalent with Hölder Radon-Nykodym derivative.

A shift-invariant transverse measure class is defined as for the diffeomorphism case. Note that the shift map preserves the class of transversals. A shift-invariant transverse Hölder measure class defines a Hölder Radon-Nykodym cohomology class over the action of the shift, just as in the foliation case.

7.3 Radon-Nykodym realization for subshifts of finite type.

The following theorem follows easily from the standard Gibbs theory described in Section 5.

Theorem 7 (Radon-Nykodym realization for Σ_A.) *Let Σ_A be a transitive subshift of finite type. Let $< \phi >_*$ be a reduced Hölder cohomology class over the shift map. Then there is a unique shift-invariant transverse measure class $\boldsymbol{\mu}$ to the stable foliation such that the associated reduced Radon-Nykodym class is $< \phi >_*$.*

Proof. We are given a Hölder cohomology class $< \phi >$ on Σ_A. We apply Lemma 1 of Section 5 to obtain a Hölder function ϕ^+ in the cohomology class

$< \phi >$, which we can view as a function on the one-sided shift Σ_A^+. We can also view Σ_A^+ as a subset of the two-sided shift, in fact as a transversal to the stable foliation, as follows. For each symbol $i \in \{1, \ldots, r\}$, pick a point $\mathbf{y}^i$ with $y_0^i = i$. Then $\tau = \cup_{i=1}^r W^u(\mathbf{y^i}, i)$ is canonically isomorphic to Σ_A^+, and is a union of small transversals to the stable foliation. Moreover, τ meets every stable set. We define the measure class on τ to be the Hölder measure class containing the measure μ_{ϕ^+} determined by the Bowen-Ruelle-Sinai theorem applied to ϕ^+. We define the measure class on any small transversal to be the pull-back of μ_{ϕ^+} by a holonomy transformation to the transversal τ (which exists since τ meets every stable set). To see that this defines a transverse Hölder measure class, it suffices to check that the holonomy transformations between local stable sets in τ preserve the measure class of μ_{ϕ^+}, with Hölder Radon-Nykodym derivative. But this is precisely the "one-sided Gibbs" property of μ_{ϕ^+}. Finally we note that the Radon-Nykodym class associated to this transverse measure class is the reduced cohomology class of $\phi^+ - P$, as desired.

□

7.4 Pushing the transverse measure class forward.

Let $W_\epsilon^u(x)$ be the ϵ-ball about x in $W^u(x)$. If i is a rectangle in the Markov partition, let $W^u(x, i) = W_\epsilon^u(x) \cap R$, where ϵ is chosen so that this is a single horizontal slice of R. Let $[i]^+ \subset \Sigma_A^+$ be the set of sequences $y_0 y_1 y_2 \ldots$ with $y_0 = i$. If $x \in M$ and $x = \pi(\mathbf{x})$ where $\mathbf{x} = \ldots x_{-1} x_0 x_1 \ldots$ with $x_0 = R$, then we define a quotient map:

$$\pi_{\mathbf{x},R} : [R]^+ \to W^u(x, R)$$

by $\pi_{\mathbf{x},R}(\mathbf{y}) = \pi(\ldots x_{-2} x_{-1} y_0 y_1 y_2 \ldots)$.

For each rectangle R and $x \in R$ with $x = \pi(\mathbf{y})$, the measure class on the transversal $W^u(x, R)$ is defined to be the image by $\pi_{\mathbf{y},R}$ of the measure class $\boldsymbol{\mu}_\phi$ on $[R]^+$.

We need to check that if parts of $W^u(x, R)$ and $W^u(x', R')$ correspond under local projection along the stable foliation, then the measure classes defined by $\pi_{\mathbf{x},R}$ and $\pi_{\mathbf{x}',R'}$ also correspond. If $R = R'$, this follows from the fact that, if $p_{x,x'}$ is the projection along local stable leaves from $W_\epsilon^u(x)$ to

$W^u_\epsilon(x')$, and if $\pi(\mathbf{x}) = x$ and $\pi(\mathbf{x}') = x'$, then

$$\pi_{\mathbf{x},R} = p_{x,x'} \circ \pi_{\mathbf{x}',R}$$

If $R \neq R'$ we consider representative measures on the $W^u(x, R)$ and $W^u(x', R')$. Let $\mu_{x,R}$ denote the image of μ_{ϕ^+} restricted to $[R]^+$ by $\pi_{\mathbf{x},R}$.

Let $U \subset W^u(x, R)$ and $V \subset W^u(x', R')$ be such that $p_{x,x'} : U \to V$ is a homeomorphism. There are two main points:

1. Let $y \in U$, and $y' = p_{x,x'}(y)$. We can make sense of the expression $\Phi^+(y \to y')$ *on* U as follows. Recall

$$\Phi(y \to y') = \Sigma_{k=0}^{\infty}(\phi \circ f^k(y') - \phi \circ f^k(y)).$$

is defined and Hölder on U. Note that the transfer function $u : \Sigma_A \to \mathbf{R}$ which makes ϕ cohomologous to ϕ^+ pushes forward to a well-defined and Hölder function u_R on each of the individual quotients $W^u(x, R)$ Therefore we *define* a Hölder function:

$$\Phi^+(y \to y') = \Phi(y \to y') + u_{R'}(y') - u_R(y).$$

2. Let μ_U be $\mu_{\mathbf{x},R}$ restricted to U. Define μ_V similarly. Then

$$\frac{d((p_{x,x'})_*(\mu_V))}{d\mu_U} = \exp(\Phi^+(y \to y')).$$

We need to prove the second statement. We define a measure ν supported on V by $\nu = p^*_{x,x'}(\exp(\Phi^+)\mu_U)$. We want to pull back to Σ_A^+ both ν and μ_V, where we will show them to be equal. For this to make sense, and imply the second statement, we need the following lemma. Let $\tilde{V} = \pi^{-1}_{x,R'}(V)$.

Lemma 2 *$\pi_{\mathbf{x},R'} : \tilde{V} \to V$ is one-to-one over a set of full measure with respect to both $(p_{x,x'})^*\mu_U$ and μ_V.*

Proof. Let $Y \subset V$ be the set of points with more than one preimage by $\pi_{\mathbf{x},R'}$. We need to show that $\mu_U(p^{-1}_{x,x'}(Y)) = 0$ and $\mu_V(Y) = 0$

Let μ be the invariant Gibbs measure on M associated to the cohomology class $< \phi >$. We need the following two facts. The technical proofs are included in the appendix.

1. Let $R = A^s \times A^u$ in the local product structure. Let $x \in R$, and $Y \subset W^u(x,R)$. Then $\mu(A^s \times Y) = 0$ implies $\mu_{x,R}(Y) = 0$.

2. $\mu(\partial\mathcal{C}) = 0$.

Now we prove the lemma. If $y \in Y$, then $d(f^n(y), \partial\mathcal{C}) \to 0$ as $n \to \infty$. The same is therefore true of points in $Y' := p_{x,x'}^{-1}(Y)$. Let $R = A^s(R) \times A^u(R)$ and $R' = A^s(R') \times A^u(R')$. Let $Z = A^s(R) \times Y$ and $Z' = A^s(R') \times Y'$. Then $d(f^n(Z), \partial\mathcal{C}) \to 0$ and $d(f^n(Z', \partial\mathcal{C}) \to 0$ as $n \to \infty$. Since $\mu(\partial\mathcal{C}) = 0$, and μ is invariant, $\mu(Z) = 0$ and $\mu(Z') = 0$, and we conclude that $\mu_V(Y) = 0$ and $\mu_U(Y') = 0$.

$\square$

We return to the proof of statement 2 relating the measures on U and V. We want to show that the measure $\tilde{\nu} = (\pi_{x,R'})_*\nu$ coincides with $(\pi_{x,R'})_*\mu_V$. The latter is simply μ_{ϕ^+} restricted to $\tilde{V}$. By the Local Uniqueness corollary to the Bowen-Ruelle-Sinai theorem, it suffices to show that $\tilde{\nu}$ has Radon-Nykodym derivative $\exp(-\sum_{k=0}^{n-1}\phi^+ \circ \sigma^k + nP)$ under $\mathbf{x} \to \sigma^n(\mathbf{x})$ whenever $\mathbf{x}$ and $\sigma^n(\mathbf{x})$ are both in $\tilde{V}$.

Recall $\nu = p_{x,x'}^*(\exp(\Phi^+)\mu_U)$. So

$$\frac{d(\tilde{\nu}(\sigma(\mathbf{y})))}{d\nu(\mathbf{y})} = \pi^*(p_{x,x'})_* \frac{d(\exp(\Phi^+)\mu_U)(f(y))}{d(\exp(\Phi^+)\mu_U)(y)}$$

The main point in the calculation is the following. ϕ^+ is well-defined on the individual quotients $W^u(x,R)$, namely $\phi_R^+ = \phi + u_R \circ f - u_R$. Then

$$\Phi^+(f(y) \to f(y')) - \Phi^+(y \to y') = -\phi_{R'}^+(y') + \phi_R^+(y)$$

where $y' = p_{x,x'}(y)$ and $f(y') = f(p_{x,x'}) = p_{x,x'}(f(y))$.

Thus

$$\begin{aligned} \frac{d(\exp(\Phi^+)\mu_U)(f(y))}{d(\exp(\Phi^+)\mu_U)(y)} &= \exp(-\phi_{R'}^+(y') + \phi_R^+(y)) \cdot \exp(-\phi^+) && (2) \\ &= \exp(-\phi_{R'}^+(p_{x,x'}(y)) + P) && (3) \end{aligned}$$

which is the desired result.

8 Gibbs charts

A Markov partition for a hyperbolic automorphism L of T^2 can be constructed as follows [AW]. Let E^s and E^u be the stable and unstable eigenspaces respectively. Project into the torus a segment in E^s through the origin, and a segment in E^u through the origin. Extend these segments until they cut the torus into parallelograms. The segment in the stable direction should map into itself under L, and the segment in the unstable direction should map into itself under L^{-1}. This decomposition of the torus is a Markov partition. Let A be the transition matrix of the partition, and let $\pi : \Sigma_A \to T^2$ be the quotient map from the subshift of finite type defined by A. The unstable segment τ_u (which is also the unstable boundary of the partition) is the image by π of a copy of the *one-sided shift* Σ_A^+ specified by fixing the backward part of a sequence to be the backward part of some fixed pre-image of the origin. Similarly, the stable segment τ_s is the image of a copy of the backward one-sided shift defined by A, or equivalently of the one-sided shift defined by the A^t, the transpose of A.

The smooth structure determined by a pair of reduced cohomology classes $< \phi_u >_*$ and $< \phi_s >_*$ has the following explicit description. We can assume that the functions ϕ_u and ϕ_s have pressure 0. If not, we can just subtract the pressure, which will not change the reduced cohomology class. Let $\tilde{\phi}_u$ be the pull-back of ϕ_u to Σ_A, by π. $\tilde{\phi}_s$ is defined similarly. Now change $\tilde{\phi}_u$ by a coboundary to get a function ϕ_u^+ that depends only on the forward part of a sequence. Similarly, one gets a function ϕ_s^- which is cohomologous to $\tilde{\phi}_s$ and depends only on the backward part of a sequence. We can regard these as functions on Σ_A^+ and $\Sigma_{A^t}^+$, respectively. Now let μ_+ be the unique probability measure on Σ_A^+ satisfying

$$\frac{d\mu_+(\sigma(\mathbf{x}))}{d\mu_+(\mathbf{x})} = \exp(\phi_u^+(\mathbf{x}))$$

Let μ_- be the unique probability measure on $\Sigma_{A^t}^+$ with

$$\frac{d\mu_-(\sigma(\mathbf{y}))}{d\mu_-(\mathbf{y})} = \exp(\phi_s^-(\mathbf{y}))$$

These measures push-forward to measures ν^+ and ν^- on τ_u and τ_s respectively.

Smooth coordinate charts (in the new structure) are obtained by integrating the measures ν^+ and ν^- along two side of a "rectangle" in the partition, and taking the product structure. The main point in the proof of the theorem is that if the segment τ_u is presented *differently* as the image of Σ_A^+, i.e. by choosing a different pre-image of the origin, with different backward part, then the smooth coordinate along τ_u determined by pushing forward the measure ν^+ by this different presentation is smoothly equivalent to the original one.

It is now clear from Proposition 2 of Section 5 that f is Anosov in the new smooth structure. We simply use the expanding measure in the Hölder measure class associated to $< \phi_u >_*$ to define the coordinate along τ_u. Similarly, use the expanding measure associated to $< \phi_s >_*$ to define the coordinate along τ_s.

Remark. The two maps $\pi_1 : \Sigma_A^+ \to \tau_u$ and $\pi_2 : \Sigma_A^+ \to \tau_u$, determined by viewing τ_u from the "clockwise" side or the "counterclockwise" side, determine a comparison map $\alpha : \Sigma_A^+ \to \Sigma_A^+$ on a set of full measure, defined by $\pi_2(\alpha(\mathbf{x})) = \pi_1(\mathbf{x})$. The measure μ^+ has Hölder Radon-Nykodym derivative under α.

Appendix

A Quasisymmetric equivalence of Gibbs structures

We show that the conjugacy between the linear map in a topological conjugacy class, and the map constructed from a Hölder cocycle, is quasisymmetric along the leaves of the stable and unstable foliations. The proof is adapted from [Ji].

A homeomorphism $h : I \to I$ is *quasisymmetric* if there exists a $K > 0$ such that for every pair of adjacent intervals I and J of equal length,

$$1/K \leq \frac{|h(I)|}{|h(J)|} \leq K.$$

The number K is the *quasisymmetry constant* of the map, and h is called $K-$*quasisymmetric*. The composition of a quasisymmetric map with a C^1 map is again quasisymmetric.

We recall Proposition 2 from Section 5. Let Σ_A^+ be a mixing one-sided subshift of finite type. Let $\pi : \Sigma_A^+ \to I$ be a Hölder map onto an interval $I \subset \mathbf{R}$, with the property that $\pi(\sigma(\mathbf{x})) = \pi(\sigma(\mathbf{y}))$ whenever $\pi(\mathbf{x}) = \pi(\mathbf{y})$. In addition we assume that the image by π of a cylinder set C_n is an interval. Let ϕ' be a Hölder function on I, and let ϕ be the pull-back of ϕ' to Σ_A^+ by π. Let μ_0 be the measure of maximal entropy on Σ_A^+, and let μ_ϕ be the measure associated to ϕ as constructed in the Bowen-Ruelle-Sinai theorem, that is, the unique probability measure with Radon-Nykodym derivative $\phi - P(\phi)$ where $P(\phi)$ is the pressure of ϕ. Assume that π is injective on a set of full measure, with respect to both μ_0 and μ_ϕ. There are two metrics d_0 and d_ϕ on I, defined by the push-forward by π of μ_0 and μ_ϕ, respectively.

Proposition 4 (Quasisymmetric equivalence of Gibbs structures) *The identity map $\iota : (I, d_0) \to (I, d)$ is quasisymmetric.*

We note that the hypotheses of the proposition are satisfied by the map $\pi : \Sigma_A^+ \to \tau_u$ defined in section 8.

The proof given in [Ji] applies in a more general context. The basic idea is the following. The image of the partitions into cylinder sets of the subshift of finite type defines a nested sequence of partitions of the interval I. The identity map of course preserves these partitions. The quasisymmetry estimate follows from bounds on the geometry of this sequence of partitions, in both the d_0 and the d_ϕ metrics.

We can assume that ϕ is in fact the Radon-Nykodym derivative of the expanding (equilibrium) measure on Σ_A^+ associated to the reduced cohomology class $< \phi >_*$. This is because the expanding measure, and the measure μ_ϕ are equivalent with a Hölder Radon-Nykodym derivative, and therefore the corresponding metrics on I are $C^{1+\alpha}$ equivalent.

Let $\mathcal{C}_n$ denote the partition of Σ_A^+ by cylinder sets of size n. Let $\mathcal{D}_n$ be the corresponding sequence of partitions of I.

In the following lemma, we consider I with the d_0 metric, and show how to approximate the intervals defined by an equally spaced triple of points by elements of the partitions $\mathcal{D}_n$.

Lemma 3 *There exists a positive integer $N = N(\Sigma_A^+)$ with the following property. Let $x, y \in I$, and let z be the midpoint of the interval $[x, y]$. We will write $R = [x, z]$ and $S = [z, y]$. Let n be the smallest integer such that there exists $D \in \mathcal{D}_n$ with $R \cup S \subset D$. Then there are $D_R, D_S \in \mathcal{D}_{n+N}$ with $D_R \subset R$ and $D_S \subset S$.*

Outline of the proof. Let h be the topological entropy of Σ_A^+, and let M be the mixing time, that is, A^M has all positive entries. Then $N \geq 4M + 1 + \frac{log2}{logh}$ has this property. This follows easily from the following three geometric properties of the partitions $\mathcal{D}_n$ in the d_0 metric. Let $\lambda = \exp h$. We denote the length of an interval D in the d_0 metric by $|D|_0$.

1. **Exponentially decreasing geometry.** Let $D_{n+m} \in \mathcal{D}_{n+m}$, $D_n \; in \mathcal{D}_{np}$, with $D_{n+m} \subset D_n$. Then

$$\frac{|D_{n+m}|_0}{|D_n|_0} \leq \lambda^{m-M}.$$

2. **Bounded ratio geometry.** Let $D_n \in \mathcal{D}_n$, $D_{n+1} \in \mathcal{D}_{n+1}$, with $D_{n+1} \subset D_n$. Then

$$\frac{|D_{n+1}|_0}{|D_n|_0} \geq \lambda^{-(M+1)}.$$

3. Bounded nearby geometry. Let $D, E \in \mathcal{D}_n$ be adjacent intervals. Then

$$\lambda^{-M} \leq \frac{|D|_0}{|E|_0} \leq \lambda^M.$$

$\square$

Outline of proof of the proposition. Analagous geometric properties hold for the partitions $\mathcal{D}_n$ in the d_ϕ metric. The quasisymmetry estimate follows from these. We define some preliminary quantities.

Let $\mathrm{S}_n\phi(\mathbf{x}) = \sum_{k=0}^{n-1}(\phi(\sigma^k(\mathbf{x}))$. Let

$$\mathrm{var}_n\phi = \sup\{|\mathrm{S}_n\phi(\mathbf{x}) - \mathrm{S}_n\phi(\mathbf{y})| \text{ where } \mathbf{x}, \mathbf{y} \in C_n \text{ for some } C_n \in \mathcal{C}_n\}.$$

Since ϕ is Hölder, there exist $c > 0$ and $\beta < 1$ such that $\mathrm{var}_n\phi < c\beta^n$. Therefore

$$\mathrm{var}\phi =: \sum_{k=0}^{\infty} \mathrm{var}_k\phi < \infty.$$

Let $\|\phi\| = \sup\{|\phi(\mathbf{x})| : \mathbf{x} \in \Sigma_A^+\}$. Define $L = \exp(2\mathrm{var}\phi) \cdot \exp(M\|\phi\|)$.

In what follows, all lengths are with respect to the d_ϕ metric, where ϕ is the Radon-Nykodym derivative of the expanding measure on Σ_A^+ associated to the reduced cohomology class $< \phi >_*$. The d_ϕ length of an interval D is denoted $|D|_\phi$.

1. Bounded ratio geometry. Let $D_{n+1} \in \mathcal{D}_{n+1}$, $D^n \in \mathcal{D}_n$, with $D_{n+1} \subset D_n$. Then

$$\frac{|D_{n+1}|_\phi}{|D_n|_\phi} \geq L^{-1} \cdot \exp(-\|\phi\|).$$

2. Bounded nearby geometry. Let $D, E \in \mathcal{D}_n$ be adjacent intervals. Then

$$\frac{1}{L} \leq \frac{|D|_\phi}{|E|_\phi} \leq L.$$

These properties follow from Bowen's estimate for the $\mu_\phi-$measure of a cylinder set C_n, when ϕ is the Radon-Nykodym derivative of the invariant measure.[B2] For any $\mathbf{x} \in C_n$,

$$c_1 \leq \frac{\mu_\phi(C_n)}{exp(S_n\phi(\mathbf{x}))} \leq c_2$$

where $c_1 = \exp(-M\|\phi\|) \cdot \exp(-\text{var}\phi)$, and $c_2 = \exp(\text{var}\phi)$.

Using these properties, and the Lemma, one obtains an estimate for the quasisymmetry constant K:

$$K \leq L^{(2N+2)} \cdot \exp((2N+1)\|\phi\|).$$

Remark. This estimate is not sharp, as can be seen by considering ϕ to be the constant function with pressure zero. The estimate can be improved by a more careful comparison of the partition elements lying in the pair of adjacent intervals.

B Gibbs measures and Markov partition boundaries

We prove here the technical facts needed in Section 7.4 Let $f : M \to M$ be an Anosov diffeomorphism, $\mathcal{C}$ be a Markov partition for f, and μ be the Gibbs measure associated to the cohomology class $< \phi >$.

Proposition 5 *Let $R \in \mathcal{C}$ be a rectangle, with $R = A_s \times A_u$ in the local product structure. Let $x \in R$, and $Y \subset W^u(x, R)$. Let $\mu_{x,R}$ be the push-forward of the one-sided measure μ_{ϕ^+}, as defined in Section 7.4. Then $\mu(A^s \times Y) = 0$ implies $\mu_{x,R}(Y) = 0$.*

Proof. We recall how μ on T^2 can be constructed from μ_{ϕ^+} on Σ_A^+. See [B2] for details. If $\pi : \Sigma_A \to T^2$ is the quotient map determined by the Markov partition, then μ is the push-forward by π of the measure $\tilde{\mu}$ on Σ_A, defined as follows. Let ν be the *shift-invariant* probability measure on Σ_A^+ equivalent to μ_{ϕ^+}. (The Radon-Nykodym derivative of ν with respect to μ_{ϕ^+} is given explicitly up to a constant factor as the unique positive eigenvector of the Perron-Fröbenius operator associated to ϕ^+.) The measure $\tilde{\mu}$ on Σ_A is obtained from the measure ν on Σ_A^+ as follows. If g is a continuous function on Σ_A, define g^* on Σ_A^+ by

$$g^*(\mathbf{x}) = \min\{g(\mathbf{y}) \text{ where } y_i = x_i \text{ for all } i \geq 0\}$$

Then $\lim_{n\to\infty} \nu((g \circ \sigma^n)^*)$ exists, and we define $\tilde{\mu}(g)$ to be this limit. This linear functional defines the measure $\tilde{\mu}$. The proposition now follows easily.

□

Proposition 6 $\mu(\partial\mathcal{C}) = 0$.

Proof. The following proof is adapted from Bowen's in the case μ is the measure associated to the constant cocycle. [B1] The proof relies on the *variational principle*, and the fact that the topological pressure always decreases when the dynamics is restricted to an invariant subset.
Variational Principle. If a cohomology class over f, say $<\phi>$ has been fixed, we define the *measure theoretic pressure* of an invariant probability measure ν to be $h_\nu + \int \phi d\nu$, where h_ν is the measure theoretic entropy of f. We denote this $P_\nu(\phi, f)$. The topological pressure of ϕ is denoted $P(\phi, f)$. Note that these depend only on the cohomology class of ϕ. The variational principle states:

$$P_\nu(\phi, f) \leq P(\phi, f) \text{ and } \sup_\nu P_\nu(\phi, f) = P(\phi, f)$$

where the supremum is over all invariant probability measures. The variational principle is true for any homeomorphism of a compact metric space. If the mapping is Anosov, and the cocycle is Hölder then supremum is achieved precisely at the Gibbs measure μ associated to $<\phi>$. Recall that if W is a compact f-invariant subset, then $P(\phi_{|W}, f_{|W}) < P(\phi, f)$.

Let μ be the Gibbs measure associated to the cohomology class $<\phi>$. We will show that $\mu(\partial^s\mathcal{C}) = 0$. A similar argument shows that $\mu(\partial^u\mathcal{C}) = 0$. Let $W = \cap_{n\geq 0} f^n(\partial^s\mathcal{C})$. Suppose that $\mu(\partial^s\mathcal{C}) = a > 0$. Then $\mu(W) = a$, and $\mu(\cup f^n(\partial\mathcal{C}) \setminus W) = 0$. Define ν_1 on W by $\nu_1 = \frac{1}{a}\mu$, and ν_2 on M by $\nu_2(E) = \frac{1}{1-a}\mu(E \setminus W)$. Then ν_1 and ν_2 are f-invariant, have disjoint support, and $\mu = a\nu_1 + (1-a)\nu_2$. Therefore

$$P_\mu(\phi, f) = aP_{\nu_1}(\phi, f) + (1-a)P_{\nu_2}(\phi, f).$$

The variational principle implies that $P_{\nu_2}(\phi, f) \leq P(\phi, f)$ and

$$\begin{aligned} P_{\nu_1}(\phi, f) &= P_{\nu_1}(\phi_{|W}, f_{|W}) && (4)\\ &\leq P(\phi_{|W}, f_{|W}) && (5)\\ &< P(\phi, f). && (6) \end{aligned}$$

But then $P_{\mu}(\phi, f) < P(\phi, f)$, a contradiction since μ achieves the supremum.

□

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