

Critical Circle Maps Near Bifurcation

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Abstract

We estimate harmonic scalings in the parameter space of a one-parameter family of critical circle maps. These estimates lead to the conclusion that the Hausdorff dimension of the complement of the frequency-locking set is less than 1 but not less than $1/3$. Moreover, the rotation number is a Hölder continuous function of the parameter.

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1 Preliminaries

1.1 Introduction

This paper will present results about circle maps and families of circle maps that we were able to obtain during the past couple of years. We will not discuss diffeomorphisms, which by far and large are the best understood class of

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circle maps. In the present paper, we will deal with critical homeomorphisms. Some methods and estimates can be carried over to the non-invertible maps, but we only mention [3] here.

We will provide rigorous proofs of our claims using only analytical tools and without resorting to the help of a computer (other than for typesetting this text.)

Let us start by defining the class of maps we consider.

The objects that we intend to investigate. Points of the real line can be projected onto the unit circle in the complex plane by means of the map

$$x \rightarrow \exp(2\pi ix) .$$

Maps from the real line project on the circle if they satisfy

$$F(x + 1) - F(x) \in \mathbf{Z}$$

for every real x . Obviously, for a continuous map this difference must be constant, and is the topological degree of the circle map.

Unless necessary, we will not make a strong distinction between objects that live on the circle and their lifts to the universal cover. Whenever we want to make a strong point of something being actually on the circle, we will write (mod 1) near the formula.

If x and y are objects on the circle, $|x - y|$ is supposed to mean the distance in the natural metric.

Hypotheses. *We consider a family a circle maps given by*

$$F_t(x) = F_0(x) + t \pmod{1}$$

where t is a real parameter which ranges on the interval $(0, 1)$ and $F_0 \pmod{1}$ is a degree one circle homeomorphism which in addition satisfies:

- *It is at least three times continuously differentiable.*
- *The derivative vanishes in exactly one point which is identified with 0. Also, 0 is fixed by F_0 .*

- The function is differentiable enough times so as to satisfy

$$\frac{dF_0^l}{dx^l}(0) \neq 0$$

for some l where the l -th derivative exists and is continuous everywhere.

In addition, we consider the corresponding family f_t of lifts of maps F_t . We denote with $\rho(t)$ the rotation number of F_t .¹

A generalization is possible. Most of our estimates would also work under a more general kind of parameter dependence which only assumes that the family f_t grows with t and there is some control over how the critical point moves. However, we did not find that generalization worth the cost of additional complications.

Theorem A. *Under the hypotheses listed above, the function $\rho(t)$ is Hölder continuous.*

Theorem B. *Consider the set*

$$\Omega' := \rho^{-1}(\mathbf{R} \setminus \mathbf{Q}) .$$

Under our hypotheses the Hausdorff dimension of Ω' satisfies

$$1/3 \leq HD(\Omega') < 1 .$$

Remark B. *The upper estimate of Theorem B can be improved as follows: there is a number $\alpha < 1$, independent of the choice of F_0 which determines our family, so that*

$$HD(\Omega') \leq \alpha .$$

While we do not formally prove Remark B, we give a justification for it.

¹We define the rotation number a little later.

A comment on the results. Theorem A is the first result that we know about concerning regularity of the rotation function for critical families. Theorem B is a refinement of the result that the measure of Ω' is zero. However, it still falls well short of numerically established universality of the Hausdorff dimension (equal to about 0.87 for cubic families).

It worth noting that the lower estimate in Theorem B contradicts a certain conjecture based on extrapolating numerical data. The work [1] gave an asymptotic formula for the fractal dimension (which is essentially another name for Hausdorff dimension, see [2]) of Ω' which was expected to tend to 0 as the critical exponent (l in our notations) grew to infinity. This is, of course, contradicted by our result. In fact, the behavior of the family near bifurcation is what one tends to miss when doing numerics and one of our hopes is that our paper will help to place extrapolation of numerical results on a sounder basis.

The results have three roots. One is bounded geometry of critical maps which has been known for awhile. Another is much stronger estimates of the geometry near a bifurcation point which are new in the sharp form in which we use them.

Finally, there is a way of establishing similarity between objects in the phase space and in parameters. We show an easy technique to achieve that which, however, is much stronger than complicated and rather crude estimates of this kind used in [12].

1.2 Topological Description of Dynamics

The *rotation number* of a circle map F is given by

$$\rho(F) = \lim_{n \rightarrow \infty} \frac{f^n(x)}{n} \pmod{1}$$

where x is any point and f any lift of F to the universal cover. For the maps we consider, the limit always exists and is independent of the choice of x or a particular lift f . If the rotation number is irrational, it is a full topological invariant; and even if it is rational, still a lot of information about the underlying dynamics can be read.

The structure on rotation numbers. There is no universal agreement on what is the best way to organize rotation numbers. The reason for that

may be that what “best” means may depend on an author’s particular objective. One way is to use the so-called *Farey tree*, and another is based on *continued fractions*. We will base our approach on the Farey tree structure and only comment marginally on the connection with continued fractions. We will use lots of properties of the rotation numbers in conjunction with the dynamics without proofs. All proofs would be elementary. Clues to the arithmetic part of the issue can be found in [6], while the dynamical properties are scattered throughout the literature. Some proofs and hints for others can be found in [12].

Farey trees.

Definition 1.1 *We define the structure of a directed graph whose vertices are exactly all rational numbers from $(0, 1)$.*

By definition, each vertex p/q has exactly two outgoing edges. One leads to a smaller number called the “left daughter”, and the other to a greater number called the “right daughter”.

If p/q is in the lowest terms, we determine $u < p/q < v$ defined as the closest neighbors in $[0, 1]$ with denominators not larger than q . Then, the left daughter is the rational number with the smallest denominator contained in $(u, p/q)$, while the right daughter is the number with the smallest denominator contained in $(p/q, v)$ (they are unique.)

It is known that the graph of this relation is a connected binary tree and $1/2$ is the root. This tree is called the Farey tree.

Coding. Thus, there is a one-to-one correspondence between a rational number mod 1 and a finite symbolic sequence (a_i) (the symbols are L for the left daughter and R for the right daughter) which tells us how to go to the number from the root.

Also, there is a unique infinite symbolic representation of every irrational number defined by the property that the rationals which correspond to finite initial segments of the code tend to the irrational number. The reader may try to compute the value of the alternating code $LRLR\dots$

The degree. For every symbolic sequence, we define inductively its *turning points* m_i . The first turning point m_1 is defined to be the least i so that $a_i \neq a_{i+1}$. If it does not exist, there are no turning points. Once m_j

has been found, m_{j+1} is the least $i > m_j + 1$ so that $a_i \neq a_{i+1}$. Again, if it does not exist, the turning point sequence ends. This definition implies that if what one would intuitively call a “turning point” occurs immediately after another turning point, it does not count as a turning point. This is required for compatibility with the “harmonic” description given a little later.

The *degree* of a rational number u (denoted $\text{deg}(u)$) is the number of turning points of the corresponding symbolic sequence plus one.

The reader is kindly asked to determine the degree of the number coded by ²

$$LRLRRRLRLR .$$

Closest returns. Consider an infinite symbolic sequence \mathcal{A} which codes an irrational number ρ . We consider the sequence q_i defined as denominators of consecutive rationals which correspond to symbolic sequences

$$a_1, \dots, a_{t_i}$$

where t_i are turning points of \mathcal{A} .

This sequence has a transparent interpretation in terms of the dynamics of the rotation by $2\pi\rho$. Namely, fix a point on the circle and consider the sequence of iterates which map this point closer to itself than any previous iterate. This turns out to be exactly the sequence q_i defined above. Because of this interpretation we will refer to q_i as the sequence of *closest returns* for ρ .

Farey domains and the harmonic subdivision.

Definition 1.2 *The interval $(P/Q, P'/Q') \subset [0, 1]$ is called a Farey domain if and only if either there is an edge between P/Q and P'/Q' in the Farey tree or it is one the three: $[0, 1]$, $[0, 1/2]$, or $[1/2, 1]$.*

Also, we will say that P/Q and P'/Q' are Farey neighbors if and only if $(P/Q, P'/Q')$ is a Farey domain.

Fact 1.1 *If P/Q and P'/Q' are Farey neighbors, then $|PQ' - P'Q| = 1$ and $1/2 \leq Q/Q' \leq 2$.*

²Five is the answer.

For every Farey domain we consider a sequence u_n where n ranges over all integers. If n is positive,

$$u_n := \frac{(n+1)P + P'}{(n+1)Q + Q'}.$$

For n non-positive

$$u_n := \frac{P + (1-n)P'}{Q + (1-n)P'}.$$

It is an elementary check that (u_n, u_{n+1}) are all Farey domains. We will call the collection of Farey domains of this form the *harmonic subdivision* of $(P/Q, P'/Q')$. The numbers u_n themselves can be called endpoints of the harmonic subdivision.

Harmonic coding. Start with the unit interval and consider its subdivision of level 1, that is its harmonic subdivision. Next, harmonically subdivide every domain of the subdivision of the previous level and so on. The Farey domains obtained on the k -th level will be called *fundamental domains* of level k . Now, consider the set of symbols $S(n, n+1)$ and $E(n)$ where n is an integer. If $S(n, n+1)$ occupies the k -th slot of the code, it refers to the interval (u_n, u_{n+1}) of the harmonic subdivision of the fundamental domain defined by the preceding part of the code. So, a fundamental domain of level k can be coded by a sequence of symbols of type $S(\cdot, \cdot)$ of length k . Endpoints of the domains of level k have a similar coding which ends with an $E(\cdot)$ symbol. Finally, irrational numbers can be coded by infinite sequences of symbols $S(\cdot, \cdot)$. This coding will be called *harmonic coding* and sometimes is more useful than the Farey coding. There is a close correspondence between the symbols in the harmonic code of a number and the coefficients of its continued fraction expansion. In particular, irrational numbers of constant type (i.e. with all continued fraction coefficients uniformly bounded) correspond to harmonic codes which consist of symbols $S(n, n+1)$ with $|n|$ uniformly bounded.

Miscellaneous properties. All numbers in the same fundamental domain of level k have the same closest returns up to q_{k-1} .

What all this means for our family of maps. Since our assumptions guarantee that the rotation number is a non-decreasing function of the parameter, all those objects from the realm of rotation numbers can be transported back to the parameter space. So we will talk of Farey domains and harmonic subdivisions in the parameter space as well.

There is a caveat, though. The rational numbers form an insignificant countable subset of the set of rotation numbers, but their preimage in the parameter space, called the *frequency-locking set*, is a huge set of full measure, as it was demonstrated in [12]. What happens is that the rotation number does not always grow with the parameter. It makes stops on all rational numbers but never on irrationals.

1.3 Harmonic Scalings

With the comments we have made so far, we hope to have explained the main purpose of the paper as stated in the abstract.

The main technical lemma of the paper will concern the harmonic scalings in the parameter space. To explain the notion we have to go back to our construction of the harmonic subdivision. The elements of the subdivision accumulate to the endpoints of the parent Farey domain. Exactly how fast their sizes decrease is the question of harmonic scalings.

It has long been known that the decrease is governed by a cubic law. The earliest mention we found in the literature is [9]. The first mathematically rigorous work which established the result was [8]. However, the estimates were non-uniform, i.e. it was proved that the scalings are indeed asymptotically cubic near every frequency-locking interval, but no estimate on how long one should wait to see the asymptotics take over in each particular case.

The saddle-node phenomenon. To fix the notations, let us concentrate on a Farey domain $(P/Q, P'/Q')$.

We can assume $Q < Q'$. Indeed so, because the map is symmetric with respect to the choice of an orientation. More precisely, instead of our family f_t we could consider a family ϕ_t given by

$$\phi_t(x) = -f_{-t}(-x)$$

It is easy to check that this operation means changing the direction in the parameter space and the orientation on the circle. The rotation number of ϕ_{-t} is going to be equal to $1 - \mathbf{rot}(f_t)$. The new family ϕ_t still satisfies our assumptions, but because the rotation numbers have been flipped around one half, so has been the Farey tree.

What happens near the lower extreme of $\rho^{-1}(P/Q, P'/Q')$. Directly below $\rho^{-1}(P/Q, P'/Q')$ there is a frequency-locking interval which belongs to P/Q . The most interesting point for us is the upper boundary of this frequency-locking. This parameter value will be denoted by t_0 . According to the general theory (see [8]) f_{t_0} is structurally unstable, even within the family. It still has a periodic point of period Q , which must be neutral, but any increase of the parameter value will destroy it, while any decrease will create two periodic points in its place.

The graph of f_{t_0} is tangent to the diagonal so that all non-periodic orbits are attracted to the neutral orbits. When the parameter value increases, the graph is pushed up and a funnel opens between the graph and the diagonal.

How to measure the scalings. Let us take a closer look at the situation for a parameter value t just a little above t_0 and on the interval between 0 and the nearest critical point of f_t^Q on the right of 0. This critical point must be a preimage of 0 and we denote it $f_t^{-q}(0)$ where $q = Q' - Q$ is the previous closest return common to all maps from $\rho^{-1}(P/Q, P'/Q')$.

Since its rotation number is a little greater than P/Q , f_t^Q moves the points a little to the right. Thus, the critical point, for example, moves to the right every Q iterates. Finally, it will leave the interval $(0, f_t^{-q}(0))$ and the number of steps it takes tells us exactly which domain of the harmonic subdivision we are in. So one way to determine the scalings could be to measure the interval in the parameter space between t_k for which the image of 0 by $f_{t_k}^Q$ hits $f_{t_k}^{-q}(0)$ after k steps and t_{k+1} where the same requires $k + 1$ steps.

This is the idea we will follow.

The scalings near the upper endpoint of $\rho^{-1}(P/Q, P'/Q')$ must follow the same rules. This is a rather trivial reduction. We can consider

the Farey domain $((P + P')/(Q + Q'), P'/Q')$ and then flip the Farey tree as described above. What we get is a Farey domain $(1 - P'/Q', 1 - \frac{P+P'}{Q+Q'})$ and now what used to be the scalings near the top of $\rho^{-1}(P/Q, P'/Q')$ now is equal to corresponding scalings at the bottom of the new domain.

So, will only consider the scalings near the lower extreme of the Farey domain, but the results will automatically extend to the upper scalings as well.

The crucial role of the funnel. The key observation made by the authors of the earlier works is that the decisive factor in estimating the scalings is the time it takes the image of 0 to go through the funnel. There are two reasons for that. The first is that the image of 0 spends most of its time in the funnel; the other is that as we consider the scalings in a very close proximity of the end of $\rho^{-1}(P/Q, P'/Q')$, the corresponding changes of the parameter are so tiny that they only bring about minute modifications to the orbit of 0 in the region away from the funnel. The main factor which effects the orbit is the change in the funnel clearance.

To prove what has been said here and study the effect of the funnel clearance on the orbit was the main achievement of both [8] and [4]. The work [8] studied this effect for the critical maps, but at that moment it was very hard technically to get uniform estimates, in particular independent of the degree of the Farey domain $(P/Q, P'/Q')$. In the meanwhile, [4] provided estimates which were uniform in this sense, but only applied to families of diffeomorphisms. Now we are finally able to give uniform estimates for the critical maps as well.

1.4 Notations

Uniform constants. Letters K with a subscript will be reserved for “uniform constants.” If we claim a statement which involves such constants we mean precisely that for each occurrence of a constant K , a positive value can be inserted which will make the statement true. The choice is uniform in the sense that once f has been fixed, there is a choice of values which makes the statement true in all cases covered.

We do not claim that the same value should always be inserted for the same constant. To avoid complete confusion, we only promise that in each

particular statement we will mean to substitute the same value for all occurrences of a constant. Thus, a statement $K_1 < K_1$ is formally true in our convention, but not in accord with our promise.

Finally, if $a < b < c < d$, the *cross-ratio* of these points is defined by

$$\mathbf{Cr}(a, b, c, d) := \frac{|b - a||d - c|}{|c - a||d - b|}.$$

The rate of change of f^Q depending on the parameter.

Proposition 1 *We define a set $S \subset T \times \mathbf{S}^1$ as follows. The t component ranges between the lower extreme of the frequency-locking P/Q and the lower endpoint of the frequency-locking of P'/Q' . Once t has been fixed, x must belong to $[0, F_t^{-q-Q}(0)]$. Then, if (t_1, x_1) and (t_2, x_2) belong to S , the ratio*

$$\frac{\frac{\partial f^Q}{\partial \tau}(t_1, x_1)}{\frac{\partial f^Q}{\partial \tau}(t_2, x_2)}$$

is uniformly bounded and bounded away from 0.

Comment. This is a new estimate. It was apparently unknown when [12] was written, as that paper instead uses a very complicated and round-about method in order to obtain inequalities in the parameter space. The main source of a tremendous technical progress that has been made in the past four years is the “real Koebe lemma” of [5] and its derivatives like the Distortion Lemma that we use here.

The role of negative Schwarzian derivative. We do not want to assume that our function f has negative Schwarzian derivative. However, there is a remarkable, though not hard, fact that high iterates of our functions already have negative Schwarzian. We will use this fact in some of our future estimates, which will, therefore, be valid only for a large enough number of iterates.

The idea that high iterates become negative Schwarzian maps is certainly not new and has been known to people working in the field. However, we are not aware of any proof in the literature. One reason for that may be that it is unclear how to formulate this result in reasonable generality. Our lemma does not pretend to be general, but the reader will see from the proof that an analogous argument will work in many other situations.

Rescaling. There are few uniform estimates on higher order derivatives for high iterates. However, it is often possible to get estimates if the map is properly rescaled. On the formal level that means that we take an arc of the circle and an iterate of the function which maps a part of this arc into the arc. Next, we conjugate it affinely, usually so that the length of the arc becomes one. We will refer to this operation most frequently as to “changing the unit of length”. In most cases that is all we want to say, and we will preserve original notations.

Proposition 2 *Assume that the degree of P/Q is larger than a uniform K_1 . Consider a parameter value $t > t_0$ from the Farey domain domain $u(P/Q, P'/Q')$ and change the unit of length so that the length of $(0, f_t^{-q}(0))$ becomes 1.*

Then, $Sf_t^Q \leq -K_2$ on

$$(F_t^{-Q}(0), F_t^{-q}(0)) \setminus \{0\} .$$

Corollary to Proposition 2.

Fact 1.2 *Consider a map F_t whose rotation number belongs to a fundamental domain of degree l . Assume that for some $i \leq q_l$, the iterate F_t^i does not have critical points on an open interval J . Then, F_t^i on J can be as*

$$F_t^i = F_t^{i_1} \circ N \circ F_t^{i_2}$$

where i_1 and i_2 are uniformly bounded while the Schwarzian of N is negative.

Proof:

Choose k so that for

$$\left(\frac{p_k}{q_k}, \frac{p_{k-1} + p_k}{q_{k-1} + q_k} \right)$$

Proposition 2 holds. This can be done in a uniform fashion.

To preserve familiar notations, denote $Q = q_{k+1}$ and $q = q_k$. Then, we consider intervals $I_1 = (F_t^{-Q}(0), F_t^{-q-Q}(0))$ and $I_2 = (F_t^{-q-Q}(0), F_t^{-q}(0))$. On $I_1 \cup I_2$, the first return time of F_t is either the Q on I_1 , or q on I_2 .

We pick i_1 as the first moment when the image of J hits $I_1 \cup I_2$. Then, i_2 is the largest so that $F_t^{i_2}(J)$ contains one of the following points: 0, an endpoint of I_1 , or an endpoint of I_2 . It is clear that i_1 and i_2 are uniformly

bounded as k was uniformly bounded. Then, all other iterates can be accounted for by composing the pieces of the first return map on $I_1 \cup I_2$, whose Schwarzian is negative by Proposition 2.

□

The nonlinearity lemma. For a function h , we introduce a quantity $nh := h''/h'$, also called *nonlinearity* of h .

Proposition 3 *Choose a parameter t from the Farey domain and change the unit of length as in Proposition 2, i.e. so that $F_t^{-q}(0)$ is in the unit distance from 0. Then, we obtain the following estimate on the rescaled map:*

$$nF_t^Q < K_1$$

on $(F_t^Q(0), F_t^{Q-q}(0))$.

If, in addition, we also assume that the claim of Proposition 2 holds, we get two more estimates

•

$$SF_t^Q > -K_2, K_2 > 0$$

on the same interval,

• If $|F_t^Q(x) - x| < K_3$, then

$$nF_t^Q(x) > K_2 > 0.$$

1.5 Prerequisites

We will review basic facts on which the proof will be based.

Bounded geometry.

Fact 1.3 *Let F be a map from our family with the rotation number in $(P/Q, P'/Q')$ so that $0 < q = Q' - Q$. There is a uniform positive constant K_1 for which the following estimates hold:*

- For any x ,

$$K^{-1} < \frac{|x - F^Q(x)|}{|x - F^q(x)|} < K_1 ,$$

and the same holds with Q replaced by Q' .

- For $x = F^i(0)$, $i = -q, \dots, q$,

$$K^{-1} < \frac{|x - F^Q(x)|}{|x - F^q(x)|} < K .$$

and the same holds with Q replaced by Q' .

Unfortunately, this fact belongs to the “folk wisdom”, and there is no clear reference to the proof. If the orbit of the critical point is periodic, Fact 1.3 was proved in [12]. Then, it was shown by M. Herman how to generalize the argument so as to include maps with irrational dynamics as well (see [7].)

The Distortion Lemma.

Fact 1.4 *Let f belong to our family. Consider f^{-m} for some $m > 0$ restricted to an interval $J = (a, b)$ and rescaled so that $|a - b|$ becomes the unit of length. If the following are satisfied:*

-

$$\frac{df^m}{dx} > 0 ,$$

- intervals $J, f(J), \dots, f^{m-1}(J)$ are all disjoint,

then for $x \in J$, the uniform estimate

$$\left| \frac{f''(x)}{f'(x)} \right| < K_1 \max(1/|a - x|, 1/|b - x|)$$

holds.

Proof:

This estimate follows from the Uniform Bounded Nonlinearity Lemma of [13].

□

2 Proofs of Propositions

2.1 Proof of Proposition 1

The key tool that we will use more than once in this paper is an approximate representation of the parameter derivative in terms of the lengths of dynamically defined intervals in the phase space.

Lemma 2.1 *If $(t, x) \in S$, then*

$$\left| \log\left(\frac{\partial f_t^Q}{\partial t}(x)\right) - \log\left(\sum_{i=1}^Q \frac{|F^{q+Q}(x) - F^Q(x)|}{|f^i(x) - f^{i+q}(x)|}\right) \right| \leq K_1 .$$

Proof:

The time derivative has the form

$$\sum_{i=1}^Q \frac{df^{Q-i}}{dx}(f^i(x)) .$$

By the Distortion Lemma, we can replace the derivatives by the ratios of nearby small intervals with a bounded error and that is all that the lemma claims. That the Distortion Lemma can be used to give uniform estimates can be checked with bounded geometry. □

Lemma 2.2 *Let t_1 and t_2 belong to a Farey domain $(P/Q, P'/Q')$ union with two adjacent frequency-lockings, and x be any point. Then*

$$K_1^{-1} < \frac{|F_{t_1}^q(x) - x|}{|F_{t_2}^q(x) - x|} < K_1 .$$

Proof:

Let us assume that $t_1 < t_2$. We fixed the configuration so that the q -th iterate moves points a little to the left. We only need to prove that $|F_t^q(x) - x|$ does not grow to much as t decreases from t_2 to t_1 . Suppose that indeed it does grow a lot, and we will show a bound. There is the point $F_{t_2}^{3q}(0)$ somewhere in the distance from x which is comparable to the distance from x to $F_{t_2}^q(x)$. For combinatorial reasons, $F_{t_1}^q(x)$ must be on the right of $F_{t_2}^{3q}(x)$. Indeed, suppose that x needs k iterates of $F_{t_2}^q$ to overcome the previous closest return. Then, only $k - 2$ or less iterates of $F_{t_2}^q$ would be needed which clashes with our assumption about rotation numbers.

□

Lemma 2.3 *Let t belong to $\rho^{-1}(P/Q, P'/Q')$. Choose x and y so that*

$$y \in [x, F_t^Q(x)] .$$

Then, the ratio

$$\frac{|F_t^Q(x) - x|}{|F_t^Q(y) - y|}$$

is uniformly bounded and bounded away from 0.

Proof:

This follows by the first statement of bounded geometry. We observe the ordering of points $F_t^{-Q}(x), F_t^{-Q}(y), x, y, F_t^Q(x), F_t^Q(y)$ which can only be preserved if the claim of the Lemma holds.

□

Finally, we get this estimate:

Lemma 2.4 *Let t_1 and t_2 be contained in $\rho^{-1}[P/Q, P'/Q']$. Then, the time derivatives of $f_{t_1}^Q(0)$ and $f_{t_2}^Q(0)$ are comparable within uniform constants.*

Proof:

We first use Lemma 2.1 to convert the parameter derivatives to sums of ratios of intervals. Next, we need to show that any the lengths of any two corresponding intervals in both sums are comparable within uniform constants.

Thus, consider $(F_{t_1}^i(0), F_{t_1}^{i+q}(0))$ and $(F_{t_2}^i(0), F_{t_2}^{i+q}(0))$. First, we can use Lemma 2.2 to compare the lengths of $(F_{t_1}^i(0), F_{t_1}^{i+q}(0))$ and $(F_{t_2}^i(0), F_{t_2}^{i+q}(0))$ instead where t is conveniently chosen in $\rho^{-1}(P/Q, P'/Q')$. But those intervals overlap, so Lemma 2.3 concludes the proof.

□

The proof of Proposition 1 In view of Lemma 2.4, the only thing that remains to be shown is the uniformity of change with respect to x with t fixed. By Lemma 2.1 this comes down to estimating the ratios of

$$\frac{|F_t^{i+q}(x) - F_t^i(x)|}{|F_t^{i+q}(0) - F_t^i(0)|}.$$

We need t in the preimage of the a Farey domain bounded by p/q to conclude the argument by Lemma 2.3. Since the left daughter of p/q is at least P/Q , we can do this unless $t \in \rho^{-1}(P/Q)$. If so, we can repeat the argument of Lemma 2.4 of first replacing t and using Lemma 2.2, and then Lemma 2.3.

2.2 Proofs of propositions about the Schwarzian

Basic lemmas. The Schwarzian derivative of a C^3 local diffeomorphism is given by:

$$Sf := (f''/f')' - \frac{1}{2}(f''/f')^2.$$

There is this remarkable formula for the Schwarzian of a composition:

$$S(f \circ g) = Sf \circ g \cdot (g')^2 + Sg$$

which for iterates of f becomes:

$$Sf^n = \sum_{i=0}^{n-1} (Sf \circ f^i) \cdot ((f^i)')^2.$$

Let F be one map from our family. Consider an iterate F^n on an interval (a, b) . For any point $x \in (a, b)$ we look at S_+F^n defined to be

$$S_+F^n := \sum_{i=0}^{n-1} \max(0, SF \circ F^i)(F^i)'^2.$$

In this situation a have a lemma:

Lemma 2.5 *If the following conditions are satisfied:*

- *there is a larger interval $(a', b') \supset (a, b)$ so that the derivative of F^n does not vanish on (a', b') ,*

- all intervals $(a, b), \dots, F^{n-1}(a, b)$ are disjoint,
- $|(a, b)|$ is taken as the unit of length,

then S_+F^n on (a, b) is bounded by a constant which only depends on $\mathbf{Cr}(a', a, b, b')$ times

$$\max\left\{\frac{|F^i(a, b)|}{|C|} : 0 \leq i \leq n - 1\right\}$$

where $|C|$ is the length of the whole circle.

Proof:

We will now use the length of the circle as our unit. Then, we want to use the affine map A which maps the unit interval to (a, b) and consider

$$S_+(F^n \circ A) .$$

By the Distortion Lemma, each derivative in the formula defining S_+ is comparable with the square of the length of the corresponding image of (a, b) , and the error is bounded proportional to $\mathbf{Cr}(a', a, b, b')$. On the other hand, the Schwarzian derivatives are bounded. Since the sum of lengths of all images is less than 1, the lemma follows.

□

Proof of Proposition 2 For simplicity, we will denote $F := F_t$. We will use the length of the circle as a unit.

First, we will consider F^q on the interval $(F_t^{-Q}(0), F_t^{-q}(0))$. F clearly has negative Schwarzian derivative where defined which is bounded away from 0. Then, $G := F^{q-1}$ on the image satisfies the assumptions of Lemma 2.5 (which follows from bounded geometry.) If A_1 and A_2 are affine and map the unit interval onto $(0, F^{-q}(0))$ and on its image by F respectively, we know that

$$S_+(G \circ A_2)$$

is small, thus we need to bound the derivative of $A_2^{-1} \circ F \circ A_1$, which clearly is bounded by something close to l (the critical point order.) The quantity $S_+(G)$ can be bounded by Lemma 2.5. Since the lengths of images of

$(0, F^{-q}(0))$ are exponentially small in terms of the degree of P/Q , the composition $G \circ F \circ A_1$ has negative Schwarzian bounded away from zero provided that the degree is sufficient. The same argument works for the composition of F followed by F^{Q-q-1} on $F^q(0, F^{-q}(0))$. The Proposition follows.

Proof of Proposition 3. The first statement follows immediately from the Koebe distortion lemma and bounded geometry.

The key to the other two statements is the identity

$$Dng = Sg + 1/2(ng)^2 .$$

Since by the first claim nf_t^Q is bounded, Sf_t^Q must satisfy some lower bound at least for some points from $(f_t^Q(0), f_t^{Q-q}(0))$, or the nonlinearity would be strongly decreasing, and, therefore, could not be uniformly bounded. However, the values of Sf_t^Q are comparable within uniform multiplicative factors for all point of that interval. That can best be seen from the chain expansion

$$Sf_t^Q(x) = \sum_{i=1}^Q Sf_t^i(f_t^{i-1}(x))(Df_t^{i-1}(x))^2 .$$

By rescaling using affine maps of bounded slope, we can assume that the x is at 0, while $f_{t_0}^{-q}(0)$ is at 1. We then consider the class \mathcal{G} of diffeomorphisms defined on $[0, 1)$ which satisfy these conditions:

1. Their Schwarzian derivatives are negative and bounded away from 0 by some $-\beta$.
2. For any $g \in \mathcal{G}$, $g(0) = |F_t^Q(x) - x|$ and $\lim_{y \rightarrow 1} g(y) = 1$.
3. If $g(x) \geq x$.

We will prove that $ng(0)$ is uniformly bounded away from 0 in \mathcal{G} . The class \mathcal{G} can be identified with the set of pairs. One element of every pair is a function bounded from above by $-\beta$, and the other is a non-negative number. The function gives the Schwarzian derivative, and the number can be taken as $ng(0)$. By the classical Schwarz result, these together with the second condition will determine a unique function which is in \mathcal{G} if and only if the last condition is satisfied. We then see that if g_1, n is in \mathcal{G} , then g_2, n is also in \mathcal{G} provided that $g_1 \leq g_2 \leq -\beta$. That is because the corresponding

solutions to the Schwarz problem will also satisfy the inequality. So, we can restrict our attention to maps $-\beta, n$. By a compactness argument, unless n is uniformly bounded away from 0, there is a map in \mathcal{G} with $n = 0$. But it can be explicitly shown, that such a map cannot satisfy the third condition if K_3 was chosen small enough in the statement of the Proposition.

3 Scalings rules

3.1 Notations and the main result

Throughout this chapter, we fix a Farey domain $(P/Q, P'/Q')$. As explained in the Introduction, we may adopt the convention $0 < q = Q' - Q$. We will denote the frequency-locking interval which belongs to a rational number u with $\alpha(u)$. The frequency-locking intervals which bound $\rho^{-1}(P/Q, P'/Q')$ from below and from above are $\alpha(P/Q)$ and $\alpha(P'/Q')$. For every rational u , within $\alpha(u)$ there is a unique point $c(u)$, called the *center* of the corresponding frequency-locking, and characterized by the property that the critical point is periodic.

Next, we consider the sequence u_n of endpoints of the harmonic subdivision of $(P/Q, P'/Q')$.

We define

$$J_n := (u_{n+1}, u_n), \quad \text{and}$$

$$J := (c(P/Q), c(P'/Q')).$$

Definition 3.1 Harmonic scalings, are the ratios

$$h_n = \frac{|J_n|}{|J|}$$

for $n \in \mathbf{Z}$.

Our main result about harmonic scalings is contained in the following proposition:

Proposition 4 *Harmonic scalings decrease no faster than according to a uniform cubic law, i.e.*

$$h_n \geq \frac{K_1}{|n|^3 + 1},$$

where K_1 does not depend on \mathcal{D} .

If, in addition, the claim of Proposition 2 holds, then

$$h_n \leq \frac{K_2}{|n|^3 + 1},$$

The rest of this section will be devoted to the proof of Proposition 4.

3.2 First estimates

As it was noticed in section 1, it is enough to estimate h_n for n positive, since we can use the Farey domain $((P + P')/(Q + Q'), P'/Q')$ and then flip the Farey tree. We define

$$t_n := c(u_n)$$

and $t_\infty = c(P/Q)$, $t_{-\infty} = c(P'/Q')$.

As t moves from t_{n+1} to t_n , $F_t^{(n+1)Q}(0)$ travels from $F_{t_{n+1}}^{-q-Q}(0)$ to $F_{t_n}^{-q}(0)$. Thus,

$$|J_n| \frac{df_{t=z}^{(n+1)Q}(0)}{dt} = |F_{t_{n+1}}^{-q-Q}(0) - F_{t_n}^{-q}(0)|$$

where z is given by the Mean Value Theorem, thus $z \in (t_{n+1}, t_n)$. An analogous argument shows that

$$|J| \frac{df_{t=w}^Q(0)}{dt} = |F_{t_{-\infty}}^{-q}(0) - 0|$$

this time with $w \in \alpha(P/Q) \cup \rho^{-1}(P/Q, P'/Q')$.

These equalities allow us to express the harmonic scalings in terms of ratios of time derivatives and lengths of relevant intervals. We will now work to make this relation as simple as possible.

Lemma 3.1 *The ratio of lengths*

$$\frac{|F_{t_{n+1}}^{-q-Q}(0) - F_{t_n}^{-q}(0)|}{|F_{t_{-\infty}}^{-q}(0) - 0|}$$

is uniformly bounded and bounded away from 0.

Proof:

The tools of the simple proof are Lemma 2.2 and bounded geometry. Since t_{n+1} and t_n are in frequency-lockings adjacent to a Farey domain (u_{n+1}, u_n) which belongs to the harmonic subdivision of $(P/Q, P'/Q')$, Lemma 2.2 can be applied with Q to see that $|F_{t_{n+1}}^{-q-Q}(0) - F_{t_n}^{-q}(0)|$ is uniformly comparable to $|F_{t_n}^{-q-Q}(0) - F_{t_n}^{-q}(0)|$. By bounded geometry, that is in a uniform relation to $|F_{t_n}^{-q}(0) - 0|$, and another application of Lemma 2.2 with q gives the claim. \square

In view of Lemma 3.1, h_n is uniformly comparable to

$$\frac{\frac{df^Q}{dt}|_{t=w}(0)}{\frac{df^{(n+1)Q}}{dt}|_{t=z}(0)}. \quad (1)$$

We now concentrate on estimating expression 1.

Estimates of time derivatives. We first consider the denominator. By bounded geometry and the Distortion Lemma, we see that

$$\frac{df^{(n+1)Q}}{dt}|_{t=z}(0)$$

is uniformly comparable to

$$\sum_{k=0}^{n-1} \frac{\partial f^Q}{\partial t}(z, f_z^{kQ}(0)) \frac{|F_z^{nQ}(0) - F_z^{(n+1)Q}(0)|}{|F_z^{kQ}(0) - F_z^{(k+1)Q}(0)|}.$$

Here, we replaced the spatial derivatives with ratios of intervals like in the proof of Lemma 2.1. Next, we use Proposition 1 to replace all parameter derivatives in this expression by

$$\frac{\partial f^Q}{\partial t}(z, 0),$$

again preserving comparability by uniform constants.

We see that expression 1 is uniformly comparable to

$$\frac{\frac{\partial f^Q}{\partial t}(w, 0)}{\frac{\partial f^Q}{\partial t}(z, 0)}$$

divided by

$$\sum_{k=0}^{n-1} \frac{|F_z^{nQ}(0) - F_z^{(n+1)Q}(0)|}{|F_z^{kQ}(0) - F_z^{(k+1)Q}(0)|}$$

The first factor is uniformly bounded and bounded away from 0 as a consequence of Lemma 2.4.

The results of this section are summarized by the following proposition:

Proposition 5 *For n positive, the harmonic scaling h_n is uniformly comparable to*

$$\frac{1}{\sum_{k=0}^{n-1} \frac{|F_z^{-q}(0) - 0|}{|F_z^{kQ}(0) - F_z^{(k+1)Q}(0)|}}.$$

Proof:

The proposition has almost been proven apart from the fact that

$$|F_z^{nQ}(0) - F_z^{(n+1)Q}(0)|$$

is comparable to $|F_z^{-q}(0) - 0|$. That, however, follows from bounded geometry and Lemma 2.3

□

3.3 Saddle-node estimates

In this section we will get uniform estimates for harmonic scalings depending on n . To this end, we will use Proposition 5 which makes this task equivalent to estimating the geometrically given sum.

It is sufficient to consider h_n with n large. This is a simple corollary to proposition 5. The formula given there gives values bounded away from 0 and infinity in a uniform fashion for any bounded n . Thus, the claim of Proposition 4 can be satisfied by choosing the uniform constants appropriately.

Normalization. Consider F_z^Q with $z \in J_n$ as in Proposition 5. Consider $\chi \in (0, F_z^{-q}(0))$, which does not have to be unique, so that

$$|F_z^Q(\chi) - \chi| = \inf\{|F_z^Q(x) - x| : x \in (0, F_z^{-q}(0))\}.$$

We change the coordinates by an affine map so that χ goes to 0 and $F_z^{-q}(0)$ goes to 1. The critical point then lands at some point c whose distance from 0 is uniformly bounded and bounded away from 0. In these coordinates F_z^Q becomes a map ϕ . By the Distortion Lemma, the second derivative of ϕ is bounded on a uniform neighborhood of 0.

Approximation rules. We say that ϕ satisfies the (α, κ) *upper approximation rule* if ϕ is not greater than the map

$$x \rightarrow x + \alpha x^2 + \phi(0)$$

on some interval $(-\kappa, \kappa)$.

Analogously, ϕ satisfies the (α, κ) *lower approximation rule* if there is the converse inequality.

Since the second derivative of ϕ is bounded on a uniform neighborhood of 0, there is a uniform choice of (α, κ) so that the upper approximation rule is always satisfied. In this sense we will say that ϕ satisfies the uniform upper approximation rule.

If the claim of Proposition 2 holds, there is also a uniform way to satisfy the lower approximation rule as a result of the estimate of Proposition 3. So, we will consider maps which satisfy the uniform lower approximation rule.

The approximating quadratic maps chosen in this uniform way will be called ϕ_u for the map given by the upper approximation rule, and ϕ_l for the other map.

The advantage of approximation rules is the orbits under quadratic maps can be examined more or less explicitly and the interesting quantities simply calculated. This was basically the idea of the authors who previously contributed to the subject, (see [9], also [8] and [4], which by no means exhaust the list as this trick was discovered independently a couple of times.) We will give only essential arguments and the reader will be able to find complete technical explanations in [4].

Lemmas about quadratic maps. Consider a function Φ

$$x \rightarrow x + \alpha x^2 + \epsilon$$

defined on $(-\kappa, \kappa)$. Also assume that $\alpha > \beta > 0$ for some β , while $\kappa > \gamma > 0$ for some γ , and $\epsilon > 0$. A *maximal orbit* is a sequence $(y_i)_{0 \leq i \leq l}$ for which $y_{i+1} = \Phi(y_i)$ and y_0 has no preimage while y_l is no longer in the domain. We have two facts about maximal sequences which are proved, though not explicitly stated, in [4].

Fact 3.1 *The length of a maximal sequence l and ϵ are related by*

$$K_1(\beta, \gamma)^{-1} \epsilon \leq \frac{\alpha}{l^2} \leq K_1(\beta, \gamma) \epsilon$$

where $K_1(\cdot, \cdot)$ is a positive function of β, γ only.

Fact 3.2 *Let l_0 be the number of points y_i from the maximal orbit which satisfy $y_{i+1} - y_i < 2\epsilon$. Then*

$$\frac{l_0}{l} > K_2(\beta)$$

where again $K_2(\cdot)$ is a positive function of β only.

Relating $\phi(0)$ and n .

Lemma 3.2 *If ϕ satisfies the uniform upper approximation rule,*

$$\frac{K_1^{-1}}{n^2} \leq \phi(0) .$$

Proof:

The number n is at least the length of a maximal orbit by ϕ_u . The claim then follows by Fact 3.1.

□

Proof of the first estimate of Proposition 4. There are n terms in the denominator of the formula given by Proposition 5. By Lemma 3.2, they are at most of the order of n^{-2} . The estimate on h_n from below follows.

The negative Schwarzian case. The other estimate of Proposition 4 is harder, but since we only claim it in the negative Schwarzian case, we can be aided by the strong claim of Proposition 3. Among other things, we now know that ϕ satisfies both uniform approximation rules. Also, ϕ' has exactly one minimum which must be attained on the right of 0. Consider $(-\kappa, \kappa)$ so that both uniform approximation rules hold.

Only what's inside $(-\kappa, \kappa)$ counts. We prove this lemma:

Lemma 3.3 *The number of iterates k so that $0 < k < n$ and $\phi^k(c) \notin (-\kappa, \kappa)$ is uniformly bounded. Moreover, for all such values of k , $\phi^{k+1}(c) - \phi^k(c)$ is uniformly bounded away from 0.*

Proof:

Actually, the first part of the claim obviously follows from the second. This is only a problem if there are many iterates k so that $\phi^k(c) \in (-\kappa, \kappa)$, otherwise $\phi(0)$ is bounded away from 0. Thus, by the approximation rules, if k_1 and k_2 are the smallest and the largest k so that $\phi^k(c) \in (-\kappa, \kappa)$, the distances $\phi_{k_1+1}(c) - \phi_{k_1}(c)$ and $\phi_{k_2+1}(c) - \phi_{k_2}(c)$ are uniformly large. But they are still smaller than the analogous distances for the values of k for which $\phi^k(c) \notin (-\kappa, \kappa)$ by the negative Schwarzian property.

□

That means that in the sum of Proposition 5 the contribution from the terms corresponding to values of k with the property that $\phi^k(c) \notin (-\kappa, \kappa)$ is uniformly bounded. On the other hand, it is clear that the whole expression grows at least as n^2 . Thus, for values of n uniformly sufficiently large, only the points of the orbit which are contained in $(-\kappa, \kappa)$ can be considered, and the result will approximate the whole sum up to a uniform multiplicative factor.

Essential estimates. Consider the smallest k so that $\phi^k(c) \in (-\kappa, \kappa)$, and denote this point with x_0 . Correspondingly, let x_l be the last point of the orbit still in $(-\kappa, \kappa)$, and in between we get a sequence which satisfies $x_{i+1} = \phi(x_i)$ for $i = 0, \dots, l-1$. We are interested in estimating the sum

$$\sum_{i=0}^{l-1} \frac{1}{x_{i+1} - x_i}. \quad (2)$$

from below.

Lemma 3.4 *In our situation,*

$$\frac{K_1^{-1}}{n^2} \leq \phi(0) \leq \frac{K_1}{n^2} .$$

Proof:

This is a stronger version of Lemma 3.2 under stronger assumptions, and the proof is really the same.

□

Estimating the sum 2 from below. Let y be the largest point of the orbit of c by ϕ still negative. Then (y_i^l) and (y_i^u) denote maximal orbits for ϕ_l and ϕ_u respectively which also contain y . The sum given by 2 is larger than the corresponding sum for (y_i^u) , as the intervals occurring in the latter are fewer and longer. An analogous argument shows that the sum for (y_i^l) bounds the interesting expression from above.

To estimate

$$\sum_i \frac{1}{y_{i+1}^u - y_i^u}$$

from below we have to use Lemma 3.4 which then asserts simply that $\phi(0)$ is comparable to n^{-2} . Then, Fact 3.1 implies that n is comparable to the length of the maximal orbit of both ϕ_u and ϕ_l .

Then, by Fact 3.2 we see that the number of intervals for ϕ_u with lengths not greater than $2\phi(0)$ is still comparable with the length of the maximal orbit, thus with n .

Thus, we get a uniform cubic estimate from below. Again, Proposition 5 immediately enables us to derive the second claim of Proposition 4.

4 Hölder continuity of the rotation number

In this section, we will prove Theorem A announced in the Introduction.

4.1 Some consequences of scalings rules

We will draw certain “Hölder type” estimates as consequences of Proposition 4. We want to emphasize that we need estimates everywhere on the parameter space and we are unwilling to assume that the denominators of Farey domains that we work with are large enough. So, only the first claim of Proposition 4 holds, i.e.

$$h_n \geq \frac{K_{1,P,4}}{|n|^3}.$$

Two estimates. We fix our attention on a Farey domain $(P/Q, P'/Q')$ subject to our usual convention $0 < q = Q' - Q$. We consider the harmonic subdivision by points u_n . Also, the centers of mode-locking intervals are denoted with t_n (n may be infinite) as in the Scalings Rules section.

Lemma 4.1 *Let $u = u_n$, $v = u_{n+1}$. Then,*

$$|u - w||c(u) - c(w)|^{-\alpha} \leq \beta |P/Q - P'/Q'| |c(P/Q) - c(P'/Q')|^{-\alpha}$$

with uniform $0 < \alpha, \beta < 1$

Proof:

We have the following estimate as a consequence of Proposition 4:

$$\frac{|\rho(t_{n+1}) - \rho(t_n)|}{|\rho(t_{-\infty}) - \rho_{\infty}|} \frac{|t_{-\infty} - t_{\infty}|^{\alpha}}{|t_{n+1} - t_n|^{\alpha}} \leq \frac{n^{3\alpha} Q Q'}{K_{1,P,4}^{\alpha} (nQ + Q')(n+1)Q + Q'}$$

(recall that $|QP' - PQ'| = 1$ as Farey neighbors.)

Consequently,

$$\frac{|\rho(t_{n+1}) - \rho(t_n)|}{|\rho(t_{-\infty}) - \rho_{\infty}|} \frac{|t_{-\infty} - t_{\infty}|^{\alpha}}{|t_{n+1} - t_n|^{\alpha}} < \frac{2(n+2)^{3\alpha-2}}{K_{1,P,4}^{\alpha}},$$

where we used $Q < Q' < 2Q$. By choosing α sufficiently close to 0 we can get the ratio on the right-side smaller than some number less than 1, say β .

□

Next, we want to generalize Lemma 4.1 to u, v arbitrary endpoints of the harmonic subdivision of $(P/Q, P'/Q')$.

Lemma 4.2 *Let u, v be arbitrary two endpoints of the harmonic subdivision of $(P/Q, P'/Q')$. or P/Q or P'/Q' . Then,*

$$|u - w||c(u) - c(w)|^{-\alpha} \leq K_1 |P/Q - P'/Q'| |c(P/Q) - c(P'/Q')|^{-\alpha}$$

with α uniform and positive.

Proof:

First, we note that it suffices to prove the lemma when $u = u_n$ and $w = u_{n'}$ with $n \cdot n' \geq 0$. Indeed, in the situation when both u and v are endpoints, but $n \cdot n' < 0$, we can consider n and 0 as well as n' and 0 separately. Then, if we sum up resulting inequalities and use convexity of $\frac{1}{x^\alpha}$, we can infer the claim of the lemma.

If, for example, u is P/Q , we can take the limit with u_n where n tends to $+\infty$. By continuity of the rotation function, this would give us almost the estimate of the Lemma, except that on the left-hand side $c(u)$ is replaced with t_0 which is the upper endpoint of $\rho^{-1}(P/Q)$. However, this is stronger than the estimate claimed by the lemma.

Furthermore, we can restrict our attention to $n, n' \geq 0$. We define m by $0 < m = n' - n$.

Again, we use Proposition 4:

$$\begin{aligned} & \frac{|\rho(t_{n+m}) - \rho(t_n)|}{|\rho(t_{-\infty}) - \rho_\infty|} \frac{|t_{-\infty} - t_\infty|^\alpha}{|t_{n+m} - t_n|^\alpha} \\ & \leq \frac{n^{3\alpha} Q Q'}{K_{1,P,4}^\alpha (nQ + Q')((n+m)Q + Q')} \left(\frac{1}{\sum_{k=n}^{n+m-1} 1/k^3} \right)^\alpha \leq \\ & \frac{2}{K_{1,P,4}^\alpha} \frac{m}{(m+n+1)(n+1)} \left(\frac{1}{1/n^2 - 1/(m+n)^2} \right)^\alpha \leq \\ & \frac{2}{n K_{1,P,4}^\alpha} \frac{n^{2\alpha} (n+m)^{2\alpha}}{m^\alpha (2n+m)^\alpha} \leq \frac{2^{2\alpha+1}}{K_{1,P,4}^\alpha} n^{2\alpha-1}. \end{aligned}$$

This expression is bounded by some K_1 if $\alpha \leq 1/2$.

□

4.2 Global estimates

Then, we let $u < w$ be arbitrary rational numbers from the unit interval. There is a unique simple path in the Farey tree from u to w . It contains the “highest” node V . This splits the path into two parts: from u to V and from V to w .

Then, we define maps μ and ν on the Farey tree. Given a rational number v , $\mu(v)$ is the rational number that corresponds to the initial segment of the symbolic code of v cut off at the last turning point (i.e. if the turning points are m_1, \dots, m_k , the symbolic sequence of $\mu(v)$ is a_1, \dots, a_{m_k} .)

Then, $\nu(v)$ is the mother of $\mu(v)$.

Clearly,

$$\deg(\mu(v)) = \deg(\nu(v)) = \deg(v) - 1 .$$

Furthermore, v lies strictly between $\mu(v)$ and $\nu(v)$, and v is in fact an end-point of the harmonic subdivision of $(\nu(v), \mu(v))$.³

The function ζ . If x and y are two rationals from the unit interval, we define

$$\zeta(x, y) := |x - y| \cdot |c(x) - c(y)|^{-\alpha} .$$

Lemma 4.2 gives us a fundamental estimate

$$\zeta(x, y) \leq K_1 \zeta(\mu(x), \nu(x)) \tag{3}$$

provided $\mu(x) = \mu(y)$.

If we iterate μ and ν , we get nested sequence of growing fundamental domains bounded by $\mu^i(x)$ and $\nu^i(x)$. For any i we have

$$\zeta(\mu^i(x), \nu^i(x)) \leq \beta \zeta(\mu^{i+1}(x), \nu^{i+1}(x)) \tag{4}$$

by Lemma 4.1.

Finally, we define a sequence $u_0 = v$ and u_i is equal to the greater of $\mu^i(v)$ and $\nu^i(v)$. Similarly, we define the sequence w_i so that $w_0 = w$ and w_i is the minimum of $\mu_i(w)$ and $\nu_i(w)$. Also, let l be the largest so that $w_l \geq V$.

We want to bound $\zeta(u, u_k)$. As $\zeta(v, v + v')$ is convex as a function of v' , and the sequence u_i is growing, clearly

$$\zeta(u, u_k) \leq \sum_{i=1}^k \zeta(u_{i-1}, u_i) . \tag{5}$$

³In this section, if we write (a, b) , we perhaps mean (b, a) when $a > b$.

Since u_i and u_{i+1} are contained between $\mu(u_{i+1})$ and $\nu(u_{i+1})$, we can estimate

$$\zeta(u_i, u_{i+1}) \leq K_1 \zeta(\mu(u_{i+1}), \nu(u_{i+1}))$$

by inequality 3. Then we iterate μ and ν on $\mu(u_{i+1})$ as many times as possible, which at least $\max(k - i - 1, 0)$. By the repeated use of inequality 4, we get

$$\zeta(\mu(u_{i+1}), \nu(u_{i+1})) \leq \beta^{\max(0, k-i-1)} \zeta(0, 1) .$$

Since $\zeta(0, 1) = 1$, we can finally get from estimate 5

$$\zeta(u, u_k) \leq K_2$$

by simply adding up a geometric progression.

The same argument shows that

$$\zeta(w_l, w) \leq K_2 .$$

Finally, by their definition $\mu(w_l) = \mu(u_k)$ so that they both are in the fundamental domain bounded by $\mu(u_k)$ and $\nu(u_k)$.

Again, we get

$$\zeta(u_k, w_l) \leq K_1 \zeta(\mu(u_k), \nu(u_k))$$

from estimate 3, and

$$\zeta(\mu(u_k), \nu(u_k)) \leq 1$$

by the repeated use of estimate 4.

Since

$$\zeta(u, w) \leq \zeta(u, u_k) + \zeta(u_k, w_l) + \zeta(w_l, w)$$

by convexity, we have proved that

$$\zeta(u, w) \leq K_3$$

with uniform K_3 , which means the Hölder estimate. That is, we have proven that

$$|\rho(x) - \rho(y)| \leq K_3 |x - y|^\alpha$$

provided that x and y are both centers of frequency-lockings.

The general Hölder estimate. By continuity, we also get the same Hölder estimate if x and y belong to the closure of the set of centers, i.e. to the complement of the union of interiors of all frequency-locking intervals. If $x < y$ are arbitrary, we consider x' which is the infimum of the set of centers which are between x , and y' is the supremum of the same set. They are well-defined unless x and y are in the same frequency-locking interval, in which case the estimate is evident. The Hölder estimate holds for x' and y' . Moreover, $\rho(x') = \rho(x)$ and $\rho(y') = \rho(y)$, while $|x' - y'| \leq |x - y|$. The Hölder estimate follows again.

5 Hausdorff dimension

We will prove Theorem B. In this section, we will use the harmonic formalism of explained in Introduction section. That is, we have a one-to-one coding of fundamental domains in the parameter space by finite sequences of S-type symbols. The length of the code will be called the *degree* of the corresponding fundamental domain.

The fundamental domain which corresponds to the code (n_1, \dots, n_r) will be denoted with $\mathcal{D}(n_1, \dots, n_r)$.

Let Ω' be $T \setminus \bigcup_{w \in \mathbf{Q}} \rho^{-1}(w)$.

It is sufficient to prove that the Hausdorff dimension of

$$\Omega' \cap \mathcal{D}(n_1, \dots, n_r)$$

satisfies our bounds for r large enough. That means that we can assume that Proposition 2 holds and, consequently, use both claims of Proposition 4.

5.1 The estimate of $\text{HD}(\Omega')$ from above

Take a cover of Ω' which consists of all fundamental domains of degree r .

Then

$$\sum_{n_1, \dots, n_r \in \mathbf{Z}, |n_r| > k} |\mathcal{D}(n_1, \dots, n_r)|^\beta \leq K_2(k) \sum_{n_1, \dots, n_{r-1}} |D(n_1, \dots, n_{r-1})|^\beta$$

if $\beta > 1/3$ as a consequence of scalings rules (Proposition 4.) Here, $K_2(\cdot)$ is a uniform positive function with limit 0 at infinity.

Since for $|n_r| \leq k$

$$K_1 < \frac{|\mathcal{D}(n_1, \dots, n_r)|}{|\mathcal{D}(n_1, \dots, n_{r-1})|}$$

by Proposition 4, by Young's inequality

$$\sum_{n_1, \dots, n_r \in \mathbf{Z}, |n_r| \leq k} |\mathcal{D}(n_1, \dots, n_r)|^\beta \leq k^{1-\beta} K_1^\beta \sum_{n_1, \dots, n_{r-1}} |D(n_1, \dots, n_{r-1})|^\beta .$$

Thus,

$$\sum_{n_1, \dots, n_r \in \mathbf{Z}} |\mathcal{D}(n_1, \dots, n_r)|^\beta \leq (k^{1-\beta} K_1^\beta + K_2(k)) \sum_{n_1, \dots, n_{r-1}} |D(n_1, \dots, n_{r-1})|^\beta .$$

We claim that $k^{1-\beta} K_1^\beta + K_2(k)$ can be made less than 1 by choosing β sufficiently close to, but less than, one. Indeed, remember that $K_1 < 1$. So, we first choose k so large that $K_2(k)$ is less than $0.5(1 - K_1)$. Then, by adjusting β we can make $k^{1-\beta} K_1^\beta$ arbitrarily close to K_1 .

Since the diameters of the fundamental domains tend to 0 with the degree, this β is not larger than the Hausdorff dimension of Ω' .

5.2 Estimate from below by 1/3.

The proof is based on the following Frostman's Lemma which we borrowed from [10]:

Fact 5.1 *Suppose that μ is a probabilistic Borel measure on the interval and that for μ -a.e. x*

$$\liminf_{\epsilon \rightarrow 0} \log(\mu(x - \epsilon, x + \epsilon)) / \log(\epsilon) \geq \lambda .$$

*Then the Hausdorff dimension of μ is not less than λ .*⁴

Take $\eta < 1/3$. By the scalings rules it is clear that a number k can be found independently of the fundamental domain $\mathcal{D}(n_1, \dots, n_r)$ so that

$$\sum_{n_1, \dots, n_r \in \mathbf{Z}, |n_r| \leq k} |\mathcal{D}(n_1, \dots, n_r)|^\eta \geq |\mathcal{D}(n_1, \dots, n_{r-1})|^\eta .$$

⁴Frostman's Lemma remains true if the \geq signs are replaced with \leq signs.

We now define μ as a limit of probabilistic measures. The measure μ_0 is just the Lebesgue measure on $\rho^{-1}(0, 1)$ properly scaled. To obtain μ_{i+1} , we consider all fundamental domains of degree i . If $\mathcal{D}(n_1, \dots, n_i)$ is one of those, the density of μ_{i+1} with respect to μ_i on $\mathcal{D}(n_1, \dots, n_i)$ equals

$$\frac{|\mathcal{D}(n_1, \dots, n_{i+1})|^{\eta-1}}{\sum_{n_{i+1}=-k}^{n_{i+1}=k} |\mathcal{D}(n_1, \dots, n_{i+1})|^\eta}$$

on fundamental domains of the harmonic subdivision with $|n_{j+1}| \leq k$, and is zero the mode-locking intervals which belong to the endpoints of the subdivision. The sequence has a limit which is supported on a set Ω_0 contained in Ω' . The set Ω_0 consists of the preimages in the parameter space of all irrationals with harmonic code symbols $S(n, n+1)$ with $n, n+1$ bounded by k as to absolute value. Moreover, by our choice of η we see that the denominator is not less than $|\mathcal{D}(n_1, \dots, n_{i-1})|^\eta$, thus the the density is not greater than

$$\frac{|\mathcal{D}(n_1, \dots, n_{i+1})|^{\eta-1}}{|\mathcal{D}(n_1, \dots, n_i)|^\eta},$$

therefore

$$\frac{\mu_{i+1}(\mathcal{D}(n_1, \dots, n_{i+1}))}{\mu_i(\mathcal{D}(n_1, \dots, n_i))} \leq \frac{|\mathcal{D}(n_1, \dots, n_{i+1})|^\eta}{|\mathcal{D}(n_1, \dots, n_i)|^\eta}.$$

So, by induction,

$$\mu_{i+1}(\mathcal{D}(n_1, \dots, n_{i+1})) \leq |\mathcal{D}(n_1, \dots, n_{i+1})|^\eta,$$

and since clearly

$$\mu(\mathcal{D}(n_1, \dots, n_{i+1})) = \mu_{i+1}(\mathcal{D}(n_1, \dots, n_{i+1})),$$

the same estimate holds for μ itself.

Take any small $\epsilon > 0$ and an $x \in \Omega_0$ and look for the largest r so that

$$\mu((x - \epsilon, x + \epsilon) \setminus \mathcal{D}(n_1, \dots, n_r)) = 0$$

for some n_1, \dots, n_r . Note that a finite r with this property always exists by topology.

Then

$$\mu((x - \epsilon, x + \epsilon)) \leq |\mathcal{D}(n_1, \dots, n_r)|^\eta$$

and 2ϵ is greater than the length of the some gap between domains of the harmonic subdivision of $\mathcal{D}(n_1, \dots, n_r)$ which have non- zero measure. Indeed, by the definition of r , the interval $(x - \epsilon, x + \epsilon)$ must be straddled between at least two such domains. If the size of the gap is denoted with γ , we get

$$\frac{\log(\mu(x - \epsilon, x + \epsilon))}{\log(\epsilon)} \geq \eta \frac{\log(|\mathcal{D}(n_1, \dots, n_r)|)}{\log(\gamma)}.$$

By Proposition 4, the lengths of both domains and $|\mathcal{D}(n_1, \dots, n_r)|$ are all related by uniform constants. But γ is not much smaller than either of them as a result of the estimates of [12]. So the logarithms differ by a bounded amount and their ratio tends to 1 as ϵ shrinks to 0.

So, if we pass to the limit with $\epsilon \rightarrow 0$ we see that the assumptions of Frostman's Lemma are satisfied, therefore the Hausdorff dimension of Ω_0 , which is larger than Ω' , is at least η . But η could have be chosen anything less than $1/3$. So, the estimate follows.

5.3 Justification of Remark B

Remark B follows from the fact that our estimates can be made uniform with respect to the family if we consider Farey domains of sufficiently large degree. Namely, it can be checked that all our estimates ultimately depend on bounded geometry and the Distortion Lemma.

We claim that as far as bounded geometry is concerned, for every family a uniform K_2 can be chosen so that if k which occurs in the statement of bounded geometry is greater than K_2 , the estimate becomes uniform with respect to the family. This is a known fact which was observed and explained in [11]. The reason is that “cross-ratio inequalities” (see [12]) which are the source of bounded geometry estimates become uniform if applied to very short intervals. This fact can be seen immediately from the “pure singularity property” of [13], but also follows easily from the much simpler Corollary to Proposition 2.

The Distortion Lemma also becomes uniform if applied to maps with rotation number of degree sufficiently large. This follows from the Corollary to Proposition 2. Indeed, the nonlinearity of the N part is uniformly bounded by the classical estimate of [5], while the other parts are also bounded if J is small, simply because the joint length of i_1 or i_2 images of J is small.

Thus, for every family the scalings rules become uniform with respect to the family on Farey domains of large degree. We want to emphasize that how large that degree should be depends on the family. However, since the Hausdorff dimension is an asymptotic quantity which we bound from scalings rules, it will be uniformly bounded away from 1.

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