

Periodic Orbits for Hamiltonian systems in Cotangent Bundles

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Abstract: We prove the existence of at least $cl(M)$ periodic orbits for certain time dependant Hamiltonian systems on the cotangent bundle of an arbitrary compact manifold M . These Hamiltonians are not necessarily convex but they satisfy a certain boundary condition given by a Riemannian metric on M . We discretize the variational problem by decomposing the time 1 map into a product of “symplectic twist maps”. A second theorem deals with homotopically non trivial orbits in manifolds of negative curvature.

0 Introduction

The celebrated theorem of Poincaré-Birkhoff states the existence of at least two fixed points for an area preserving map of the annulus $\mathbb{S}^1 \times [0, 1]$ which “twists” the boundaries in opposite directions.

In the 60’s, Arnold proposed a generalization of this theorem for a time 1 map F of a time dependent Hamiltonian of $\mathbb{T}^n \times B^n$ (where B^n is the closed ball in \mathbb{R}^n). While the Hamiltonian condition naturally generalizes the preservation of area, a linking of the boundary of each fiber (a sphere in \mathbb{R}^n) with its image by F in the boundary of $\mathbb{T}^n \times B^n$ was to replace the twist condition. Arnold [Ar1] conjectured that such a map has at least as many fixed points as a real valued function \mathbb{T}^n has critical points. The philosophy was that fixed points for symplectic maps should arise from Morse theory and not, say, from Lefschetz theory.

Later, in [Ar2], he explained how fixed points theorems on the annulus could be derived from theorems on the 2-torus, by glueing carefully two annuli together (see also [Ch1]). He thus transformed the problem to one of fixed points of symplectic maps on a **compact** symplectic manifold. This last conjecture, which asserts that the number of fixed point for the map is at

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least equal to the minimum number of critical points of a real valued function on the manifold, is what got to be well known as the Arnold conjecture.

However, it is unclear whether the glueing construction can be done (symplectically) in higher dimensions. Even if it could, one would (if one could) have to use existing proofs of the Arnold conjecture (e.g. [F2]), which we think are substantially harder than the techniques we use here (and do not deal with homotopically non trivial orbits as our Theorem 2 does).

In 1982, Conley and Zehnder [CZ 1] gave a first proof of the Arnold conjecture for the torus \mathbb{T}^{2n} . In the same article, they also gave a direct proof of Arnold's original conjecture on $\mathbb{T}^n \times B^n$.

However, they were not able to use the linking of spheres in its full generality. Their result remains crucial since it was the first non perturbational one in this direction. The boundary condition that they used is expressed on the Hamiltonian in the following way. Letting (q, p) be the coordinates on $\mathbb{T}^n \times B^n$ which is endowed with the canonical symplectic structure $dp \wedge dq$, they set:

$$(0.1) \quad H(q, p, t) = \langle Ap, p \rangle + \langle b, p \rangle \text{ for } \|p\| \geq K,$$

where $A = A^t$ is a non degenerate $n \times n$ matrix and $b \in \mathbb{R}^n$. This condition implies the linking of spheres at the boundary.

We propose here a version of this theorem on the cotangent bundle of an arbitrary compact manifold. We also find, in a second theorem, orbits of all free homotopy classes (and large enough period).

The bulk of this work was done as I was on a Postdoctoral position at the Forschungsinstitut für Mathematik, E.T.H. Zürich. I would like to express my deep gratitude to Prof. Moser and Prof. Zehnder for inviting me there. I had some invaluable discussions with them as well as with my companions Fredy Künzle, Boris Hasselblat, Frank Josellis, to whom I extend my thanks. I am very much indebted to Patrice LeCalvez, whose work is the starting point of mine.

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Finally, were it not for the narrow mindedness of the French immigration office, this work would have been joint with Augustin Banyaga. I dedicate this work to him.

1 Results and basic ideas

Let (M, g) be a compact Riemannian manifold. Define

$$B^*M = \{(q, p) \in T^*M \mid g(q)(p, p) = \|p\|^2 \leq C^2 < R^2\},$$

where R is the radius of injectivity of (M, g) . Let π denote the canonical projection $\pi : B^*M \rightarrow M$.

Theorem 1 *Let $F : B^*M \rightarrow B^*M$ be the time 1 map of a time dependent Hamiltonian H on B^*M , where H satisfies the boundary condition:*

$$H(q, p, t) = g(q)(p, p) \text{ for } \|p\| = C.$$

Then F has $cl(M)$ distinct fixed points and $sb(M)$ if they are all non degenerate. Moreover, these fixed points can all be chosen to correspond to homotopically trivial closed orbits of the Hamiltonian flow.

We remind the reader that $cl(M)$ is the cup length of M , which is known to be a lower bound for the number of critical points of any real valued function on M . Non degenerate means that no Floquet multiplier is equal to one. $sb(M)$ is a lower bound for the number of critical points for a Morse function on M .

Remark 1.2 It is important to note that, in the case where M has \mathbb{R}^n as covering space, Theorem 1 can be expressed for a lift of F . In this case, the radius of injectivity may be ∞ (e.g. for a metric close to a flat metric on the torus, or when M has a metric of negative curvature), and the set B^*M can be as big as one wants. Theorem 1 can then serve as a starting point to study Hamiltonian systems with asymptotic boundary conditions.

Theorem 2 *Let F be as in Theorem 1. If (M, g) is of negative curvature, then F has at least two periodic orbits of period d in any given free homotopy class, provided d is big enough. In particular, F has infinitely many periodic orbits in B^*M .*

Exactly how big d should be in Theorem 2 depends only on the metric. For a more precise statement, see section 7. Note also that if H is 1-periodic in time, periodic orbits of F correspond to periodic orbits of the Hamiltonian flow of H . Such a Hamiltonian system will then have infinitely periodic solutions in B^*M .²

² If H is not 1-periodic, periodic orbits of F will correspond to orbits of the Hamiltonian flow that come back to their starting point, but generally at an angle. One can find infinitely many of these orbits from Theorem 1, by applying it to time t map, $t \in (0, C)$, rescaling the metric each time.

Note the difference in the boundary conditions (1.1) and that of Conley-Zehnder (0.1) : theirs allow basically all pseudo Riemannian metrics that are completely integrable and constant. Ours only deals with Riemannian metrics, but with no further condition. Note also that the orbits they find are homotopically trivial. We refer the reader to [G1,2], [J] for the study of the homotopically nontrivial case for $M = T^*\mathbb{T}^n$ (the former with a method akin to that of this paper, the latter in the spirit of [CZ 1]).

The method used to prove Theorem 1 and 2 is quite different from that of Conley and Zehnder: whereas they use cut-offs on Fourier expansions, we decompose the time 1 map into “symplectic twist maps” to get a finite dimensional variational problem.

Symplectic twist maps are a natural generalization of monotone twist maps of the cotangent bundle of the circle (i.e. the annulus).

In short, a symplectic twist map is a diffeomorphism F from some neighborhood U of the zero section of T^*M onto itself with the property that $F^*pdq - pdq = dS$ for some S and that $(q, p) \rightarrow (q, Q)$ is a change of coordinates, where $F(q, p) = (Q, P)$.

To give an example, we make the following trivial remark. The shear map of the annulus :

$$(q, p) \rightarrow (q + p, p),$$

which is a key model in the twist map theory, is nothing more than the time 1 of the Hamiltonian $H_0(q, p) = \frac{1}{2}p^2$ and in fact, its first coordinate map:

$$\begin{aligned} T^*\mathbb{S}^1 &\rightarrow \mathbb{S}^1 \\ (q, p) &\rightarrow q + p \end{aligned}$$

is just the exponential map for the standard (flat) metric on $\mathbb{S}^1 \times \mathbb{R} = T^*\mathbb{S}^1$.

This suggest that the key model for symplectic twist maps on the cotangent bundle T^*M of a general compact manifold M should be the time one map of a metric. The twist condition is given in that case by the fact that the exponential map is a diffeomorphism of a neighborhood of zero in each fiber T_q^*M and a neighborhood of q in M . Of course, most of the time, such a map is not completely integrable.

If F is a symplectic twist map, we have a simple proof of the original conjecture of Arnold:

Theorem 3 (*Banyaga, Golé [BG]*) *Let F be a symplectic twist map of B^*M . Suppose that each sphere ∂B_q^*M links with its image by F in ∂B^*M . Then the fixed points of F are given by the critical points of a real valued function on M .*

In Appendix B, we reproduce the proof of [BG], for the convenience of the reader. As in all these questions about fixed points, the major task is to make the argument global: symplectic twist maps should be seen as

local objects (even though they should not be seen as perturbations) and the problem is to piece them together to form global ones. Here is one fundamental principle involved in this.

Suppose we have two “exact symplectic maps”:

$$F^*pdq - pdq = dS \text{ and } G^*pdq - pdq = dS'$$

Then it is simple to see that:

$$(F \circ G)^*pdq - pdq = d(S \circ G + S')$$

which we express as : generating functions add under compositions of maps. This simple fact is key to the method in this paper: the functional we use is a sum of generating functions of a finite sequence of twist maps that decompose the time 1 map we study.

This additivity property is the common thread between the method exposed here and that of “broken geodesics” reintroduced in symplectic geometry by Chaperon [Ch2]. The essential difference is in the choice of coordinates in which one expresses the generating function $:(p, Q)$ in the method of Chaperon, (q, Q) in the twist map method. In this sense the twist map method is closer to the original method of broken geodesics as described in [Mi]. It even coincides with it in the case of the geodesic flow.

Whereas Moser [Mo2] noticed that the time 1 map of a two dimensional convex Hamiltonian can be decomposed into a product of twist maps, the idea of decomposition of a time 1 map of a general Hamiltonian stems from the work of LeCalvez [L] on twist maps of the annulus. We generalize his simple but extremely efficient construction to any cotangent bundle (Lemma (3.4)).

There are various theorems on the suspension of certain classes of symplectic twist maps by Hamiltonian flows ([D], [Mo2], [P-B]). In this sense, one might decide to forget about symplectic twist maps and concentrate on Hamiltonian systems instead. In this paper, we take the opposite point of view: we think that symplectic twist maps are a very useful tool to study Hamiltonian systems on cotangent bundles (see also the work of LeCalvez [L] on the torus).

The rest of the paper is organized as follows:

In section 2, we review some facts about geodesic flows and exponential maps. We prove a lemma which is crucial for the construction of an isolating block in section 4.

In section 3 we give a precise definition of symplectic twist maps and prove the Decomposition Lemma (3.4).

In section 4, we use this decomposition and the additive property of generating functions to construct a finite dimensional variational problem, i.e. a functional W on a finite dimensional space. This method is basically Aubry’s ([Au], [Ka]), when seen on maps of the annulus.

In section 5, we construct an isolating block for the functional W . For this, the boundary condition in the theorem is crucial.

In section 6, we make use of a theorem of Floer [F1] on global continuation of normally hyperbolic invariant sets: we exhibit such an invariant set for the time 1 map of H_0 whose cohomology survives under a deformation to our H . We then use the Conley-Zehnder Morse theory to finish the proof of Theorem 1.

In section 7, we show how to adapt the proof of theorem 1 to the case of non trivial homotopy classes, and prove theorem 2.

In Appendix A, we outline the connection that there is between the index of the Hessian of W and the Floquet multipliers along a closed orbit of F . This is used in sections 5 and 7 to prove normal hyperbolicity of the invariant set.

In Appendix B, we reproduce the proof of Theorem 3, given in [BG].

1 A few facts about the geodesic flow

We start with some notation. Let (M, g) be a Riemannian manifold. Both the tangent fiber $T_q M$ and the cotangent fiber $T_q^* M$ are endowed with bilinear forms:

$$\begin{aligned} (v, v') &\rightarrow g(q)(v, v') \text{ for } v, v' \in T_q M, \text{ and} \\ (p, p') &\rightarrow g^\#(q)(p, p') \text{ for } p, p' \in T_q^* M. \end{aligned}$$

We will denote by

$$\|v\| := \sqrt{g(q)(v, v)}, \text{ and } \|p\| := \sqrt{g^\#(q)(p, p)},$$

hoping that the context will make it clear whether we speak about a vector or a covector.

The relation between g and $g^\#$ is better understood in local coordinates: If $A(q)$ denotes the matrix for $g^\#$ then $A^{-1}(q)$ is the matrix for g . The terms of these matrices are usually denoted g^{ij} and g_{ij} respectively. The matrix $A(q)$ also gives the standard (although g -dependent) isomorphism between $T_q^* M$ and $T_q M$, which is an isometry for the above metrics. We will use the same notation “ $A(q)$ ” for this isomorphism, even though it is coordinate independent, whereas the matrix is not.

We want to outline here some connections between the geodesic flow for the metric g , the exponential map and the Hamiltonian flow for the Hamiltonian:

$$H_0(q, p) = \frac{1}{2} \|p\|^2.$$

Let T^*M be given the usual symplectic structure $dp \wedge dq$, and canonical projection π . Let h_0^t denote the time t map of the Hamiltonian flow of H_0 . Then:

$$\exp_q(tA(q)p) = \pi(h_0^t(q, p)),$$

This is basically a rewording of the equivalence of Hamilton's and Lagrange's equations under the Legendre transformation. Here H_0 and $L_0(q, \dot{q}) = \frac{1}{2}\|\dot{q}\|^2$ are Legendre transforms of one another under the change of coordinate $\dot{q} = \partial H_0 / \partial p = A(q)p$ ([Arnold], section 15 or [Abraham-M] theorem 3.7.1 and 3.6.2). This change of coordinate we will refer to as the Legendre transformation as well.

What is usually called the geodesic flow is just the flow h_0^t restricted to the (invariant) energy level $\{(q, p) \in T^*M \mid H_0(q, p) = 1 = \|p\|\}$ (the unit sphere bundle).

Because the exponential map :

$$(2.1) \quad \begin{aligned} \exp : TM &\rightarrow M \times M \\ (q, v) &\rightarrow (q, Q) := (q, \exp_q(v)) \end{aligned}$$

defines a diffeomorphism between a neighborhood of the 0-section in TM and some neighborhood of the diagonal in $M \times M$ ([Mi], Lemma 10.3) , we also have, via the Legendre transformation:

$$\begin{aligned} \exp : T^*M &\rightarrow M \times M \\ (q, p) &\rightarrow (q, Q) := (q, \exp_q(A(q)p)) \end{aligned}$$

which gives a diffeomorphism between a neighborhood of the 0-section in T^*M and some neighborhood of the diagonal in $M \times M$. Just how big these neighborhoods are is measured by the radius of injectivity R .

Because the Legendre transformation $A(q)$ is an isometry, equation (2.1) gives a relation between distances between points in M and norms of vectors in T^*M :

$$(q, Q) = \exp(q, p) \Rightarrow \|p\| = \text{Dis}(q, Q)$$

It will be of interest for us to know the differential of the map "Dis".

Lemma 2.2 *If $(q, Q) = \exp(q, p)$, and $h_0^1(q, p) = (Q, P)$, then:*

$$(2.2) \quad \partial_1 \text{Dis}(q, Q) = -\frac{p}{\|p\|} \quad \text{and} \quad \partial_2 \text{Dis}(q, Q) = \frac{P}{\|P\|}$$

Proof. Let $v = A(q)p$. We have $\text{Dis}(q, Q) = \|v\|$. The point $\exp_q(-tv)$ is on the same geodesic as the one running from q to Q , namely $\{\exp_q(tv) \mid t \in [0, 1]\}$. For all small and positive t we must then have:

$$\text{Dis}(\exp_q(-tv), Q) = (1 + t) \|v\|.$$

Differentiating with respect to t at $t = 0$ yields:

$$(2.3) \quad -\partial_1 \text{Dis}(q, Q) \cdot v = \|v\|.$$

On the other hand, by Gauss' lemma ([Mi], Lemma 10.5), the geodesic through q and Q must be orthogonal to the sphere centered at Q and of radius $\text{Dis}(q, Q)$. This sphere is just the level set of the function:

$$q' \rightarrow \text{Dis}(q', Q)$$

whose gradient $A(q)\partial_1 \text{Dis}(q, Q)$ at q must be colinear to v . Equation (2.3) yields:

$$A(q)\partial_1 \text{Dis}(q, Q) = \frac{-v}{\|v\|}$$

which immediately translates to the first equation we wanted to prove.

For the proof of the second equation, one must remember that $V = A(Q)P$ is tangent at Q to the geodesic between q and Q and has same norm as v . It is, more precisely, the parallel transport of v along this geodesic. Thus:

$$\frac{d}{dt} \text{Dis}(q, \exp_q((1+t)v)) \Big|_{t=0} = \|V\| = \partial_2 \text{Dis}(q, Q) \cdot V$$

and the rest of the reasoning is the same as for the first equation. \square

3 Symplectic twist maps and the decomposition lemma

If $H(q, p, t)$ is an optical Hamiltonian function (i.e. H_{pp} is convex), then its flow has many similar features to that of $H_0(q, p) = \frac{1}{2}\|p\|^2$. In particular if F is its time ϵ , and $F(q, p) = (Q, P)$, the correspondance $(q, p) \rightarrow (q, Q)$ is a diffeomorphism between suitable neighborhoods of the 0-section in T^*M and the diagonal in $M \times M$ (compare equation 2.1). This can be seen in a chart, looking at the Taylor series of the solution with respect to small time:

$$\begin{aligned} Q &= q(\epsilon) = q(0) + \epsilon \cdot H_p + o(\epsilon^2) \\ P &= p(\epsilon) = p(0) - \epsilon \cdot H_q + o(\epsilon^2) \end{aligned}$$

From this we see that $\frac{\partial Q}{\partial p}(z(0))$ is non degenerate. This remark was made by Moser in the dimension 2 case ([Mo2]).

Another feature enjoyed by Hamiltonian flows is that they are exact symplectic.

These two properties put together give us the following:

Definition 3.1 A **symplectic twist map** F is a diffeomorphism of a neighborhood U of the 0-section of T^*M onto itself satisfying the following:

- (1) F is **exact symplectic**: $F^*pdq - pdq = dS$ for some real function S on U .
- (2) (**Twist**) if $F(q, p) = (Q, P)$, then the map $\psi : (q, p) \rightarrow (q, Q)$ is embedding of U in $M \times M$.

The function $S(q, Q)$ is then called the **generating function** for F .

Remark 3.2 Of course ([G1,2,3]), monotone twist maps of the annulus (i.e. of $T^*\mathbb{S}^1$) are symplectic twist maps in the sense of this definition. U is usually taken to be either the whole cylinder, or the subset $\mathbb{S}^1 \times [0, 1]$. Note that one way to express the twist condition is by saying that the image by F of a (vertical) fiber in U intersects any fiber in at most one point.

To my knowledge, the term symplectic twist map was introduced by McKay, Meiss and Stark. Their definition ([MMS]) is a little more restrictive than the above, in that they work on $T^*\mathbb{T}^n$ and ask that $\partial Q/\partial p$ be definite positive. Our condition only implies that $\det(\partial Q/\partial p) \neq 0$. Similar maps have also been studied extensively by Herman ([H]): he called them monotone. We also have used this terminology ([G1,2]) but in the end found it misleading because we were also dealing with monotone flows [G2], the two concepts being only related in certain cases.

Remark 3.3 Equation (2.1) tells us that the time 1 map h_0^1 of H_0 is also a symplectic twist map on some neighborhood U of the 0-section. Note that for the time 1 of an Hamiltonian H , the function S is (when defined) the action:

$$S(q, Q) = \int_{(p,q)}^{(P,Q)} pdq - Hdt$$

taken along the unique solution of the Hamiltonian flow between (p, q) and (P, Q) . If L is the Legendre transform of H , the above integral is just

$$S(q, Q) = \int_0^1 L(q, \dot{q}, t) dt$$

along the solution. In the case where $H = H_0$, $L(q, \dot{q}, t) = \frac{1}{2} \|\dot{q}\|^2$, i.e. S is the energy of the (unique) geodesic between q and Q .

As noted in the introduction, h_0^1 should be our model map, the way the shear map is the model map in the theory of monotone twist maps.

The reason why twist maps can be so useful lies in the following fundamental lemma, due to LeCalvez [L] in the case of diffeomorphisms of the annulus isotopic to the Id :

Lemma 3.4 (Decomposition) (*LeCalvez, Banyaga, Golé*):

Let F be the time 1 map of a (time dependent) Hamiltonian on a compact neighborhood U of the 0-section. Suppose that F leaves U invariant. Then, F can be decomposed into a finite product of symplectic twist maps:

$$F = F_{2N} \circ \dots \circ F_1$$

Remark 3.5 No convexity is assumed of the Hamiltonian, nor any closeness to an integrable one.

Proof. Let G_t be the time t map of the Hamiltonian, starting at $t = 0$. We can write :

$$F = G_1 \circ G_{\frac{N-1}{N}}^{-1} \circ \dots \circ G_{\frac{k}{N}} \circ G_{\frac{k-1}{N}}^{-1} \circ \dots \circ G_{\frac{1}{N}} \circ Id$$

and each of the $G_{\frac{k}{N}} \circ G_{\frac{k-1}{N}}^{-1}$ is an exact symplectic map, which we can make as close as we want to the Id by increasing N . If H_{pp} is positive definite, each of these maps are twist, by Moser's remark, and we are done (F is the product of N twist maps in this case) . In general, we do the following. The twist condition (2) in Definition (3.1) of symplectic twist is an open condition. Hence, if h_0^t is the time t map of H_0 , the map $F_{2k-1} := h_0^{-1} \circ G_{\frac{k}{N}} \circ G_{\frac{k-1}{N}}^{-1}$ must satisfy (2) for N big enough (here, the compactness of U is needed). We then set $F_{2k} = h_0^1$ for all k to get the decomposition advertized. \square

Remark 3.6 We leave it to the reader to check that Lemma 3.4 is also valid for lifts of maps to the covering space of M .

4 The discrete variational setting

Let F be as in Theorem 1. From the previous section, we can write

$$F = F_{2N} \circ \dots \circ F_1,$$

with the further information that F_{2k} restrained to the boundary ∂B^*M of B^*M is the time 1 map of H_0 , that we have called h_0^1 . Likewise, F_{2k-1} is $h_0^{\frac{1}{N}-1}$ on ∂B^*M , by the proof of the decomposition Lemma (3.4) and the boundary condition (1.1) imposed on F .

Let S_k be the generating function for the twist map F_k and ψ_k the diffeomorphism $(q, p) \rightarrow (q, Q)$ induced by F_k . We can assume that ψ_k is defined on a neighborhood U of B^*M in T^*M . Let

$$(4.1) \quad O = \{\bar{q} = (q_0, \dots, q_{2N-1}) \in M^{2N} \mid (q_k, q_{k+1}) \in \psi_k(U) \text{ and} \\ (q_{2N}, q_0) \in \psi_{2N-1}(U)\}$$

O is an open set in M^{2N} , containing a copy of M (the elements \bar{q} such that $q_k = q_0$, for all k).

Next, define :

$$(4.2) \quad W(\bar{q}) = \sum_{q=0}^{2N-1} S_k(q_k, q_{k+1}),$$

where we have set $q_{2N} = q_0$. Let p_k be such that $\psi_k(q_k, p_k) = (q_k, q_{k+1})$ and let P_k be such that $F_k(q_k, p_k) = (q_{k+1}, P_k)$. P_k and p_k are well defined functions of (q_k, q_{k+1}) .

We claim:

Lemma 4.3 *The sequence \bar{q} of O is a critical point of W if and only if the sequence $\{(q_k, p_k)\}_{k \in \{0, \dots, 2N, 0\}}$ is an orbit under the successive F_k 's, that is if and only if (q_0, p_0) is a fixed point for F .*

Proof. Because the twist maps are exact symplectic and using the definitions of p_k, P_k , we have:

$$(4.4) \quad P_k dq_{k+1} - p_k dq_k = dS_k(q_k, q_{k+1}),$$

and hence

$$dW(\bar{q}) = \sum_{k=0}^{2N-1} (P_{k-1} - p_k) dq_k$$

which is null exactly when $P_{k-1} = p_k$, i.e. when $F_k(q_{k-1}, p_{k-1}) = (q_k, p_k)$. Now remember that we assumed that $q_{2N} = q_0$. \square

Hence, to prove Theorem 1, we need to find enough critical points for W . For this, we will study the gradient flow of W (where the gradient will be given in terms of the metric g) and use the boundary condition to find an isolating block.

We now indicate how this variational setting is related to the classical method of broken geodesics, and how to modify it to deal with homotopically non trivial solutions.

Because each F_k is close to $h_0^{t_k}$ for some positive or negative t_k , we have that:

$$q \in \psi_k(B^* M_q)$$

and, since $B_q^* M \rightarrow \psi_k(B^* M)$ is a diffeomorphism, we can define a path $c_k(q, Q)$ between q and a point Q of $\psi_k(B_q^* M)$ by taking the image of the oriented line segment between $\psi_k^{-1}(q)$ and $\psi_k^{-1}(Q)$ in $B_q^* M$. In the case where $F_k = h_0^1$, this amounts to taking the unique geodesic between q and Q in $\psi_k(B_q^* M)$.

If we look for periodic orbits of period d and of a given homotopy type, we decompose F^d into $2Nd$ twist maps, by decomposing F into $2N$. Analogously to (4.1), we define :

$$O_d = \{\bar{q} = (q_0, \dots, q_{2Nd-1}) \in M^{2Nd} \mid (q_k, q_{k+1}) \in \psi_k(U) \text{ and} \\ (q_{2Nd}, q_0) \in \psi_{2Nd-1}(U)\},$$

remarking that the ψ_k 's here correspond to the decomposition of F^d into $2Nd$ steps (U is as before a neighborhood of B^*M).

To each element \bar{q} in O_d , we can associate a closed curve, made by joining up each pair (q_k, q_{k+1}) by the unique curve $c_k(q_k, q_{k+1})$ defined above. This loop $c(\bar{q})$ is piecewise differentiable and it depends continuously on \bar{q} , and so does its derivatives (left and right). In the case of the decomposition of h_0^1 , taking $F_k = h_0^1$, this is exactly the construction of the broken geodesics ([Mi], §16). Now any closed curve in M belongs to a free homotopy class m .

To any d periodic point for F , we can associate a sequence $\bar{q}(x) \in O_d$ of q coordinates of the orbit of this point under the successive F_k 's in the decomposition of F^d .

Definition 4.5 Let x be a periodic point of period d for F . Let \bar{q} be the sequence in O_d corresponding to x . We say that x is an (m, d) point if $c(\bar{q}(x))$ is in the free homotopy class m .

To look for (m, d) orbits in (Theorem 2 in section 7), we will work in:

$$(4.6) \quad O_{m,d} = \{\bar{q} \in O \mid c(\bar{q}) \in m\}$$

Since $c(\bar{q})$ depends continuously on $\bar{q} \in O$, we see that $O_{m,d}$ is actually a connected component of O .

The functional W will be given this time by:

$$W(\bar{q}) = \sum_{k=0}^{2Nd-1} S_k(q_k, q_{k+1})$$

defined on $O_{m,d}$. Again, as in Lemma 4.3, critical points of W in $O_{m,d}$ correspond to (m, d) periodic points.

Remark 4.7 The reader that wants to make sure that, in the proof of Theorem 1, the orbits found are homotopically trivial, should check that throughout the proof, one can work in the component $O_{e,1}$ of $O_1 = O$ of sequences \bar{q} which have $c(\bar{q}) \in e$, where e is the identity element of $\pi_1(M)$.

5 The isolating block

In this section we prove that the set B defined as follows:

$$(5.1) \quad B = \{\bar{q} \in O \mid \|p_k(q_k, q_{k+1})\| \leq C\}$$

is an isolating block for the gradient flow of W , where O is defined in (4.1), C is as in (1.1) and p_k is the function defined in the previous section (see below (4.2)). To try to visualize this set in M^{2N} , the reader should realize that the twist condition on F_k and the fact that F_k coincides with the time 1 or time $\frac{1}{n} - 1$ of the hamiltonian H_0 at the boundary of B^*M implies that:

$$(5.2) \quad Dis(q_k, q_{k+1}) = a_k \|p_k\| \quad \text{where} \quad \begin{cases} a_k = 1 & \text{if } k \text{ is even} \\ a_k = \frac{1-N}{N} & \text{if } k \text{ is odd} \end{cases}$$

Note that B still contains a copy of M (the constant sequences).

We will define an **isolating block** for a flow to be a compact neighborhood with the property that the solution through each boundary point of the block goes immediately out of the block in one or the other time direction ([C], 3.2). Sometimes, more refined definitions are made, but this one is sufficient to ensure that the maximal invariant set for the flow contained in the block is actually contained in its interior: a block in this sense is an isolating neighborhood, which is really the only property we need here.

Proposition 5.3 *B is an isolating block for the gradient flow of W .*

Proof. Suppose the point \bar{q} of U is in the boundary of B . this means that $\|p\|_k = C$ for at least one k . As noted in (5.2), this means that $Dis(q_k, q_{k+1}) = a_k C$ for some factor a_k only depending on the parity of k . We want to show that this distance increases either in positive or negative time under the gradient flow of W . This flow is given by:

$$(5.4) \quad \dot{q}_k = A_k(P_{k-1} - p_k) = \nabla W_k(\bar{q})$$

Where $A_k = A(q_k)$ is the duality morphism associated to the metric g at the point q_k (see beginning of section 2). Remember that we have put the product metric on O , induced by its inclusion in M^{2N} .

Let us compute the derivative of the distance along the flow at a boundary point of B , using Lemma 2.1:

$$(5.5) \quad \begin{aligned} \frac{d}{dt} Dis(q_k, q_{k+1}) \Big|_{t=0} &= \partial_1 Dis(q_k, q_{k+1}) \cdot \nabla W_k(\bar{q}) \\ &\quad + \partial_2 Dis(q_k, q_{k+1}) \cdot \nabla W_{k+1}(\bar{q}) \\ &= \left(\frac{a_k}{|a_k|} \right) \frac{-p_k}{\|p_k\|} \cdot A_k \cdot (P_{k-1} - p_k) \\ &\quad + \left(\frac{a_k}{|a_k|} \right) \frac{P_k}{\|P_k\|} \cdot A_{k+1} \cdot (P_k - p_{k+1}) \end{aligned}$$

We now need a simple linear algebra lemma to treat this equation.

Lemma 5.6 *Let $\langle \cdot, \cdot \rangle$ denote a metric form in \mathbb{R}^n , and $\|\cdot\|$ its corresponding norm. Suppose that p and p' are in \mathbb{R}^n , that $\|p\| = C$ and that $\|p'\| \leq C$. Then :*

$$\langle p, p' - p \rangle \leq 0.$$

Moreover, equality occurs if and only if $p' = p$.

Proof. From the positive definiteness of the metric, we get:

$$\langle p' - p, p' - p \rangle \geq 0,$$

with equality occurring if and only if $p' = p$ (call this last assertion *). From this, we get:

$$2\langle p, p' \rangle \leq \langle p', p' \rangle + \langle p, p \rangle$$

with *. Finally,

$$\langle (p' - p), p \rangle = \langle p', p \rangle - \langle p, p \rangle \leq 0$$

with *. □

Applying Lemma 5.6 to each of the right hand side terms in (5.5), we can deduce that $\frac{d}{dt} Dis(q_k, q_{k+1})$ is positive when k is pair, negative when k is odd. Indeed, because of the boundary condition in the hypothesis of the theorem, we have $\|P_k\| = \|p_k\|$ whenever $\|p_k\| = C$: the boundary ∂B^*M is invariant under F and all the F_k 's. On the other hand $\bar{q} \in B \Rightarrow \|p_l\| \leq C$ and $\|P_l\| \leq C$, for all C , by invariance of B^*M . Finally, a_k is positive when k is even, negative when k is odd.

But what we really want is this derivative to be of a definite sign, not zero. It is certainly the case when at least one of $\nabla W_k(\bar{q})$, $\nabla W_{k+1}(\bar{q})$ is not zero. Suppose they are both zero. Then k is in an interval $\{l, \dots, m\}$ such that, for all j in this interval, $\|p_j\| = C = \|P_j\|$ and $\nabla W_j(\bar{q}) = 0$.

It is now crucial to notice that $\{l, \dots, m\}$ can not cover all of $\{0, \dots, 2N\}$: this would mean that \bar{q} is a critical point corresponding to a fixed point of h_0^1 in ∂B^*M . But such a fixed point is forbidden by our choice of C : geodesics in that energy level can not be fixed loops ($C > 0$), and they can not close up in time one either (C is less than the injectivity radius).

We now let $k = m$ in (5.5) and see that the flow must definitely escape the set P at \bar{q} in either positive or negative time, from the the m^{th} face of P . □

Remark 5.7 If we have decomposed the time 1 map of a Hamiltonian that is positive definite into a product of N twist maps, all the F_k 's coincide with $h_0^{\frac{1}{N}}$ on the boundary of B^*M . In that case,

$$\|p_k\| = \frac{1}{N} Dis(q_k, q_{k+1}), \text{ for all } k$$

and the a_k 's in the above proof are always positive. Following the argument through, we find that B is a repeller block in this case: all points on ∂B exit in positive time.

Remark 5.8 LeCalvez ([L]) provides a more detailed analysis of the behavior of the flow at “corner” points of his analog of the set B . He indicates an induction to show that the flow enters or exits the j^{th} face (j is in $\{1, \dots, m\}$ as in the above proof) at different orders in small time. Such a reasoning could be made in our context also, but we find it unnecessary, given our working definition of an isolating block.

6 Proof of Theorem 1

To finish the proof of Theorem 1 we will be using a refinement of the Conley Index continuation proved by Floer ([F1]). The homology group of the invariant set G^λ appearing in this lemma bears the germs of what became later Floer Cohomology (see e.g. [F2], and also [McD]), and in the case that we study, it is probable that it is one and the same thing. The present approach enables us to avoid the problem of infinite dimensionality in [F2], i.e. all the analysis.

Lemma 6.1 (Floer) *Let ϕ_λ^t be a one parameter family of flows on a C^2 manifold M . Suppose that G^0 is a compact C^2 submanifold invariant under the flow ϕ_0^t . Assume moreover that G^0 is **normally hyperbolic**, i.e. there is a decomposition:*

$$TM|_{G^0} = TG^0 \oplus E^+ \oplus E^-$$

which is invariant under the covariant linearization of the vector field V_0 corresponding to ϕ_0^t with respect to some metric $\langle \cdot, \cdot \rangle$, so that for some constant $m > 0$:

$$6.1 \quad \begin{aligned} \langle \xi, DV_0 \xi \rangle &\leq -m \langle \xi, \xi \rangle \text{ for } \xi \in E^- \\ \langle \xi, DV_0 \xi \rangle &\geq m \langle \xi, \xi \rangle \text{ for } \xi \in E^+ \end{aligned}$$

Suppose that there is a retraction $\alpha : M \rightarrow G^0$ and that there is a compact neighborhood B which is isolating for all λ . Then, if G^λ denotes the maximum invariant set for ϕ_λ^t in B , the map:

$$(\alpha|_{G^\lambda})^* : H^*(G^0) \rightarrow H^*(G^\lambda)$$

in Čech cohomology is injective.

In this precise sense, normally hyperbolic invariant sets continue *globally*: their topology can only get more complicated as the parameter varies away from 0. Note that we have given here a watered down version of Floer's

theorem. He uses the notion of Conley continuation of invariant sets. He also works in the equivariant case. But the above, taken from his Theorem 2 in [F1], is what we need here.

The family of flows we consider is ζ_λ^t , the flow solution of

$$\frac{d}{dt}\bar{q} = \nabla W^\lambda(\bar{q}),$$

and W^λ is defined as in 4.2 for the map F_λ , time 1 map of the Hamiltonian:

$$H_\lambda = (1 - \lambda)H_0 + \lambda H$$

We can assume that this construction is well defined, i.e., that we make the decomposition in the Decomposition Lemma 3.4 fine enough to fit any F_λ , λ in $[0, 1]$. The manifold on which we consider these (local) flows is O , an open neighborhood of B in M^{2N} . Of course, each of the F_λ satisfies the hypothesis of Theorem 1, and thus Proposition 5.3 applies to ζ_λ^t for all λ in $[0, 1]$: B is an isolating block for each one of these flows.

The part of Floer's lemma that we are missing so far is the normally hyperbolic invariant manifold for ζ_0^t .

Lemma 6.2 *Let $G^0 = \{\bar{q} \in B \mid q_k = q_0, \forall k\}$. Then G^0 is a normally hyperbolic invariant set for ζ_0^t . It is a retract of O and is the maximal invariant set in B .*

Proof. All the F_k 's in the decomposition of h_0^1 are time a_k maps of the Hamiltonian H_0 , for a_k as in (5.2). But for this Hamiltonian, the 0-section of T^*M is made out of fixed points. These translate, in terms of sequences, to points in G^0 . Moreover, these are the *only* periodic orbits for the Hamiltonian flow of H_0 in B^*M , by the definition of this set. (e.g. in the case $M = \mathbb{S}^n$ with the standard metric, the orbits corresponding to great circles would not be fixed points of h_0^1 in B^*M).

This implies that G^0 is the maximum invariant set for ζ_0^t in B . Indeed, since ζ_0^t is a gradient flow, such an invariant set should be formed by critical points and connections between them. We saw that there are no other critical points but the points of G^0 . If there were a connection orbit entirely lying in B , it would have to connect two points in G^0 , which is absurd since by continuity any two points of G^0 give the same value for W^0 , whereas W^0 should increase along non constant orbits.

G^0 is a retract of M^{2N} under the composition of the maps:

$$\bar{q} = (q_1, \dots, q_{2N}) \rightarrow q_1 \rightarrow (q_1, q_1, \dots, q_1) = \alpha(\bar{q})$$

which is obviously continuous and fixes the points of G^0 .

We are left to show that G^0 is normally hyperbolic. For this, we are going to appeal to a relationship between the linearized flow of ζ_λ^t and that

of H_λ . The following lemma was proven by McKay and Meiss in the twist map of the annulus case. We present their proof in Appendix A: it holds in the setting of general cotangent bundles.

Lemma 6.3 (*[M-M]*) *Let F be the time 1 map of a Hamiltonian and let W be its associated functional. If \bar{q} is a critical point corresponding to the orbit of (q_0, p_0) , the set of eigenvectors of eigenvalue 1 of $DF_{(q_0, p_0)}$ are in 1-1 correspondence with the set of eigenvectors of eigenvalue 0 of $HessW(\bar{q})$*

To use this lemma, we remark that since G^0 is made out of critical points, saying that it is normally hyperbolic is equivalent to saying that $HessW^0(\bar{q})$ has exactly $n = \dim G^0$ eigenvalues equal to zero for any point \bar{q} in G^0 . These eigenvalues have to correspond to eigenvectors in TG^0 , the normal space of which must be spanned by eigenvectors with non zero eigenvalues ($HessW^0$ is symmetric). Hence, from Lemma 6.3, it is enough to check that at a point $(q_0, 0) \in B^*M$ corresponding to \bar{q} , 1 is an eigenvalue of multiplicity exactly n for $Dh_0^1(q_0, 0)$. Let us compute $Dh_0^1(q_0, 0)$ in local coordinates. It is the solution at time 1 of the linearized equation:

$$\dot{U} = JHessH_0(q_0, 0)U$$

along the constant solution $(q(t), p(t)) = (q_0, 0)$, where J denotes the usual symplectic matrix $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. An operator solution for the above equation is given by $\exp(tJHessH_0(q_0, 0))$. On the other hand:

$$HessH_0(q_0, 0) = \begin{pmatrix} 0 & 0 \\ 0 & A(q_0) \end{pmatrix}$$

which we computed from $H_0(q, p) = A(q)p.p$, the zero terms appearing at $p = 0$ because they are either quadratic or linear in p . From this,

$$Dh_0^1(q_0, 0) = \exp(JHessH_0(q_0, 0)) = \begin{pmatrix} I & A(q_0) \\ 0 & I \end{pmatrix}$$

is easily derived. This matrix has exactly n eigenvectors of eigenvalue 1 (it has in fact no other eigenvector). Hence, from Lemma 6.3, $HessW(\bar{q})$ has exactly n vectors with eigenvalue 0, as was to be shown. \square

We now conclude the proof of Theorem 6.3. We have proved that the flow ζ^t , which is gradient, has an invariant set $G = G^1$ with $H^*(M) \hookrightarrow H^*(G)$. From this we get in particular:

$$cl(G) \geq cl(M) \text{ and } sb(G) \geq sb(M).$$

The corollary of Theorem 5 in [CZ 1] tells us that ζ^t must have at least $cl(G)$ rest points in the set G , whereas the generalized Morse inequalities in Theorem 3.3 of [CZ 2] tell us that, if they are all assumed to be non

degenerate, ζ^t must have $sb(G)$ rest points. But Lemma 6.3 tells us that non degeneracy for $HessW$ at a critical point is the same thing as nondegeneracy of a fixed point for F (no eigenvector of eigenvalue 1).

As was stated in Remark 4.7, we could have worked in $O_{e,1}$ all along to guarantee that the orbits found are homotopically trivial. The only thing that one should check is that $G_{m,d}^0$ is indeed in this component of O , which is the case. This concludes the proof of Theorem 1. \square

7 Negative curvature and orbits of different homotopy types

7.1 Setting the problem

We are going to sketch here the changes needed in the proof of Theorem 1 in order to prove Theorem 2 on the existence of orbits of different homotopy types.

It is a known (see e.g. [GHL], 2.98) that on a compact Riemannian manifold there exists in any nontrivial free homotopy class m a smooth and closed geodesic which is of length minimal in m .³

Moreover, a theorem of Cartan asserts that if the manifold is of negative curvature, there is in fact one and only one geodesic in each class m (m not containing the point curves ([Kl], Theorem 3.8.14)).

Let M be of negative curvature and let $l(m)$ denote the length of the geodesic in m in that case.

In 4.5, we have defined (m, d) orbits by saying that a certain curve that the orbit defines in M is of class m . We could also use a favored lift \overline{F} of F to the covering space \overline{M} of M to define such orbits, by asking:

$$\overline{F}^d(x) = m.x$$

where $m.x$ denotes the action of m seen as a deck transformation in $T^*\overline{M}$ (the favored lift is the one corresponding to lifting the solution curves of the Hamiltonian flow). But definition 4.5 turns out to be more convenient to use here (both are equivalent, of course). We now restate:

Theorem 2 *Let (M, g) be a Riemannian manifold of negative curvature. and H be as in Theorem 1. Then, whenever (M, g) has a geodesic whose class in $\pi_1(M)$ is m , F has at least 2 (m, d) orbits in B^*M when $l(m) < dC$.*

³ We remind the reader that free homotopy classes of loops differ from elements of $\pi_1(M)$ in that no base point is kept fixed under the homotopies. As a result, free homotopy classes can be seen as conjugacy classes in $\pi_1(M)$, and thus can not be endowed with a natural algebraic structure. Two elements of a free class give the same element in $H_1(M)$. Hence free homotopy classes form a set smaller than $\pi_1(M)$, bigger than $H_1(M)$. All these sets coincide if $\pi_1(M)$ is abelian.

Remark 7.1.1 The fact that we find two orbits and not a number given by the topology of the manifold is not an artifact of the proof, but derives from the unicity of the closed geodesic in a given class. Note also that we do not guarantee that an orbit of the form (m^k, kd) is not actually an (m, d) orbit. We should then ask for (m, d) to be prime, in that very sense.

Note also, there are a priori more of these pairs (m, d) than there are rational homology directions.

The proof of Theorem 2 has the same broad outline as that of Theorem 1.

We decompose $F = F_{2N} \circ \dots \circ F_0$ as before, which gives us a decomposition of F^d into $2Nd$ twist maps.

We would like to claim, in analogy to Proposition 5.3 that

$$B = \{\bar{q} \in O_{m,d} \mid \|p_k(q_k, p_k)\| \leq C\}$$

is an isolating block for the gradient flow of W .

But this will not be enough for our purpose. To make sure that two critical points correspond to points that are actually on 2 distinct orbits, one should do the following: to decompose F^d , we have decomposed F in $2N$ steps. Define:

$$\sigma : O_{m,d} \rightarrow O_{m,d} \text{ by setting } (\sigma \bar{q})_k = q_{k+2N}$$

where we identify: $q_{k+2Nd} = q_k$. It is clear that 2 critical sequences corresponding to points in the same orbit by F get identified in the quotient by the action of σ . So our candidate for isolating block will be given by the quotient B/σ of B by the action of σ (note that σ leaves B invariant, so that the quotient makes sense).

It can be seen that if m is non trivial, then the action of σ is without fixed points. Since it is also periodic, the action is then properly discontinuous ([Gr], Chapter 5), and hence the quotient map $O_{m,d} \rightarrow O_{m,d}/\sigma$ is a covering map.

We now describe the candidate for normally hyperbolic invariant set. It will be the quotient by σ of the set $G_{m,d}^0$ made of the critical sequences corresponding to the continuum of (m, d) orbits that form the closed geodesic of class m , parametrized so that the Hamiltonian flow goes through it in time d . Call this orbit γ .

Note that $G_{m,d}^0$ contains all the possible critical points for W^0 in $O_{m,d}$ since a critical point for W^0 must be contained in a continuum of critical points: if (q_0, p_0) is an (m, d) point, so is $h_0^t(q_0, p_0)$, for any t . But we know, in the case of H_0 that there is one and only one such set, namely γ .

Writing $F_k^0 \circ \dots \circ F_1^0 = \phi_k$, where the F_k^0 's decompose the map h_0^d , we can write:

$$G_{m,d}^0 = \{\bar{q}(t) \in O_{m,d} \mid q_k = \pi \circ \phi_k(\gamma(t))\}$$

where $\pi(q, p) = q$ is the canonical projection. Again, since σ restricted to $G_{m,d}^0$ actually corresponds to the action of F on γ , $G_{m,d}^0$ is σ invariant

and hence the quotient $G_{m,d}^0/\sigma$ makes sense. Since the quotient map is a covering map and $G_{m,d}^0 \cong \gamma$, we have :

$$(7.1.2) \quad G_{m,d}^0/\sigma \cong \mathbb{S}^1$$

7.2 Proof of Theorem 2

Lemma 7.2.1 *B/σ is an isolating block*

Proof. Because we have assumed $l(m) < dC$, F cannot have any (m, d) orbits confined to ∂B^*M : since F coincides with F_0 on this set, such an orbit would have to correspond to a closed geodesic of free homotopy class m , but of length dC , which is absurd. This in turn implies that W has no critical points on ∂B , and the reasoning of Proposition 5.3 applies without change to show that points in ∂B must exit B in positive or negative time. Since the covering map is a local diffeomorphism, this is also true in B/σ , which is then an isolating block. \square

Lemma 7.2.2 *$G_{m,d}^0/\sigma$ is a normally hyperbolic invariant set for ζ_0^t . It is a retract of $O_{m,d}/\sigma$.*

Proof. We prove the statement “upstairs”, taking the quotient by σ only at the end.

According to Lemma 6.3, and the reasoning in the proof of Lemma 6.2, it is enough to show that the differential of h_0^d on a point of γ has no other eigenvector of eigenvalue 1 than the vector tangent to γ .

To compute the differential of h_0^d at the point $(q_0, p_0) = \gamma(0)$ we are going to choose a coordinate system (z, t, s) around (q_0, p_0) in the following way: z, t will be a coordinate system for a tubular neighborhood in the energy surface containing (q_0, p_0) , t being in the direction of γ . We will define s by the following: a point (q, p) on the energy level of γ will be assigned coordinates $(z, t, 1)$ and the point (q, sp) will be assigned coordinates (z, t, s) . It is clear that in an interval $s \in (a, b)$, $a > 0$, this gives a system of coordinates.

It is interesting to notice that $(0, t, 1)$ is a parametrization of γ , whereas the cylinder $(0, t, s)$ is foliated by circles $s = c$ invariant under the flow h_0^t : each one corresponds to a reparametrization of γ , by rescaling the velocity by s .

The map h_0^d leaves the cylinder invariant and in fact induces a monotone twist map on it :

$$h_0^d(0, t, s) = (0, t + (s - 1)d, s)$$

Now, remember that the geodesic flow of a manifold with strictly negative curvature is Anosov. This translates into: in the subspace tangent at a point

$(0, t, s)$ to the z coordinate, Dh_0^d has no eigenvalue equal to 1 (we can assume the splitting tangent to t, s to be invariant by Dh_0^t). Hence, in the (z, t, s) coordinates:

$$Dh_0^d(0, s, t) \begin{pmatrix} A & & \\ & 1 & d \\ & 0 & 1 \end{pmatrix}$$

where A has no eigenvalue 1. Hence $Dh_0^d(0, t, s)$ has only the vector tangent to γ as eigenvector with eigenvalue 1 (I am indebted to Leonid Polterovitch for giving me the idea of this argument).

We now have to prove that $G_{m,d}^0$ is a retract of $O_{m,d}$. Define the following map

$$\begin{aligned} \rho : O_{m,d}(2N) &\rightarrow O_{m,d}(N) \\ (q_0, q_1, q_2, \dots, q_{2Nd}) &\rightarrow (q_0, q_2, \dots, q_{2k} \dots, q_{2Nd}) \end{aligned}$$

It is not hard to see that ρ induces a diffeomorphism on $G_{m,d}^0$: the projection on the 0th factor would itself give a diffeomorphism. We claim that the image \underline{G} of $G_{m,d}^0$ under ρ is a deformation retraction of the image \underline{Q} of $O_{m,d}$ in $O_{m,N}$. Call r this retraction. Then $\rho_{\underline{G}}^{-1} \circ r \circ \rho$ is a retract of $O_{m,d}$ to $G_{m,d}^0$, as we want to prove.

We now construct the map r . Decompose $h_0^d = (h_0^{\frac{1}{N}})^{Nd}$. Since $h_0^{\frac{1}{N}}$ is a symplectic twist map , we can rig up the variational setting relative to this decomposition. Call

$$\underline{W}(\bar{q}) = \sum_{k=1}^{k=Nd} S(q_k, q_{k+1})$$

where S is the generating function of $h_0^{\frac{1}{N}}$. In this case, the isolating block $\underline{B} = \rho B$ is a repeller block for the gradient flow of \underline{W} (see Remark 5.7). Hence \underline{W} has attains a minimum value, say a , in the interior of \underline{B} . It has to be at a point in \underline{G} , which contains all the critical points of \underline{W} , as we remarked above for $G_{m,d}^0$. Hence on all of \underline{G} , \underline{W} must equal a . Since we can choose \underline{Q} to be exhausted by an increasing sequence of repeller neighborhoods of the same type as \underline{P} , a is actually a global minimum for the function \underline{W} in \underline{Q} .

The same argument as for $G_{m,d}^0$ shows that \underline{G} is normally hyperbolic. In particular, this implies that the set $\underline{W} \leq a + \epsilon$ forms a tubular neighborhood of \underline{G} ([DNF],§20). Then a standard argument ([Mi] Theorem 3.1) in Morse theory shows that, since there are no other critical points but those in the level $\underline{W} = a$, the set $\underline{W} \leq a + \epsilon$ must be a deformation retraction of \underline{Q} . Finally, \underline{G} is a deformation retraction of $\underline{W} \leq a + \epsilon$, since the latter is a tubular neighborhood of \underline{G} . This finishes the construction of r .

Finally, we indicate how all these features go through in the quotient by σ .

To check that $G_{m,d}^0/\sigma$ is normally hyperbolic, we just note that this notion is a local one, in the tangent space, and the quotient map is a local

diffeomorphism. It can be checked that the above construction of the retraction map is σ invariant. And, finally, the quotient of our invariant set $G_{m,d}^0$ (see 7.3) is a circle, as noted in 7.1.2. This concludes the proof of Lemma 7.2.2. \square

To finish the proof of Theorem 2, we use Floer's Lemma, as in the proof of Theorem 1, to find that there is an invariant set $G_{m,d}$ for the flow ζ^t in $O_{m,d}/\sigma$ which is such that:

$$H^*(G_{m,d}^0/\sigma) = H^*(\mathbb{S}^1) \hookrightarrow H^*(G_{m,d})$$

Since $cl(\mathbb{S}^1) = sb(\mathbb{S}^1) = 2$, in all cases, we will get at least 2 distinct *orbits* of type (m, d) . \square

Appendix A Linearized gradient flow vs.linearized Hamiltonian flow

Suppose that $(q_0, p_0) = x_0$ is a fixed point for F . We want to solve the equation:

$$(A.1) \quad DF_{x_0}(v) = \lambda v$$

with $v \in T(T^*M)_{x_0}$. In terms of Hamiltonian flow, we want to find the Floquet multipliers of the periodic orbit corresponding to x_0 .

In the (q_k, q_{k+1}) coordinates, we want to express a condition on the orbit $(\delta q_k, \delta q_{k+1})$ of a tangent vector $(\delta q_1, \delta q_2)$ under the successive differentials of the maps F_{k-1} along the given orbit. A way to do it is the following ([M-M]): If \bar{q} corresponds to the orbit of x_0 under the successive F_k 's, it must satisfy:

$$\frac{\partial W(\bar{q})}{\partial q_k} = \partial_2 S_{k-1}(q_{k-1}, q_k) + \partial_1 S_k(q_k, q_{k+1}) = 0$$

(see 4.3). Therefore, a "tangent orbit" $\delta \bar{q}$ must satisfy:

$$(A.2) \quad S_{21}^{k-1} \delta q_{k-1} + (S_{11}^k + S_{22}^{k-1}) \delta q_k + S_{12}^k \delta q_{k+1} = 0$$

where we have abbreviated:

$$S_{ij}^k = \partial_{ij} S_k(q_k, q_{k+1}).$$

When \bar{q} corresponds to a fixed point (q_0, p_0) . Equation A.1 translates, in terms of the $\delta \bar{q}$, to:

$$(A.3) \quad \delta q_{2N} = \lambda \delta q_0$$

Equations (A.2) and (A.3) can be put in matrix form as $M(\lambda)\delta\bar{q} = 0$ where $M(\lambda)$ is the following $2Nn \times 2Nn$ tridiagonal matrix:

$$M(\lambda) = \begin{pmatrix} S_{22}^0 + S_{11}^1 & S_{12}^1 & 0 & \dots & 0 & \frac{1}{\lambda}S_{21}^0 \\ S_{21}^1 & S_{22}^1 + S_{11}^2 & S_{12}^2 & \ddots & & 0 \\ 0 & S_{12}^2 & & & & \vdots \\ \vdots & \ddots & & & & 0 \\ 0 & \dots & 0 & & & S_{12}^{2N-1} \\ \lambda S_{12}^0 & 0 & \dots & 0 & S_{21}^{2N-1} & S_{22}^{2N-1} + S_{11}^0 \end{pmatrix}$$

Hence the eigenvalues of DF_{x_0} are in one to one correspondence with the values λ for which $\det M(\lambda) = 0$. More precisely, to each vector v solution of (A.1) corresponds one and only one vector $\delta\bar{q}$ solution of $M(\lambda)\delta\bar{q} = 0$. Setting $\lambda = 1$, this proves Lemma 6.3.

Remark (A.4) This construction can be given a symplectic interpretation: the Lagrangian manifolds $\text{graph}(dW)$ and $\text{graph}(F)$ are related by symplectic reduction. Lemma 6.3 can then be restated in terms of the invariance of a certain Maslov index under reduction ([V]).

Appendix B: Twist maps and linking of spheres

In this appendix, we present the proof given in [BG] of the original conjecture of Arnold in the restrictive case of symplectic twist maps (Theorem 3 in the introduction).

To that effect, we have to give our interpretation of what linking of spheres in ∂B^*M is.

Call Δ_q the fiber of B^*M over q , and $\partial\Delta_q$ its boundary in ∂B^*M . Then $\partial\Delta_q$ is an n dimensional sphere. It make sense to talk about its linking with its image $F(\partial\Delta_q)$ in ∂B^*M : the latter set has dimension $2n - 1$ and the dimensions of the spheres add up to $2n - 2$.

We first restrict ourselves to the case when the two spheres $\partial\Delta_q$ and $F(\partial\Delta_q)$ are in a trivializing neighborhood in ∂B^*M , say $U \times \mathbb{S}^n = E$.

The type of linking of $F(\partial\Delta_q)$ with $\partial\Delta_q$ should be given by the class $[F(\partial\Delta_q)] \in H_{n-1}(\partial E \setminus \partial\Delta_q)$

More precisely, we have:

$$(B.1) \quad \begin{aligned} H_{n-1}(\partial E \setminus \partial\Delta_q) &\cong H_{n-1}(\mathbb{S}^{n-1} \times (\mathbb{R}^n - \{0\})) \\ &\stackrel{\text{Kunneth}}{\cong} H_{n-1}(\mathbb{R}^n - \{0\}) \oplus H_{n-1}(\mathbb{S}^n) \end{aligned}$$

Thus, taking $\partial\Delta_q$ from ∂E creates a new generator in the $n - 1$ st homology, i.e. the generator b of $H_{n-1}(\mathbb{R}^n - \{0\})$.

Definition (Linking condition) We say that the spheres $F(\partial\Delta_q)$ and $\partial\Delta_q$ link in ∂E if they do not intersect and if the decomposition of $[F(\partial\Delta_q)]$ in the direct sum in (B.1) has a non zero term in its $H_{n-1}(\mathbb{R}^n - \{0\})$ factor. We will say that the symplectic twist map F satisfies the linking condition if for all $q \in M$ these spheres link in ∂E for some trivializing neighborhood E ⁴

If F is a symplectic twist map, it turns out that this is a well defined characterization of linking: we can always construct a trivializing neighborhood containing both $\partial\Delta_q$ and its image. Indeed, take $T^*(\pi \circ F(\Delta_q))$ (homeomorphic to $B^n \times \mathbb{R}^n$ since F is twist) if q is in $\pi \circ F(\Delta_q)$. If not join q to this set by a path, and fatten this path. The union of the set and the fattened path is homeomorphic to a ball. Hence the bundle over this ball is trivial.

Moreover, it turns out that if the spheres link in one trivializing neighborhood, they do in all of them, as a consequence of the following

Lemma B.2 *If F is a symplectic twist map, the following are equivalent:*

- a) *The spheres $\partial\Delta_q$ and $F(\partial\Delta_q)$ link in some trivializing neighborhood in ∂B^*M*
- b) *The fiber Δ_q and its image $F(\Delta_q)$ intersect in one point of their interior.*

Remark B.3 We can also define the linking condition for a map F of B^*M which is not necessarily symplectic twist. If the covering space of M is \mathbb{R}^n , we say that F satisfies the linking condition if at least one of its lifts does (the trivializing neighborhood is taken to be $\overline{M} \times \mathbb{R}^n \cong \mathbb{R}^{2n}$ in this case.) If M is not covered by \mathbb{R}^n , Lemma B.2 suggests that we may take as a linking condition that the intersection number $\sharp(\Delta_q \cap F(\Delta_q))$ is ± 1 .

Proof. Suppose that E is a trivializing neighborhood containing the 2 spheres. We complete B.1 into the following commutative diagram:

$$\begin{array}{ccc} H_{n-1}(\partial E \setminus \partial\Delta_q) & \cong & H_{n-1}(\mathbb{R}^n - \{0\}) \oplus H_{n-1}(\mathbb{S}^n) \\ \downarrow i_* & & \downarrow j_* \\ H_{n-1}(E \setminus \Delta_q) & \cong & H_{n-1}((\mathbb{R}^n - \{0\}) \times B^n) \end{array}$$

where i, j are inclusion maps. It is clear that j_*b generates

$$H_{n-1}((\mathbb{R}^n - \{0\}) \times B^n) \cong H_{n-1}((\mathbb{R}^n - \{0\}) \times \mathbb{R}^n).$$

The last group measures the (usual) linking number of a sphere with the fiber $\pi^{-1}(q)$ in $T^*E \cong \mathbb{R}^{2n}$. But it is well known that such a number is

⁴ Here, as a convention, a trivializing neighborhood will always be homeomorphic to $B^n \times \mathbb{R}^n$.

the intersection number of any ball bounded by the sphere with the fiber $\pi^{-1}(q)$, counted with orientation. In our case, where the sphere considered is $F(\partial\Delta_q)$ this number can only be 0 or 1 or -1, because of the twist condition (see Remark 3.2).

Conversally, if Δ_q and $F(\Delta_q)$ intersect in their interior, then their bounding spheres must lie on the trivializing neighborhood over $F(\Delta_q)$, and must link. \square

Remark B.4 If all fibers intersect their image under a twist map, i.e. if the linking condition is satisfied, then the intersection number must be uniformly 1 or -1: we could call F a positive twist map in the first case, a negative twist map in the second case. Of course, this corresponds to the same notion in dimension 2.

We can now prove Theorem 3.

From Lemma 4.3, fixed points of F correspond to critical points of $q \rightarrow S(q, q)$. This function only make sense for all q in M if the diagonal in $M \times M$ is in the image of U by the embedding ψ (see Definition 3.1). This is exactly the case when $q \in F(\Delta_q)$ for all q , i.e., from Lemma B.2, exactly when the linking condition is satisfied. Hence F has as many fixed points as the function $q \rightarrow S(q, q)$ has critical points on M . \square

To our knowledge, Arnold's original conjecture is still open, even in the case $M = \mathbf{T}^n$.

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