On the realization of Fixed Point Portraits.
(An addendum to "Fixed Point Portraits" by Goldberg and Milnor)

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Abstract.

We establish that every formal critical portrait (as defined in [GM]), can be realized by a postcritically finite polynomial.

1. Preliminaries.

1.1. Let P be a polynomial of degree $d \geq 2$ with connected Julia set $J(P)$. The *rational type $T(z)$* of a fixed point z, is by definition the set of all angles of (rational) external rays which land at z. The *fixed point portrait of P* is the collection $\mathcal{T}(P) = \{T_1, \ldots, T_k\}$ consisting of all rational types $T_i \neq \emptyset$ of its fixed points.

In the work [GM], Goldberg and Milnor gave combinatorial conditions on the family $\mathcal{T}(P)$ and conjectured that those conditions where also sufficient. The purpose of this note is to prove this conjecture.

1.2 Rational Rotation Sets (See also [G].) We start by parametrizing the unit circle $S^1 = \mathbf{R}/\mathbf{Z}$ by the interval $[0, 1)$. Let $d \geq 2$ and consider the d-fold covering map $f_d : \theta \mapsto d\theta \ (mod\ 1)$. We will adopt the convention throughout that an indexed subset $\Theta = \{\theta_0, \ldots, \theta_{n-1}\}$ of $\mathbf{R}/\mathbf{Z}$ satisfies $0 \leq \theta_0 < \ldots < \theta_{n-1} < 1$.

Definition. A finite subset $\Theta = \{\theta_0, \ldots, \theta_{n-1}\}$ of $\mathbf{R}/\mathbf{Z}$ is a *degree d-rotation set* if there exists a positive integer m so that $f_d(\theta_i) = \theta_{i+m} \ (mod\ 1)$ for $i = 0, \ldots, n-1$. Note that in this definition m and n need not be relatively prime. The ratio $m/n \ (mod\ 1)$ is called the *rotation number*.

Theorem. (See [G, Theorem 7].) *Let Θ, Θ' be degree d rotation sets with the same rotation number m/n. Then $\Theta = \Theta'$ if and only if for all $i = 0, \ldots, d-2$*

$$\#\big(\Theta \cap [\frac{i}{d-1}, \frac{i+1}{d-1})\big) = \#\big(\Theta' \cap [\frac{i}{d-1}, \frac{i+1}{d-1})\big).$$

In other words a d−rotation set is uniquely determined by, its rotation number, its cardinality, and the relative position of its elements with respect to the $d-1$ roots of unity.

1.3 Unlinked sets. We will say that two subsets T and T' of the circle $\mathbf{R}/\mathbf{Z}$ are unlinked if they are contained in disjoint connected subsets of $\mathbf{R}/\mathbf{Z}$, or equivalently, if T' is contained in just one connected component of the complement $\mathbf{R}/\mathbf{Z} - T$. (In particular T and T' must be disjoint.) If we identify $\mathbf{R}/\mathbf{Z}$ with the boundary of the unit disk, an equivalent condition would be that the convex closures of these sets are pairwise disjoint.

As an example, if T and T' are the types for any two distinct fixed points of P, then evidently T and T' are unlinked.

1.4 We fix an integer $d \geq 2$ and a family $\mathcal{T} = \{T_1, \ldots, T_k\}$ to which we impose the following conditions

P1. Each T_j is a degree d-rotation set.

P2. The T_j are disjoint and pairwise unlinked.

P3. The union of those T_j which have rotation number zero is precisely equal to the set $\{0, \frac{1}{d-1}, \ldots, \frac{d-2}{d-1}\}$ consisting of all angles which are fixed by f_d.

P4. Each pair $T_i \neq T_j$ with non-zero rotation number is separated by at least one T_ℓ with zero ration number. That is, T_i and T_j must belong to different connected components of the complement $\mathbf{R}/\mathbf{Z} - T_\ell$.

The importance of the above conditions is shown by the following Theorem proved by Goldberg and Milnor.

Theorem. ([GM, Theorem 3.8]) *If $\mathcal{T}(P) = \{T_1, \ldots, T_k\}$ is the fixed point portrait for some polynomial P with connected Julia set $J(P)$, then conditions P1-P4 above are satisfied.*

The main result of this note is the sufficiency of these conditions.

Theorem A. *Given a family $\mathcal{T} = \{T_1, \ldots, T_k\}$, satisfying conditions P1-P4 above, there is a postcritically finite polynomial P such that $\mathcal{T}(P) = \mathcal{T}$.*

Goldberg and Milnor proved this theorem only for some special cases. Our proof is based on the construction of a unique smallest abstract Hubbard tree which realizes the given fixed point portrait. (For a different approach, based on Thurston laminations see [HJ].)

1.5 Abstract Hubbard Trees. By an *(angled) tree H* will be meant a finite connected acyclic m-dimensional simplicial complex ($m = 0, 1$), together with a function $\ell, \ell' \mapsto \angle(\ell, \ell') = \angle_v(\ell, \ell') \in \mathbf{Q}/\mathbf{Z}$ which assigns a rational modulo 1 to each pair of edges ℓ, ℓ' which meet at a common vertex v. This angle $\angle(\ell, \ell')$ should be skew-symmetric, with $\angle(\ell, \ell') = 0$ if and only if $\ell = \ell'$, and with $\angle_v(\ell, \ell'') = \angle_v(\ell, \ell') + \angle_v(\ell', \ell'')$ whenever three edges are incident at a vertex v. Such an angle function determines a preferred isotopy class of embeddings of H into $\mathbf{C}$.

Let V be the set of vertices. We specify a mapping $\tau : V \to V$ and call it the *vertex dynamics*, and require that $\tau(v) \neq \tau(v')$ whenever v and v' are endpoints of a common edge ℓ. We consider also a *local degree function* $\delta : V \to \mathbf{Z}$ which assigns an integer $\delta(v) \geq 1$ to each vertex $v \in V$. We require that $d(\delta) = 1 + \sum_{v \in V}(\delta(v) - 1)$ be greater that 1. By definition a vertex v *is critical* if $\delta(v) > 1$ and *non-critical* otherwise. The *critical set* $\Omega(\delta) = \{v \in V : v \text{ is critical}\}$ is thus not empty.

The maps τ and δ must be related in the following way. Extend τ to a map $\tau : H \to H$ which carries each edge homeomorphically onto the shortest path joining the images of its

endpoints. We require then that $\angle_{\tau(v)}(\tau(\ell), \tau(\ell')) = \delta(v)\angle_v(\ell, \ell')$ whenever ℓ, ℓ' are incident at v (in this case $\tau(\ell)$ and $\tau(\ell')$ are incident at the vertex $\tau(v)$ where the angle is measured).

A vertex v is *periodic* if for some $n > 0$, $\tau^{\circ n}(v) = v$. Given $W \subset V$, we define its orbit $\mathcal{O}(W) = \cup_{n=0}^{\infty}\tau^{\circ n}(W)$. The orbit of a periodic critical point is a *critical cycle*. We say that a vertex v is of *Fatou type or a Fatou vertex* if it eventually maps into a critical cycle. Otherwise, if it eventually maps to a non critical cycle, it is of *Julia type or a Julia vertex*.

We define the distance $d_H(v, v')$ between vertices in H as the number of edges in a shortest path γ between v and v'. We say that (H, V, τ, δ) is *expanding* if the following condition is satisfied. For any edge ℓ whose end points v, v' are Julia vertices, there is an $n \geq 1$ such that $d_H(\tau^{\circ n}(v), \tau^{\circ n}(v')) > 1$. Note that angles are not needed in this definition.

The angles at Julia vertices are rather artificial, so we normalize them as follows. If m edges $\ell_1, \ldots, \ell_m$ meet at a periodic Julia vertex v, then we assume that the angles $\angle_v(\ell_i, \ell_j)$ are all multiples of $1/m$. (It follows that the angles at a periodic Julia vertex convey no information beyond the cyclic order of these m incident edges.)

Definition. By an *abstract Hubbard tree* we mean an angled tree $\mathbf{H} = ((H, V, \tau, \delta), \angle)$ such that the angles at any periodic Julia vertex where m edges meet are multiples of $1/m$. We define isomorphism between abstract Hubbard Trees in an obvious way.

Douady and Hubbard showed in [DH] that a postcritically finite polynomial P and a finite invariant set M containing the critical set $\Omega(P)$ of P naturally defines an abstract Hubbard tree $\mathbf{H}_{P,M}$. To define the angle function we note the following facts. Near a Fatou vertex the edges of the tree are by definition segments of constant argument in the Böttcher coordinate, we define the angle between two such edges as the difference in such coordinates. For a (periodic or preperiodic) Julia set point v, $J(P) - \{v\}$ consists of a finite number (say m) of components. We define the 'angle' between consecutive components around v to be $1/m$. As each edge in the tree correspond locally to one of these components, we have an angle function between them. (This procedure is well defined and compatible with the definition above, see [P].) It is easy to prove that this abstract Hubbard tree is expanding (see [P]).

The main result for Hubbard trees is the following.

Theorem 1. (See [P].) *Let $\mathbf{H}$ be an abstract Hubbard tree. Then there is a postcritically finite polynomial P and an invariant set $M \supset \Omega(P)$ such that $\mathbf{H}_{P,M} \cong \mathbf{H}$ if and only if $\mathbf{H}$ is expanding. Furthermore, P is unique up to affine conjugation.*

This abstract Hubbard tree also gives information about external rays as the following theorem essentially due to Douady and Hubbard shows (see [DH, Chap VII] or [P]).

Theorem 2. *The number of rays which land at a periodic Julia vertex is equal to the number of incident edges of the tree, and in fact, there is exactly one ray landing between each pair of consecutive edges. Furthermore, the ray which lands at v between ℓ and ℓ' maps to the ray which lands at $\tau(v)$ between $\tau(\ell)$ and $\tau(\ell')$.*

2. Proof of Theorem A.

We identify $\mathbf{R}/\mathbf{Q}$ with ∂D via the exponential map $e(\theta) = e^{2\pi i\theta}$. For each element $T_j = \{\theta_1, \ldots, \theta_n\}$ consider the baricenter $v(T_j)$ of all elements of T_j. In other words define

$$v(T_j) = \frac{1}{n}\sum_{i=1}^{n} e(\theta_i).$$

Next we join each element $e(\theta) \in T_j$ to $v(T_j)$ by a straight segment (these segments will not be part of our tree). This construction clearly divides the closed unit disk into a finite number of components or *regions* (see condition P2 in §1 and compare Figure 1).

For each of these regions U_i define *the critical capacity* $CC(U_i)$ as the number of vertices $v(T_j)$ with zero rotation number belonging to the boundary of U_i. Clearly the sum of critical capacities must be equal to $d-1$ (this is an easy induction using condition $P3$).

Insert inside each region U_i a vertex $w(U_i)$, which we join to every vertex $v(T_j)$ in the boundary of U_i. The union of these joining edges together with the vertices $w(U_i)$ and $v(T_j)$ will form the required topological tree (compare with Figure 2).

We proceed now to construct the local degree and angle functions of the tree. For every vertex $w(U_i)$ we define its degree $\delta(w(U_i)) = CC(U_i) + 1$. For the vertices $v(T_j)$ define $\delta(v(T_j)) = 1$. At every vertex where $m \geq 1$ edges come together we define the angle between consecutive edges to be $1/m$. At a vertex $w(U_i)$ this number equals to the number of vertices $v(T_j) \in \partial U_i$, while at a vertex $v(T_j)$ it is equal to the number of elements in T_j.

To define the vertex dynamics we consider first those vertices $w(U_i)$ for which the region U_i has a (necessarily unique by condition $P4$) vertex $v(T_l)$ with non zero rotation number on its boundary. Then, there are exactly two elements $\theta, \theta' \in T_l$ such that $e(\theta), e(\theta') \in \partial U_i$. In fact, we can order these two elements so that $e(\theta + \epsilon), e(\theta' - \epsilon) \in \partial U_i$ for small $\epsilon > 0$. Then there is a unique U_j ($\neq U_i$) such that $e(d\theta + \epsilon), e(d\theta' - \epsilon) \in \partial U_j$, and we define $\tau(w(U_i)) = w(U_j)$. (Note that in this way $w(U_i)$ has the same period as any $\theta \in T_l$ under multiplication by d modulo 1.) For all other vertices v define $\tau(v) = v$.

Note that by construction all $v(T_l)$ are of Julia type (fixed and non critical), while all $w(U_l)$ are of Fatou type. Also, between any two different $v(T_i)$ and $v(T_j)$ there is a vertex $w(U_l)$, and so, the expanding condition is trivially satisfied. Furthermore, at non fixed Fatou vertices the angle between consecutive edges is $1/\delta$, while at fixed Fatou vertices it is $1/(\delta - 1)$. From this it is easy to see that the angle condition is satisfied.

Thus, there is a unique (up to affine conjugation) polynomial of degree d which realizes this abstract Hubbard Tree. We must still verify that this polynomial (or tree) has the required fixed point portrait. We begin by locating all fixed points.

Lemma. *The abstract Hubbard tree constructed above has exactly d fixed points.*

Proof. This is just a matter of counting. Let k be the number of rotation sets, and let $\ell = \sum \#\{T_i : T_i \text{ has non zero rotation number}\}$. Note that by condition $P3$

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$d - 1 = \sum \#\{T_i : T_i$ has zero rotation number$\}$. By induction it is easy to see that there are exactly $\ell + d - k = (1 + \sum(\#(T_j) - 1))$ regions U_l. There are k Julia fixed points (as many as rotation sets). By construction there are ℓ regions without an interior fixed point, so there are $d - k$ Fatou fixed points.

$$\#\#\#$$

To verify that this tree has the required fixed point portrait we use theorem 2 in section 1.5. We first note that by construction all edges incident at a vertex $v(T_l)$ where T_l has rotation number zero are 'fixed'. Thus by theorem 2, only fixed rays land there, and every fixed ray must land at one of those points. Now suppose $0 \in T_\ell$, then by construction (if T_ℓ is not a singleton) there is a segment (the one which is not part of the tree!) joining $e(0)$ to $v(T_\ell)$. As topologically this segment is located between two consecutive fixed edges of the tree, it corresponds to a fixed ray of the polynomial map (this is true even if T_ℓ is a singleton). After affine conjugation if necessary we assume that this is the zero ray. Now, if we walk counterclockwise around the tree, it follows from conditions $P2$ and $P3$ that given T_j with zero rotation number, all rays with argument in T_j land at $v(T_j)$.

If T_l has rotation number m/n, clearly the set of rays which land at $v(T_l)$ has also rotation number m/n. The result then follows from Theorem 1.2, which asserts that a m/n rotation set is uniquely determined by the relative position of its elements respect to $\frac{i}{d-1}$, $i = 0, \dots, d - 2$.

3. An example.

We will illustrate the proof of the above theorem by taking the degree 5 fixed point portrait determined by $T_1 = \{0, 3/4\}$, $T_2 = \{1/8, 5/8\}$, $T_3 = \{1/4\}$, $T_4 = \{1/2\}$, with rotation numbers 0, 1/2, 0, 0 respectively. The Julia set of the actual polynomial is shown in figure 3.
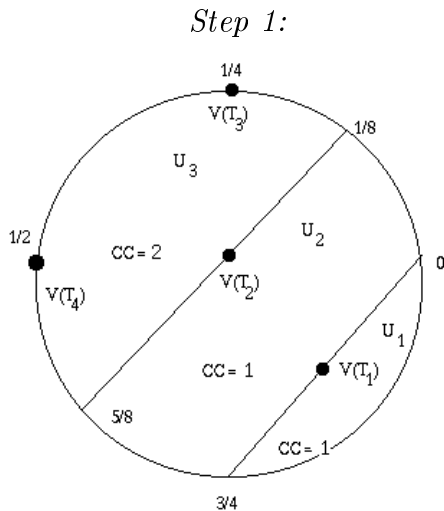
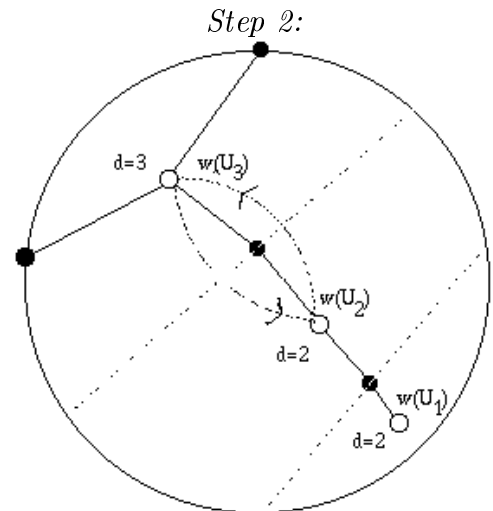*Step 1:* *Step 2:*

Figure 1

Figure 2
all vertices are fixed except
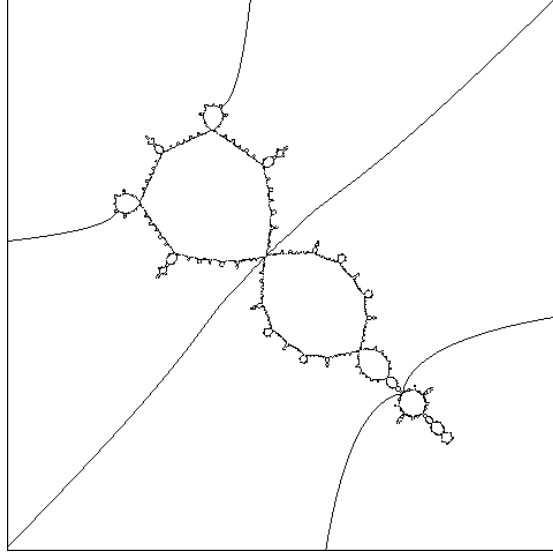$w(U_2)$, $w(U_3)$ *which are interchanged.*

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Figure 3

Julia set of the polynomial $P(z) = z^5 + Az^3 + Bz^2 + Cz + D$, where
$A \approx 2.714670827i$, $B \approx 0.693957313(1+i)$, $C \approx -1.608651885$
$D \approx -0.355745059(1-i)$.
The rays $0, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}$ shown.

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