Dynamics of Certain Smooth One-dimensional Mappings
I. The $C^{1+\alpha}$-Denjoy-Koebe distortion lemma

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Abstract

We prove a technical lemma, the $C^{1+\alpha}$-Denjoy-Koebe distortion lemma, estimating the distortion of a long composition of a $C^{1+\alpha}$ one-dimensional mapping $f : M \mapsto M$ with finitely many, non-recurrent, power law critical points. The proof of this lemma combines the ideas of the distortion lemmas of Denjoy and Koebe.

Contents

§1 Introduction

There are two techniques in studying the distortion of a long composition of a one-dimensional smooth mapping.

"Denjoy Principle": One technique goes back to Denjoy. Many people have contributed to this technique [D], [S], [N1], [N2], [N3], [H], [M], etc.. We call one of the formulations of this technique the naive distortion lemma because its proof is straightforward – any one, who has been trained in Calculus, will understand the proof ([J1], p25–26, or Lemma 3 in this paper). The naive distortion lemma is one of the key lemmas in studying a long composition of a one-dimensional $C^{1+\alpha}$-endomorphism.

"Koebe principle": The second technique was found in recent years in studying a long composition of a mapping with critical points from a one-dimensional manifold to itself. Many people formulated this principle in different ways, [GS], [Su1], [Su2], [WS], [Sw], etc.. We call one version the C^3-Koebe distortion lemma (see also [J1, p26–27] for a complete proof). I learned this from Sullivan, who invented the name "Koebe principle" in analogy with the Koebe lemma in one variable complex analytic functions. We consider the nonlinearity of a C^2-function f on an interval I as a one-form

$$n(f) = \frac{f''}{f'}dx.$$

If the nonlinearity of the function f is integrable on I, then the distortion $|f'(x)/f'(y)|$ of f at any pair x and y in I is bounded. The problem is that the nonlinearity of f may be non-integrable if f has a critical point. The C^3-Koebe distortion lemma estimates the nonlinearity of a one-dimensional C^3-mapping f with nonnegative Schwarzian derivative. This property, nonnegative Schwarzian derivative, is preserved under composition, which makes the C^3-Koebe distortion lemma a very useful technique in studying a long composition of a one-dimensional C^3-mapping with nonpositive Schwarzian derivative (its inverse branches have nonnegative Schwarzian derivatives). However, the assumption of nonpositive Schwarzian derivative is a very strong one.

What we would like to say in this paper. We prove a technical lemma, the $C^{1+\alpha}$-Denjoy-Koebe distortion lemma, estimating the distortion of a long composition of a one-dimensional $C^{1+\alpha}$-mapping with finitely many non-recurrent critical points of certain types. The formulation and the proof of this lemma combine the ideas of the distortion lemmas of Denjoy and Koebe.

Suppose M is an oriented connected one-dimensional C^2-Riemannian manifold with Riemannian metric dx^2 and associated length element dx. Suppose $f : M \mapsto M$ is a continuous mapping. A critical point c of f is a point in M such that either f is not differentiable at this point or f is differentiable at this point with zero derivative. We always assume that f is C^1 at a noncritical point p, namely there is a neighborhood U_p of p such that the restriction of f to U_p is differentiable and the derivative $(f|U_p)'$ is continuous. We call the image of a critical point under f a critical value of f. We say a critical point c of f is a power law critical point if it is an isolated critical point and there is a number $\gamma \geq 1$ such that the limits of ratio, $f'(x)/|x-c|^{\gamma-1}$, exist and equal nonzero numbers as x goes to c from below and from above. We call the number γ the exponent of f at the power law critical point c. We will assume that $f : M \mapsto M$ is a C^1-mapping for we are only interested in a smooth critical point of f.

For a C^1-mapping $f : M \mapsto M$ with only power law critical points such that the set of critical points and the set of critical values of f are disjoint, we define a new differentiable structure on the underlying space M. This new differentiable structure associated with the mapping f has the local parameter $\int dx/|x|^{\tau}$, where $\tau = 1 - 1/\gamma$, on a neighborhood of a critical value of f if the corresponding critical point has the exponent γ. On a neighborhood of any other point, the new differentiable structure has the local parameter $\int \rho(x)dx$ where $\rho(x)$ is a positive C^2-function. *With respect to the new differentiable structure, the left and the right derivatives of f at any critical point exist and equal nonzero numbers* (see Figure 1). We call the original differentiable structure the old one.

We use the oriented connected one-dimensional smooth manifolds M and $\tilde{M}$, which are the same topological space but with the old and the new differentiable structures , respectively, to study the dynamics of the mapping $f : M \mapsto M$: the distortions of a long composition of a one-dimensional $C^{1+\alpha}$-mapping $f : M \mapsto M$ with only finitely many,

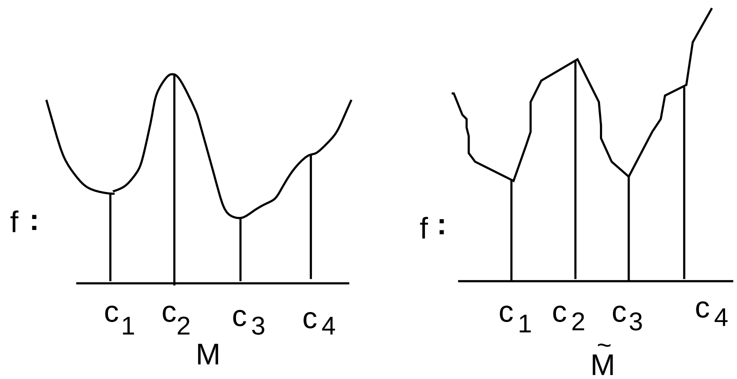

Figure 1

non-recurrent, power law critical points has an estimate like that in the naive distortion lemma and that in the C^3-Koebe distortion lemma.

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§2 A Very Good Mappings

Suppose M is an oriented connected one-dimensional C^2-Riemannian manifold with Riemannian metric dx^2 and associated length element dx. Suppose $f : M \mapsto M$ is a continuous mapping. A critical point c of f is a point in M such that

(a) f is not differentiable at this point or

(b) f is differentiable at this point but the derivative of f at this point is zero.

We always assume that f is C^1 at a noncritical point p, namely there is a neighborhood U_p of p such that the restriction of f to U_p is differentiable and the derivative $(f|U_p)'$ is continuous. We call the image of a critical point under f a critical value of f.

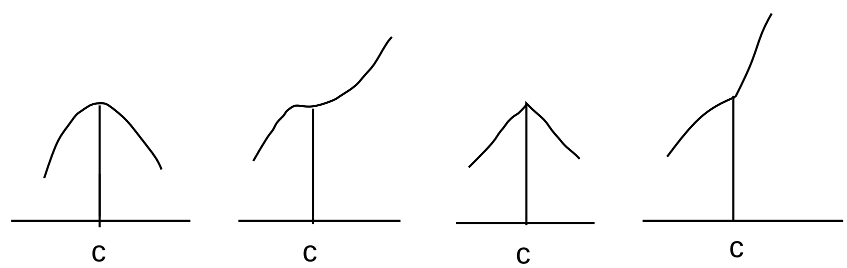
§2.1 A power law critical point.

We give a definition of a power law critical point for the one-dimensional mapping $f : M \mapsto M$ as follows.

Definition 1. *Suppose c is an isolated critical point of f and suppose there are γ^-, $\gamma^+ \geq 1$ such that*

$$\lim_{x\mapsto c-} f'(x)/|x-c|^{\gamma^- -1} = A \text{ and } \lim_{x\mapsto c+} f'(x)/|x-c|^{\gamma^+ -1} = B$$

exist and equal nonzero numbers. Then we say that c is a power law critical point with the left and right exponents γ^- and γ^+.



Examples of power law critical points

Figure 2

The following is essentially proved in [J4] (see [J5], too).

Preliminary Lemma. *Suppose $f : M \mapsto M$ is a continuous mapping and c is a power law critical point with the left and right exponents γ^- and γ^+. Then there is a continuous mapping $\tilde{f} : M \mapsto M$ and a real number $\sigma \neq 0$ such that*

(a) the mapping $\tilde{f}$ either has the form

$$\tilde{f} = \begin{cases} -\sigma|x-c|^{\gamma^-} + f(c) & x \leq c, \\ |x-c|^{\gamma^+} + f(c) & x \geq c \end{cases} \quad \text{or} \quad \tilde{f} = \begin{cases} \sigma|x-c|^{\gamma^-} + f(c) & x \leq c, \\ -|x-c|^{\gamma^+} + f(c) & x \geq c \end{cases}$$

where x is in a small neighborhood of c,

(b) the mapping f is semi-conjugate to the mapping $\tilde{f}$. This means that there is a monotone and continuous mapping h from M onto M such that

$$h \circ f = \tilde{f} \circ h$$

and h is differentiable at c with $h'(c) > 0$.

Moreover,

(i) if f is $C^{1+\alpha}$ on $x \leq c$ and on $x \geq c$ for some $0 < \alpha \leq 1$, and $r_-(x) = f'(x)/|x-c|^{\gamma^- -1}$, $x \leq c$, and $r_+(x) = f'(x)/|x-c|^{\gamma^+ -1}$, $x \geq c$, are C^β for some $0 < \beta \leq 1$, where x is in a small neighborhood of c, then the mapping h can be an orientation-preserving C^1-diffeomorphism.

(ii) The mapping h can be an orientation-preserving $C^{1,1}$ or C^2-diffeomorphism if and only if f is $C^{1,1}$ or C^2 on $x \leq c$ and $x \geq c$, and $r_-(x)$ and $r_+(x)$ are Lipschitz or C^1, where x is in a small neighborhood of c.

Remark. For a power law critical point of f, the left and right exponents are C^1-invariants. By this we mean that they are the same numbers for f and for $h \circ f \circ h^{-1}$ whenever h is an orientation-preserving C^1-diffeomorphism. When the left and right exponents are the same, we then have an important C^1-invariant

$$\sigma = \lim_{x \mapsto c-} \frac{f'(x)}{f'(-x+2c)}$$

which we call the asymmetry of f at c. The number σ in Preliminary Lemma is the asymmetry. In the paper [J1], we showed that the asymmetry is an independent C^1-invariant.

§2.2 The new differentiable structure associated with a semi-good mapping.

Although the results in the rest of the paper hold for a mapping f with both smooth and non-smooth critical points, but we are only interested in a smooth critical point of f. Henceforth we will assume that $f : M \mapsto M$ is a C^1-mapping. Moreover we will assume that the left and right exponents of f at a power law critical point are the same.

DEFINITION 2. *We say f is a semi-good mapping if*

(I) the mapping f has only finitely many power law critical points,

(II) the set of critical points and the set of critical values of f are disjoint, and if

(III) the exponents of f at two critical points are the same whenever the images of these two points under f are the same.

Suppose $f : M \mapsto M$ is a semi-good mapping. Let $CP = \{c_1, \cdots, c_d\}$ be the set of critical points of f and $\Gamma = \{\gamma_1, \cdots \gamma_d\}$ be the set of corresponding exponents. We define a new differentiable structure associated with f as follows.

Suppose $\Phi = \{(w_j, W_j)\}_{j\in\Lambda}$ is a C^2-atlas of M, this means that $\{W_j\}_{j\in\Lambda}$ is a cover of open sets of M and $\{w_j : W_j \mapsto \mathbf{R}^1\}_{j\in\Lambda}$ is a set of homeomorphisms such that every $w_{jk} = w_j \circ w_k^{-1}$ is a C^2-function whenever W_j and W_k are overlap. Suppose every critical value $v_i = f(c_i)$ is in one and only one chart (w_i, W_i) and w_i maps the critical value v_i to 0. For every critical value $v_i = f(c_i)$, we use $k_i(x)$ to denote the homeomorphism $\int_0^x dx/|x|^{\tau_i} : \mathbf{R}^1 \mapsto \mathbf{R}^1$ where $\tau_i = 1 - 1/\gamma_i$. Let $\tilde{w}_j = k_i \circ w_i$ if $W_j = W_i$ contains a critical value $v_i = f(c_i)$ and $\tilde{w}_j = w_j$ if W_j does not contain any critical values. The set $\tilde{\Phi} = \{(\tilde{w}_j, W_j)\}$ is another C^2-atlas of M. We call the maximal C^2-atlas of M which contains the set $\tilde{\Phi} = \{(\tilde{w}_j, W_j)\}$ the new differentiable structure associated with f on M. We denote the topological space M equipped with this new differentiable structure as a differentiable manifold $\tilde{M}$.

It is often convenient to think the new differentiable structure associated with f as a singular metric $\rho(x)dx$ with respect to dx and the mapping $h = \int \rho(x)dx : M \mapsto M$ as the corresponding change of coordinate on M. The mapping $\tilde{f} = h \circ f \circ h^{-1} : M \mapsto M$ is the representation of the mapping $f : \tilde{M} \mapsto \tilde{M}$.

LEMMA 1. *Suppose $f : M \mapsto M$ is a semi-good mapping and CP is the set of critical points of f. Then the mapping $f : \tilde{M} \mapsto \tilde{M}$ is a continuous mapping and at every point $c_i \in CP$, the left and right derivatives of $f : \tilde{M} \mapsto \tilde{M}$ exist and equal nonzero numbers.*

Proof. The proof of this lemma is easy. The reader may do it as an exercise or refer to the proof in [J1, p21].

§2.3 The definition of a very good mapping.

We define a very good mapping. Before to give the definition of a very good mapping, we define the term $C^{1+\alpha}$ for a real number $0 < \alpha \leq 1$ and a semi-good mapping.

Suppose $f : M \mapsto M$ is a semi-good mapping. Let CP be the set of critical points of f. Suppose η_0 is the set of the closures of the intervals of the complement of CP. We say a homeomorphism $g : I \mapsto J$ is a $C^{1+\alpha}$-embedding for some $0 < \alpha \leq 1$ if g and g^{-1} are both differentiable with α-Hölder continuous derivatives.

DEFINITION 3. *we say f is $C^{1+\alpha}$ for some $0 < \alpha \leq 1$ if*

(1) *the restriction of f to every interval in η_0 is differentiable with α-Hölder continuous derivative,*

(2) *for every critical point c_i, there is a neighborhood U_i of c_i such that the restrictions of $f : \tilde{M} \mapsto \tilde{M}$ to the intersection of U_i and $\{x \leq c_i\}$ and the intersection of U_i and $\{x \geq c_i\}$ are $C^{1+\alpha}$-embeddings.*

We will assume that U_i is a closed interval for every $i = 1, \cdots, d$. Suppose $\mathcal{U}$ be the union $\cup_{i=1}^{d} U_i$ and $\mathcal{V}$ be the closure of the complement of $\mathcal{U}$ in M.

DEFINITION 4. *A C^1-mapping $f : M \mapsto M$ is a very good $C^{1+\alpha}$-mapping (or a very good mapping) for some $0 < \alpha \leq 1$ if it is a semi-good mapping and satisfies*

(IV) f is $C^{1+\alpha}$,

(V) the set CP of critical points and the closure of the post-critical orbits $\cup_{n=1}^{\infty} f^{\circ n}(CP)$ are disjoint and

(VI) there are two constants $K > 0$ and $\nu > 1$ such that for any $\mathcal{O}_{x,n} = \{x, f(x), \cdots, f^{\circ(n-1)}(x)\}$ with $\mathcal{O}_{x,n} \cap \mathcal{U} = \emptyset$, $|(f^{\circ k})'(x)| \geq K\nu^k$ for any $1 \leq k \leq n$.

The space of good mappings is a quite large one, for example, it contains all C^3 semi-good mappings with nonpositive Schwarzian derivative and finitely many non-recurrent critical points (see, for example, [Mi], [MS] and [J2]).

§3 The Distortion Of A Long Composition Of A Very Good Mapping

Suppose $f : M \mapsto M$ is a very good $C^{1+\alpha}$-mapping for some $0 < \alpha \leq 1$. We always assume that $\mathcal{U}$, the union of all U_i in Definition 3, is disjoint with the closure of the post-critical orbits $\cup_{n=1}^{\infty} f^{\circ n}(CP)$. We use U_{i-} to denote the subset consisting of all points x in U_i with $x \leq c_i$ and use U_{i+} to denote the subset consisting of all points x in U_i with $x \geq c_i$. Let $\mathcal{W}$ be the collection of all U_{i-} and U_{i+}. Remember that $\mathcal{V}$ is the closure of the complement of $\mathcal{U}$ in M. We say a sequence

$\mathcal{I} = \{I_j\}_{j=0}^n$ of intervals of M is suitable if

(i) I_j is the image of I_{j+1} under f for $j = 0, \cdots n-1$ and

(ii) either I_j is in $\mathcal{V}$ or I_j is in some interval in $\mathcal{W}$ for every $j = 0, \cdots, n$.

For a suitable sequence $\mathcal{I} = \{I_j\}_{j=0}^n$ of intervals of M, we use g_j to denote the inverse of the restriction of $f^{\circ j}$ to I_j. For a pair of points x and y in I_0, we use x_j and y_j to denote the images of x and y under g_j and call the ratio $|g_n'(x)|/|g_n'(y)|$ the distortion of f at x and y along $\mathcal{I}$. We use D_{xy} to denote the distance between $\{x, y\}$ and post-critical orbits $\cup_{n=1}^{\infty} f^{\circ n}(CP)$.

The main result of this paper is the following:

LEMMA 2 (the $C^{1+\alpha}$-Denjoy-Koebe distortion lemma). *Suppose $f : M \mapsto M$ is a very good $C^{1+\alpha}$-mapping for some $0 < \alpha \leq 1$. There are two positive constants A and B such that for any suitable sequence $\mathcal{I} = \{I_j\}_{j=0}^n$ of intervals of M and any pair x and y in I_0, the distortion of f at x and y along $\mathcal{I}$ satisfies*

$$\frac{|g_n'(x)|}{|g_n'(y)|} \leq \exp\Big(A \sum_{i=0}^n |x_i - y_i|^\alpha + \frac{B|x-y|}{D_{xy}}\Big).$$

§3.1. The naive distortion lemma.

Before to prove Lemma 2, we state the naive distortion lemma. Suppose $g : U \mapsto M$ is a $C^{1+\alpha}$-mapping for some $0 < \alpha \leq 1$ where U is an interval of M. Let K be the α-Hölder constant of the derivative of g, this means that K is the smallest positive constant such that

$$|g'(x) - g'(y)| \leq K|x-y|^\alpha$$

for all x and y in U. Suppose $\{I_j\}_{j=1}^n$ is a sequence of intervals of U and x_i and y_i are two points in I_j for $1 \leq j \leq n$. We also call the product of ratios $\prod_{j=1}^n |g'(x_j)|/|g'(y_j)|$ the distortion of g at $\{x_j\}_{j=1}^n$ and $\{y_j\}_{j=1}^n$. Let β be the minimum of $|g'|$ on $\cup_{j=0}^n I_j$.

LEMMA 3 (the naive distortion lemma). *The distortion of g at $\{x_j\}_{j=1}^n$ and $\{y_j\}_{j=1}^n$ satisfies*

$$\prod_{j=1}^n \frac{|g'(x_j)|}{|g'(y_j)|} \leq exp\Big(\frac{K}{\beta} \sum_{j=0}^n |x_j - y_j|^\alpha\Big).$$

Proof. Take the function $\log x$ at $\prod_{j=1}^{n} |g'(x_j)|/|g'(y_j)|$, we have

$$\log\Big(\prod_{j=1}^{n}\frac{|g'(x_j)|}{|g'(y_j)|}\Big)=\sum_{j=1}^{n}\Big(\log|g'(x_j)|-\log|g'(y_j)|\Big).$$

Because $\log x$ is Lipschitz continuous with the Lipschitz constant $1/\beta$ on the interval $[\beta,+\infty)$ and the α-Hölder constant of g' on U is K, we have that

$$\Big|\sum_{j=0}^{n}\Big(\log|g'(x_j)|-\log|g'(y_j)|\Big)\Big|\leq\frac{1}{\beta}\sum_{j=0}^{n}|g'(x_j)-g'(y_j)|$$

which is bounded above by $(K/\beta)\sum_{j=0}^{n}|x_j-y_j|^{\alpha}$.

§3.2 The proof of $C^{1+\alpha}$-Denjoy-Koebe distortion lemma.

We call $\mathcal{U}$, the union of U_i for $i=1,\cdots,d$, the critical set and $\mathcal{V}$, the closure of the complement of $\mathcal{U}$ in M, the noncritical set (see Figure 3). Let η_0 be the set of the closures of the intervals of the complement of the set CP of critical points of f in M. Let $\tilde{f}=h\circ f\circ h^{-1}:M\mapsto M$ be the representation of $f:\tilde{M}\mapsto\tilde{M}$, where h is the corresponding change of coordinate. Remember that $U_{i-}=U_i\cap\{x:x\leq c_i\}$ and $U_{i+}=U_i\cap\{x:x\geq c_i\}$.

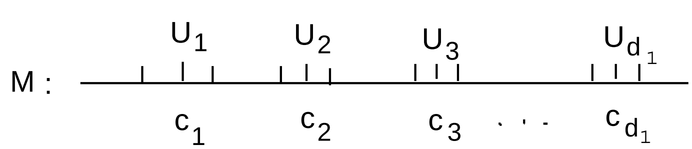


Figure 3

Let $K_1>0$ be the maximum of the α-Hölder constants of the derivatives of the restrictions of f to the intervals in η_0 and $\beta_1>0$ be the minimum of the absolute value of the restriction of the derivative f' of f to $\mathcal{V}$.

The restrictions of $\tilde{f}$ to the sets U_{i-} and U_{i+} are $C^{1+\alpha}$-embeddings for $i=1,\cdots,d$. Let $K_2>0$ be the maximum of the α-Hölder constants of the derivatives of these restrictions and $\beta_2>0$ be the minimum of the absolute value of the derivatives of these restrictions.

The restrictions of h to the intervals of $\mathcal{U}$ are $C^{1,1}$. Let $K_3 > 0$ be the maximum of Lipschitz constants of the derivatives of these restrictions and $\beta_3 > 0$ be the minimum of the absolute value of the derivatives of these restrictions.

The distortion of f along $\mathcal{I}$ at x and y satisfies

$$\frac{|g_n'(x)|}{|g_n'(y)|} = \frac{|(f^{\circ n})'(y_n)|}{|(f^{\circ n})'(x_n)|}.$$

By the chain rule, the ratio $|(f^{\circ n})'(y_n)|/|(f^{\circ n})'(x_n)|$ equals the product of ratios $|f'(y_{n-i})|/|f'(x_{n-i})|$ where i runs from 0 to $n-1$. This product can be factored into two products,

$$\prod_{x_i, y_i \in \mathcal{V}} \frac{|f'(y_i)|}{|f'(x_i)|} \quad \textit{and} \quad \prod_{x_i, y_i \in \mathcal{U}} \frac{|f'(y_i)|}{|f'(x_i)|}.$$

We note that the subscript i in the products are integers in the range $[1, n]$.

Using Lemma 3 (the naive distortion lemma), we can show that the first product

$$\prod_{x_i, y_i \in \mathcal{V}} \frac{|f'(y_i)|}{|f'(x_i)|} \leq \exp\Big(\frac{K_1}{\beta_1} \sum_{i=0}^{n} |x_i - y_i|^{\alpha}\Big).$$

The second product

$$\prod_{x_i, y_i \in \mathcal{U}} \frac{|f'(y_i)|}{|f'(x_i)|}$$

can be factored into three products

$$\prod_{x_i, y_i \in \mathcal{U}} \frac{|h'(y_i)|}{|h'(x_i)|} \cdot \prod_{x_i, y_i \in \mathcal{U}} \frac{|\tilde{f}'(h(y_i))|}{|\tilde{f}'(h(x_i))|} \cdot \prod_{x_i, y_i \in \mathcal{U}} \frac{|h'(f(x_i))|}{|h'(f(y_i))|},$$

by using the formula

$$f'(x) = \frac{h'(x)\tilde{f}'(h(x))}{h'(f(x))}.$$

By using Lemma 3 again, the first product

$$\prod_{x_i,y_i\in\mathcal{U}} \frac{|h'(y_i)|}{|h'(x_i)|} \leq \exp\Big(\frac{K_3}{\beta_3}\sum_{i=0}^{n}|x_i-y_i|\Big)$$

and the second product

$$\prod_{x_i,y_i\in\mathcal{U}} \frac{|\tilde{f}'(h(y_i))|}{|\tilde{f}'(h(x_i))|} \leq \exp\Big(\frac{K_3^{\alpha}K_2}{\beta_2}\sum_{i=0}^{n}|x_i-y_i|^{\alpha}\Big).$$

Suppose x_i, y_i and $c_{k(i)}$ are in the same set $U_{k(i)}$ and $v_{k(i)} = f(c_{k(i)})$ is the critical value. Because $h'(x) = 1/|x-v_{k(i)}|^{\tau_{k(i)}}$ on a neighborhood of $v_{k(i)}$, where $\tau_{k(i)} = 1 - 1/\gamma_{k(i)}$ and $\gamma_{k(i)}$ is the exponent of f at $c_{k(i)}$, the third product has the form

$$\prod_{x_i,y_i\in\mathcal{U}} \Big(\frac{|y_{i-1}-v_{k(i)}|}{|x_{i-1}-v_{k(i)}|}\Big)^{\tau_{k(i)}}.$$

We note that $x_{i-1} = f(x_i)$ and $y_{i-1} = f(y_i)$ are the points near the critical value $v_{k(i)}$ for x_i and y_i are in the set $U_{k(i)}$.

To control the third product we write

$$\frac{|y_{i-1}-v_{k(i)}|}{|x_{i-1}-v_{k(i)}|} = |1+\frac{y_{i-1}-x_{i-1}}{x_{i-1}-v_{k(i)}}|,$$

which is less than or equal to $1+|x_{i-1}-y_{i-1}|/|x_{i-1}-v_{k(i)}|$, for every pair x_i and y_i in $\mathcal{U}$.

Suppose l is the smallest positive integer such that x_l and y_l are in $\mathcal{U}$. We consider l in the two cases. The first case is that $l=1$ and the second case is that $l>1$.

In the first case, the images of x_l and y_l under f are x and y. We have that

$$\frac{|x-y|}{|x-v_{k(l)}|} \leq \frac{|x-y|}{D_{xy}}.$$

In the second case, suppose I_l is the smallest interval containing x_l, y_l and $c_{k(l)}$ and $I_{l-i} = f^{\circ i}(I_l)$ for $i=0,\cdots,l$. Because the intervals I_{l-i} are contained in $\mathcal{V}$ for $i=1,\cdots,l-1$ (we can always reduce to this case), by using (VI) of Definition 4 and Lemma 3, there is a constant $K_4 > 1$ such that

$$\frac{|x_{l-1} - y_{l-1}|}{|x_{l-1} - v_{k(l)}|} \leq K_4 \frac{|x-y|}{|x - f^{\circ(l-1)}(v_{k(l)})|}.$$

We note that $f^{\circ l}(x_l) = x$ and $f^{\circ l}(y_l) = y$ (see Figure 4-a). This implies that

$$\frac{|y_{l-1} - x_{l-1}|}{|x_{l-1} - v_{k(l)}|} \leq K_4 \frac{|x-y|}{|x - f^{\circ(l-1)}(v_{k(l)})|} \leq K_4 \frac{|x-y|}{D_{xy}}.$$

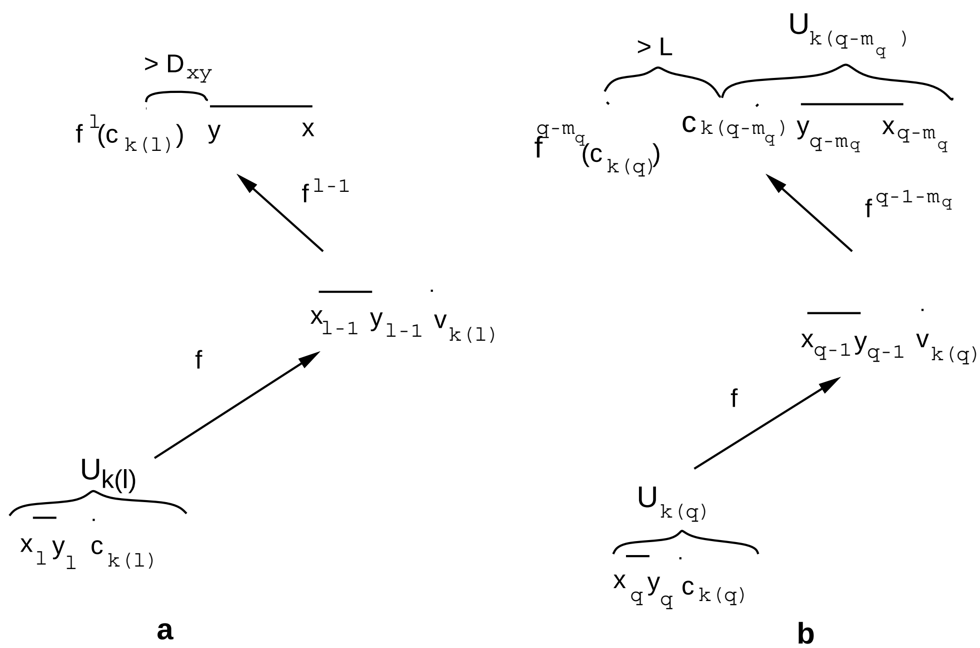


Figure 4

For any $q > l$ with x_q and y_q in $\mathcal{U}$, let m_q be the smallest positive integer such that x_{q-m_q} and y_{q-m_q} are in $\mathcal{U}$ (see Figure 4-b).

Suppose I_q is the smallest interval containing x_q, y_q and $c_{k(q)}$ and $I_{q-i} = f^{\circ i}(I_q)$ for $i = 0, \cdots, m_q$. The intervals I_{q-i} for $i = 1, \cdots, m_q - 1$ are contained in $\mathcal{V}$ (we always can reduce to this case). By using (VI) of Definition 4 and Lemma 3, there is a positive constant, we still denote it as K_4, such that

$$\frac{|y_{q-1}-x_{q-1}|}{|x_{q-1}-v_{k(q)}|} \leq K_4 \frac{|y_{q-m_q}-x_{q-m_q}|}{|x_{q-m_q}-f^{\circ(q-m_q)}(c_{k(q)})|}.$$

Because x_{q-m_q} is in $\mathcal{U}$ and $f^{\circ(q-m_q)}(c_{k(q)})$ is not in $\mathcal{U}$, the number $|x_{q-m_q}-f^{\circ(q-m_q)}(c_{k(q)})|$ is bigger than or equal to L, the distance between the set $\mathcal{U}$ and the closure of the post-critical orbits $\cup_{n=1}^{\infty} f^{\circ n}(CP)$. Hence we get

$$\frac{|x_{q-1}-y_{q-1}|}{|x_{q-1}-v_{k(q)}|} \leq K_4 \frac{|x_{q-m_q}-y_{q-m_q}|}{L}.$$

Now the third product satisfies that

$$\prod_{x_i,y_i\in\mathcal{U}} \Big(\frac{|y_{i-1}-v_{k(i)}|}{|x_{i-1}-v_{k(i)}|}\Big)^{\tau_{k(i)}} \leq \exp\Big(\frac{K_4|x-y|}{\tau D_{xy}} + \frac{K_4}{L\tau}\sum_{i=1}^{n}|x_i-y_i|\Big),$$

where τ is the maximum of $\tau_j = 1-1/\gamma_j$ for $j=1, \cdots, d$.

We now prove Lemma 2 by putting all the estimates together and $A = K_1/c_1 + (K_3^{\alpha}K_2)/c_2 + K_3/c_3 + K_4/(L\tau)$ and $B = K_4/\tau$.

§3.3 A larger class of one-dimensional mappings.

We can actually prove Lemma 2 for a wider class of one-dimensional mappings as follows.

Suppose $f : M \mapsto M$ is a C^1-mapping with only power law critical points. Let $CP = \{c_1, \cdots, c_d\}$ be the set of critical points of f and $\Gamma = \{\gamma_1, \cdots, \gamma_d\}$ be the set of corresponding exponents. Suppose η_0 be the set of the closures of the intervals of the complement of the set CP of critical points of f in M.

DEFINITION 5. *We say f is $C^{1+\alpha}$ for some $0 < \alpha \leq 1$ if*

(1) *the restriction of f to every interval in η_0 is differentiable with α-Hölder continuous derivative,*

(2) *for every critical point c_i, there is a neighborhood U_i of c_i such that the functions $r_{i,-}(x) = f'(x)/|x-c_i|^{\gamma_i-1}$ for $x < c_i$ in U_i and $r_{i,+}(x) = f'(x)/|x-c_i|^{\gamma_i-1}$ for $x > c_i$ in U_i are α-Hölder continuous.*

Suppose $\mathcal{U}$ is the union of U_i for $i = 1, \cdots, d$ and $\mathcal{V}$ is the closure of the complement of $\mathcal{U}$ in M. Let U_{i-} be the subset consisting of all points x in U_i with $x \leq c_i$ and U_{i+} be the subset consisting of all points x in U_i with $x \geq c_i$, for $i = 1, \cdots, d$. Suppose $\mathcal{W}$ be the collection of all U_{i-} and U_{i+}.

DEFINITION 6. *Suppose $f : M \mapsto M$ is a C^1-mapping. We say f is a good $C^{1+\alpha}$-mapping (or good mapping) for some $0 < \alpha \leq 1$ if*

(I) f has only finitely many power law critical points,

(II) f is $C^{1+\alpha}$,

(III) there is a positive integer N such that the set CP of critical points and the closure of the set $\cup_{n=N}^{\infty} f^{\circ n}(CP)$ are disjoint,

(IV) there are two constants $K > 0$ and $\nu > 1$ such that for any $\mathcal{O}_{x,n} = \{x, f(x), \cdots, f^{\circ(n-1)}(x)\}$ with $\mathcal{O}_{x,n} \cap \mathcal{U} = \emptyset$, $|(f^{\circ k})'(x)| \geq K\nu^k$ for any $1 \leq k \leq n$.

We say a sequence $\mathcal{I} = \{I_j\}_{j=0}^{n}$ of intervals of M is suitable if

(i) I_j is the image of I_{j+1} under f for $j = 0, \cdots n-1$ and

(ii) either I_j is in $\mathcal{V}$ or I_j is in some interval in $\mathcal{W}$, for every $j = 0, \cdots, n$.

LEMMA 4 ($C^{1+\alpha}$-Denjoy-Koebe distortion lemma). *Suppose f is a good $C^{1+\alpha}$-mapping for some $0 < \alpha \leq 1$. There are positive constants A and B such that for any suitable sequence $\mathcal{I} = \{I_j\}_{j=0}^{n}$ of intervals of M and any pair x and y in I_0, the distortion of f at x and y along $\mathcal{I}$ satisfies*

$$\frac{|g_n'(x)|}{|g_n'(y)|} \leq \exp\Big(A \sum_{i=0}^{n} |x_i - y_i|^{\alpha} + \frac{B|x-y|}{D_{xy}}\Big)$$

where D_{xy} is the distance between the set $\{x, y\}$ and the post-critical orbit $\cup_{n=1}^{\infty} f^{\circ n}(CP)$.

The idea of the proof of this lemma is the same as that of Lemma 2. Details will be omitted.

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