**On the quasisymmetrical classification of infinitely renormalizable maps**

II. Remarks on maps with a bounded type topology

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## §0 Introduction

This note is a remark to the paper [1]. The aim is to show that the techniques in [1] can also be used to understand the quasisymmetrical classification of infinitely renormalizable maps of bounded type. We will use the same terms and notations as those in [1] without further notices. The result we will prove is the following theorem.

Theorem 1. *Suppose $f$ and $g$ in $\mathcal{U}$ are two infinitely renormalizable maps of bounded type and topologically conjugate. Moreover, suppose $H$ is the homeomorphism between $f$ and $g$. Then $H$ is quasisymmetric.*

Since the techniques as well as ideas of the proof are similar to those in [1], we outline the proof in the next section. The reader may refer to [1] and [2] for more details.

## §1 The outline of the proof of Theorem 1

We outline the proof of Theorem 1 by several lemmas.

Suppose $f = h \circ Q_t$, for some $t > 1$, in $\mathcal{U}$ is an infinitely renormalizable map of bounded type. We note that $Q_t(x) = -|x|^t$. Let $f_0 = f$. And inductively, let $f_k = \alpha_k \circ f_{k-1}^{\circ n_k} \circ \alpha_k^{-1}$ be the renormalization $\mathcal{R}(f_{k-1})$ of $f_{k-1}$ where $\alpha_k$ is the linear rescale from $J_k$ to $[-1, 1]$ and $n_k$ is the return time for any integer $k \geq 1$ (see [1]). We call $J_k$ a restricted interval.

Let $I_0$ be the interval $[-1, 1]$ and $I_k$ be the preimage of $[-1, 1]$ under $\alpha_1 \circ \cdots \circ \alpha_k$ for $k \geq 1$. We note that the set $\{I_k\}_{k=0}^{\infty}$ forms a sequence of nested intervals. Moreover, one of the endpoints of $I_k$, say $p_k$, is a periodic point of period $m_k = n_1 \cdots n_k$ of $f$ and the orbit $O(p_k)$ of $p_k$ under

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$f$ stays outside of the interior of $I_k$ (see Figure 1).

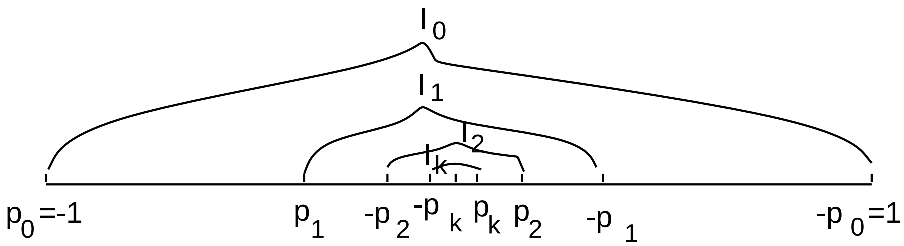


**Figure 1**

Suppose $L_k$ is the image of $I_k$ under $f^{\circ m_k}$ and $T_k$ is the interval bounded by the points $p_k$ and $p_{k+1}$. Let $M_k$ be the complement of $T_k$ in $L_k$. Then $M_k$ is the interval bounded by $p_{k+1}$ and $c_{m_k}$, where $c_{m_k} = f^{\circ m_k}(0)$ (See Figure 2).

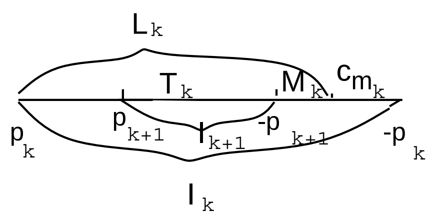


**Figure 2**

Lemma 1. *There is a constant $C_1 = C_1(f) > 0$ such that*

$$C_1^{-1} \leq |M_k|/|I_k| \leq C_1.$$

*for all the integers $k \geq 0$ .*

*Proof.* This lemma is actually proved in [3] by using the techniques such as the smallest interval and shuffle permutation on the intervals.

Lemma 2. *There is a constant $C_2 = C_2(f) > 0$ such that*

$$C_2^{-1} \leq |I_k|/|I_{k-1}| \leq C_2$$

*for all the integers $k \geq 0$.*

We first prove a more general result, as that in [1], as follows. Let $\mathcal{K} = \mathcal{K}(t, N, K)$, for fixed numbers $t > 1$, $N \geq 2$ and $K > 0$, be the subspace of renormalizable maps $f = h \circ Q_t$ in $\mathcal{U}$ such that $|\Big(N(h)\Big)(x)| \leq K$ for all $x$ in $[-1, 0]$ and all the return times $n_k$ of $\mathcal{R}^{\circ k}(f)$ are less than or equal to $N$.

Lemma 3. *There is a constant* $C_3 = C_3(t, N, K) > 0$ *such that*

$$C_3^{-1} \leq f(0) = c_1(f) \leq C_3$$

*for all* $f$ *in* $\mathcal{K}$.

*Proof.* The proof of this lemma is similar to the proof of Lemma 3 in [1] but needs little more work to solve a little more complicated equation.

Remember that $f_k$ is the $k^{th}$-renormalization of $f = f_0$. Let $f_k = h_k \circ Q_t$. We note that the graph of $f_k$ is the rescale of the graph of the restriction of $f^{\circ m_k}$ to $I_k$.

Lemma 4. *There is a constant* $C_4 = C_4(f) > 0$ *such that*

$$|\Big(N(h_k)\Big)(x)| \leq C_4$$

*for all* $x$ *in* $[-1, 0]$ *and all the integer* $k \geq 0$.

*Proof.* It is the a prior bound proved in [3].

*Proof of Lemma 2.* It is now a direct corollary of Lemma 1, Lemma 3 and Lemma 4 for $K = C_4$ and $N = \max_{0 \leq k < \infty}\{n_k\}$.

The set of the nested intervals $\{I_0, I_1, \cdots, I_k, \cdots\}$ gives a partition of $[-1, 1]$ as follows. Let $p_k$ be one of the endpoints of $I_k$ and $O_{k,f}(p_k)$ be the intersection of $I_{k-1}$ and the orbit $O_{k,f}(p_k)$ of $p_k$ under $f$ for $k \geq 1$. Suppose $M_{k-1,1}, \cdots, M_{k-1,n_k+1}$ are the connected components of $I_{k-1} \setminus (O_{k,f}(p_k) \cup I_k)$ for $k \geq 1$. Then the set $\eta_0 = \{M_{k,i}\}$ for $i = 1, \cdots, n_k + 1$ and $k = 1, 2, \cdots$ forms a partition of $[-1, 1]$ (see Figure 3).

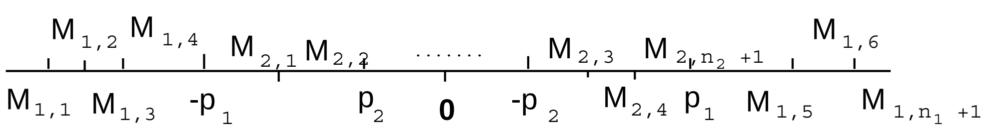


**Figure 3**

Now we are going to define a Markov map $F$ induced by $f$. Let $F$ be a function of $[-1, 1]$ defined by

$$F(x) = \begin{cases} f(x), & x \in M_{1,1} \cup M_{1,2} \cup \cdots \cup M_{1,n_1+1}, \\ f^{\circ n_1}(x), & x \in M_{2,1} \cup \cdots \cup M_{2,n_2+1}, \\ \vdots & \\ f^{\circ n_1 n_2 \cdots n_k}(x), & x \in M_{k,1} \cup \cdots \cup M_{k,n_k+1}, \\ \vdots & \end{cases}$$

It is clearly that $F$ is a Markov map in the sense that the image of every $M_{k,i}$ under $F$ is the union of some intervals in $\eta_0$ (Figure 4).

Let $g_{k,i} = (F|M_{k,i})^{-1}$ for $k = 1, \cdots$, and $i = 1, \cdots n_k + 1$ be the inverse branches of $F$ with respect to the Markov partition $\eta_0$. Suppose $w = i_0 i_1 \cdots i_{l-1}$ is a finite sequence of the set $\mathcal{I} = \{(k,i),\ k = 1, \cdots \text{ and } i = 1, \cdots, n_k + 1\}$. We say it is admissible if the range $M_{i_s}$ of $g_{i_s}$ is contained in the domain $F_{i_{s-1}}(J_{i_{s-1}})$ of $g_{i_{s-1}}$ for $s = 1, \cdots, l-1$. For an admissible sequence $w = i_0 i_1 \cdots i_{l-1}$, we can define $g_w = g_{i_0} \circ g_{i_1} \circ \cdots \circ g_{i_{l-1}}$. We use $D(g_w)$ to denote the domain of $g_w$ and use $|D(g_w)|$ to denote the length of the interval $D(g_w)$.

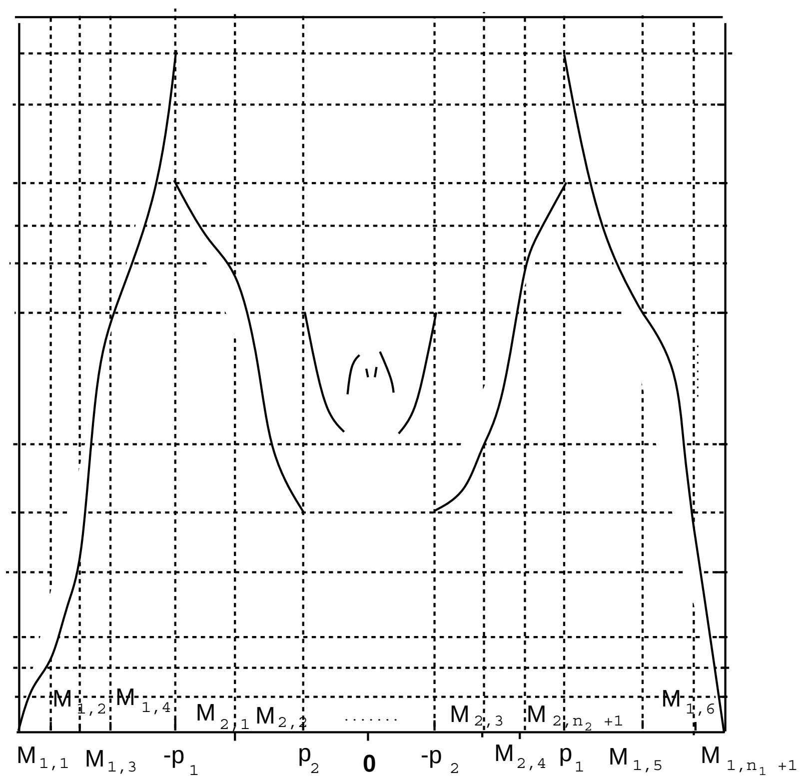


**Figure 4**

Definition 1. *We say the induced Markov map $F$ has bounded distortion property if there is a constant $C_5 = C_5(f) > 0$ such that*

*(a)* $C_5^{-1} \leq |M_{k,i}|/|M_{k,i+1}| \leq C_5$ *for* $k = 1, 2, \cdots$ *and* $i = 1, \cdots, n_k$,
*(b)* $C_5^{-1} \leq |M_{k,i}|/|I_k| \leq C_5$ *for* $k = 1, 2, \cdots$ *and* $i = 1, \cdots, n_k + 1$, *and*
*(b)* $|\big(N(g_w)\big)(x)| \leq C_5/|D(g_w)|$ *for all* $x$ *in* $D(g_w)$ *and all admissible* $w$.

The reason we give this definition is the following lemma as that in [1].

LEMMA 5. *Suppose $f$ and $g$ in $\mathcal{U}$ are two infinitely renormalizable maps of bounded type and $H$ is the conjugacy between $f$ and $g$. If both of the induced Markov maps $F$ and $G$ have the bounded distortion property, then $H$ is quasisymmetric.*

*Proof.* It can be proved by almost the same arguments as that we used in the paper [2]. For more details of the proof, the reader may refer to [2].

Now the proof of Theorem 1 concentrates on the next lemma.

LEMMA 6. *Suppose $f = h \circ Q_t$, for some $t > 1$, in $\mathcal{U}$ is an infinitely renormalizable map of bounded type and $F$ is the Markov map induced by $f$. Then $F$ has the bounded distortion property.*

*Proof.* Let $I_{k,j} = \left(f^{\circ m_{k-1}}|I_{k-1}\right)^{\circ j}(I_k)$ for $j = 0, 1, \cdots, n_k$ and $\{G_{k,i}\}$ are all the connected components of $I_k \setminus \cup_{j=0}^{n_k} I_{k,j}$ (Figure 5).

Each $M_{k,j}$ is either a single $G_{k,i}$ or $I_{k,j} \cup G_{k,i}$ for some $j$ and some $i$. By the bounded geometry [3] of $\{I_{k,j}\}$ and $\{G_{k,i}\}$, there is a constant $C_6 > 1$ such that all the ratios $|I_{k,j}|/|I_{k,j'}|$, $|G_{k,i}|/|G_{k,i'}|$ and $|G_{k,i}|/|I_{k,j}|$ are in the interval $[C_6^{-1}, C_6]$. We note that $C_6$ does not depend on $k$ as well as $i$, $i'$, $j$ and $j'$. This fact and Lemma 3 imply the condition (a) in Definition 1.

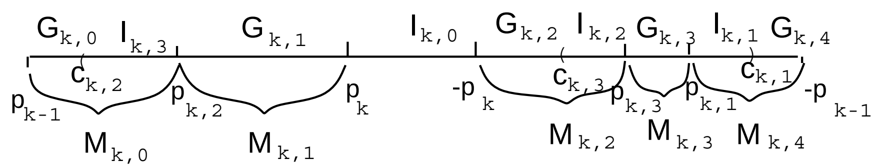


**Figure 5**

The condition (b) in Definition 1 is assured by Lemma 2 and the condition (a). The proof of the condition (c) in Definition 1 is similar to that in [1].

The arguments in Lemma 1 to Lemma 6 give the proof of Theorem 1.

# References


[1] Y. Jiang, On quasisymmetrical classification of infinitely renormalizable maps – Maps with Feigenbaum's topology, preprint in this issue, IMS at SUNY at Stony Brook.

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[3] D. Sullivan, Bounds, quadratic differentials, and renormalization conjectures, preprint, 1991 and American Mathematical Society Centennial Publications, Volume 2: Mathematics into the Twenty-first Century, to appear.

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