On the quasisymmetrical classification of infinitely renormalizable maps

II. Remarks on maps with a bounded type topology

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§0 Introduction

This note is a remark to the paper [1]. The aim is to show that the techniques in [1] can also be used to understand the quasisymmetrical classification of infinitely renormalizable maps of bounded type. We will use the same terms and notations as those in [1] without further notices. The result we will prove is the following theorem.

THEOREM 1. Suppose \( f \) and \( g \) in \( \mathcal{U} \) are two infinitely renormalizable maps of bounded type and topologically conjugate. Moreover, suppose \( H \) is the homeomorphism between \( f \) and \( g \). Then \( H \) is quasisymmetric.

Since the techniques as well as ideas of the proof are similar to those in [1], we outline the proof in the next section. The reader may refer to [1] and [2] for more details.

§1 The outline of the proof of Theorem 1

We outline the proof of Theorem 1 by several lemmas.

Suppose \( f = h \circ Q_t \) for some \( t > 1 \), in \( \mathcal{U} \) is an infinitely renormalizable map of bounded type. We note that \( Q_t(x) = -|x|^t \). Let \( f_0 = f \). And inductively, let \( f_k = \alpha_k \circ f_{k-1}^\circ \circ \alpha_k^{-1} \) be the renormalization \( R'(f_{k-1}) \) of \( f_{k-1} \) where \( \alpha_k \) is the linear rescale from \( J_k \) to \([-1, 1]\) and \( n_k \) is the return time for any integer \( k \geq 1 \) (see [1]). We call \( J_k \) a restricted interval.

Let \( I_0 \) be the interval \([-1, 1]\) and \( I_k \) be the preimage of \([-1, 1]\) under \( \alpha_1 \circ \cdots \circ \alpha_k \) for \( k \geq 1 \). We note that the set \( \{ I_k \} \) forms a sequence of nested intervals. Moreover, one of the endpoints of \( I_k \), say \( p_k \), is a periodic point of period \( m_k = n_1 \cdots n_k \) of \( f \) and the orbit \( O(p_k) \) of \( p_k \) under
f stays outside of the interior of \( I_k \) (see Figure 1).

![Figure 1](image)

Suppose \( L_k \) is the image of \( I_k \) under \( f^{\circ m_k} \) and \( T_k \) is the interval bounded by the points \( p_k \) and \( p_{k+1} \). Let \( M_k \) be the complement of \( T_k \) in \( L_k \). Then \( M_k \) is the interval bounded by \( p_{k+1} \) and \( c_{m_k} \), where \( c_{m_k} = f^{\circ m_k}(0) \) (See Figure 2).

![Figure 2](image)

**Lemma 1.** There is a constant \( C_1 = C_1(f) > 0 \) such that

\[
C_1^{-1} \leq |M_k|/|I_k| \leq C_1.
\]

for all the integers \( k \geq 0 \).

**Proof.** This lemma is actually proved in [3] by using the techniques such as the smallest interval and shuffle permutation on the intervals.

**Lemma 2.** There is a constant \( C_2 = C_2(f) > 0 \) such that

\[
C_2^{-1} \leq |I_k|/|I_{k-1}| \leq C_2
\]

for all the integers \( k \geq 0 \).

We first prove a more general result, as that in [1], as follows. Let \( \mathcal{K} = \mathcal{K}(t, N, K) \), for fixed numbers \( t > 1 \), \( N \geq 2 \) and \( K > 0 \), be the subspace of renormalizable maps \( f = h \circ Q_t \) in \( \mathcal{U} \) such that \( |(N(h))(x)| \leq K \) for all \( x \) in \([-1, 0]\) and all the return times \( n_k \) of \( \mathcal{R}^k(f) \) are less than or equal to \( N \).
Lemma 3. There is a constant $C_3 = C_3(t, N, K) > 0$ such that

$$C_3^{-1} \leq f(0) = c_1(f) \leq C_3$$

for all $f$ in $\mathcal{K}$.

Proof. The proof of this lemma is similar to the proof of Lemma 3 in [1] but needs little more work to solve a little more complicated equation.

Remember that $f_k$ is the $k^{th}$-renormalization of $f = f_0$. Let $f_k = h_k \circ Q_t$. We note that the graph of $f_k$ is the rescale of the graph of the restriction of $f^{\circ m_k}$ to $I_k$.

Lemma 4. There is a constant $C_4 = C_4(f) > 0$ such that

$$|\left(N(h_k)\right)(x)| \leq C_4$$

for all $x$ in $[-1, 0]$ and all the integer $k \geq 0$.

Proof. It is the a priori bound proved in [3].

Proof of Lemma 2. It is now a direct corollary of Lemma 1, Lemma 3 and Lemma 4 for $K = C_4$ and $N = \max_{0 \leq k < \infty} \{n_k\}$.

The set of the nested intervals $\{I_0, I_1, \cdots, I_k, \cdots\}$ gives a partition of $[-1, 1]$ as follows. Let $p_k$ be one of the endpoints of $I_k$ and $O_{k,f}(p_k)$ be the intersection of $I_{k-1}$ and the orbit $O_{k,f}(p_k)$ of $p_k$ under $f$ for $k \geq 1$. Suppose $M_{k-1,1}, \cdots, M_{k-1,n_k+1}$ are the connected components of $I_{k-1} \setminus (O_{k,f}(p_k) \cup I_k)$ for $k \geq 1$. Then the set $\eta_0 = \{M_{k,i}\}$ for $i = 1, \cdots, n_k + 1$ and $k = 1, 2, \cdots$ forms a partition of $[-1, 1]$ (see Figure 3).

![Figure 3](image-url)

Now we are going to define a Markov map $F$ induced by $f$. Let $F$ be a function of $[-1, 1]$ defined by

$$F(x) = \begin{cases} 
  f(x), & x \in M_{1,1} \cup M_{1,2} \cup \cdots \cup M_{1,n_1+1}, \\
  f^{\circ m_1}(x), & x \in M_{2,1} \cup \cdots \cup M_{2,n_2+1}, \\
  \vdots \\
  f^{\circ m_{n_2-1} \cdots n_k}(x), & x \in M_{k,1} \cup \cdots \cup M_{k,n_k+1}, \\
  \vdots 
\end{cases}$$
It is clearly that $F$ is a Markov map in the sense that the image of every $M_{k,i}$ under $F$ is the union of some intervals in $\eta_0$ (Figure 4).

Let $g_{k,i} = (F|_{M_{k,i}})^{-1}$ for $k = 1, \ldots$, and $i = 1, \ldots n_k + 1$ be the inverse branches of $F$ with respect to the Markov partition $\eta_0$. Suppose $w = i_0i_1 \cdots i_{l-1}$ is a finite sequence of the set $\mathcal{I} = \{(k,i), k = 1, \ldots \text{ and } i = 1, \ldots n_k + 1\}$. We say it is admissible if the range $M_{i_s}$ of $g_{i_s}$ is contained in the domain $F_{i_{s-1}}(J_{i_{s-1}})$ of $g_{i_{s-1}}$ for $s = 1, \ldots, l - 1$. For an admissible sequence $w = i_0i_1 \cdots i_{l-1}$, we can define $g_w = g_{i_0} \circ g_{i_1} \circ \cdots \circ g_{i_{l-1}}$. We use $D(g_w)$ to denote the domain of $g_w$ and use $|D(g_w)|$ to denote the length of the interval $D(g_w)$.

**Figure 4**

**Definition 1.** We say the induced Markov map $F$ has bounded distortion property if there is a constant $C_5 = C_5(f) > 0$ such that

(a) $C_5^{-1} \leq |M_{k,i}|/|M_{k,i+1}| \leq C_5$ for $k = 1, 2, \ldots$ and $i = 1, \ldots, n_k$,

(b) $C_5^{-1} \leq |M_{k,i}|/|I_k| \leq C_5$ for $k = 1, 2, \ldots$ and $i = 1, \ldots, n_k + 1$, and

(b) $\left|\left(\mathcal{N}(g_w)\right)(x)\right| \leq C_5/|D(g_w)|$ for all $x$ in $D(g_w)$ and all admissible $w$.

The reason we give this definition is the following lemma as that in [1].
Lemma 5. Suppose $f$ and $g$ in $\mathcal{U}$ are two infinitely renormalizable maps of bounded type and $H$ is the conjugacy between $f$ and $g$. If both of the induced Markov maps $F$ and $G$ have the bounded distortion property, then $H$ is quasisymmetric.

Proof. It can be proved by almost the same arguments as that we used in the paper [2]. For more details of the proof, the reader may refer to [2].

Now the proof of Theorem 1 concentrates on the next lemma.

Lemma 6. Suppose $f = h \circ Q_t$, for some $t > 1$, in $\mathcal{U}$ is an infinitely renormalizable map of bounded type and $F$ is the Markov map induced by $f$. Then $F$ has the bounded distortion property.

Proof. Let $I_{k,j} = (f^{\circ n_j})^{-1}(I_k)$ for $j = 0, 1, \ldots, n_k$ and $\{G_{k,i}\}$ are all the connected components of $I_k \setminus \bigcup_{j=0}^{n_k} I_{k,j}$ (Figure 5).

Each $M_{k,j}$ is either a single $G_{k,i}$ or $I_{k,j} \cup G_{k,i}$ for some $j$ and some $i$. By the bounded geometry [3] of $\{I_{k,j}\}$ and $\{G_{k,i}\}$, there is a constant $C_0 > 1$ such that all the ratios $|I_{k,j}|/|I_{k,j'}|$, $|G_{k,i}|/|G_{k,i'}|$ and $|I_{k,j}|/|I_{k,j'}|$ are in the interval $[C_0^{-1}, C_0]$. We note that $C_0$ does not depend on $k$ as well as $i$, $i'$, $j$ and $j'$. This fact and Lemma 3 imply the condition (a) in Definition 1.

Figure 5

The condition (b) in Definition 1 is assured by Lemma 2 and the condition (a). The proof of the condition (c) in Definition 1 is similar to that in [1].

The arguments in Lemma 1 to Lemma 6 give the proof of Theorem 1.

References


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