Periods implying almost all periods, trees with snowflakes and zero entropy maps

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August 1990, revised August 1991

Abstract. Let X be a compact tree and $f : X \longrightarrow X$ be a continuous map; denote by $End(X)$ the number of endpoints of X and by $Edg(X)$ the number of edges of X. In Section 2 we prove the following statements:

(1) if $n > 1$ is an integer with no prime divisors less than $End(X) + 1$ and f has a cycle of period n, then f has cycles of all periods greater than $2End(X)(n-1)$, and if $h(f)$ is its topological entropy then $h(f) \geq \dfrac{\ln 2}{nEnd(X)-1}$;

(2) if $0 < n < End(X) + 1$ and E is the set of all periods of cycles of some interval map then there exists a continuous map $g : X \longrightarrow X$ such that the set of all periods of cycles of g is $nE \cup \{1\}$ where $nE = \{nk : k \in E\}$.

This implies that if p is the least prime number greater than $End(X)$ and f has cycles of all periods from 1 to $2End(X)(p-1)$, then f has cycles of all periods (for tree maps this verifies the conjecture of M. Misiurewicz, made in Bratislava in 1990). Combining the spectral decomposition theorem for graph maps (see [B1-B3]) with our results, we prove the equivalence of the following statements for tree maps:

(1) there exists n such that f has a cycle of period mn for any m;

(2) $h(f) > 0$.

Note that the Misiurewicz conjecture and the last result are true for graph maps ([B4,B5]).

In Section 3 we study properties of tree maps with zero entropy. Namely let C be a periodic orbit of a tree map $g : X \longrightarrow X$; we call it *a snowflake* if it has certain properties related to those of Block's *simple periodic orbits* for interval maps [Bl]. We prove that the following statements are equivalent:

(1) $h(f) = 0$;

(2) $(f, orb\, x)$ is a snowflake for every $x \in Per\, f$;

(3) the period of every cycle of f is of the form $2^l m$ where $m \leq Edg(X)$ is an odd integer and all prime divisors of m are less than $End(X) + 1$.

0. Introduction

Let us call one-dimensional compact branched manifolds *graphs*; we call them *trees* if they are connected and do not contain sets homeomorphic to the circle. *In what follows we consider only continuous tree maps.* One of the well-known and impressive results about dynamical properties of one-dimensional maps is the Sharkovskii theorem [S1] about the

This paper was partly written while I was visiting MPI für Mathematik in Bonn and SUNY at Stony Brook; I would like to thank both institutions for their kind hospitality

co-existence of periods of cycles for maps of the real line. To formulate it let us introduce the following *Sharkovskii ordering* for positive integers:

$$(*) \qquad 3 \prec 5 \prec 7 \prec \dots \prec 2 \cdot 3 \prec 2 \cdot 5 \prec 2 \cdot 7 \prec \dots \prec 8 \prec 4 \prec 2 \prec 1$$

Denote by $S(k)$ the set of all integers m such that $k \prec m$ or $k = m$ and by $S(2^\infty)$ the set $\{1, 2, 4, 8, \dots\}$. Also denote by $P(\varphi)$ the set of periods of cycles of a map φ.

Theorem[S1]. *Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous map. Then either $P(g) = \emptyset$ or there exists $k \in \mathbb{N} \cup 2^\infty$ such that $P(g) = S(k)$. Moreover, for any such k there exists a map $g : [0, 1] \to [0, 1]$ with $P(g) = S(k)$ and there exists a map $g_0 : \mathbb{R} \to \mathbb{R}$ with $P(g_0) = \emptyset$.*

Clearly, one can apply the Sharkovskii theorem to an interval map; in this case $P(g) \neq \emptyset$ and hence there exists $k \in \mathbb{N} \cup 2^\infty$ such that $P(g) = S(k)$. So one can consider the Sharkovskii theorem as the first result describing possible sets of periods of cycles for tree maps. Other information about these sets for tree maps is contained in [ALM] for maps of the letter Y and [Ba] for maps of the n-od.

The Sharkovskii theorem implies that if a map $f : \mathbb{R} \to \mathbb{R}$ has a cycle of period 3 then it has cycles of all periods. The following conjecture, formulated by M. Misiurewicz at the Problem Session at Czecho-Slovak Summer Mathematical School near Bratislava in 1990, is related to the aforementioned property of maps of the real line.

Misiurewicz Conjecture. *For a graph X there exists an integer $L = L(X)$ such that for a map $f : X \to X$ the inclusion $P(f) \supset \{1, 2, \dots, L\}$ implies that $P(f) = \mathbb{N}$.*

In Section 2 we verify the Misiurewicz conjecture for tree maps and get more information about sets of periods of cycles for tree maps. The general verification of this conjecture for arbitrary continuous graph maps may be found in [B4,B5]. Note that all results of the paper are true in the same formulations for finite unions of connected trees; the corresponding extension of our results is left to the reader.

Fix a tree X. We use the terms "vertex", "edge" and "endpoint" in the usual sense. Denote the number of edges of X by $Edg(X)$ and the number of endpoints of X by $End(X)$. An integer n is said to have *the ap-property for X* or to be *an ap-number for X* if there exists an integer $j(n, X)$ such that any map $f : X \to X$ with a cycle of period n also has cycles of all periods greater than $j(n, X)$. Theorem 1 describes the set of all ap-integers for X.

Theorem 1. *Let X be a tree. Then the following holds.*

(1) *Let $n > 1$ be an integer with no prime divisors less than $End(X) + 1$. If a map $f : X \to X$ has a cycle of period n, then f has cycles of all periods greater than $2End(X)(n-1)$. Moreover, $h(f) \geq \dfrac{\ln 2}{nEnd(X) - 1}$.*

(2) *Let $1 \leq n \leq End(X)$ and E be the set of all periods of cycles of some interval map. Then there exists a map $f : X \to X$ such that the set of all periods of cycles of f is $\{1\} \cup nE$, where $nE \equiv \{nk : k \in E\}$.*

Thus integers with no prime divisors less than $End(X) + 1$ form the set of ap-integers for X. For interval maps this implies that the set of all ap-integers coincides with the set of all odd numbers greater than 1; clearly, this also may be deduced from Sharkovskii's theorem.

Let us formulate some corollaries of Theorem 1.

Corollary 1 (cf. [BF]). *Let $f : X \to X$ be a cycle of period $n = pk$ where $p > 1$ has no prime divisors less than $End(X) + 1$. Then $h(f) \geq \dfrac{\ln 2}{k[pEnd(X) - 1]} > \dfrac{\ln 2}{nEnd(X) - n}$.*

The next corollary verifies the Misiurewicz conjecture for tree maps.

Corollary 2. *Let p be the least prime number greater than $End(X)$. If $f : X \to X$ has cycles of all periods from 1 to $2End(X)(p-1)$ then f has cycles of all periods.*

Theorem 1 and the spectral decomposition theorem for graph maps ([B1-B3]) imply

Corollary 3. *The following two statements are equivalent:*

(1) *there exists n such that f a cycle of period mn for any m;*

(2) *$h(f) > 0$.*

Note that in fact Corollary 3 is true for arbitrary graph maps ([B4,B5]; the different proof may be found in [LM]).

In Section 3 we study properties of tree maps with zero entropy. It was proved in the papers [BF], [MS] and [M] that for an interval map g the fact that $h(g)=0$ is equivalent to $P(g)\subset\{1,2,2^2,\dots\}$. The structure of cycles for zero entropy interval maps was studied in [Bl]. In Section 3 we generalize the above mentioned results to the case of tree maps. First we need some definitions.

Suppose that $A\subset X$; denote by $[A]$ the smallest connected set containing A and call $[A]$ *the connected hull* of A. The definition makes sense because X is a tree. Let us give a definition of *a cycle of sets.* Namely suppose that there are connected sets $\{C_i\}_{i=0}^{n-1}$ such that $fC_i\subset C_{i+1}$ for $0\le i\le n-1, fC_{n-1}\subset C_0$ and the sets $C_i,\ 0\le i\le n-1$ are pairwise disjoint. Then we call the collection of sets $\{C_i\}_{i=0}^{n-1}$ *a cycle of sets (of period n).* Note that by the definition speaking of a cycle of sets we always mean *connected sets.* It is convenient to order sets in a cycle in accordance with the way they are permuted, so usually we will index them in this order. Finally suppose that $Z\subset X$ and $[Z]\setminus Z$ is a connected set; then we call the set Z *surrounding.*

Let us pass to the definition of a snowflake. Suppose that $f:X\to X$ is continuous and $C\subset X$ is a connected invariant set. We say that *$f|C$ is a snowflake map (of type $m_0=1<m_1<\dots<m_k$)* if there exists a collection of nested cycles of sets

$$C=Y_0^0\supset\bigcup_{r=0}^{m_1-1}Y_r^1\supset\dots\supset\bigcup_{r=0}^{m_k-1}Y_r^k,\quad k>0$$

such that the following properties hold:

1) $Y_0^0\supset Y_0^1\supset\dots\supset Y_0^k$, so by the definition of cycle of sets it is clear that if $s\equiv t \pmod{m_i}$ with $0\le t<m_i,\ 0\le s<m_{i+1}$, then $Y_t^i\supset Y_s^{i+1}$ (in what follows we will say that the sets Y_r^i and the cycle of sets $\bigcup_{r=0}^{m_i-1}Y_r^i$ are *of level* i);

2) the set $\bigcup_{s\equiv t \pmod{m_i}}Y_s^{i+1}$ is surrounding for $0\le t<m_i$ (in other words, all sets of the next level belonging to some set of the previous level form a surrounding set); in particular, $\bigcup_{r=0}^{m_1-1}Y_r^1$ is a surrounding set.

The cycles of sets $\{\bigcup_{r=0}^{m_i-1} Y_r^i\}_{i=0}^k$ are called *generating*. Now suppose that A is a cycle of sets of period n. Clearly, $\bigcup_{i\geq 0} f^i[A] = C$ is a connected invariant set; if $f|C$ is a snowflake map (of type $m_0 = 1 < m_1 < \cdots < m_k = n$) and each generating set on the last level contains exactly one set from A, then we say that (f, A) is *a snowflake (of type $m_0 = 1 < m_1 < \cdots < m_k$)*. Note that in fact C is the minimal by inclusion connected invariant set containing A; the connected hull $[A]$ itself may not be invariant, so in order to get such a set we need to consider the whole orbit of $[A]$.

Let us give some examples of snowflakes for tree maps. To begin with let us consider maps of the interval. Then the only possible non-connected surrounding sets Z are those having exactly two connected components, because otherwise $[Z]\setminus Z$ is not connected which contradicts to the definition of surrounding sets. Now let $f : [0,1] \to [0,1]$ be a continuous interval map, let A be a cycle of sets and (f, A) be a snowflake of type $m_0 = 1 < m_1 < \cdots < m_k$. Set $C = \bigcup_{i\geq 0} f^i[A]$ and suppose that $\{\bigcup_{r=0}^{m_i-1} Y_r^i\}_{i=0}^k$ are generating sets (so every Y_r^i is an interval). Then by the definition $\bigcup_{r=0}^{m_1-1} Y_r^1$ is a surrounding set so $m_1 = 2$ and Y_0^1, Y_1^1 are simply intervals which are interchanged by f. Similarly considering other levels of the snowflake (f, A) we can see that the picture on each level is close to that on the first one; in other words, for any $0 \leq i < k$ we have $m_{i+1} = 2m_i$, for any $0 \leq t < m_i$ the intervals $Y_t^{i+1}, Y_{t+m_i}^{i+1}$ are the only intervals of level $i+1$ lying inside Y_t^i and they exchange their places under the appropriate iterations of f (namely $f^{m_i} Y_t^{i+1} \subset Y_{t+m_i}^{i+1}$, $f^{m_i} Y_{t+m_i}^{i+1} \subset Y_t^{i+1}$). So we see that the definition of a snowflake generalizes the definition of a simple periodic orbit (see [Bl]) to tree maps.

A similar situation takes place in the general case. Inside a set of any level the sets from the next one form a surrounding set, so the pictures on each level inside each set are analogous; one can consider this as a sort of self-similarity. Let us also point out that if Z is a surrounding set then by the definition $A = [Z] \setminus Z$ is connected, and geometrically $[Z]$ may be obtained by "sticking" components of Z to the endpoints of A.

The most important case which we consider is the one when a cycle of sets A is in fact an orbit of a periodic point. The main result is the following

Theorem 2. *Let X be a tree. Then the following statements are equivalent:*

(1) $h(f)=0$;

(2) $(f, orb\, x)$ *is a snowflake for every* $x \in Per\, f$;

(3) *every* $n \in P(f)$ *is of form* $n = 2^l m$ *where* $m \leq Edg(X)$ *is an odd integer and all prime divisors of* m *are less than* $End(X)+1$.

Notation

f^n is the n-fold iterate of a map f;

$\overline{Z}$ is the closure of Z;

$\mathrm{orb}\, x \equiv \{f^n x\}_{n=0}^{\infty}$ is the *orbit (trajectory)* of x;

$\mathrm{Per}\, f$ is the set of all periodic points of a map f;

$P(f)$ is the set of all periods of periodic points of a map f;

$h(f)$ is the topological entropy of a map f.

1. Preliminary lemmas

We first give some definitions. Let X be a tree. By *an interval* we mean a homeomorphic image $h[0,1]$ of an interval $[0,1]$ in X regardless of whether it contains vertices of X or not; we also consider degenerate intervals, i.e. points. Note that the notation we use for an interval corresponds to that from the definition of a connected hull of a set. Points $h(0)=a, h(1)=b$ are called *endpoints* of the interval $h[0,1] \equiv [a,b]$; clearly, there exists a unique interval $[a,b]$ with given endpoints a,b. Moreover, let us denote intervals of different types in the following way: $(a,b] \equiv [a,b] \setminus \{a\}, [a,b) \equiv [a,b] \setminus \{b\}, (a,b) \equiv [a,b] \setminus \{a,b\}$. Furthermore, let $h : [0,1] \to [a,b],\ h(0)=a,\ h(1)=b$ be a homeomorphism; if $x \in [a,b], y \in [a,b]$ and $h^{-1}(x) < h^{-1}(y)$ then say that *x is closer to a than y (or y is further from a than x) on the interval* $[a,b]$ (in fact we will not mention the interval if it is clear which one we mean). Clearly, the definition is correct. In the similar way we will speak about subsets of intervals in X; in this case by *C is closer to a than D* we mean that for any $c \in C, d \in D$ either $h^{-1}(c) < h^{-1}(d)$ or $c=d$. In what follows we consider a continuous map $f : X \to X$.

Lemma 1. *Let $[a, b], [c, d]$ be intervals and $f[a, b] \supset [c, d]$, $(fa, c) \cap (c, d) = \emptyset$, $(d, fb) \cap (c, d) = \emptyset$. Suppose also that $I_0, I_1, \dots, I_k \subset [c, d]$ are intervals with pairwise disjoint interiors containing no vertices of X and that I_{i+1} is further from c than I_i for $0 \le i \le k-1$. Then there exist intervals $J_0, J_1, \dots, J_k \subset [a, b]$ with pairwise disjoint interiors such that J_{i+1} is further from a than J_i for $0 \le i \le k-1$ and $fJ_i = I_i$, $0 \le i \le k$.*

PROOF: Clearly, for any $0 \le i \le k$ there exist intervals $L \subset [a, b]$ such that $fL = I_i$. Indeed, let $I_i = [x, y]$ where x is closer to c than y. Choose the closest to a preimage of y and denote it by y_{-1}. Then choose the preimage of x closest to y_{-1} in $[a, y_{-1}]$, and denote it by x_{-1}. It is easy to see that $f[x_{-1}, y_{-1}] = [x, y]$. Say that an interval L is *good* if $fL = I_i$ for some i and inclusion $M \subsetneqq L$ implies that $fM \ne I_i$. Choose for $0 \le i \le k$ the closest to a good interval J_i such that $fJ_i = I_i$. The relations $(fa, c) \cap (c, d) = \emptyset, (d, fb) \cap (c, d) = \emptyset$ easily imply now that J_i is closer to a than J_{i+1} for $0 \le i \le k-1$ which completes the proof. ∎

Lemma 2. *Let $J_0 = [c_0, d_0], J_1 = [c_1, d_1], \dots, J_k = [c_k, d_k]$ be intervals and $0 = n_0 < n_1 < \dots < n_{k+1}$ be integers. Suppose also that for any $0 \le i \le k-1$, we have $f^{n_{i+1}-n_i} J_i \supset J_{i+1}, (d_{i+1}, f^{n_{i+1}-n_i} d_i) \cap (c_{i+1}, d_{i+1}) = \emptyset$ and similarly $f^{n_{k+1}-n_k} J_k \supset J_0, (d_0, f^{n_{k+1}-n_k} d_k) \cap (c_0, d_0) = \emptyset$. Then there exists $z \in J_0$ such that $f^{n_i} z \in J_i (0 \le i \le k)$ and $f^{n_{k+1}} z = z$.*

Remark. In particular, if $f^{n_{i+1}-n_i} J_i \supset J_{i+1}$, $f^{n_i} d_0 = d_i$ for $0 \le i \le k$ and also $f^{n_{k+1}-n_k} J_k \supset J_0, (d_0, f^{n_{k+1}-n_k} d_k) \cap (c_0, d_0) = \emptyset$ then there exists $z \in J_0$ such that $f^{n_i} z \in J_i (0 \le i \le k)$ and $f^{n_{k+1}} z = z$ (note that $(d_0, f^{n_{k+1}-n_k} d_k) = (d_0, f^{n_{k+1}} d_0)$).

PROOF: We divide the proof into several steps.

STEP 1. *There exist a number M and intervals $L_0, L_1, \dots, L_M \subset J_0$ such that the following holds:*

1. *the interiors of intervals $L_0, L_1, \dots, L_M$ are pairwise disjoint;*
2. *$f^{n_i} L_j$ is an interval $(0 \le i \le k+1, 0 \le j \le M)$;*
3. *$f^{n_i} L_j \subset J_i$ $(0 \le i \le k, 0 \le j \le M)$;*
4. *$f^{n_{k+1}} L_0 \cup f^{n_{k+1}} L_1 \cup \dots \cup f^{n_{k+1}} L_M = J_0$ and interiors of intervals $f^{n_{k+1}} L_j, 0 \le j \le M$ are pairwise disjoint;*

(5) *for any* $0 \le i \le M-1$, *the interval* L_i *is closer to* c_0 *than* L_{i+1} *and the interval* $f^{n_{k+1}} L_i$ *is closer to* c_0 *than* $f^{n_{k+1}} L_{i+1}$.

First choose intervals $N_0, N_1, \dots, N_m$ in such a way that their union is J_0 and their interiors are pairwise disjoint and do not contain vertices of X; moreover, we may assume that N_i is closer to c_0 than N_{i+1} for $0 \le i \le m-1$. Furthermore, choose a point $x_k \in J_k$ such that $f^{n_{k+1}-n_k} x_k = c_0$. Now by Lemma 1 we can find intervals $T_0, T_1, \dots, T_s \subset [x_k, d_k]$ with pairwise disjoint interiors in such a way that $f^{n_{k+1}-n_k} T_i = N_i, 0 \le i \le s$, and T_i is closer to x_k than $T_{i+1}, 0 \le i \le s-1$. It remains to divide the intervals T_i into subintervals in such a way that these new subintervals have pairwise disjoint interiors, do not contain vertices of X, and still are ordered in the sense of "closer - further" ordering on the interval $[x_k, d_k]$. Going on with this construction and making use of Lemma 1, we will find intervals $L_0, L_1, \dots, L_M$ with the required properties.

STEP 2. *In the situation of Step 1, there exists a point* $z \in \bigcup_{i=0}^{M} L_i$ *such that* $f^{n_{k+1}} z = z$.

Denote $f^{n_{k+1}}$ by g. We may assume that $J_0 = [0,1]$ and intervals $L_0, L_1, \dots, L_M$ and $gL_0, gL_1, \dots, gL_M$ increase in the usual sense. Now the fact that $\bigcup_{i=1}^{M} gL_i = [0,1] \supset \bigcup_{i=1}^{M} L_i$ easily implies that $\sup g|L_M = 1 \ge L_M$, $\inf g|L_0 = 0 \le L_0$ and so there exists i such that $gL_i \supset L_i$. Indeed, we need to find i such that $\sup g|L_i \ge L_i$ and $\inf g|L_i \le L_i$. Clearly, the fact that $\inf g|L_{j+1} > L_{j+1}$ implies that $\sup g|L_j > L_j$ (because the intervals $\{L_j\}$ are ordered by increasing and at the same time $\bigcup_{i=1}^{M} gL_i = [0,1]$). Now take the maximal i such that $\inf g|L_i \le L_i$. If $i = M$ then $gL_M \supset L_M$ and we are done. If $i < M$ then $\inf g|L_{i+1} > L_{i+1}$ and so $\sup g|L_i > L_i$ and $gL_i \supset L_i$ which completes the proof. ∎

A connected closed set $Y \subset X$ is called a *subtree*.

Lemma 3. *Let* X *be a tree,* $Y \subset X$ *be a subtree and* $f : Y \to X$ *be a continuous map such that if* $a \in Y$ *then* $(a, fa] \cap Y \ne \emptyset$. *Then there exists* $z \in Y$ *such that* $fz = z$.

PROOF: Let us construct a map $g : X \to X$ in the following way. First define a map $h : X \to Y$ so that if $x \in Y$ then $hx = x$ and if $x \notin Y$ then $hx = y$ where $y \in Y$ is the unique point with $(y, x] \cap Y = \emptyset$. Now consider a map $g = f \circ h : X \to X$. Then there exists $z \in X$ such that $gz = z$. If $z \in Y$ then $hz = z = fz$ and we are done. Let $z \notin Y$.

Then $hz = y$ where $y \in Y$ and $(y, z] \cap Y = \emptyset$; at the same time $gz = f(hz) = fy = z$, so $(y, fy] \cap Y = \emptyset$ which is a contradiction. ∎

Lemma 4. *Let $Y \subset X$ be a subtree, $f : X \to X$ be a continuous map. Then there exists a point $y \in Y$ such that for any $z \in Y$ the relation $fz \in Y$ implies the inclusion $f[y, z] \supset [y, fz]$ and either $fy = y$ or $fy \notin Y$ and $(y, fy] \cap Y = \emptyset$.*

PROOF: Consider the case when there is no fixed point in Y. Then by Lemma 3 $(y, fy] \cap Y = \emptyset$ for some $y \in Y$. Now the properties of trees imply the conclusion. ∎

In what follows we call the point $y \in Y$ existing by Lemma 4 *the basic point for* (f, Y).

1. The description of ap-numbers for tree maps

First we need the following definition: if $x \in Per f$ then we call points $a, b \in orb\, x$ *neighboring* if $(a, b) \cap orb\, x = \emptyset$.

Theorem 1. *Let X be a tree. Then the following holds.*

(1) *Let $n > 1$ be an integer with no prime divisors less than $End(X) + 1$. If a map $f : X \to X$ has a cycle of period n then f has cycles of all periods greater than $2End(X)(n - 1)$. Moreover, $h(f) \geq \dfrac{\ln 2}{nEnd(X) - 1}$.*

(2) *Let $1 \leq n \leq End(X)$ and E be the set of all periods of cycles of some interval map. Then there exists a continuous map $f : X \to X$ such that the set of all periods of cycles of f is $\{1\} \cup nE$, where $nE \equiv \{nk : k \in E\}$.*

PROOF: We start with statement 1). Let x be a periodic point of period $n > 1$ where n has no prime divisors less than $End(X) + 1$. Let y be a basic point for $(f, [orb\, x])$; then $y \in [orb\, x] \setminus orb\, x$. Consider the connected component Z of $[orb\, x] \setminus orb\, x$ such that $y \in Z$. If $z_1, z_2, \dots, z_l$ are endpoints of Z then $z_i \in orb\, x$ and $(y, z_i) \cap orb\, x = \emptyset, 1 \leq i \leq l$. Denote by Z_i the connected component of the set $[orb\, x] \setminus Z$ containing z_i and let $Y_i = Z_i \cap orb\, x$. We divide the rest of the proof into steps, but first let us make the following two quite simple remarks: 1) $l \leq End(X)$; 2) $n \geq 3$. We also need the following easy property which we formulate without proof.

Property A. *If $\{A_1, A_2, \ldots, A_n\}$ are sets and $B = \bigcup_{i=1}^n [A_i]$ is connected then $B = [\bigcup_{i=1}^n A_i]$.*

STEP 1. *There exist two neighboring points $a, b \in orb\, x$ such that $b \in (a, y)$ and $y \in f^{l-1}(a, b)$.*

Let us describe the following procedure. Let $F_1, \ldots, F_m$ be pairwise disjoint subsets of $orb\, x = \bigcup_{i=1}^m F_i$ such that $[F_1], \ldots, [F_m]$ are pairwise disjoint subtrees of X; denote $\bigcup_{i=1}^m [F_i]$ by D_0. Now consider the set $D_1 = \bigcup_{i=1}^m ([fF_i] \cup [F_i])$; let $G_1, \ldots, G_u$ be the connected components of D_1. Denoting $H_1 = G_1 \cap orb\, x, \ldots, H_u = G_u \cap orb\, x$, we can easily see that $G_i = [H_i], 1 \leq i \leq u$. Indeed, denote by $\mathcal{A}_1$ the family of all sets of type $f^r F_i, 1 \leq i \leq m, r = 1, 2$. Now consider the set G_j. Then by the definition there is a subfamily $\mathcal{B}^j \subset \mathcal{A}_1$ such that $G_j = \bigcup_{E \in \mathcal{B}^j} [E]$, $H_j = \bigcup_{E \in \mathcal{B}^j} E$ and by Property A we have $G_j = [H_j]$. Thus the procedure of constructing the pairwise disjoint subtrees may go on.

Let us show that if we start the procedure in question with $m \leq End(X)$ subtrees then after at most $m - 1$ steps we get the set $[orb\, x]$ (in other words we are going to show that $D_{m-1} = [orb\, x]$). Indeed, by the properties of the number n we see that n and m have no common divisors. Hence in the first step of the procedure we see that there is at least one set, say F_1, such that fF_1 intersects with at least two of the sets $F_1, \ldots, F_m$ and so the number of connected components of D_1 is less than or equal to $m - 1$. Repeating this argument we get the conclusion.

It is quite easy to give the exact formula for sets D_i. However we need here only to show that $D_j \subset \bigcup_{i=1}^m \bigcup_{s=0}^j f^s[F_i] \equiv S_j$. Clearly, it is true for $j = 0, 1$. Suppose that it is the case for some j;we show that $D_{j+1} \subset \bigcup_{i=1}^m \bigcup_{s=0}^{j+1} f^s[F_i]$. Indeed, by the construction $D_{j+1} \subset D_j \cup fD_j \subset S_j \cup fS_j = S_{j+1}$ and we are done. Finally we have that $[orb\, x] = D_{m-1} \subset \bigcup_{i=1}^m \bigcup_{s=0}^{m-1} f^s[F_i]$. Now let us start our procedure with the sets $[Y_1] = Z_1, \ldots, [Y_l] = Z_l$; then after $l - 1$ steps we get the set $[orb\, x]$. In other words, $[orb\, x] \subset \bigcup_{i=1}^l \bigcup_{s=0}^{l-1} f^s Z_i$. Thus there exist $s \leq l-1$ and two neighboring points $a, b \in orb\, x$ such that $b \in (a, y)$ and $y \in f^s(a, b)$; by the properties of basic points (see Lemma 4) this implies Step 1.

Choose a point $\zeta \in (a, b)$ such that $f^{l-1}\zeta = y$; let for definiteness $f^{l-1}[a, \zeta] \supset [y, z_1]$.

STEP 2. *There exist integers p, q and r such that $f^p[y, z_1] \supset [y, z_q]$, $f^r[y, z_q] \supset [y, z_q]$ where $1 \le r$, $p + r \le l \le End(X)$.*

Consider for any $j \le l$ an integer $s(j)$ such that $[y, fz_j] \supset [y, z_{s(j)}]$. Then Lemma 4 easily implies Step 2 ($s^p(1) = q = s^r(q)$ is an r-periodic point of the map s).

Denote by D the set $orb_s(q) = \{q, s(q), \dots, s^{r-1}(q)\}$.

STEP 3. *For any $v \ge (n-1)r$ and $t \in D$ we have $f^v[y, z_t] \supset [orb\, x]$.*

Clearly, if $B_j = f^{rj}[y, z_t] \cap orb\, x$ then $B_j \cup f^r B_j \subset B_{j+1}$ $(\forall j)$. Thus $\bigcup_{j=0}^{n-1} f^{rj} z_t \subset f^{(n-1)r}[y, z_t]$. But $r \le End(X)$ and hence r and n have no common divisors. Therefore $\bigcup_{j=0}^{n-1} f^{rj} z_t = orb\, x$ which proves Step 3.

Now suppose that $End(X) = c, N \ge 2c(n-1)$ and make use of Lemma 2. Consider the following sequence of intervals and iterates of f (points ζ, a have been chosen in Step 1):

0) $J_0 = [\zeta, a], n_0 = 0$;

1) $J_1 = [y, z_1], n_1 = l - 1$;

2) $J_2 = [y, z_{s(1)}], n_2 = l$;

$\vdots$

k) $J_k = [y, z_{s^k(1)}], n_k = N - (n-1)r$ where $k = N - (n-1)r - l + 2$;

k+1) $n_{k+1} = N$.

It is easy to see that the inequalities $n \ge 3, N \ge 2c(n-1), r \ge 1$ and $c \ge l \ge p + r$ imply that $k = N - (n-1)r - l + 2 \ge (2c - r)(n-1) - l + 2 \ge 2(l+p) - l + 2 \ge l$. Hence $s^k(1) \in D$ and by Step 3, $f^{(n-1)r}[y, z_{s^k(1)}] \supset [orb\, x] \supset [\zeta, a] = J_0$. So by Lemma 2 (see also Remark after Lemma 2), there is a point $\alpha \in [\zeta, a]$ such that $f^{n_i}\alpha \in J_i$ $(0 \le i \le k), f^N\alpha = \alpha$.

Let us prove that N is a period of α. Indeed, otherwise α has a period m which is a divisor of N. Consider all iterates of α of type $f^{n_i}\alpha, 1 \le i \le k$. Clearly, $\dfrac{N}{3} \ge \dfrac{2c(n-1)}{3} \ge \dfrac{4l}{3} > l - 1 = n_1$ since $c \ge l$ and $n \ge 3$. Furthermore, $n_k = N - (n-1)r \ge \dfrac{N}{2}$ because $N \ge 2c(n-1) \ge 2r(n-1)$. So $l - 1 = n_1 \le \dfrac{N}{3} < \dfrac{N}{2} \le n_k = N - (n-1)r$. At the same time, there exists i such that $n_1 \le \dfrac{N}{3} \le mi \le \dfrac{N}{2} \le n_k$. Hence $f^{mi}\alpha = \alpha \in [\zeta, a]$, but on the other hand, $f^{mi}\alpha \in \bigcup_{j=1}^{l}[y, z_j] \equiv S$ where $S \cap [\zeta, a] = \emptyset$. This contradiction shows that α has a period N.

To estimate $h(f)$ it is enough to observe that $f^{l-1+p+r(n-1)}[\zeta,a] \supset [\zeta,a]\cup S$, $f^{l-1+p+r(n-1)}S \supset [\zeta,a]\cup S$ and at the same time $l-1+p+r(n-1)\le nc-1$. By usual arguments, this implies that $h(f)\ge \frac{\ln 2}{nc-1}$.

Let us pass to statement 2) of Theorem 1. Let $l\le m\le End(X)$ and $g:[0,1]\to[0,1]$ be a map with $P(g)=E$. We may assume that $g(0)=0, g(1)=1$. Let us construct a map $f:X\to X$ such that $P(f)=1\cup mE$ where $mE\equiv\{mk:k\in E\}$. First fix m endpoints $z_1,\dots,z_m$ of X. For any $1\le i\le m$, there exists a single edge $[z_i,y_i]$ containing z_i; choose a point $x_i\in(z_i,y_i)$. Then choose $x\in X\setminus\bigcup_{i=1}^m[z_i,y_i)$. Now construct a continuous map f with the following properties.

(1) The map f outside $\bigcup_{i=1}^m[z_i,y_i)$ is identity.

(2) For any $1\le i\le m-1$ the map f is injective on $[y_i,z_i]$ and maps the edge $[y_i,z_i]$ onto the union of the intervals $[y_i,x]\cup[x,z_{i+1}]$ in such a way that $fy_i=y_i, fx_i=x_{i+1}, fz_i=z_{i+1}$; so $f[z_i,x_i]=[z_{i+1},x_{i+1}]$.

(3) Define $f|[z_m,y_m]$ in such a way that the following holds:
   (a) $f|[x_m,y_m]$ is injective, $fy_m=y_m, fx_m=x_1, f[x_m,y_m]=[x_1,y_m]$;
   (b) $f[x_m,z_m]=[x_1,z_1]$ (which implies that $f^m[x_1,z_1]=[x_1,z_1]$) and moreover, $f^m|[x_1,z_1]$ is topologically conjugate to the map g.

It is easy to see now that $P(f)=\{1\}\cup mE$ where $mE\equiv\{mk:k\in E\}$. ∎

Corollary 1 (cf.[BF]). *Let $f:X\to X$ be a cycle of period $n=pk$ where $p>1$ has no prime divisors less than $End(X)+1$. Then $h(f)\ge\frac{\ln 2}{k[pEnd(X)-1]}>\frac{\ln 2}{nEnd(X)-n}$.*

PROOF: It is enough to consider the map f^k and apply Theorem 1. ∎

Corollary 2. *Let p be the least prime number greater than $End(X)$. If $f:X\to X$ has cycles of all periods from 1 to $2End(X)(p-1)$ then f has cycles of all periods.*

PROOF: The proof is left to the reader. ∎

Corollary 3 follows from Theorem 1 and the spectral decomposition theorem for graph maps (see [B1-B3]).

Corollary 3. *Let $f : X \to X$ be continuous. Then the following two statements are equivalent:*

(1) *there exists n such that f has a cycle of period mn for any m;*

(2) *$h(f) > 0$.*

PROOF: Statement 1) implies statement 2) by Corollary 1. The inverse implication follows from the spectral decomposition theorem for graph maps (see [B1-B3]) and some properties of maps with the specification property.

First we need the following definition: a graph map $\varphi : M \to N$ is called *monotone* if for any connected subset of N its φ-preimage is a connected subset of M. We also need the definition of the specification property. Namely, let $T : X \to X$ be a map of a compact infinite metric space (X, d) into itself. A dynamical system (X, T) is said to have *the specification property* or simply *the specification* [DGS] if for any $\varepsilon > 0$ there exists such integer $M = M(\varepsilon)$ that for any $k > 1$, for any k points $x_1, x_2, \dots, x_k \in X$, for any integers $a_1 \le b_1 < a_2 \le b_2 < \dots < a_k \le b_k$ with $a_i - b_{i-1} \ge M$, $2 \le i \le k$ and for any integer p with $p \ge M + b_k - a_1$ there exists a point $x \in X$ with $T^p x = x$ such that $d(T^n x, T^n x_i) \le \varepsilon$ for $a_i \le n \le b_i, 1 \le i \le k$. Maps with the specification have a lot of nice properties. The one we need may be easily obtained by methods similar to those from [DGS], Section 21; it states that if ψ is a map with the specification then there exists N such that $P(\psi) \supset \{i : i > N\}$.

Now by the results of [B1-B3], the fact that the map $f : X \to X$ has a positive topological entropy implies that there exist a subtree $Y \subset X$, an integer n, a tree Z, a continuous map $g : Z \to Z$ with the specification, and a monotone map $\varphi : Y \to Z$ such that $f^n Y = Y$ and $f^n|Y \circ \varphi = \varphi \circ g$ (i.e. φ monotonically semiconjugates $f^n|Y$ to g). The aforementioned property of maps with the specification easily implies now that there exists a number k such that g has cycles of period mk for any m. On the other hand, properties of monotone graph maps and of continuous tree maps imply that then $f^n|Y$ has cycles of the same periods which completes the proof. ∎

2. Trees and snowflakes

The main result of Section 2 is the following

Theorem 2. *Let X be a tree. Then the following statements are equivalent:*

1) $h(f) = 0$;

2) $(f, orb\, x)$ is a snowflake for every $x \in Per\, f$;

3) every $n \in P(f)$ is of form $n = 2^l m$ where $m \leq Edg(X)$ is an odd integer and all prime divisors of m are less than $End(X) + 1$.

The definitions and notation for objects we are going to deal with in Section 2 (such as *a connected hull, a cycle of sets, a snowflake,* etc.) may be found in the Introduction.

First we prove Proposition 1 which is in fact a part of Theorem 2.

Proposition 1. *Let X be a tree, $h(f) = 0, a \in Per\, f$ is a periodic point of period M. Then $(f, orb\, a)$ is a snowflake.*

PROOF: Set $A = \bigcup_{i \geq 0} f^i[orb\, a]$ and let $y \in [orb\, a]$ be a basic point for $(f, [orb\, a])$. Consider the family $\mathcal{R}$ of all cycles of sets of periods greater than 1 which contain $orb\, a$ and belong to A. Let n be the smallest period of a cycle of sets from $\mathcal{R}$. Then by Zorn lemma there exists an element of $\mathcal{R}$ which is maximal by inclusion in $\mathcal{R}$ and has a period n. Clearly, $y \notin B$ (otherwise, the period of B is 1). Let Z be a connected component of the set $A \setminus B$ containing y. We will show that $A \setminus B = Z$. First we prove the following

PROPERTY 1. *If $C \subset A$ is connected and strictly contains some component of B, then $\bigcup_{i \geq 0} f^i C = orb\, C = A$.*

Indeed, if $y \in orb\, C = \bigcup_{n=0}^{\infty} f^n C$, then by the definition of A we have $A = orb\, C$. Suppose that $y \notin orb\, C$. Then $orb\, C \neq A$ and so $orb\, C$ is not a connected set. This easily implies that $orb\, C$ is in fact a cycle of sets (these are exactly the components of $orb\, C$) and at the same time $orb\, C \supsetneq B$, which is a contradiction and completes the proof of Property 1.

Consider several cases. First suppose that there are components B_i and B_j of B such that $D = \overline{B_i} \cap \overline{B_j} \neq \emptyset$. By the definition of cycle of sets we have $B_i \cap B_j = \emptyset$, so the properties of trees imply that D consists of one point $x \in Per\, f$. Clearly, $x \in A$ (otherwise

$x \in \overline{A} \setminus A$, i.e. x is one of the endpoints of A which is impossible because $B_i \cap B_j = \emptyset$ and at the same time $\{x\} = \overline{B_i} \cap \overline{B_j}$). Thus $orb\,(B \cup x) = A$ by Property 1 and so $y \in orb\,x$ is a periodic point. Now the properties of basic points imply that y is a fixed point. Indeed, otherwise by Lemma 4 the interval $[y, fy]$ belongs to the closures of several components of the set B which is impossible. Hence $y = x$ is a fixed point and we see that $\{y\} = \{x\} = Z = A \setminus B$, which completes the proof of Property 1 in this case.

Now suppose the closures of components of B are pairwise disjoint. Let us prove that if E is the maximal component of $A \setminus Z$ containing some component F of B, then $E = F$. Indeed, suppose that $E \supsetneqq F$. Clearly, we may find a point $x \in E$ such that $[x, y] \cap F = [b, c]$ and $[x, y] = [x, b] \cup [b, c] \cup [c, y]$ where $[x, b) \cap (c, y] = \emptyset$; if we consider the natural ordering on the interval $[x, y]$ (see definitions in the beginning of Section 1), we see that the point x lies further from y than the set $[x, y] \cap F = [b, c]$ on the interval $[x, y]$.

We are going to construct (using Property 1) a sort of "symbolic dynamics" for the map f which guarantees that $h(f) > 0$. Indeed, by Property 1 $orb\,(F \cup (c, y]) = A$. Thus there exists a point $u \in (c, y]$ and an integer L such that $f^L u = x$. It implies (by Lemma 2) that $f^L[u, y] \supset [y, x]$. Similarly, considering the set $F \cup [x, b)$ and making use of Property 1 one can find a point $v \in [x, b)$ and an integer K such that $f^K[v, b] \supset [y, x]$. So we see that

(1) $f^L[u, y] \supset [y, x] \supset [y, u] \cup [b, v]$;

(2) $f^K[v, b] \supset [y, x] \supset [y, u] \cup [b, v]$;

(3) $[b, v] \cap [y, u] = \emptyset$.

As usual, this implies that $h(f) > 0$ which is a contradiction; thus $F = E$. Hence components of B are exactly components of the set $A \setminus Z$, i.e. the set $A \setminus B = [B] \setminus B = Z$ is connected. The cycle of sets $B_1 = B = \bigcup_{i=0}^{n-1} Y_i^1$, where Y_i^1 are components of B_1, is of the first level in the construction of sets generating a snowflake $(f, orb\,a)$. Using the terminology from Section 0, we would say that B is a surrounding set. Let $m_1 = n, A_1 = A$.

Now set $A_2 = \bigcup_{i \geq 0} f^{in}[orb\,a \cap Y_0^1] \subset B$. Clearly, A_2 is a cycle of sets of period n with components belonging to the corresponding components of B. Consider a family $\mathcal{P}$ of all cycles of sets which belong to A_2, contain $orb\,a$ and have periods greater than n; then choose the minimal period of sets from $\mathcal{P}$, denote it by m_2 and then choose the maximal (by inclusion) cycle of sets $B_2 \in \mathcal{P}$ with the period m_2. Repeating the arguments we used finding the set B_1, one can easily prove that the set-theoretic difference between a component G of A_2 and all components of B_2 belonging to G is connected as it is required in the definition of a snowflake. In other words, if G is a connected component of A_2 then all the components of B_2 belonging to G form a surrounding set and their connected hull coincides with G.

Going on with the procedure of finding appropriate sets A_i and B_i we see that periods m_i of sets B_i increase strictly monotonically but cannot exceed M which is a period of a. Hence the procedure we have just described is finite. At the same time by the construction the procedure stops on the step k if and only if $m_k = M$. This means that a generating cycle of sets B_k is of period M and consists of m_k components $\{Y_i^k\}_{i=0}^{m_k-1}$; each of these components contains exactly one point from $orb\, a$. So by the definition we see that $(f, orb\, a)$ is a snowflake, which completes the proof. ∎

Theorem 2. *Let X be a tree. Then the following statements are equivalent:*

(1) $h(f) = 0$;

(2) $(f, orb\, x)$ *is a snowflake for every* $x \in Per\, f$;

(3) *every* $n \in P(f)$ *is of the form* $n = 2^l m$, *where* $m \leq Edg(X)$ *is an odd integer and all prime divisors of* m *are less than* $End(X) + 1$.

Proof: By Proposition 1 statement 1) implies statement 2). Let us show that statement 3) follows from statement 2). Suppose that statement 2) holds. Let $a \in Per\, f$ have an odd period n;we shall show that n has the required properties (i.e. $n \leq Edg(X)$ and all prime divisors of n are less than $End(X) + 1$).

Indeed, let $(f, orb\, a)$ be a snowflake of type $(m_0 = 1, m_1, \dots, m_k = n)$. Then $n = m_0 \cdot (\frac{m_1}{m_0}) \cdot (\frac{m_2}{m_1}) \dots (\frac{m_k}{m_{k-1}})$. By definition, $\frac{m_i}{m_{i-1}}$ is the number of endpoints of a connected subset of X. Namely if $\{Y_j^i\}_{j=0}^{m_i-1}$ are components of the generating for $(f, orb\, a)$ cycle of sets of level i, then the set $[\bigcup_{j \equiv r \pmod{m_{i-1}}} Y_j^i] \setminus \bigcup_{j \equiv r \pmod{m_{i-1}}} Y_j^i$ is connected and has $\frac{m_i}{m_{i-1}}$ endpoints. So $\frac{m_i}{m_{i-1}} < End(X) + 1$ which implies that all prime divisors of n are less than $End(X) + 1$.

Let us show that there is no edge of X containing more than one point of $orb\, a$. Suppose that there exist an edge $[x, y]$ and points $a, b \in [x, y] \cap orb\, a$ such that $(a, b) \cap orb\, a = \emptyset$. Let $orb\, a = B_k \subset B_{k-1} \subset \cdots \subset B_0 = \bigcup_{i \geq 0} f^i[orb\, a]$ be generating sets for $(f, orb\, a)$. Denote by Y_0^i the component of B_i containing a. Choose i such that $b \notin Y_0^i$ and $b \in Y_0^{i-1}$. Then the fact that $T = [\bigcup_{j \equiv 0 \pmod{m_{i-1}}} Y_j^i] \setminus \bigcup_{j \equiv 0 \pmod{m_{i-1}}} Y_j^i$ is a connected set and that $Y_0^i \subset Y_0^{i-1}$ implies that T is a subinterval of (a, b); thus $\dfrac{m_i}{m_{i-1}} = 2$ is the number of endpoints of T. But n is odd and at the same time n is a multiple of $\dfrac{m_i}{m_{i-1}}$, which is a contradiction. So there is no edge of X containing more than one point from $orb\, a$ and thus $n \leq Edg(X)$. Now suppose that $n \in P(f)$ is of form $n = 2^l m$ where m is an odd integer. Then we can consider the map f^{2^l} and apply to it the property we have just proved (clearly, f^{2^l} has a periodic point of odd period m), which completes the verification of statement 3).

Let us prove that statement 3) implies statement 1). Indeed, if $h(f) > 0$ then by Corollary 3 (or, as in the proof of Corollary 3, by the spectral decomposition theorem for graph maps [B1-B3]) there exists k such that $P(f) \supset \{ki : i > 0\}$. Clearly, this contradicts statement 3) and proves that if statement 3) holds then $h(f) = 0$. This completes the proof of Theorem 2. ∎

Note that if C is a cycle of sets, then one may consider the restriction $f|C$ from the view of relative positions of sets in C ignoring the behavior of the map outside or inside C (one could call this approach *combinatorial*). Namely, suppose that there are a map g and a g-cycle of sets $C = \bigcup_{i=0}^{N-1} A_i$ (note that the map g may be defined only on some part of X). We say that $g|C$ is *a combinatorial snowflake (of type $m_0 = 1 < m_1 < \cdots < m_k = N$)* if the following properties hold:

(1) let $B_r^i = \bigcup_{\substack{s \equiv r \pmod{m_i} \\ 0 \leq s < N}} A_s$, $0 \leq r < m_i$, then the sets $[B_r^i]$ are pairwise disjoint (here $0 \leq i \leq k$) and we will say that sets B_r^i are *of level* i;

(2) the set $\bigcup_{\substack{s \equiv r \pmod{m_i} \\ 0 \leq s < m_{i+1}}} [B_s^{i+1}]$, $0 \leq r < m_i$ is surrounding for $0 \leq i < k$ (in other words, the set-theoretic difference between a connected hull of a set on some level and connected hulls of sets on the next level belonging to it is connected).

Remark. Obviously, for interval maps combinatorial snowflakes are exactly simple periodic orbits introduced by Block in [Bl].

It is easy to see that if (f, C) is a snowflake then $f|C$ is a combinatorial snowflake. The following Proposition 2 shows that from combinatorial point of view snowflakes are exactly those cycles which zero entropy tree maps may have.

Proposition 2. *Let $A \subset X$ is finite and $g : A \to A$ is a map such that $g|A$ is a combinatorial snowflake. Then there exists a continuous map $f : X \to X$ such that $h(f) = 0$ and $f|A = g|A$.*

Proof: Let $N = card\, A$ and $g|A$ be a combinatorial snowflake of type $m_0 = 1 < m_1 < \cdots < m_k = N$. By definition the following properties hold:

(1) let $B^i_r = \bigcup_{\substack{s\equiv r \pmod{m_i}\\ 0\le s<N}} A_s$, $0 \le r < m_i$, then the sets $[B^i_r]$ are pairwise disjoint (here $0 \le i \le k$) and all the sets B^i_r are *of level* i;

(2) the set $\bigcup_{\substack{s\equiv r \pmod{m_i}\\ 0\le s<m_{i+1}}} [B^{i+1}_s]$, $0 \le r < m_i$ is surrounding for $0 \le i < k$.

We will construct a map $f : X \to X$ by induction. The map f will have a finite number of f-cycles with periods $m_0 = 1, m_1, \ldots, m_k = N$ and every point of X will tend to one of these cycles. The map is already defined on the set $A = \bigcup_{j=0}^{N-1} B^k_j$; namely it is the map g. So we need only to explain how to make a step in the construction, i.e. how to extend the map f from the set $A^i = \bigcup_{j=0}^{m_i-1}[B^i_j]$ to the set $A^{i-1} = \bigcup_{j=0}^{m_{i-1}-1}[B^{i-1}_j]$.

Suppose that the map f is defined on the set $A^i = \bigcup_{j=0}^{m_i-1}[B^i_j]$ such that A^i is a cycle of the sets $[B^i_j]$ and the properties required in the previous paragraph hold, i.e. $f|A^i$ has finite number of cycles of periods $m_i, \ldots, m_k$ and every point from A^i tends to one of them. Consider sets $B^{i-1}_r, 0 \le r < m_{i-1}$ and define the map f on their connected hulls, i.e. on the sets $[B^{i-1}_r] = \bigcup_{\substack{s\equiv r \pmod{m_{i-1}}\\ 0\le s<m_i}}[B^i_s] \cup Z^{i-1}_r$ where Z^{i-1}_r is connected by the definition.

The map f is already defined on the sets $[B^i_s]$ which form a cycle of sets and we need only to extend the map f to the union of connected sets $Z^{i-1}_r, 0 \le r < m_{i-1}$ so that the sets $[B^{i-1}_r], 0 \le r < m_{i-1}$ form a cycle of sets A^{i-1}, there exists only finite number of cycles of period m_{i-1} belonging to $\bigcup_{r=0}^{m_{i-1}-1} Z^{i-1}_r$, and every f-orbit from A_{i-1} which does not enter A^i tends to one of these cycles of period m_{i-1}. Taking into account that the map f is already defined only on the endpoints of the sets $Z_r^{i-1}, 0 \le r < m_{i-1}$ and that by the definition these endpoints are mapped into $[B_{r+1}^{i-1}] \setminus Z_{r+1}^{i-1}$, one can easily construct the required extension of the map f. For the sake of completeness we give a sketch of the construction.

1. Let us denote by z_r the endpoint of the set B_r^i which is common for this set and the corresponding set Z_j^{i-1} where $j \equiv r \pmod{m_{i-1}}$. Then find a point x_r such that $(z_r, x_r]$ does not contain vertices of X and $(z_r, x_r] \subset Z_j^{i-1}$. Then find a point $y_j \in Z_j^{i-1} \setminus \bigcup_{r \equiv j \pmod{m_{i-1}}} (z_r, x_r] = P_j$ for any $0 \le j < m_{i-1}$.

2. Set $f(P_j) = y_{j+1} (0 \le j \le m_{i-1} - 2), f(P_{m_{i-1}-1}) = y_0$.

3. Define $f|[z_r, x_r], 0 \le r < m_i$ in such a way that $f[z_r, x_r] = [z_{r+1}, y_{r+1}], f|[z_r, x_r]$ is injective and if $D_r \subset [z_r, x_r]$ consists of all points ζ such that $f^{m_i}(\zeta) \in (z_r, x_r]$ then D_r is an interval, $f^{m_i} D_r = (z_r, x_r]$, there is only one periodic point belonging to $(z_r, x_r]$ and this point is of period m_i.

It is easy to check that the construction in question is possible and that this way we will construct a map with the required properies. This completes the proof of Proposition 2. ∎

Using methods similar to those from the proof of statement 2) from Theorem 1 or Proposition 2 one can easily prove the following

Proposition 3. *If $m \le End(X)$ and $k \ge 0$ then there exists a continuous map $f : X \to X$ and an f-periodic point a such that $h(f) = 0$ and the period of a is $2^k m$.* ∎

Acknowledgments. I would like to thank E. M. Coven for informing me about Baldwin's paper [Ba] and J. Milnor and S. Sutherland for looking through this paper and providing useful comments.

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