Dynamics of Certain Smooth One-dimensional Mappings

III. Scaling function geometry

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Dedicated to Professor Shantao Liao on the occasion of his seventieth birthday

Abstract

We study scaling function geometry. We show the existence of the scaling function of a geometrically finite one-dimensional mapping. This scaling function is discontinuous. We prove that the scaling function and the asymmetries at the critical points of a geometrically finite one-dimensional mapping form a complete set of C^1-invariants within a topological conjugacy class.

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§1 Introduction

Smooth classification. Two smooth mappings f and g from a one-dimensional manifold M to itself are topologically conjugate if there is a homeomorphism h from M to itself such that $f \circ h = h \circ g$. The homeomorphism h is not usually a smooth diffeomorphism; if is is, then all the eigenvalues of f and g at the corresponding periodic points have to be the same. We say f and g are smoothly conjugate if the homeomorphism h is a smooth diffeomorphism,

A celebrated theorem first proved by M. Herman [H] says that any circle diffeomorphism with a diophantine rotation number is smoothly conjugate to the rigid rotation by that number. In this case, the circle diffeomorphism lacks periodic points. Thus the topological invariant, the rotation number, is the complete smooth invariant.

A theorem proved by M. Shub and D. Sullivan [SS] says that any two smooth orientation-preserving expanding endomorphisms of the circle are smoothly conjugate if they are topologically conjugate and the conjugacy is absolutely continuous. D. Sullivan [S1] also showed that the set of eigenvalues at periodic points of a smooth orientation-preserving expanding endomorphism of the circle forms a complete set of smooth invariants within a topological conjugacy class.

A recent theorem proved by D. L. Llave and R. Moriyón [LM] says that any two Anosov diffeomorphisms on the torus are smoothly conjugate if they are topologically conjugate and have the same eigenvalues at corresponding periodic points. Thus the set of eigenvalues at periodic points of an Anosov diffeomorphism on the torus forms a complete set of smooth invariants within a topological conjugacy class.

A recent work in [J1], I provided a smooth classification of the space of generalized Ulam-von Neumann transformations. These are certain smooth interval mappings topologically conjugate to the mapping $q(x) = -x^2 + 2$ of the interval $[-2, 2]$. I classified this space up to smooth equivalence by showing that all the eigenvalues at periodic points, the type of power law at the critical point, and a quantity which we call the asymmetry at the critical point form a complete and optimal set of smooth invariants.

What we would like to say in this paper. We study continuously geometrically finite one-dimensional mappings (see [J3]). These are a subspace of $C^{1+\alpha}$ one-dimensional mappings with finitely many, critically finite power law critical points. We concentrate on scaling function geometry. We show the existence of the scaling function of a geometrically finite one-dimensional mapping. We study the rigidity on the space of geometrically finite one-dimensional mappings.

Suppose M is an oriented connected compact one-dimensional C^2-Riemannian manifold with Riemannian metric dx^2 and associated length element dx. Suppose f is a mapping from M into M. A <u>Markov partition</u> η_1 of M by f is a set $\{I_0, \cdots, I_m\}$ of closed intervals of M such that (a) $I_0, \cdots, I_m$ have pairwise disjoint interiors, (b) the union $\cup_{i=0}^m I_i$ of the intervals is M, (c) the restriction of f to each interval I_i is injective and continuous, and (d) the image of I_i under f is the union of some intervals in η_1. We say f is a <u>Markov mapping</u> if it has a Markov partition η_1 of M by f.

Suppose f is a Markov mapping and η_1 is a fixed Markov partition of M by f. We use the symbols $0, \cdots, m$ to name the intervals in the partition η_1 and use g_i to denote the inverse of the restriction of f to the interval with name i. Let symbol r_i be $+i$ if g_i is orientation-preserving and be $-i$ if g_i is orientation-reversing. Suppose $w_n = r_{i_0} \cdots r_{i_n}$ is a sequence of the symbols $\{r_0, \cdots, r_m\}$. We say it is a <u>suitable sequence</u> of length $n+1$ if $I_{i_k} \subset f(I_{i_{k-1}})$ for any $k = 0, \cdots, n$. In the other words, it is suitable if the interval I_{i_k} is in the domain of $g_{i_{k-1}}$ for any $k = 0, \cdots, n$. For a suitable sequence $w_n = r_{i_0} \cdots r_{i_n}$, we use g_{w_n} to denote the composition $g_{i_0} \circ \cdots \circ g_{i_n}$ and use I_{w_n} to denote the image of $f(I_{i_n})$ under g_{w_n}. We call w_n the name of the interval I_{w_n}. We may read the name w_n either from the left to the right or from the right to the left.

Suppose we read all the names from the left to the right. Then we get the set $\Sigma_f = \{a = r_{i_0} r_{i_1} \cdots\}$ of infinite suitable sequences which start from the left and extend to the right. Suppose $\sigma_f : \Sigma_f \mapsto \Sigma_f$ is the shift mapping which knocks off the first symbol in the left of an infinite suitable sequence in Σ_f. The space (Σ_f, σ_f) is the phase space of the Markov mapping f.

Let us now read all the names from the right to the left. We then get the set $\Sigma_f^* = \{a^* = \cdots r_{i_1} r_{i_0}\}$ of infinite suitable sequences which start from the right and extend to the left. Suppose $\sigma_f^* : \Sigma_f^* \mapsto \Sigma_f^*$ is the shift mapping which knocks off the first symbol in the right of an infinite suitable sequence in Σ_f^*. We call the space (Σ_f^*, σ_f^*) the dual space of the Markov mapping f.

The mapping $sign : \eta_n \mapsto \{-1, 1\}$ is defined by $sign(I_{w_n})$ where $sign(I_{w_n})$ is 1 if the number of $-$ in the sequence w_n is even and $sign(I_{w_n})$ is -1 if the number of $-$ in the sequence w_n is odd. For an infinite suitable sequence $a^* = \cdots w_n$ in Σ_f^*, let $\sigma_f^*(a^*) = \cdots v_{n-1}$ where $w_n = v_{n-1} r_{i_0}$. We use $s(w_n)$ to denote the ratio

$$\frac{sign(I_{w_n})|I_{w_n}|}{sign(I_{v_{n-1}})|I_{v_{n-1}}|}$$

and call it the signed scale at $w_n = v_{n-1} r_{i_0}$. We also call the absolute value of the signed scale the scale. If the limit $s_f(a^*)$ of the sequence $\{s(w_n)\}_{n=0}^{\infty}$ of the signed scales exists as n goes to infinity, then we say there is the signed scale at a^*. If there is the signed scale at every point in Σ_f^*, then we define a function $s_f : \Sigma_f^* \mapsto \mathbf{R^1}$ as $s_f(a^*)$. We call this function the signed scaling function of f and its absolute value the scaling function of f.

The scaling function was first defined by M. Feigenbaum [F] to describe the universal geometric structure of the attractors obtained by period doubling. D. Sullivan [S2] defined in mathematics the scaling function for a Cantor set which is the maximal invariant set of a $C^{1+\alpha}$-expanding mapping for some $0 < \alpha \leq 1$. He gave a complete $C^{1+\alpha}$-classification of these Cantor sets on the line by their scaling functions and used this classification in the study of the universal geometric structure of the attractors obtain by period doubling. The definition of a signed scaling function in this paper (also see [J4]) generalizes Sullivan's idea to a Markov mapping.

We show some basic properties of the signed scaling function of a Markov mapping in §2.4 (Proposition 1 to Proposition 4).

A geometrically finite one-dimensional mapping f has been defined in [J3], which is a certain Markov mapping with finitely many, critically finite power law critical points (see §3 for a definition). The fixed Markov partition η_1 of a geometrically finite one-dimensional mapping f is the set of the closures of the intervals of the complement of the critical orbits of f. One of the main theorems in this paper is the following:

THEOREM A. *Suppose f is a geometrically finite one-dimensional mapping. Then there is the signed scaling function $s_f : \Sigma_f^* \mapsto \mathbf{R}^1$ of f.*

The proof of this theorem is an application of the $C^{1+\alpha}$-Denjoy-Koebe distortion lemma in [J2] (see also [J3]).

Suppose f is a geometrically finite one-dimensional mapping. A critical point c of f is a power law critical point of f if there is a number $\gamma > 1$ such that the limits of $f'(x)/|x-c|^{\gamma-1}$ exist and equal nonzero numbers as x goes to c from below and from above. We call the number γ the exponent of f at the power law critical point c. Two corollaries of Theorem A are the following:

COROLLARY A1. *Suppose f is a geometrically finite one-dimensional mapping. Then the scaling function $|s_f| : \Sigma_f^* \mapsto \mathbf{R}^1$ of f is discontinuous.*

COROLLARY A2. *Suppose f is a geometrically finite one-dimensional mapping. Then the exponent γ of f at a power law critical point c can be calculated by the scaling function $s_f : \Sigma_f^* \mapsto \mathbf{R}^1$.*

Suppose f is a geometrically finite one-dimensional mapping. We say an object is a <u>C^1-invariant</u> of f if it is the same for f and for $h \circ f \circ h^{-1}$ whenever h is an orientation-preserving C^1-diffeomorphism. The <u>asymmetry</u> of f at a power law critical point c of f is the limit of $f'(x)/f'(-x+2c)$ as x tends to c from below. It is a C^1-invariant of f (see [J1]). The signed scaling function s_f of f is a C^1-invariant too (see Proposition 1 in §2.4). Another main result in this paper is that the scaling function s_f of f and the asymmetries of f at all the critical points of f form a complete set of C^1-invariants within a topological conjugacy class as follows.

THEOREM B. *Suppose f and g are geometrically finite and topologically conjugate by an orientation-preserving homeomorphism h. Then f and g are C^1-conjugate if and only if they have the same signed scaling function and the same asymmetries at the corresponding critical points.*

Actually, we can say more on the smoothness of the conjugating mapping h as follows.

COROLLARY B1. *Suppose f and g are $C^{1+\alpha}$-geometrically finite one-dimensional mappings for some $0 < \alpha \leq 1$. Furthermore suppose they are topologically conjugate by an orientation-preserving homeomorphism h. If f and g have the same signed scaling function and the same asymmetries at the corresponding critical points, then h is a $C^{1+\alpha}$-diffeomorphism.*

Suppose $f : M \mapsto M$ is a geometrically finite one-dimensional mapping and $s_f : \Sigma_f \mapsto \Sigma_f$ is the signed scaling function of f. The eigenvalue $e_f(p) = (f^{\circ n})'(p)$ of f at a periodic point p of period n and the exponent γ of f at a critical point c can be calculated by the signed scaling function s_f of f showed by Proposition 2 and by Corollary A2. Both of them are then clearly C^1-invariants. Moreover, in the case that the set of periodic points of f is dense in M, we show that the eigenvalues of f at periodic points and the exponents and the asymmetries of f at critical points form a complete C^1-invariants within a topologically conjugate class as follows.

THEOREM C. *Suppose f and g are $C^{1+\alpha}$-geometrically finite one-dimensional mappings for some $0 < \alpha \leq 1$. Furthermore, suppose f and g are topologically conjugate by an orientation-preserving homeomorphism h and suppose the set of periodic points of f is dense in M. If f and g have the same eigenvalues at the corresponding periodic points and the same exponents at the corresponding critical points, then they have the same scaling functions. Moreover, if f and g have also the same asymmetries at the corresponding critical points, then h is a $C^{1+\alpha}$-diffeomorphism.*

In the case that the set of periodic points of f is not dense in M, it seems that the eigenvalues of f at periodic points and the exponents and the asymmetries of f at critical points are not enough to form a complete C^1-invariants within a topologically conjugate class for there may be a Cantor set in M which is an invariant set of some iterate of f (see the results in [CP]). But we are still interested in the following question.

QUESTION 1. *Suppose $f : M \mapsto M$ is a geometrically finite one-dimensional mapping. Do the eigenvalues of f at the periodic points, the exponents and the asymmetries of f at the critical points of f form* *a complete set of C^1-invariants within a topologically conjugate class ?*

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§2 Scaling Structure of a Markov Mapping

Suppose M is an oriented connected compact one-dimensional C^2-Riemannian manifold with Riemannian metric dx^2 and associated length element dx. Suppose $f : M \mapsto M$ is a piecewise continuous mapping.

§2.1 A Markov partition.

A Markov partition η_1 of M by f is a set $\{I_0, \cdots, I_m\}$ of closed intervals of M such that (a) $I_0, \cdots, I_m$ have pairwise disjoint interiors, (b) the union $\cup_{i=0}^{m} I_i$ of the intervals is M, (c) the restriction of f to each interval I_i is injective and continuous, and (d) the image of I_i under f is the union of some intervals in η_1.

Definition 1. *The mapping $f : M \mapsto M$ is a Markov mapping if there is a Markov partition of M by f.*

§2.2 The symbolic and the dual symbolic spaces.

Suppose f is a Markov mapping and η_1 is a fixed Markov partition of M by f. We use $0, \cdots, m$ to name the intervals in η_1 and use g_i to denote the inverse of the restriction of f to the interval with name i. Let symbol r_i be $+i$ if g_i is orientation-preserving and be $-i$ if g_i is orientation-reversing.

Suppose $w_n = r_{i_0} \cdots r_{i_n}$ is a sequence of the symbols $\{r_0, \cdots, r_m\}$. We say it is a suitable sequence of length $n+1$ if $I_{i_k} \subset f(I_{i_{k-1}})$ for any $k = 0, \cdots, n$. In the other words, it is suitable if the interval I_{i_k} is in the domain of $g_{i_{k-1}}$ for any $k = 0, \cdots, n$. Suppose $w_n = r_{i_0} \cdots r_{i_n}$ is a suitable sequence. Let $g_{w_n} = g_{r_{i_0}} \circ \cdots \circ g_{r_{i_n}}$ be the composition of g_{i_0} to g_{i_n} and let $I_{w_n} = g_{w_n}(f(I_{i_n}))$ be the image of $f(I_{i_n})$ under g_{w_n}. We call w_n the name of the interval I_{w_n}. Suppose η_n is the set of the intervals I_{w_n} for all suitable sequences w_n of length n. This set is also a Markov partition of M by f. We call it the n^{th}-partition of M by f. Let λ_n be the maximum of the lengths of the intervals in η_n. We always assume that λ_n tends to zero as n goes to infinity.

Suppose Γ_n is the set of all the names w_n of the intervals in η_n. We define a (n,k)-left cylinder for $0 \leq k \leq n$ as

$$[w_n^0] = \{w_n = r_{i_0} \cdots r_{i_n} | \in \Gamma_n, r_{i_l} = r_{i_l}^0, \ for \ l = 0, \cdots, k\}$$

where $w_n^0 = r_{i_0}^0 \cdots r_{i_n}^0$ is a fixed suitable sequence in Γ_n. All the (n,k)-left cylinders form a topological basis of Γ_n. We still use Γ_n to denote the set Γ_n with this topological basis. The sequence $\{\Gamma_n\}_{n=0}^{\infty}$ of the topological spaces Γ_n with the inclusions $I_n : \Gamma_n \mapsto \Gamma_{n-1}$ forms an inverse limit set. Its inverse limit $\Sigma_f = \{a = r_{i_0} r_{i_1} \cdots\}$ with the shift mapping $\sigma_f : \Sigma_f \mapsto \Sigma_f$ which is defined as $\sigma_f(r_{i_0} r_{i_1} \cdots) = r_{i_1} \cdots$ is the phase space of the dynamical system $f : M \mapsto M$ as follows:

LEMMA 1. *There is a continuous mapping h from Σ_f onto M such that*

$$f \circ h = h \circ \sigma_f$$

and the fiber $h^{-1}(x)$ contains at most two points for every $x \in M$.

We now consider a (n,k)-right cylinder for $0 \leq k \leq n$ as

$$[w_n^0] = \{w_n = r_{i_n} \cdots r_{i_0} | \in \Gamma_n, r_{i_l} = r_{i_l}^0 \ for \ l = 0, \cdots, k\}$$

where $w_n^0 = r_{i_n}^0 \cdots r_{i_0}^0$ is a fixed suitable sequence in Γ_n. All the (n,k)-right cylinders form another topological basis of Γ_n. Let Γ_n^* be the set Γ_n with this topological basis. The sequence $\{\Gamma_n^*\}_{n=0}^{\infty}$ of the topological spaces Γ_n^* with the inclusions $I_n^* : \Gamma_n^* \mapsto \Gamma_{n-1}^*$ forms an inverse limit set. Its inverse limit $\Sigma_f^* = \{a^* = \cdots r_{i_1} r_{i_0}\}$ with the shift mapping $\sigma_f^* : \Sigma_f^* \mapsto \Sigma_f^*$ which is defined as $\sigma_f^*(\cdots r_{i_1} r_{i_0}) = \cdots r_{i_1}$ is not the phase space of the dynamical system $f : M \mapsto M$ any more. We call it the dual space of f. The scaling invariants will be defined on this dual space as follows.

Let $sign : \eta_n \mapsto \{-1, \ 1\}$ be the mapping defined by $sign(I_{w_n})$ where $sign(I_{w_n})$ is 1 if the number of $-$ in the sequence w_n is even and $sign(I_{w_n})$ is -1 if the number of $-$ in the sequence w_n is odd. Suppose $a^* = \cdots w_n$ is a point in Σ_f^* and $\sigma_f^*(a^*) = \cdots v_{n-1}$ where $w_n = v_{n-1} r_{i_0}$.

We define the signed scale at w_n as

$$s(w_n) = \frac{sign(I_{w_n})|I_{w_n}|}{sign(I_{v_{n-1}})|I_{v_{n-1}}|}.$$

We call the absolute value $|s(w_n)|$ of the signed scale $s(w_n)$ at w_n the scale at w_n. If the limit

$$s_f(a^*) = \lim_{n \mapsto \infty} s(w_n)$$

exists, then we say there is the signed scale at a^*. We call the absolute value $|s_f(a^*)|$ of the signed scale $s(a^*)$ at a^* the scale at a^*.

Definition 2. *Suppose there is the signed scale at every point in Σ_f^*. Then we call the function $s_f : \Sigma_f^* \mapsto \mathbf{R^1}$ defined as the signed scale $s_f(a^*)$ the signed scaling function of f and call its absolute value $|s_f|$ the scaling function of f.*

§2.4 Some properties of a signed scaling function.

We show some properties of a signed scaling function (if it exists) of a Markov mapping

Definition 3. *Suppose $f : M \mapsto M$ is a continuous mapping. We say an object associated with f is a C^1-invariant of f if it is the same for f and for $h \circ f \circ h^{-1}$ whenever h is an orientation-preserving C^1-diffeomorphism.*

The following proposition is immediately from the definition of a signed scaling function.

Proposition 1. *Suppose $f : M \mapsto M$ is a Markov mapping. Then the signed scaling function $s_f : \Sigma_f^* \mapsto \mathbf{R}^1$ (if it exists) is a C^1-invariant of f.*

Suppose f is a Markov mapping and (σ_f^*, Σ_f^*) is the dual space of f. Let $P(\sigma_f^*)$ and $P(f)$ be the sets of the periodic points of σ_f^* and f, respectively.

Proposition 2. *There is a surjective mapping $Q : P(\sigma_f^*) \mapsto P(f)$ such that every fiber $Q^{-1}(p)$ contains at most two points.*

Proof. Suppose a^* is a point in $P(\sigma_f^*)$. It can be written in $a^* = (w_n)^\infty = \cdots w_k w_k$ where $w_k = r_{i_{k-1}} \cdots r_{i_0}$ is a finite sequence. The intervals with the names $(w_k)^l$ satisfy that

$$\cdots \subset I_{(w_k)^l} \subset I_{(w_k)^{l-1}} \subset \cdots \subset I_{w_k} \subset I_{r_{i_{k-1}}}.$$

Let $\{p\} = \cap_{l=0}^{\infty} I_{(w_n)^l}$, it is easy to check that $f^{\circ k}(p) = p$. We define $Q : P(\sigma_f^*) \mapsto P(f)$ as $Q((w_k)^\infty) = p$ where $p = \cap_{l=1}^{\infty} I_{(w_k)^l}$. It is easy to check that the mapping Q is a surjective mapping and there are at most two points in $P(\sigma_f^*)$ being mapped to a same point under Q.

Proposition 3. *Suppose $f : M \mapsto M$ is a Markov mapping. Furthermore suppose there is the signed scaling function $s_f : \Sigma_f^* \mapsto \mathbf{R}^1$ of f. Then for an a^* in $P(\sigma_f^*)$ and $p = Q(a^*)$ in $P(f)$, the inverse of the eigenvalue $e_f(p) = (f^{\circ k})'(p)$ of f at p can be calculated by*

$$\frac{1}{e_f(p)} = \prod_{l=0}^{k-1} s_f((\sigma_f^*)^{\circ l}(a^*)).$$

Proof. The scale $s((w_k)^l)$ at $(w_k)^l$ equals

$$\frac{sign(I_{(w_k)^l}) I_{(w_k)^l}}{sign(I_{(w_k)^{(l-1)} r_{i_{k-1}} \cdots r_{i_1}}) I_{(w_k)^{(l-1)} r_{i_{k-1}} \cdots r_{i_1}}}.$$

By using $f^{\circ k}(I_{(w_k)^l}) = I_{(w_k)^{l-1}}$ and the mean value theorem,

$$s((w_k)^l) = \frac{1}{(f^{\circ k})'(\xi_{(w_k)^l})} \cdot \frac{sign(I_{(w_k)^{l-1}}) I_{(w_k)^{l-1}}}{sign(I_{(w_k)^{l-1} r_{i_{k-1}} \cdots r_{i_1}}) I_{(w_k)^{l-1} r_{i_{k-1}} \cdots r_{i_1}}}$$

$$= \frac{1}{(f^{\circ k})'(\xi_{(w_k)^l})} \cdot \frac{1}{s((w_k)^{l-1} r_{i_{k-1}}) \cdots s((w_k)^{l-1} r_{i_{k-1}} \cdots r_{i_1})}$$

where $\xi_{(w_k)^l} \in I_{(w_k)^l}$.

Because the maximum λ_{kl} of the lengths of the intervals in η_{kl} tends to 0 as l goes to infinity, we have that $\xi_{(w_k)^l}$ tends to p as l goes to infinity. Let l tends to infinity, we get

$$s(a^*) = \frac{1}{(f^{\circ k})'(p)} \cdot \frac{1}{s((w_k)^\infty r_{i_{k-1}}) \cdots s((w_k)^\infty r_{i_{k-1}} \cdots r_{i_1})}$$

$$= \frac{1}{(f^{\circ k})'(p)} \cdot \frac{1}{\prod_{l=1}^{k-1} s((\sigma_f^*)^{\circ l}(a^*))}.$$

This implies Proposition 3.

Suppose $f : M \mapsto M$ is a Markov mapping and $\{\eta_n\}_{n=1}^{\infty}$ is the induced sequence of nested partitions of M by f. We say the restriction of f to an interval in η_1 is a $C^{1+\alpha}$-embedding for some $0 < \alpha \leq 1$ if this restriction and its inverse are C^1 with α-Hölder continuous derivatives. We say the n^{th}-partition of M by f η_n goes to zero exponentially with n if there are positive constants K and $\mu < 1$ such that $\lambda_n \leq K\mu^n$ for every integer $n > 0$.

DEFINITION 4. *Suppose $f : M \mapsto M$ is a Markov mapping. We say f is a good Markov mapping if*

(*a*) *the restriction of f to every interval in the first partition η_1 is a $C^{1+\alpha}$-embedding for some $0 < \alpha \leq 1$,*

(*b*) *the n^{th}-partition of M by f η_n goes to zero exponentially with n.*

Suppose $f : M \mapsto M$ is a Markov mapping and Σ_f^* is the dual Cantor set of f. We say a function $s : \Sigma_f^* \mapsto \mathbf{R}^1$ is Hölder continuous if there are positive constants K and $0 < \mu < 1$ such that

$$|s(a_1^*) - s(a_2^*)| \leq K\mu^n$$

whenever the first n digits of a_1^* and a_2^* in Σ_f^* are the same.

PROPOSITION 4. *Suppose $f : M \mapsto M$ is a good Markov mapping. Then the signed scaling function $s_f : \Sigma_f^* \mapsto \mathbf{R}^1$ exits and is Hölder continuous.*

Proof. The proof of this proposition is the use of the naive distortion lemma (see [J1] or [J2]) and is similar to the proof of Theorem A in §3. We outline the proof as follows.

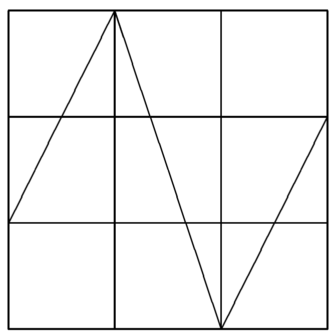
Suppose $a_1^* = \cdots w_n$ and $a_2^* = \cdots w_n$ are two points in Σ_f^* with the same first n symbols w_n (from the right). Then following the proof of Theorem A and using the naive distortion lemma (see [J1] or [J2]),

$$|s_f(a_1^*) - s_f(a_2^*)| \leq K|I_{w_n}|^{\alpha}$$

where K is a positive constant. It implies that s_f is Hölder continuous.

EXAMPLE 1. Suppose M is the unit interval $[0,1]$ and l_0, l_1 and l_2 are positive numbers satisfying that $l_0+l_1+l_2=1$. Let $K_0=[0,l_0]$, $K_1=[l_0,l_0+l_1]$ and $K_2=[l_0+l_1,1]$ are the subintervals in M and define a Markov mapping $f: M\mapsto M$ as

$$f(x)=\begin{cases} \frac{l_1+l_2}{l_0}x+l_0 & x\in K_0, \\ -\frac{1}{l_1}(x-l_0)+1 & x\in K_1, \\ \frac{l_0+l_1}{l_2}(x-l_0-l_1) & x\in K_2. \end{cases}$$



K_0 K_1 K_2

The graph of f

Figure 1

Suppose A is the induced matrix by the Markov mapping f (see [B]). Then

$$A=(a_{ij})_{3\times 3}=\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Then Σ_f^* is $\Sigma_A=\{a^*=(\cdots r_{i_1}r_{i_0})|a_{i_k i_{k-1}}=1$ for all $k=1,\cdots,\infty\}$ and the scaling function s_f is

$$s_f(w)=\begin{cases} l_0 & a^*=(\cdots r_{i_2}-1+0), \\ \frac{l_0}{l_0+l_1} & a^*=(\cdots r_{i_2}+2+0), \\ -\frac{l_1}{l_1+l_2} & a^*=(\cdots r_{i_2}+0-1), \\ -l_1 & a^*=(\cdots r_{i_2}-1-1), \\ -\frac{l_1}{l_0+l_1} & a^*=(\cdots r_{i_2}+2-1), \\ \frac{l_2}{l_1+l_2} & a^*=(\cdots r_{i_2}+0+2), \\ l_2 & a^*=(\cdots r_{i_2}-1+2). \end{cases}$$

§3 The Scaling Function of a Geometrically Finite One-dimensional Mapping

A geometrically finite one-dimensional mapping is defined in the paper [J3]. Let me review this definition here.

Suppose M is an oriented connected compact one-dimensional C^2-Riemannian manifold with Riemannian metric dx^2 and associated length element dx. Suppose $f : M \mapsto M$ is a C^1-mapping. We say a point $c \in M$ is a critical point if the derivative of f at this point is zero. We say a critical point c of f is a power law critical point if there is a $\gamma \geq 1$ such that

$$\lim_{x \mapsto c+} \frac{f'(x)}{|x-c|^{\gamma-1}} \ \textit{and} \ \lim_{x \mapsto c-} \frac{f'(x)}{|x-c|^{\gamma-1}}$$

have nonzero limits A and B. We call the numbers γ and $\tau = A/B$ the exponent and the asymmetry of f at the power law critical point c (see [J1] and [J2]). We say a critical point c of f is critically finite if the orbit $\{c, f(c), \cdots\}$ is a finite set.

Remember that an object associated with f is a C^1-invariant of f if it is the same for f and for $h \circ f \circ h^{-1}$ whenever h is an orientation-preserving C^1-diffeomorphism. We have the following proposition.

PROPOSITION 5. *Suppose $f : M \mapsto M$ is a C^1-mapping and c is a power law critical point of f. Then the exponent γ and the asymmetry τ of f at c are C^1-invariants of f.*

Henceforth, without loss generality, we will assume that f maps the boundary of M (if it is not empty) into itself and the one-sided derivatives of f at all boundary points of M are not zero. We note that in the general case, a boundary point of M should count as a critical point anyway.

Suppose f has only power law critical points. We use $CP = \{c_1, \cdots, c_d\}$ to denote the set of critical points of f and use $\Gamma = \{\gamma_1, \cdots, \gamma_d\}$ to denote the corresponding exponents of f. Suppose η_0 is the set of the intervals in the complement of the set CP of critical points of f in M.

DEFINITION 5. *We say the mapping f is $C^{1+\alpha}$ if*

$(*)$ *the restrictions of f to the intervals in η_0 are C^1 with α-Hölder continuous derivatives and*

$(**)$ *for every critical point* c_i *of* f *, there is a small neighborhood* U_i *of* c_i *in* M *such that* $\delta_{-,i}(x) = f'(x)/|x-c|^{\gamma_i - 1}$ *for* $x < c$ *in* U_i *and* $\delta_{+,i}(x) = f'(x)/|x-c|^{\gamma_i - 1}$ *for* $x > c$ *in* U_i *are* α*-Hölder continuous.*

Suppose the set of the critical orbits $\cup_{n=0}^{\infty} f^{\circ n}(CP)$ is finite. Then the set of the closures of the intervals of the complement of the critical orbits $\cup_{n=0}^{\infty} f^{\circ n}(CP)$ is a Markov partition of M by f. We always take this Markov partition of M by f as the first partition η_1 of M by f in this case. Suppose λ_n is the maximum of lengths of the intervals in the n^{th}-partition η_n of M by f. Remember that the n^{th}-partition η_n tends to zero exponentially with n if there are constants $K > 0$ and $0 < \mu < 1$ such that $\lambda_n \leq K\mu^n$ for any n. The definition of a geometrically finite one-dimensional mapping is the following.

Definition 6. *A* C^1*-mapping* $f : M \mapsto M$ *with only power law critical points is* $C^{1+\alpha}$*-geometrically finite for some* $0 < \alpha \leq 1$ *(geometrically finite) if it satisfies the following conditions:*

Smooth condition: f *is* $C^{1+\alpha}$*.*

Finite condition: the set of critical orbits $\cup_{i=0}^{\infty} f^{\circ}(CP) \neq \emptyset$ *is finite.*

No cycle condition: no critical point is periodic.

Exponential decay condition: the n^{th}*-partition* η_n *tends to zero exponentially with* n*.*

A technical lemma, the $C^{1+\alpha}$-Denjoy-Koebe distortion lemma, has been developed in [J2] to study the certain mappings with finitely many nonrecurrent critical points. For a geometrically finite one-dimensional mapping $f : M \mapsto M$, this lemma can be written in the following simple form (see of §3.3 in [J2]).

Lemma 2 (The $C^{1+\alpha}$-Denjoy-Koebe distortion lemma) *Suppose* $f : M \mapsto M$ *is geometrically finite. There are two positive constants* A *and* B *and a positive integer* n_0 *such that for any inverse branch* g_n *of* $f^{\circ n}$ *and any pair* x *and* y *in the intersection of one of the intervals in* η_{n_0} *and the domain of* g_n*, the distortion* $|g_n(x)/g_n(y)|$ *of* g_n *at these two points satisfies*

$$\frac{|g_n(x)|}{|g_n(y)|} \leq \exp\Big(\Big(A + \frac{B}{D_{xy}}\Big)|x-y|^{\alpha}\Big)$$

where D_{xy} *is the distance between* $\{x,\, y\}$ *and the boundary of the do-*

main of g_n.

§3.1 The existence of the signed scaling function.

One of the main results in this paper , which is an application of Lemma 2 (the $C^{1+\alpha}$-Denjoy-Koebe distortion lemma), is the following:

THEOREM A. *Suppose f is a geometrically finite one-dimensional mapping. Then there is the signed scaling function $s_f : \Sigma_f^* \mapsto \mathbf{R}^1$ of f.*

Proof. Suppose $U_1, \cdots, U_d$ are the neighborhoods of the critical points $c_1, \cdots, c_d$ of f in Definition 5. We say an interval I in η_n is a critical interval if one of its endpoints is a critical point of f. Suppose n_0 is the integer in Lemma 2. Let $n_1 > n_0$ be an integer such that every critical interval I in η_{n_1} is contained in one of $U_1, \cdots, U_d$ and one of its endpoints is not in the critical orbits $\cup_{n=0}^{\infty} f^{\circ n}(CP)$. Let $\mathcal{U}$ be the union of the critical intervals in η_{n_1} and $\mathcal{V}$ be the union of the non-critical intervals in η_{n_1}.

For a point $a^* = \cdots w_n$ in Σ_f^*, let $\sigma_f^*(a^*) = \cdots v_{n-1}$ where $w_n = v_{n-1} r_{i_0}$. Suppose I_{w_n} and $I_{v_{n-1}}$ are the intervals with names w_n and v_{n-1}, respectively. We note that I_{w_n} is a subinterval of $I_{v_{n-1}}$. We discuss the sequence $\{I_{v_{n-1}}\}_{n=1}^{\infty}$ in the two cases. One is that there is a positive integer N such that $I_{v_{n-1}}$ is contained in $\mathcal{V}$ for every $n > N$. The other is that there is an increasing subsequence $\{n_k\}_{k=2}^{\infty}$ of the integers such that $I_{v_{n_k-1}}$ is contained in $\mathcal{U}$ for every $k \geq 2$. Suppose $n_2 \geq n_1$.

In the first case, we use the naive distortion lemma (see [J1] and [J2]) to obtain the following estimate:

For any integers $m > n > N$, the intervals I_{w_n} and $I_{v_{n-1}}$ are the images of I_{w_m} and $I_{v_{m-1}}$ under $f^{\circ(m-n)}$ and the intervals I_{w_N} and $I_{v_{N-1}}$ are the images of I_{w_n} and $I_{v_{n-1}}$ under $f^{\circ(n-N)}$. We note that the signs of $s(w_m)$ and $s(w_n)$ are the same. These imply that

$$|s(w_m) - s(w_n)| = |\frac{f^{\circ(m-n)}(\xi_1)}{f^{\circ(m-n)}(\xi_2)} - 1| \cdot \frac{|I_{w_n}|}{|I_{v_{n-1}}|}$$

$$= |\frac{(f^{\circ(m-n)})'(\xi_1)}{(f^{\circ(m-n)})'(\xi_2)} - 1| \cdot |\frac{(f^{\circ(n-N)})'(\xi_3)}{(f^{\circ(n-N)})'(\xi_4)}| \cdot \frac{|I_{w_N}|}{|I_{v_{N-1}}|}$$

which is less than $K|I_{w_n}|^\alpha$ for a positive constant K where ξ_1 and ξ_2 are two points in $I_{v_{m-1}}$ and ξ_3 and ξ_4 are two points in $I_{v_{n-1}}$. This estimate says that the sequence $\{s(w_n)\}_{n=1}^{\infty}$ is a Cauchy sequence and the limit $s(a^*) = \lim_{n\mapsto\infty} s(w_n)$ exists.

In the other case, we have that the intervals $I_{w_{n_k}}$ and $I_{v_{n_k-1}}$ are the images of I_{w_n} and $I_{v_{n-1}}$ under $f^{\circ(n-n_k)}$ for any $n > n_k$ and the intervals $I_{w_{n_2}}$ and $I_{v_{n_2-1}}$ are the images of $I_{w_{n_k}}$ and $I_{v_{n_k-1}}$ under $f^{\circ(n_k-n_2)}$. Then we get

$$|s(w_n) - s(w_{n_k})| = |\frac{f^{\circ(n-n_k)}(\xi_1)}{f^{\circ(n-n_k)}(\xi_2)} - 1| \cdot \frac{|I_{w_{n_k}}|}{|I_{v_{n_k-1}}|}$$

$$= |\frac{(f^{\circ(n-n_k)})'(\xi_1)}{(f^{\circ(n-n_k)})'(\xi_2)} - 1| \cdot |\frac{(f^{\circ(n_k-n_2)})'(\xi_3)}{(f^{\circ(n_k-n_2)})'(\xi_4)}| \cdot \frac{|I_{w_{n_2}}|}{|I_{v_{n_2-1}}|}$$

where ξ_1 and ξ_2 are two points in $I_{v_{n-1}}$ and ξ_3 and ξ_4 are two points in $I_{v_{n_k-1}}$. Suppose L is the minimum of lengths of the intervals in η_{n_1}. Then $D_{\xi_1\xi_2}$ and $D_{\xi_3\xi_4}$ in Lemma 2 (the $C^{1+\alpha}$-Denjoy-Koebe distortion lemma), respectively, are both greater than or equal to L (we can always reduce to this situation). By using Lemma 2, there is a positive constant K such that $|s(w_n) - s(w_{n_k})| \leq K|I_{w_{n_k}}|^\alpha$. From this estimate, $|s(w_n) - s(w_m)| \leq 2K|I_{w_{n_k}}|^\alpha$ for any $m > n \geq n_k$. Again the sequence $\{s(w_n)\}_{n=1}^{\infty}$ is a Cauchy sequence and the limit $s(a^*) = \lim_{n\mapsto\infty} s(w_n)$ exists. We proved Theorem A.

Corollary A1. *Suppose f is a geometrically finite one-dimensional mapping. Then the scaling function $|s_f| : \Sigma_f^* \mapsto \mathbf{R}^1$ of f is discontinuous.*

Proof. Suppose p_i is the periodic point such that the critical point c_i of f lands on it under some iterates of f for every $i = 1, \cdots, d_1$. Suppose $OB(p_i) = \cup_{k=0}^{\infty} f^{\circ k}(p_i)$ be the periodic orbit of p_i under f. Let $O = \cup_{k=1}^{d_1} OB(p_i)$ be the union of the periodic orbits $OB(p_i)$. It is contained in $P(f)$. Suppose $Q : P(\sigma_f^*) \mapsto P(f)$ is the mapping in Proposition 2. Let $\mathcal{A}_0$ be the preimage of O under Q and $\mathcal{A}$ be the union of the preimages of $\mathcal{A}_0$ under the n^{th}-iterate of σ_f^* for $n = 0, 1, \cdots$. We claim that all the points in $\mathcal{A}$ are discontinuous points of the scaling function $s_f : \Sigma_f^* \mapsto \mathbf{R}^1$.

For an $a_0^* = \cdots w_n \in \mathcal{A}$, let I_{w_n} be the interval in η_n with the name w_n. There is a subsequence $\{n_k\}_{k=2}^{\infty}$ of the integers such that $I_{w_{n_k}}$ tends to a periodic point p_i of f as k goes to infinity. To simplify our arguments, let us assume that the critical point c_i is not in the post-critical orbits $\cup_{k=1}^{\infty} f^{\circ k}(CP)$ and $f(c_i) = p_i$. For a more general situation, the proof can be easily obtained by modifying the following arguments.

Suppose $I_{u_{n_k+1}}$ is an inverse branch of the interval $I_{w_{n_k}}$ under f and is contained in a critical interval I in η_{n_1} (where n_1 is the integer in the proof of Theorem A). Because the restriction of f to I is comparable with $|x - c_i|^{\gamma_i} + f(c_i)$ for some $0 \leq i \leq l$, then we can show that the scale $s(u_{n_k+1})$ at u_{n_k+1} can be calculated as follows:

$$s(u_{n_k+1}) = \frac{\xi_{n_k,1}}{\xi_{n_k,2}} s(w_{n_k}).$$

The limit μ of the sequence $\{\mu_k = \xi_{n_k,1}/\xi_{n_k,2}\}_{k=2}^{\infty}$ exists and does not equal one if $I_{w_{n_k}} \neq I_{v_{n_k-1}}$ where $w_{n_k} = v_{n_k-1} r_{i_0}$ (this is true in general). Let k go to infinity, we get that

$$\lim_{a^* = \cdots u_{n_k+1}, k \mapsto \infty} s_f(a^*) = \mu \cdot s_f(a_0^*).$$

This implies that s_f is discontinuous at the point a_0^*.

Remark. We actually can find all the continuous points and discontinuous points of s_f. Let us do it in a little simple case. Suppose the set CP of critical points of f is disjoint with the post-critical orbits $\cup_{n=1}^{\infty} f^{\circ n}(CP)$. Let $\mathcal{U}$ and $\mathcal{V}$ be the sets in the proof of Theorem A. Suppose $a^* = \cdots v_{n-1} r_{i_0}$ is a point in the dual space Σ_f^*. Let $I_{v_{n-1}}$ be the interval in η_{n-1} with the name v_{n-1}. We say a^* is recurrent if there is a subsequence $\{n_k\}_{k=2}^{\infty}$ of the integers such that $I_{v_{n_k-1}}$ is contained in $\mathcal{U}$ for every $k \geq 2$. We say a^* is totally nonrecurrent if there is an integer $N > 0$ such that the preimage of $I_{v_{N-1}}$ under $f^{\circ k}$ is contained in $\mathcal{V}$ for every $k \geq 0$. We say a^* is wandering if (a) there is an integer $N > 0$ such that $I_{v_{n-1}}$ is contained in $\mathcal{V}$ for every $n \geq N$ and (b) for every $k > N$ there is an integer n_k satisfying that the preimage of $I_{v_{k-1}}$ under $f^{\circ n_k}$ intersects with the interior of $\mathcal{U}$. Then we can prove that s_f is continuous at all the recurrent and totally nonrecurrent points and discontinuous at all the wandering points.

COROLLARY A2. *Suppose f is a geometrically finite one-dimensional mapping. Then the exponent γ of f at a power law critical point c can be calculated by the scaling function $s_f : \Sigma_f^* \mapsto \mathbf{R}^1$.*

Proof. Suppose $c_{i_1}, \cdots, c_{i_n}$ are critical points of f. We say they form a chain if there are integers $l_1, \cdots, l_{n-1}$ such that $f^{\circ l}(c_{i_k})$ is not a critical point of f for $0 < l < l_k$ and $f^{\circ l_k}(c_{i_k}) = c_{i_{k+1}}$.

Suppose $c_{i_1}, \cdots, c_{i_n}$ form a maximum chain. Let I_{w_m} is an interval in η_m which has c_{i_1} as an endpoint. Then $I_{w_{m-m_k}} = f^{\circ m_k}(I_{w_m})$ has c_{i_k} as an endpoint where $m_k = l_1 + \cdots + l_k$ for $1 \leq k < n$. Suppose l_n is the smallest integer such that $p = f^{\circ l_n}(c_{i_n})$ is a periodic point of f and $I_{w_{m-m_n}} = f^{\circ m_n}(I_{w_m})$ is an interval which has p as an endpoint where $m_n = l_1 + \cdots + l_n$. Suppose $\gamma_{i_1}, \cdots, \gamma_{i_n}$ are the corresponding exponents of these critical points and $a^* = (w_{m-m_n})^\infty \in \Sigma_f^*$. Then we have that

$$\gamma_n = \frac{\log |s_f(a^*)|}{\lim_{m \mapsto \infty} \log |s(w_{m-m_{n-1}})|},$$

and

$$\gamma_k = \frac{\lim_{m \mapsto \infty} \log |s(w_{m-m_k})|}{\lim_{m \mapsto \infty} \log |s(w_{m-m_{k-1}})|}$$

for $1 \leq k < n$.

§3.2 The $C^{1+\alpha}$-classification.

Suppose f and g are geometrically finite and topologically conjugate by an orientation-preserving homeomorphism h. We say f and g are C^1-conjugate if h is a C^1-diffeomorphism. One of the corollaries of Proposition 1 and Proposition 5 is that the signed scaling functions of f and g and the asymmetries of f and g at the corresponding critical points are the same if f and g are C^1-conjugate. Another main result in this paper is that the signed scaling function and the asymmetries at critical points of a geometrically finite one-dimensional mapping form a complete set of C^1-invariants within a topological conjugacy class as follows.

THEOREM B. *Suppose f and g are geometrically finite and topologically conjugate by an orientation-preserving homeomorphism h. Then f and g are C^1-conjugate if and only if they have the same signed scaling function and the same asymmetries at the corresponding critical* *points.*

Remark. The topological conjugacy class $[f]$ is the subset of geometrically finite one-dimensional mappings which are topologically conjugate to f. The class $[f]$ equals the union of $[f]_+$ and $[f]_-$. Here $[f]_+$ is the subset of geometrically finite one-dimensional mappings which are topologically conjugate to f by orientation-preserving homeomorphisms and $[f]_-$ is the subset of geometrically finite one-dimensional mappings which are topologically conjugate to f by orientation-reversing homeomorphisms. There is a one-to-one corresponding between $[f]_+$ and $[f]_-$.

Actually, we can show more on the smoothness of the conjugating mapping h as follows.

Corollary B1. *Suppose f and g are $C^{1+\alpha}$-geometrically finite one-dimensional mappings for some $0 < \alpha \leq 1$. Furthermore suppose they are topologically conjugate by an orientation-preserving homeomorphism h. If f and g have the same signed scaling function and the same asymmetries at the corresponding critical points, then h is a $C^{1+\alpha}$-diffeomorphism.*

As we mentioned in the beginning of this subsection, "if only" part of Theorem B is a corollary of Proposition 1 and 5. We prove "if" part of Theorem B and Corollary B1 by several lemmas.

Suppose f and g are geometrically finite and topologically conjugate by an orientation-preserving homeomorphism h, that is, $h \circ f = g \circ h$. Furthermore suppose f and g are both $C^{1+\alpha}$ for some $0 < \alpha \leq 1$. We use $\eta_{n,f}$ to denote the n^{th}-partition of M by f and use $\eta_{n,g}$ to denote the n^{th}-partition of M by g for every integer $n \geq 0$. We note that the dual space Σ_f^* of $f : M \mapsto M$ and the dual space Σ_g^* of $g : M \mapsto M$ are the same.

To present a clear idea and to avoid unnecessary notations, we prove the following lemmas under the assumption that the set PC of critical points and the post-critical orbits $\cup_{n=1}^{\infty} f^{\circ n}(CP)$ of f are disjoint. We may also assume that there is an interval I_{k_0} in the first partition $\eta_{1,f}$ such that every interval I in the first partition $\eta_{1,f}$ covers I_{k_0} eventually under some iterate of f, that means, there is an integer n such that the image of I under $f^{\circ n}(I)$ contains I_{k_0}. Otherwise, we can divide M

into finitely many intervals, each of which consists of some intervals in the first partition $\eta_{1,f}$, such that the restrictions of f to these intervals are geometrically finite and satisfy this assumption.

Suppose A_f, B_f and $n_{0,f}$ are the constants in Lemma 2 for f and A_g, B_g and $n_{0,g}$ are the constants in Lemma 2 for g. Let A_0, B_0 and n_0 are the maximums of A_f and A_g, B_f and B_g, $n_{0,f}$ and $n_{0,g}$, respectively.

We say an interval I in $\eta_{n,f}$ is a critical interval if one of its endpoints is a critical points of f. Suppose $n_1 > n_0$ is an integer such that every critical interval I in $\eta_{n_1,f}$ has an endpoint which is not in the critical orbits $\cup_{n=0}^{\infty} f^{\circ n}(CP)$ of f. Suppose L_f is the minimum of lengths of the intervals in $\eta_{n_1,f}$ and L_g is the minimum of lengths of the intervals in $\eta_{n_1,g}$. Let L be the minimum of L_f and L_g. We use $\mathcal{U}$ to denote the union of the critical intervals in $\eta_{n_1,f}$ and use $\mathcal{V}$ to denote the closure of the complement of $\mathcal{U}$ in M.

Lemma B1. *There is a positive constant K such that for an interval I in $\eta_{n+n_1,f}$, if the image $I_n = f^{\circ n}(I)$ of I under $f^{\circ n}$ is a critical interval in $\eta_{n_1,f}$, then*

$$\frac{|f^{\circ m}(z)|}{|f^{\circ m}(w)|} \leq \exp\Big(K|f^{\circ n}(z) - f^{\circ n}(w)|^{\alpha}\Big),$$

$$\frac{|g^{\circ m}(h(z))|}{|g^{\circ m}(h(w))|} \leq \exp\Big(K|g^{n}(h(z)) - g^{\circ n}(h(w))|^{\alpha}\Big)$$

for any points z and w in I.

Proof. This lemma is actually a corollary of Lemma 2 (the $C^{1+\alpha}$-Denjoy-Koebe distortion lemma).

Lemma B2. *There is a positive constant K such that for an interval I in $\eta_{n+n_1,f}$, if the image $I_i = f^{\circ i}(I)$ of I under $f^{\circ i}$ is in $\mathcal{V}$ for any $0 \leq i \leq n$, then*

$$\frac{|f^{\circ m}(z)|}{|f^{\circ m}(w)|} \leq \exp\Big(K|f^{\circ n}(z) - f^{\circ n}(w)|^{\alpha}\Big),$$

$$\frac{|g^{\circ m}(h(z))|}{|g^{\circ m}(h(w))|} \leq \exp\Big(K|g^{n}\Big(h(z)\Big) - g^{\circ n}\Big(h(w)\Big)|^{\alpha}\Big)$$

for any points z and w in I.

Proof. This lemma is actually a corollary of the naive distortion lemma in [J1] (see also [J2]).

We say a homeomorphism $q : I \mapsto J$ from an interval I to an interval J is absolutely continuous if it is non-singular with respect to the Lebesgue measure m, that is, $m(X) = 0$ if and only if $m(h(X)) = 0$ for any measurable set X of I. For example, if q and q^{-1} are both Lipschitz continuous, then q is absolute continuous.

LEMMA B3. *Suppose h is absolutely continuous and has a differentiable point p_0 in I_{k_0} with nonzero derivative. Then the restriction of h to every critical interval in $\eta_{n_1,f}$ is $C^{1+\alpha}$.*

Proof. Suppose $GPI = \cup_{i=0}^{\infty} \cup_{j=0}^{\infty} f^{-j}(f^{\circ i}(p_0))$ is the grand preimage of p_0 under f. Then GPI is a dense subset of M. By the equation $h \circ f = g \circ h$ and the definition of $C^{1+\alpha}$ in this paper, h is differentiable at every point in GPI and there is a constant $K_0 > 0$ such that $h'(x) > K_0$ for all $x \in GPI$.

Suppose I_w is a critical interval in $\eta_{n_1,f}$ and $a^* = \cdots w_n w$ is a point in Σ_f^*. Let $I_{w_n w}$ be the interval in $\eta_{n_1+n,f}$ with name $w_n w$. For any pair x and y in the intersection of I_w and GPI, let x_n and y_n in $I_{w_n w}$ be the preimages of x and y under $f^{\circ n}$. Using the equation $h \circ f = g \circ h$, we have that

$$\frac{h'(x)}{h'(y)} = \prod_{n=0}^{\infty} \frac{|f'(y_n)|}{|f'(x_n)|} \prod_{n=0}^{\infty} \frac{|g'(h(x_n))|}{|g'(h(y_n))|}.$$

By using Lemma B1,

$$\prod_{n=0}^{\infty} \frac{|f'(y_n)|}{|f'(x_n)|} \leq \exp\Big(K|x-y|^{\alpha}\Big)$$

and

$$\prod_{n=0}^{\infty} \frac{|g'(h(x_n))|}{|g'(h(y_n))|} \leq \exp\Big(K|h(x)-h(y)|^{\alpha}\Big).$$

This implies that

$$\frac{h'(x)}{h'(y)} \leq \exp\Big(K\Big(|x-y|^{\alpha} + |h(x)-h(y)|^{\alpha}\Big)\Big).$$

From the last inequality, the restriction of h' to the intersection of I_w and GPI is uniformly continuous. Then it can be extended to a continuous function on I_w. Because the restriction of h to I_w is absolutely continuous, this continuous extension is the derivative of the restriction of h to I_w. Using the last inequality again, the restriction of h to I_w is $C^{1+\alpha}$.

COROLLARY B2. *Suppose h is absolutely continuous. Then the exponents of f and g at the corresponding critical points are the same.*

LEMMA B4. *Suppose h is absolutely continuous. Then the restriction of h to every interval in $\eta_{n_1,f}$ is $C^{1+\alpha}$.*

Proof. We still use the same notations as that in the proof of Lemma B3. Suppose I_w is an interval in $\eta_{n_1,f}$ and $a^* = \cdots w_n w$ is a point in Σ_f^*. Let $I_{w_n w}$ be the interval in $\eta_{n_1+n,f}$ with the name $w_n w$. Suppose x and y are any pair in I_w and x_n and y_n in $I_{w_n w}$ are the preimages of x and y under $f^{\circ n}$. Using the equation $h \circ f = g \circ h$, we have that

$$\frac{h'(x)}{h'(y)} = \prod_{n=0}^{\infty} \frac{|f'(y_n)|}{|f'(x_n)|} \prod_{n=0}^{\infty} \frac{|g'(h(x_n))|}{|g'(h(y_n))|}.$$

If I_w is a critical interval, then it is Lemma B3. Suppose I_w is not a critical interval. We consider the sequence $\{I_{w_n w}\}_{n=0}^{\infty}$ in the two cases. The first is that all $I_{w_n w}$ are contained in $\mathcal{V}$ and the other is that there is an integer n such that $I_{w_n w}$ is contained in $\mathcal{U}$.

In the first case, by using Lemma B2,

$$\prod_{n=0}^{\infty} \frac{|f'(y_n)|}{|f'(x_n)|} \leq \exp\Big(K|x-y|^{\alpha}\Big)$$

and

$$\prod_{n=0}^{\infty} \frac{|g'(h(x_n))|}{|g'(h(y_n))|} \leq \exp\Big(K|h(x)-h(y)|^{\alpha}\Big).$$

This implies that

$$\frac{h'(x)}{h'(y)} \leq \exp\Big(K\Big(|x-y|^{\alpha} + |h(x)-h(y)|^{\alpha}\Big)\Big).$$

Then by the same arguments in the proof of Lemma B1, we get that the restriction of h to I_w is $C^{1+\alpha}$.

For the other case, let k be the smallest integer such that $I_{w_k w}$ is in $\mathcal{U}$. The product

$$\prod_{n=0}^{\infty} \frac{|f'(y_n)|}{|f'(x_n)|} \prod_{n=0}^{\infty} \frac{|g'(h(x_n))|}{|g'(h(y_n))|}$$

can be written in three products

$$\prod_{n=0}^{k-1} \frac{|f'(y_n)|}{|f'(x_n)|} \prod_{n=0}^{k-1} \frac{|g'(h(x_n))|}{|g'(h(y_n))|},$$

$$\frac{|f'(y_k)|}{|f'(x_k)|} \frac{|g'(h(x_k))|}{|g'(h(y_k))|},$$

$$\prod_{n=k+1}^{\infty} \frac{|f'(y_n)|}{|f'(x_n)|} \prod_{n=k+1}^{\infty} \frac{|g'(h(x_n))|}{|g'(h(y_n))|}.$$

By using Lemma B2

$$\prod_{n=0}^{k-1} \frac{|f'(y_n)|}{|f'(x_n)|} \leq \exp\Big(K|x-y|^{\alpha}\Big)$$

and

$$\prod_{n=0}^{k-1} \frac{|g'(h(x_n))|}{|g'(h(y_n))|} \leq \exp\Big(K|h(x)-h(y)|^{\alpha}\Big).$$

By using Lemma B1,

$$\prod_{n=k+1}^{\infty} \frac{|f'(x_n)|}{|f'(y_n)|} \leq \exp\Big(K|x_{k+1}-y_{k+1}|^{\alpha}\Big)$$

and

$$\prod_{n=k+1}^{\infty} \frac{|g'(h(y_n))|}{|g'(h(x_n))|} \leq \Big(K|h(x_{k+1})-h(y_{k+1})|^{\alpha}\Big).$$

Suppose $I_{w_k w}$ is in the critical interval I in $\eta_{n_1,f}$ which has a critical point c_i of f as an endpoint. Let

$$l_f(x) = f'(x)/|x-c_i|^{\gamma_i - 1} \textit{ and } l_g(h(x)) = g'(h(x))/|h(x)-h(c_i)|^{\gamma_i - 1}$$

for x in I, where γ_i is the exponent of f at c_i. Then

$$\frac{|f'(y_k)|}{|f'(x_k)|}\frac{|g'(h(x_k))|}{|g'(h(y_k))|} = \frac{l_f(y_k)}{l_f(x_k)}\frac{l_g(h(y_k))}{l_g(h(x_k))}\frac{(\frac{h(y_k)-h(c_i)}{|y_k-c_i|})^{\gamma_i-1}}{(\frac{h(x_k)-h(c_i)}{|x_k-c_i|})^{\gamma_i-1}}.$$

By the definition of $C^{1+\alpha}$ in this paper and Lemma B3, the functions l_f, l_g and $h(x)/|x-c_i|$ are α-Hölder continuous. There is a positive constant, we still denote it as K, such that

$$\frac{|f'(x_k)|}{|f'(y_k)|}\frac{|g'(h(y_k))|}{|g'(h(x_k))|} = \exp\Big(K\Big(|x_k-c_i|^{\alpha}+|h(x_k)-h(c_i)|^{\alpha}\Big)\Big).$$

All these estimates and the same arguments as that in the proof of Lemma B1 say that the restriction of h to I_w is $C^{1+\alpha}$.

Lemma B5. *Suppose h is absolutely continuous. Furthermore suppose f and g have the same asymmetries at the corresponding critical points. Then h is $C^{1+\alpha}$.*

Proof. From Lemma B4, the restriction of h to every interval in $\eta_{n_1,f}$ is $C^{1+\alpha}$. This implies that for every interval I in $\eta_{n_1,f}$, the one-sided limits of the derivative $h'|I$ at the endpoints of I exist. We need to prove that these one-sided limits are the same at a common endpoint of any two intervals in $\eta_{n_1,f}$.

Suppose I and I' are two intervals in $\eta_{n_1,f}$ and have a common endpoint p. By the equation $h\circ f = g\circ h$,

$$h'(p-) = \lim_{x\mapsto p-}\frac{f'(x)}{g'(h(x))}h'(p_1-),$$

$$h'(p+) = \lim_{x\mapsto p+}\frac{f'(x)}{g'(h(x))}h'(p_1+)$$

where p_1 is a point in the preimage of p under f. Without loss generality, we may assume that p_1 is an interior point of an interval in η_{n_1}. Then $h'(p_1-) = h'(p_1+)$.

If p is not a critical point of f, it is easy to see that $h'(p-) = h'(p+)$.

Suppose p is a critical point c_i of f. Let

$$A_f(p) = \lim_{x\mapsto p-} f'(x)/|x-p|^{\gamma_i-1},$$

$$B_f(p) = \lim_{x \mapsto p+} f'(x)/|x-p|^{\gamma_i - 1},$$

and

$$A_g(h(p)) = \lim_{x \mapsto p-} g'(h(x))/|h(x)-h(p)|^{\gamma_i - 1},$$

$$B_g(h(p)) = \lim_{x \mapsto p+} g'(h(x))/|h(x)-h(p)|^{\gamma_i - 1}.$$

Then

$$(h'(p-))^{\gamma_i} = \frac{A_f(p)}{A_g(h(p))} h'(p_1), \ \ and \ \ (h'(p+))^{\gamma_i} = \frac{B_f(p)}{B_g(h(p))} h'(p_1).$$

The equality

$$A_f(p)/B_f(p) = A_g(h(p))/B_g(h(p))$$

implies that $h'(p-) = h'(p+)$. We proved Lemma B5.

Lemma B6. *Suppose $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are two sequences of positive numbers and there is a constant $K > 0$ such that $a_n/b_n \leq K$ for any n. Then $(\sum_{n=0}^{\infty} a_n)/(\sum_{n=0}^{\infty} b_n) \leq K$.*

The proof of this lemma is very easy, but it is very useful in the study of dynamic systems.

Lemma B7. *Suppose f and g have the same scaling function. Then the conjugating mapping h is Lipschitz continuous.*

Proof. Suppose N is the number of the intervals in the first partition $\eta_{1,f}$ and n_1 is greater than $2N$.

For every integer $n \geq 0$ and every interval I_{ww_n} in $\eta_{n+n_1,f}$ with name ww_n, let $s_f(ww_n)$ and $s_g(ww_n)$ be the scales at ww_n (with respect to f and g). Then we have an equation

$$\frac{|h(I_{ww_n})|}{|I_{ww_n}|} = \frac{|s_g(ww_n)|}{|s_f(ww_n)|} \frac{|h(I_{wv_{n-1}})|}{|I_{wv_{n-1}}|},$$

where $ww_n = wv_{n-1} r_{i_0}$.

Let $a^* = \cdots u_m w w_n$ be a point in Σ_f^* and $I_{u_m w w_n}$ be the interval in $\eta_{m+n_1+n,f}$ with the name $u_m w w_n$. We discuss the sequence $\{I_{u_m w w_n}\}_{m=0}^{\infty}$ in the three cases. The first case is that $I_{w w_n}$ is in $\mathcal{U}$ The second case is that all $I_{u_m w w_n}$ are in $\mathcal{V}$. The third case is that there is a positive integer k such that $I_{u_m w w_n}$ is in $\mathcal{V}$ for every $0 \leq m \leq k$ and $I_{u_{k+1} w w_n}$ is in $\mathcal{U}$.

In the first and the second cases, we use Lemma B2 and Lemma B1, respectively, to prove the following:

There is a constant $0 < \mu < 1$ such that

$$||s_f(a^*)| - |s_f(ww_n)|| \leq \exp(\mu^n),$$

$$||s_g(a^*)| - |s_g(ww_n)|| \leq \exp(\mu^n).$$

Because there is a constant $\beta > 0$ such that $|s_f(c^*)| \geq \beta$ for all $c^* \in \Sigma_f^*$ and $|s_g| = |s_f|$, we can find a constant, we still denote it as μ, in $(0,1)$ such that

$$\frac{|s_f(ww_n)|}{|s_g(ww_n)|} \leq \exp(\mu^n).$$

For the third case, let us suppose that $I_{u_{k+1} w w_n}$ is contained in a critical interval I in $\eta_{n_1,f}$ which has a critical point c of f as an endpoint. There is an integer $0 < m < 2N$ such that $f^{\circ m}(c)$ is a periodic point of f and the interval $f^{\circ m}(I_{u_{k+1} w w_n})$ is contained in $f^{\circ m}(I)$. We note that $f^{\circ m}(I)$, which has p as an endpoint, is an interval in $\eta_{n_1-m,f}$ and $n_1 - m > 0$. We also note that $f^{\circ m}(I_{u_{k+1} w w_n})$ is an interval in $\eta_{n_1+n+k+1-m,f}$. Now we can find a point $b^* = \cdots v_j$ in Σ_f^* such that the first $n_1 + n + k + 1 - m$ symbols of a^* and b^* (from the right) are the same and the interval I_{v_j} in η_j is contained in $\mathcal{V}$ for every $j > 0$ and tends to the periodic orbit $\cup_{i=0}^{\infty} f^{\circ i}(p)$ as j goes to infinity. By using Lemma B2, there is constant, we still denote it as μ, in $(0,1)$ such that

$$||s_f(b^*)| - |s_f(ww_n)|| \leq \exp(\mu^n),$$

$$||s_g(b^*)| - |s_g(ww_n)|| \leq \exp(\mu^n).$$

Again, because $|s_f(c^*)| \geq \beta$ for all $c^* \in \Sigma_f^*$ and $|s_g| = |s_f|$, we can find a constant, we still denote it as μ, in $(0,1)$ such that

$$\frac{|s_f(ww_n)|}{|s_g(ww_n)|} \leq \exp(\mu^n).$$

Suppose K_0 is the minimum of the ratios, $|h(I_w)|/|I_w|$, for I_w in $\eta_{n_1,f}$. By the above arguments and the induction, we find a sequence $\{K_n\}_{n=0}^{\infty}$ and a constant $\mu \in (0,1)$ such that

$$\frac{|h(I_{ww_n})|}{|I_{ww_n}|} \leq K_n$$

for every interval I_{ww_n} in $\eta_{n+n_1,f}$, $n \geq 0$, and

$$K_n \leq \exp(\mu^n) K_{n-1}$$

for every integer $n \geq 1$. This yields a positive constant K such that

$$\frac{|h(I_{ww_n})|}{|I_{ww_n}|} \leq K$$

for every $n \geq 0$ and every interval I_{ww_n} in $\eta_{n+n_1,f}$.

Because the union of the boundary points of all the intervals in $\eta_{n+n_1,f}$ for all the integer $n \geq 0$ is a dense subset in M, by using Lemma B6

$$\frac{|h(x) - h(y)|}{|x-y|} \leq K$$

for every pair x and y in M. In the other words, h is Lipschitz continuous.

LEMMA B8. *Suppose f and g have the same scaling function and the same asymmetries at the corresponding periodic points. Then the conjugating mapping h is a $C^{1+\alpha}$-diffeomorphism.*

Proof. From Lemma B7, the mapping h is Lipschitz continuous. It is then differentiable at almost every points in M. Let p_0 be a point in I_{k_0} such that h is differentiable at this point. Suppose $GPI = \cup_{i=0}^{\infty} \cup_{j=0}^{\infty} f^{-j}(f^{\circ i}(p_0))$ is the grand preimage of p_0 under f. It is a dense subset of M. If the derivative $h'(p_0)$ at p_0 is zero, then by the equation $h \circ f = g \circ h$, the derivative $h'(p)$ at every point $p \in GPI$ is zero. But h is absolutely continuous, this implies that h is a constant. So the derivative $h'(p_0)$ is not zero. Now Lemma B6 says that h is $C^{1+\alpha}$. The same arguments can be applied to h^{-1}. Hence h is a $C^{1+\alpha}$-diffeomorphism.

Lemma B1 to Lemma B8 give the proof of Theorem A.

Suppose $f : M \mapsto M$ is a geometrically finite one-dimensional mapping and $s_f : \Sigma_f \mapsto \Sigma_f$ is the signed scaling function of f. The eigenvalue $e_f(p) = (f^{\circ n})'(p)$ of f at a periodic point p of period n and the exponent γ of f at a critical point c can be calculated by the signed scaling function s_f of f showed by Proposition 2 and by Corollary A2. Both of them are then clearly C^1-invariants. In the case that the set of periodic points of f is dense in M, we can show that the eigenvalues of f at periodic points and the exponents and the asymmetries of f at critical points form a complete C^1-invariants within a topologically conjugate class as follows.

THEOREM C. *Suppose f and g are $C^{1+\alpha}$-geometrically finite one-dimensional mappings for some $0 < \alpha \leq 1$. Furthermore, suppose the set of periodic points of f is dense in M and suppose f and g are topologically conjugate by an orientation-preserving homeomorphism h If f and g have the same eigenvalues at the corresponding periodic points and the same exponents at the corresponding critical points, then they have the same scaling functions. Moreover, if f and g have also the same asymmetries at the corresponding critical points, then h is a $C^{1+\alpha}$-diffeomorphism.*

Proof. The idea of the proof of Theorem C is the same as that of the proof of Theorem B and that of the proof of Theorem 1.4 in [J1, p 63-74]. The details will be omitted.

References

[B] R. Bowen. Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms. *Springer-Verlag, Berlin, Heidelberg, New York.*

[CP] D. Cooper and T. Pignataro, [1985]. On The Shape of Cantor Sets. *Preprint, IHES, March 1985.*

[F] M. Feigenbaum. The Universal Metric Properties of Non-linear Transformations. *J. Stat. Phys. 21 669(1979).*

[H] M. R. Herman, [1979]. Sur la Conjugaision Differentiable des Diffeomorphisnes du Cercle á des Rotations. *Thesis, University de Paris, Orsay and Publ. Math. IHES, No.* **49**

[LM] R. de la Llave and R. Moriyón, [1987]. Invariant for Smooth Conjugacy of Hyperbolic Dynamical System IV. *Comm. in Math. Phys.* **109***, 369-378.*

[J1] Y. Jiang, [1990]. Generalized Ulam-von Neumann Transformations. *Thesis, the Graduate School of CUNY.*

[J2] Y. Jiang, [1990]. Dynamics of Certain Smooth One-dimensional Mappings – I. The $C^{1+\alpha}$-Denjoy-Koebe Distortion Lemma. *Preprint.*

[J3] Y. Jiang, [1990]. Dynamics of Certain Smooth One-dimensional Mappings – II. Geometrically finite one-dimensional mappings. *Preprint.*

[J4] Y. Jiang, [1989]. Ratio Geometry in Dynamical Systems. *Preprint.*

[J5] Y. Jiang, [1990]. Dynamics of Certain Smooth One-dimensional Mappings – VI. Smooth structure of one-dimensional mappings. *To appear.*

[SS] M. Shub and D. Sullivan, [1985]. Expanding Endomorphisms of the circle revisited. *Ergod. Th & Dynam. Sys., 5, 285-289.*

[S1] D. Sullivan, [1989]. Private Communication.

[S2] D. Sullivan, [1988]. Differentiable Structure on Fractal Like Sets Determined by Intrinsic Scaling Functions on Dual Cantor Sets, [1988], *The Proceedings of Symposia in Pure Mathematics, Vol.* **48***.*