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On the Lebesgue measure of the Julia set of a quadratic polynomial.

Misha Lyubich

§1. Statement of the result.

The goal of this note is to prove the following

> **Theorem.** Let $p_a : z \mapsto z^2 + a$ be a quadratic polynomial which has no irrational indifferent periodic points, and is not infinitely renormalizable. Then the Lebesgue measure of the Julia set $J(p_a)$ is equal to zero.

It was proven by McMullen that the Julia set of a cubic polynomial $f_{a,b} : z \mapsto z^3 - 3a^2z + b$ has zero measure provided it is a Cantor set (see [BH]). Our idea is to extract from the Julia set $J(p_a)$ an essential part on which a "renormalization" of p_a is a "polynomial-like map" g (in a generalized sense). The construction involves the Yoccoz partitions of the Julia set. Then using Branner-Hubbard-McMullen's method and a notion of the modulus of a multiply connected domain, one can show that the Julia set $J(g)$ has zero measure. By [L], $J(p_a)$ has zero measure as well.

As part of the proof we discuss a property of the critical point to be *persistently recurrent*, and relate our results to corresponding ones for real one dimensional maps. In particular, we will show that in the persistently recurrent case the restriction $p_a|\omega(0)$ is topologically minimal and has zero topological entropy.

Let us mention that the Douady-Hubbard-Yoccoz rigidity theorem follows from the above result:

> **Corollary.** If p_a is not infinitely renormalizable and has no attracting periodic points then it is J-unstable.

Indeed, by the Theorem, p_a has no measurable invariant line fields on $J(p_a)$ and hence has no deformations concentrated on the Julia set [MSS]. It has no deformations on the Fatou set as well since there are no attracting periodic points. So, p_a is rigid.

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§2 Polynomial-like maps.

Let us introduce a notion of a *(generalized) polynomial-like map* (compare [DH1]). Let V and V_i , i=1,...,d, be open topological disks with piecewise smooth boundaries such that the V_i are pairwise disjoint and $\mathrm{cl}\, V_i \subset V$. (We don't require $\mathrm{cl}\, V_i$ to be pairwise disjoint.) A branched covering

$$g : \bigcup V_i \to V$$

will be called a (generalized) polynomial-like map.

As usual, one can define the filled Julia set of g as

$$K(g) = \{z : f^n z \in \cup V_i, n = 0, 1, ...\},$$

and the Julia set $J(g)$ as its boundary.

In what follows we will assume that g has a unique critical point c which is non-degenerate , and moreover that $c \in K(g)$.

Let us consider the nested sequence of inverse images $V^n = f^{-n}V$. Clearly, $\mathrm{cl}\, V^{n+1} \subset V^n$. Components V_k^n are called *pieces* of level n . Denote by $V^n(x)$ a piece of level n containing the point x . The pieces $V^n(c)$ will be called *critical*.

Following Branner and Hubbard [BH], for any $x \in K(g)$ one can organize the pieces in a *tableau* $T(x) = \{V^i(g^j x)\}_{i,j=0}^{\infty}$, mark the positions of the critical pieces and state three combinatorial rules for the resulting *marked grids*.

Now let us consider a topological disk D containing a compact subset $K \subset D$. Let us assign to the domain $A = D \smallsetminus K$ the *modulus* $\mu(A)$ in the following way (see [A]). If K has zero capacity then $\mu(A) = \infty$. Otherwise

$$\mu(A) = \frac{1}{I(u)} \tag{1}$$

where u is the harmonic function in A which tends to 0 at regular points of K and tends to 1 at regular points of ∂D , and

$$I(u) = \int\int_A |\mathrm{grad}\ u|^2 dz d\bar{z}$$

is its Dirichlet integral . Clearly, if we have a d-sheeted branched covering $h : (D, K) \to (D', K')$ then $\mu(A') = d\mu(A)$.

For a tableau $T(x) = V_k^n$ denote by μ_k^n the matrix of moduli of the domains $A_k^n \equiv V_k^n \backslash V^{n+1}$.

Lemma 1.

(i). If a piece V_k^n is not critical, $n > 0$, then $\mu_{k+1}^{n-1} = \mu_k^n$. Otherwise $\mu_{k+1}^{n-1} = 2\mu_k^n$.

(ii). If the critical tableau is aperiodic then for any $x \in K(f)$ the nest of pieces $V^n(x)$ is of divergent type:

$$\sum \mu(A^n(x)) = \infty.$$

Proof. The first point is immediate from the properties of the moduli. The second point is the formal consequence of the first one and the combinatorics of marked grids [BH]. □

Corollary 1. Assume that the critical tableau is aperiodic. Then $K(g)$ is a Cantor set.

Proof. Let us consider the annulus $A(x) = V \smallsetminus \cap V^n(x)$ containing the disjoint union of domains $A^n(x)$. Since the embeddings $A^n(x) \subset A(x)$ are homotopically non-trivial, the Grötzsch inequality yields

$$\mu(A(x)) \geq \sum \mu(A^n(x)) = \infty.$$

So, $\cap V^n(x)$ is a point. □

Now let us state an analytical lemma which generalizes McMullen's one onto multiply connected domains (see [BH], §5.4). Denote by λ the Lebesgue measure on the plane.

Lemma 2. Let D be a topological disk and $K \subset D$ be its compact subset consisting of finitely many components, $A = D \setminus K$. Then

$$\frac{\lambda(D)}{\lambda(K)} \geq 1 + 4\pi\mu(A).$$

Proof. The modulus $\mu(A)$ can also be defined as the reciprocal to the extremal length $\sigma(\Gamma)$ of the family Γ of (non-connected) curves separating K from ∂D. For such a curve γ there is a decomposition of K into the union of disjoint pieces K_i surrounded by pieces γ_i of γ. Then

$$\sigma(\Gamma) \geq \inf_{\gamma\in\Gamma} \frac{|\gamma|^2}{\lambda(A)} \geq \inf_{\gamma\in\Gamma} \frac{\sum|\gamma_i|^2}{\lambda(A)} \geq 4\pi\frac{\sum\lambda(K_i)}{\lambda(A)} = 4\pi\frac{\lambda(K)}{\lambda(A)}$$

where the last inequality follows from the isoperimetric one. Now the required estimate follows. □

Corollary 2. If the critical tableau is aperiodic then the Lebesgue measure of $K(g)$ is equal to zero.

Proof. One can repeat the McMullen's argument word by word. Just for fun we will slightly modify it.

Let us organize the set of pieces V_k^n in a tree joining V_k^n with V_i^{n+1} in the case when $V_i^{n+1} \subset V_k^n$. Let us assign to each edge $[U, W]$ of the tree a number

$$\nu[U, W] \equiv \nu(U) = \min(\mu(U), 1/2),$$

and to each branch γ a number $\nu(\gamma)$ which is the sum of $\nu[U, W]$ over all edges of γ. Denote by Γ_n the family of all branches of length n, $n \leq \infty$ (saying "branch" we mean a path in the tree beginning at the root vertex V). By Lemma 1,

$$\nu(\gamma) = \infty \tag{2}$$

for any $\gamma \in \Gamma_\infty$. Let us show that

$$M_n \equiv \min_{\gamma\in\Gamma_n} \nu(\gamma) \to \infty. \tag{3}$$

Indeed, given a C, consider a subtree of vertices W such that $\nu[V, W] \leq C$ where $[V, W]$ is the branch ending at W. Now use so called König lemma: if a tree with finitely many branches at any vertex has arbitrary long branches then it has an infinite branch. Along this branch the divergent (2) condition fails.

By Lemma 2, for any vertex U of level n

$$\frac{\lambda(V^{n+1}\cap U)}{\lambda(U)} \le \exp(-b\nu(U))$$

with an appropriate constant b. Now one can easily derive from here (induction in n) that

$$\lambda(V^n) \le \exp(-bM_n)\lambda(V),$$

and by (3) this goes down to 0. □

§3 **Persistent recurrence and renormalization.**

Set $c=0$, the critical point of a quadratic polynomial $f=p_a : z \to z^2+a$. If f has an attractive periodic point or a rational indifferent periodic point then $\lambda(J(f))=0$ ([DH2], [L]). So, we will assume till the end of the paper that f has no such points (and is not infinitely renormalizable). Then we are exactly in the situation studied recently by Yoccoz. We will use the Yoccoz construction without detailed explanation (see [H]).

For (open) pieces V_k^n of the Yoccoz partitions we will use the same notations as for the Branner-Hubbard ones. By ∂V^n denote the union of ∂V_k^n. The intersection $\partial V^n \cap J(f)$ consists of finitely many points, preimages of a fixed point α. Set

$$V^n(z) = \mathrm{int} \bigcup \mathrm{cl} V_k^n$$

over the pieces V_k^n of level n whose closure contain z (if z is not a preimage of α, $V^n(z)$ is just the piece of level n containing z). The $V^n(z)$ is a neighborhood of z. It follows from the Yoccoz Theorem that

$$\mathrm{diam}\, V^n(z) \to 0 \qquad \text{as } n\to\infty \tag{4}$$

uniformly in z.

Given a domain U and an orbit $\{z_k\}_{k=0}^n$ such that $z_n \in U$, one can *pull U back* along this orbit, that is to consider the string of domains $U_k,\ k=0,1,...,n,$ such that $U_n=U$, and U_k is the component of $f^{-1}U_{k+1}$ containing z_k. In particular, if $U=V^l(z_n)$ then $U_k=V^{l+n-k}(z_k)$.

The *order* $\mathrm{ord}\,\mathbf{U}$ of the pull-back $\mathbf{U}$ is the number of domains U_k containing the critical point c. The pull-back of zero order (that is, none of U_k covers the critical point) will be called *univalent* .

Lemma 3. Let W and U be two domains intersecting the Julia set $J(f)$, and $f^n(W)\subset U$ for some $n\in\mathbf{N}$. Then

$$\mathrm{diam}\, U<\delta \Rightarrow \mathrm{diam}\, W<\epsilon_n(\delta)<\epsilon(\delta)$$

with $\epsilon(\delta)\to 0$ as $\delta\to 0$, and $\epsilon_n(\delta)\to 0$ as $n\to\infty$.

Proof. Fix $\epsilon>0$. According to (4), there is an $\ell\in\mathbf{N}$ such that

$$\mathrm{diam}\, V^\ell(z)<\epsilon \tag{5}$$

for any $z \in J(f)$. The sets $V^\ell(z)$ form an open covering of the Julia set. Let δ be its *Lebesgue number*. This means that any set U of diameter less than δ is contained in some domain $V^\ell(\zeta), \quad \zeta \in J(f)$.

Let us apply this to a given domain U, and find $z \in W \cap J(f)$ such that $f^n z = \zeta$. Pulling V^ℓ back along the orbit of z we come to a piece $V^{\ell+n}(z)$ containing W. Now (5) yields that $\operatorname{diam} W < \epsilon$, and we are done. □

Denote by $B(z,r)$ the Eucledian disk of radius r centered at z.

Lemma 4. If $\lambda(J(p_c)) > 0$ then for almost all $z \in J(p_c)$ we have

$$\omega(z) = \omega(c) \ni c.$$

So, $\lambda(J(p_c)) = 0$ provided c is non-recurrent.

Proof. The inclusion $\omega(z) \subset \omega(c)$ for almost all z follows from the Koebe Distortion Theorem (see [L]). We are going to show that $\omega(z) \supset c$ for almost all z which certainly implies the required statement. Let

$$\operatorname{dist}(\operatorname{orb}(z), c) \geq \gamma. \tag{6}$$

Find a $\delta > 0$ such that $\epsilon(\delta) < \gamma$ (see Lemma 3). Consider the disk $B(z_n, \delta)$ around $z_n \equiv f^n z$, and let $\mathbf{U} = \{U_k\}_{k=0}^n$ be its pull-back along the orbit of z. By Lemma 3, $\operatorname{diam} U_k < \gamma,\ k = 0, ..., n$, hence the pull-back $\mathbf{U}$ is univalent. By the Koebe Theorem, z is not a density point of $J(p_c)$, and the statement follows. □

Remark. Consider a set J_γ of all z satisfying (6). Then the restriction $f|J_\gamma$ is *expanding*: there exist $C > 0$ and $q > 1$ such that

$$(f^n)'(z) \geq cq^n, \qquad z \in J_\gamma,\ n = 0, 1, ...$$

Indeed, by Lemma 3, $\operatorname{diam} U_0 < \epsilon_n(\delta) \to 0$ as $n \to \infty$ (where U_0 is the set from the above proof). Hence, there is an n such that $(f^n)'(z) \geq q > 1$ for all $z \in J_\gamma$, and the statement follows. The analogous fact is well-known in one-dimensional dynamics [G].

Let us consider now the Yoccoz τ-function. For an $n \in \mathbf{N}$ it assigns the biggest $m \in [0, n-1]$ for which the piece $f^{n-m}V^n(c)$ of level m contains the critical point c (if there is no such an m, set $\tau(n) = -1$.) The critical point is called *persistently recurrent* if $\tau(n) \to \infty$ as $n \to \infty$. The following lemma was probably known to several people.

Lemma 5. If the critical point is not persistently recurrent then $\lambda(J(p_c)) = 0$.

Proof. Since c is not persistently recurrent, there is an N and arbitrary large l such that f^l is a double covering of $Q \equiv V^{N+l}(c)$ over $V^N(c)$. Let

$$z \in J(f) \setminus \bigcup_{n=0}^{\infty} f^{-n}\alpha,$$

and consider the first moment n for which $f^n z \in Q$. Let $\mathbf{Q} = Q_0, ..., Q_n$ be the pull-back of Q along the orbit of z. Since the level of Q_i is bigger than $N + l$ for $i < n$, these Q_i don't cover c which means that the pull-back is univalent.

The boundary ∂V^N consists of N-preimages of invariant rays through α and N-preimages of an equipotential level. It follows that there exists a $\delta > 0$ such that for any $u \in V^N(c)$ there is a k and a neighborhood $U \subset V^N(c)$ around u such that f^k univalently maps U onto a disk $B(f^k u, \delta)$ of radius δ centered at $f^k u$.

Set $u = f^{n+l} z$ and find corresponding U and k. Now consider two cases

(i) $f^l c$ does not belong to U or $|f^{l+k} c - f^k u| > \delta/100$. Then the pull-back T_i of the disk $B(f^k u, \delta/200)$ along the orbit $\{z_m\}_{m=0}^{n+l+k}$ is univalent. By the Koebe theorem, the density of the Julia set in T_0 is bounded away from 1. Since l is arbitrary large, z is not a density point of $J(p_c)$.

(ii) $f^l c \in U$ and $|f^{l+k} c - f^k u| < \delta/100$. Then we can find a disk B_1 centered at the critical value and such that its f^{l+k-1}-image lies in between $B(f^k(u), \delta/10)$ and $B(f^k(u), \delta/2)$. By the Koebe Theorem, the density of $J(p_c)$ at B_1 is bounded away from 1. Now let us consider the preimage B_0 of B_1. It is a disk centered at c and contained in $V^{N+l}(c)$. The squaring map in the disk cannot expand density of thin sets too much. It follows that the density of the Julia set in B_0 is bounded away from 1 as well. Now pulling a little bit smaller disk along the orbit $\{z_i\}_{i=0}^n$, we conclude again that z is not a density point of the Julia set. □

Now let us concentrate on the main case of a persistently recurrent critical point.

Lemma 6. The following properties are equivalent:

(i). The critical point c is persistently recurrent.

(ii). *Absence of long univalent pull-backs.* Given an $\epsilon > 0$, there is an $N \in \mathbf{N}$ with the following property. Let $\overline{z} = \{z, z_{-1}, ..., z_{-n}\}$ be any backward orbit in $\omega(c)$. If the pull-back of the disk $B(z, \epsilon)$ along $\overline{z}$ is univalent then $n \le N$.

Remark. Property (ii) equivalent to persistent recurrence appeared in one dimensional setting in [BL]. Note that the formulation does not involve any particular partitions of the Julia set.

Proof. (i) $\Rightarrow$ (ii). Find an ℓ so that (5) holds. Take a backward orbit $\overline{z}$ and assume that the pull-back of $B(z, \epsilon)$ along it is univalent. By (5), $B(z, \epsilon) \supset V^\ell(z)$, hence the pull-back of $V^\ell(z)$ along $\overline{z}$ is also univalent.

Since $z_{-n} \in \omega(c)$, there is an $s \in \mathbf{N}$ for which

$$c_s \in V^{\ell+n}(z_n) \qquad (7).$$

So, we can pull the piece $V^{\ell+n}(z_{-n})$ back along the orbit of c till the first moment t when it covers the critical point (if s is the first moment for which (7) holds then $t = s$). We obtain a critical piece $V^{\ell+n+t}(c)$ such that

$$f^{n+t} : V^{\ell+n+t}(c) \to V^\ell$$

is a double covering. By persistent recurrence, we get a uniform bound on n.

(ii) $\Rightarrow$ (i). If f is not persistently recurrent, we can find a critical piece $V^\ell(c)$ allowing arbitrarily long univalent pull-backs along $\omega(c)$. Since $V^\ell(c) \supset B(c, \epsilon)$ for some $\epsilon > 0$, we arrive at a contradiction. □

The following Corollary seems to be interesting by itself. It will not be used for the proof of the Theorem.

Corollary(cf [BL],§11). If f is persistently recurrent then

(i). The restriction $f|\omega(c)$ is topologically minimal (that is, all orbits are dense in $\omega(c)$)

(ii). $f|\omega(c)$ has zero topological entropy.

Proof. (i).Let $z \in \omega(c)$. We should prove that $c \in \omega(z)$. Otherwise dist(orb $(z), c) \geq \gamma > 0$. By Lemma 3, for sufficiently small δ , the pull-back of the disk $B(z_n, \delta)$ along the orbit of z is unimodal, contradicting lemma 6(ii).

(ii) If there is a measure μ of positive entropy supported on $\omega(c)$, then the Pesin theory of unstable manifolds yields the existence of a disk $B(z, \delta)$, $z \in J(f)$, having an infinite pull-back along $\omega(c)$ contradicting Lemma 6(ii) again. □

We are prepared to prove the main lemma.

Lemma 7. Let c be persistently recurrent. Then there is a polynomial-like map

$$g : \bigcup V_i \to V$$

such that

(i) $g|V_i = p_c^{n_i}$ for some n_i .

(ii) c is the unique critical point of g .

(iii) $c \in K(g)$.

Proof. Let us take a non-degenerate annulus

$$V^{n-1}(c) \smallsetminus V^n(c)$$

around the critical point (see [H]), so that $\mathrm{cl}\, V^n(c) \subset V^{n-1}(c)$. Set $V = V^n(c)$ Then

$$\partial(f^j V) \cap \mathrm{cl} V = \emptyset, \qquad j = 1, 2, ... \qquad (8).$$

Consider all returns $c_{m(i)}$ of orb c into V , and let $l(i) = m(i+1) - m(i)$. Consider all pull-backs $\{f^k V_i\}_{k=0}^{l(i)}$ of V along the strings $\{c_k\}_{k=m(i)}^{m(i+1)}$. For any i all intermediate pieces $f^k V_i$, $0 < k < l(i)$, lie outside V , so the maps $f^{l(i)-1} : fV_i \to V$ are univalent. Hence the maps $f^{l(i)} : V_i \to V$ are either univalent (if the piece V_i is not critical) or double coverings.

By Lemma 6(ii), l_i are uniformly bounded: $l_i \leq L$. So, actually we have only finitely many different sets V_i. Let us show that these sets are disjoint. Indeed, otherwise $V_i \supset V_j$ for two different sets V_i and V_j . Let us push V_i forward till the level n : $f^k V_i = V$. Then $f^k V_j \subset V$ despite the fact that all sets $f^k V_j$ of level greater than n lie outside of V . (Actually, (8) yields more: $\mathrm{cl}\, V_i \cap \mathrm{cl}\, V_j = \emptyset$ for $i \neq j$.)

Now let us show that $\mathrm{cl}\, V_i \subset V$. Indeed, otherwise $\partial V_i \cap \partial V \neq \emptyset$. Let $\ell > n$ be the level of V_i , $j = \ell - n$. Then

$$\partial V \cap \partial(f^j V) \supset f^j(\partial V_i \cap \partial V) \neq \emptyset$$

contradicting (8).

We have shown that $g|\bigcup V_i$ is a generalized polynomial-like map. Since g is non-univalent only on the critical piece, c is the only critical point of g . Since $g c_{m(i)} = c_{m(i+1)} \in V_{i+1}$, $c \in K(g)$. □

Proof of the Theorem. If f is finitely renormalizable, let us renormalize it to be non-renormalizable. It follows from Lemma 4 that the measure of the Julia set of the original polynomial and its renormalization is simultaneously positive or zero. So, we will assume in what follows that f is non-renormalizable (and c is persistently recurrent).

Consider the polynomial-like map g constructed in the previous lemma. Since p_c is non-renormalizable, g has an aperiodic tableau. By Corollaries 1 and 2, $K(g)$ is a Cantor set of zero measure. Let z be a typical point of $J(f)$, so that the conclusion of Lemma 4 holds. Consider the moments $n(i)$ when the orb(z) returns to V. Since $\omega(c) \cap V \subset \cup V_i$, eventually $z_{n(i)} \in \cup V_i$. Hence $f^n z \in K(g)$ for some n. Now the Theorem follows. □

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