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Scalings in Circle Maps (I)

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Abstract. Let f be a "flat spot" circle map with irrational rotation number. Located at the edges of the flat spot are non-flat critical points ($S: x \to Ax^{\nu}$, $\nu \geqq 1$). First, we define scalings associated with the closest returns of the orbit of the critical point. Under the assumption that these scalings go to zero, we prove that the derivative of long iterates of the critical value can be expressed in the scalings. The asymptotic behavior of the derivatives and the scalings can then be calculated. We concentrate on the cases for which one can prove the above assumption. In particular, let one of the singularities be linear. These maps arise for example as the lower bound of the non-decreasing truncations of non-invertible bimodal circle maps. It follows that the derivatives grow at a sub-exponential rate.

I. Introduction

It is, in our opinion, an important problem how for a smooth family of dynamical systems the varying dynamics in the configuration space is reflected in the parameter space. In one dimension (in the context of unimodal maps of the interval and homeomorphisms of the circle) this problem has been studied in great detail in the case renormalization on a compact set of such maps is uniformly hyperbolic. We will refer to this case as being renormalizable. The context in which we propose to investigate this problem is that of a smooth one-parameter family of bimodal maps $\bar{f}_t$ of the circle.

As for the parameter space, the quantity we are interested in is the lower bound $\varrho(t)$ of the rotation interval associated with f_t. It turns out that f_t always has an order-preserving non-wandering set Ω_t with rotation number $\varrho(t)$, such that

$$\bar{f}_t|_{\Omega_t} = f_t|_{\Omega_t}.$$

Here f_t is the lowest bound on the non-decreasing truncations of $\bar{f}_t$ [as described in Veerman (1989)]. This then reduces the setting to a more manageable one as it allows us to study families of monotone maps with a "flat spot." The precise definitions in Sect. 2 actually allow for a somewhat more general context, but this is certainly the most important case.

This simplification pays us a service in the following sense. We can assign natural (configuration) scalings $\sigma(n)$ to closest returns of the critical orbit (ratio of the distance between the critical point c and its closest return $f^{q_n}(c)$ and the distance between c and the previous closest return, see below for more precise definitions). If $\varrho(t)$ is irrational and of bounded type, these scalings tend to zero as shown in Sect. 5. (This is in contrast with the situation for maps with non-flat singularities where the scalings are bounded away from zero and one.) We will show in Sect. 4 that then distortions of high iterates (on appropriate intervals) also tend to zero. This is sufficient to show that the scalings $\sigma(n)$ are related to the long term derivatives $Df^{q_n}(fc)$ by a relation of the following type:

$$Df^{q_n}(fc) \cdot \sigma(n) \approx \text{const.}$$

(The precise relation in Theorem 4.6 is a modication of this.)

This relation in turn is sufficient to find and solve the recursion relation for scalings, which is done in Sect. 6. One obtains estimates for the scalings as well as for the derivatives of high iterates. We find that the behavior of the orbit of the critical value is not hyperbolic (Lyapunov exponent is zero), but

$$Df^{q_n}(fc) \approx \exp(C \cdot (q_n)^\gamma),$$

where $0 < \gamma \leqq 1$ depends exclusively on the singularity type.

Finally, in Sect. 7 we demonstrate once more the usefulness of the notion of scalings by utilizing it to prove that the non-wandering sets (if $\varrho(t)$ is irrational and of bounded type) have Hausdorff dimension zero. We add here that it is not clear whether any of our conclusions hold if the rotation number is irrational but not of bounded type.

The tools we use to prove our results could be collectively described by the words "Koebe principle and negative Schwarzian derivative." We capitalize on ratios, not on cross-ratios as was done before (Yoccoz 1984; De Melo and Van Strien 1986; Swiatek 1986, 1988), because that seems to be the way one has to study scalings. Our results are also valid if f_t has negative Schwarzian only in a neighborhood of the criticality and bounded distortion elsewhere (Sullivan 1987, 1988).

In the sequel to this paper we show how parameter scalings for one-parameter families are related to dynamic scalings.

II. Definitions and Notation

In this section, we define the notions that will be used throughout the work. Most important among these is the definition of close returns and a one parameter family. To simplify the formulae, we also introduce some notation.

From now on we will write D for $\frac{d}{dx}$. We consider circle maps satisfying the following requirements.

1. f has degree one and preserves orientation.
2. There is an open interval U in S^1 containing the point $x=0$ such that $Df(x)=0$ for x in U.
3. $f(x)$ is C^3 on S^1-clos(U) and has negative Schwarzian on S^1-clos(U).

4. In a half-neighborhood N_r of the right end-point of U, f has the form $S_r(x) = A_r(x^{v_r})$, where $v_r \geqq 1$. At the left end-point f has the form $S_l(x) = A_l(x^{v_l})$, where $v_1 \geqq 1$. Here A_r and A_l have bounded distortion.

Remark. The assumption that Sf be negative is very restrictive. With additional work this assumption can be relaxed to $Sf \leqq 0$ near the endpoints of U and bounded distortion elsewhere (Swiatek 1989).

We will refer to the derivative $\lim_{y \to x} [f(y) - f(x)]/(y-x)$, for both x and y in $S^1 \setminus U$ (note that this is a one-sided derivative on ∂U), simply as $Df(x)$. We will denote the points in ∂U by $r(0)$ and $l(0)$.

Unless otherwise specified, we will work with an irrational rotation number ϱ that is of bounded type. Its continued fraction coefficients will be denoted by a_n and its continued fraction approximants by p_n/q_n. We use the (for coefficients somewhat unusual) convention that:

$$p_{n+1} = a_n p_n + p_{n-1}, \qquad q_{n+1} = a_n q_n + q_{n-1},$$

and

$$p_{2n}/q_{2n} < \varrho < p_{2n-1}/q_{2n-1}.$$

Suppose f has rotation number ϱ. By closest returns we mean the collection of points $\{f^{q_i}(U)\}_{i=1}^{\infty}$. Iterates of a point x will be denoted by x_n, or, if we are dealing more particularly with closest returns, by their index only. Thus q_{2n} stands for $f^{q_{2n}}(r(0))$.

Lemma 2.1. *Let f satisfy conditions 1, 2, 3, and 4. If f has irrational rotation number ϱ then $\{q_n\}$ converges to the boundary of U as $n \to \infty$.*

Proof. Suppose that either q_{2n} or q_{2n+1} (or both) do not limit on clos(U). For definiteness, assume that $\{q_{2n}\}$ does not limit on the boundary of U. There is an interval J bounded by $r(0)$, the right endpoint of U, and $\lim q_{2n}$. This interval cannot intersect an inverse image of U, since in that case the critical point would be periodic. Therefore J has to be a wandering interval. But then we can change f on clos(J)$\cup$clos(U) so that we obtain a C^1 critical map (a homeomorphism with one non-flat critical point and with negative Schwarzian in a neighborhood of the critical point), see Fig .2.2. By construction this map has a wandering interval, namely $J \cup U$. This is in contradiction with a theorem by Yoccoz (1984) that states that such maps have no wandering intervals. □

Similarly, $U(-n)$ will denote the n^{th} inverse image of U, and its boundary points will be referred to as $r(n)$ and $l(n)$. The derivative of the n^{th} iterate of f at the critical *value* will be of special importance and we will denote it by $D(n)$.

The scalings $\sigma_i(n)$ for $i \in \{1, 2, \ldots, a_n\}$ are defined as follows. Consider the sequence of points (see Fig. 2.1):

$$q_n, 2q_n, \ldots, (a_n - 1)q_n, a_n q_n, (a_n + 1)q_n, q_{n-2}, \ldots.$$

For a point P in this sequence we denote its neighbors by $I(P)$ and $O(P)$. The symbol I is used for the neighbor closest to U; O for the other neighbor. Now we define:

$$\sigma_i(n) = \left| \frac{iq_n - I(iq_n)}{O(iq_n) - iq_n} \right|.$$

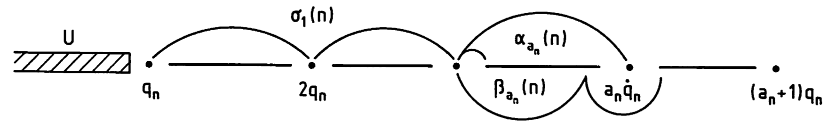

Fig. 2.1. Definition of the scalings

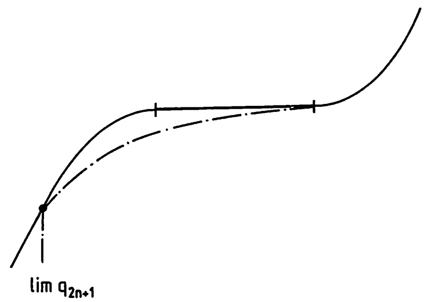

Fig. 2.2. The construction of Lemma 2.1.

We can extend the previous sequence to include the appropriate inverse images of U. This is the configuration drawn in Fig. 2.1. We define two extra scalings we need for the discourse. Note that $U(-[(a_n-i+1)q_n+q_{n-1}])$ is contained in the interval $[I(iq_n), iq_n]$. Denote by $\imath((a_n-i+1)q_n+q_{n-1})$ its inside (closest to U) boundary point, and by $o((a_n-i+1)q_n+q_{n-1})$ its outside boundary point,

$$\alpha_i(n)=\left|\frac{I(iq_n)-\imath((a_n-i+1)q_n+q_{n-1})}{I(iq_n)-iq_n}\right|$$

and

$$\beta_i(n)=\left|\frac{o((a_n-i+1)q_n+q_{n-1})-\imath((a_n-i)q_n+q_{n-1})}{o((a_n-i+1)q_n+q_{n-1})-I(iq_n)}\right|.$$

In order to express limits conveniently, we use the following notation:

$$a(n)\approx b(n)$$

means

$$\lim_{n\to\infty} a(n)/b(n)=1\,.$$

III. Calculus of Distortions

In this section, we briefly exhibit the essentials of the theory that enables one to calculate distortions for high iterates of one-dimensional circle maps with negative Schwarzian. The theory of distortions has been developed by Van Strien (1981), Yoccoz (1984), De Melo and Van Strien (1986), and Swiatek (1986, 1988, 1989). We have used Sullivan (1987, 1988) for reference.

The fundamental idea is to study how the ratio of lengths of two abutting intervals is changed under repeated iteration. So, let I be an interval $[a,b]$ that contains the point y, and $f:[a,b]\to[f(a),f(b)]$ a homeomorphism. One would like to bound the distortion of f at y defined as follows:

$$\operatorname{dist}(f, I, y)=\left|\ln\left[\frac{f(y)-f(a)}{y-a}\cdot\frac{b-y}{f(b)-f(y)}\right]\right|. \tag{3.1}$$

Define the distortion of f on I as:

$$\operatorname{dist}(f, I)=\sup_{y\in I}\operatorname{dist}(f, I, y).$$

One has:
(1) $\operatorname{dist}(f, I)=0 \Leftrightarrow f$ is affine
(2) $\operatorname{dist}(f\circ g, I)\leqq\operatorname{dist}(f, g(I))+\operatorname{dist}(g, I)$
(3) $\operatorname{dist}(f, I)=\operatorname{dist}(f^{-1}, f(I))$.
A standard estimate is:

$$\operatorname{dist}(f, I)\leqq\sup_{\eta,\xi\in I}\left|\ln\left[\frac{Df(\xi)}{Df(\eta)}\right]\right|\leqq\int_I|nf(x)|\,\mathrm{d}x, \tag{3.2}$$

where

$$nf(x)=\frac{f''(x)}{f'(x)}.$$

Combining the previous, one has:

$$\operatorname{dist}(f^n, I)\leqq\sum_{i=0}^{k}\int_{I_i}|nf(x)|\,dx\leqq\sup\{|nf|\}\sum|I_i|. \tag{3.3}$$

Here, the subscript i means f^i. We will refer to (3.3) as the estimate for the distortion based on bounded non-linearity.

Near a critical point of f, nf becomes large. Nevertheless, distortion estimates can be obtained by virtue of the Koebe principle. Assume that $I=[a,b]$ is near a critical point for f. We assume $Sf\leqq 0$ on I. Consider $g=f^{-1}:I'=f(I)\to I$. Then $Sg\geqq 0$. Integrating this differential inequality and knowing that ng is bounded on I', one obtains the

Koebe Principle. For any point $x'\in I'$,

$$ng(x')\leqq\frac{2}{|x'-\partial I'|}. \tag{3.4}$$

Here, $|x'-\partial I'|$ denotes the distance of x' to the boundary of I'. As a consequence we have that if J is a sub-interval of I and $I'=f(J)$ then:

$$\operatorname{dist}(f, J)=\operatorname{dist}(f, J')\leqq\int_{J'}\frac{2}{|x'-\partial I'|}dx'. \tag{3.5}$$

One observes that in order to show that $\operatorname{dist}(f, J)$ is small, one must find I and I' such that J' is small with respect to I'. One particular example where we will evaluate (3.5) is where J' is close to one of the endpoints of I'.

Proposition 3.1. *Assume $J'=[c', d']$ and J' is contained in $I'=[a', b']$. If $f:I\to I'$, $Sf\leqq 0$ and if for $x'\in J'$,*

$$|x'-\partial I'|=|x'-a'|,$$

then

$$\operatorname{dist}(f, J)\leqq 2\frac{d'-c'}{c'-a'}+O\left(\left[\frac{d'-c'}{c'-a'}\right]^2\right).$$

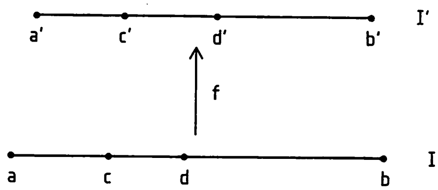

Fig. 3.1. Geometry of the Koebe principle

As a further application of the Koebe principle, we state the one-sided Koebe principle.

Proposition 3.2 (one-sided Koebe): *If (see Fig. 3.1)*

(i)
$$|c'-d'| \leqq \tfrac{1}{4}|a'-b'|,$$
and

(ii)
$$\left|\frac{d'-c'}{c'-a'}\right| \leqq 1,$$
then
$$\left|\frac{d-c}{b-c}\right| \leqq 4\left|\frac{d'-c'}{c'-a'}\right|.$$

Proof. Pick any point z with $d<z<b$ such that
$$|z'-\partial I'|=|z'-a'|.$$
Then
$$\frac{d-c}{b-c} \leqq \frac{d-c}{z-c} \leqq \frac{d'-c'}{z'-c'} \sup_{\xi,\eta\in[c,z]} \left|\frac{Df(\xi)}{Df(\eta)}\right| \leqq \frac{d'-c'}{z'-c'}$$
$$\times \exp\left\{\int_{[c',z']} \frac{2}{|x-a'|}\,dx\right\} = \frac{d'-c'}{z'-c'}\left(\frac{z'-a'}{c'-a'}\right)^2$$
$$= \frac{d'-c'}{c'-a'}\,\frac{(z'-a')^2}{(z'-c')(c'-a')}.$$

The assumptions imply that we can choose z such that the last factor in the above equation is minimized and equal to 4. □

Remark. Both propositions are proved under the assumption that the Schwarzian of f is not positive. As remarked before, this condition can be replaced by the requirement that f is non-flat near the end-points of U and has bounded distortion elsewhere. The main technical tool goes by the name of "Shuffling lemma" (Swiatek 1988a).

To calculate derivatives, one of the main purposes of this work, one uses the mean value theorem, and then Eq. (3.5) to estimate the error. In the situation of Fig. (3.1) (with Sg positive on I') we have that for η in $[c,d]$ there is a y in $[c,d]$:
$$Df(\eta)=Df(y)\frac{Df(\eta)}{Df(y)}=\left|\frac{d'-c'}{d-c}\right|(1+\varepsilon),$$
where
$$|\ln(1+\varepsilon)| \leqq 2\left|\ln\left(\frac{c'-a'}{d'-a'}\right)\right|. \tag{3.7}$$

IV. Derivatives and Scalings

In this section, we show that scalings and derivatives of high iterates along the critical orbit, provided the orbit has a rotation number of bounded type, are intimately related. According to the arguments in the previous section, estimates are accurate when the scalings $\{\sigma_i(n)\}$ go to zero as $n\to\infty$. We will assume throughout this section that the $\sigma_i(n)$ are asymptotically equal to zero. In Sect. 5, we will prove that this is indeed the case for some cases of interest. In addition, we assume the rotation number to be an irrational number of bounded type. We will not state this assumption in the results.

The first four lemmas in this section describe certain relations among various points of the non-wandering set. The purpose is to establish that the inverse images of U take up a large fraction of the interval contained between its two neighbors. In fact, this fraction converges to one. This implies that high iterates of the map in the intervals between two inverse images are (asymptotically) linear. The information in the last part of this section we will use to express the derivative in terms of the scalings.

The proofs of following results rely upon an understanding of Fig. 4.1. The first and most important observation one should make is that suitable iterates of f are (asymptotically) linear on appropriate intervals. To see this we will apply the Koebe principle as described in Sect. 3. One can include parts of U in the consideration when one uses the one-sided Koebe principle (this is done in Lemma 4.3.).

The second observation, which is used in Lemma 4.2, is that, since

$$|iq_n| = |iq_n - I(iq_n)|\left(1 + O\left(\max_i \sigma_i(n)\right)\right),$$

the distance $|iq_n - I(iq_n)|$ can be replaced by $|iq_n|$ when necessary.

Finally, we note that long compositions of f (on appropriate domains) are composed of maps with very small distortion and singular maps of the form Ax^ν.

Lemma 4.1. (i) $\operatorname{dist}(f^{-q_{2n}-1}, [r(0), a_{2n}q_{2n}]) \to 0$ *as* $n\to\infty$.
(ii) $\operatorname{dist}(f^{-q_{2n-1}-1}, [a_{2n-1}q_{2n-1}, l(0)]) \to 0$ *as* $n\to\infty$.

Proof. (i) $f^{-q_{2n}-1}$ extends as a diffeomorphism to $[l(0), (a_{2n+1}+1)q_{2n}]$. By the Koebe principle (Proposition 3.1), we obtain that

$$\begin{aligned}&\operatorname{dist}(f^{-q_{2n}-1}, [r(0), a_{2n}q_{2n}])\\ &\quad = O\left(\frac{r(0)-a_{2n}q_{2n}}{a_{2n}q_{2n}-(a_{2n+1}+1)q_{2n}}\right) = O\left(\sigma_{a_{2n}}(2n)\left(1+\max_i \sigma_i(2n)\right)\right).\end{aligned}$$

Since the scalings $\sigma_i(2n)$ are small, this proves (i). The proof of (ii) is analogous to that of (i). □

Lemma 4.2. *For all i, $1<i<a_n$, the following relations (see Fig. 4.1) hold:*

$$\left|\frac{[q_{n-1}+(i+1)q_n]-[q_{n-1}+iq_n]}{[q_{n-1}+iq_n]-[q_{n-1}+(i-1)q_n]}\right| \approx \sigma_i(n)^\nu = S(\sigma_i(n)),$$

where $\nu=\nu_l$ for n even and $\nu=\nu_r$ for n odd.

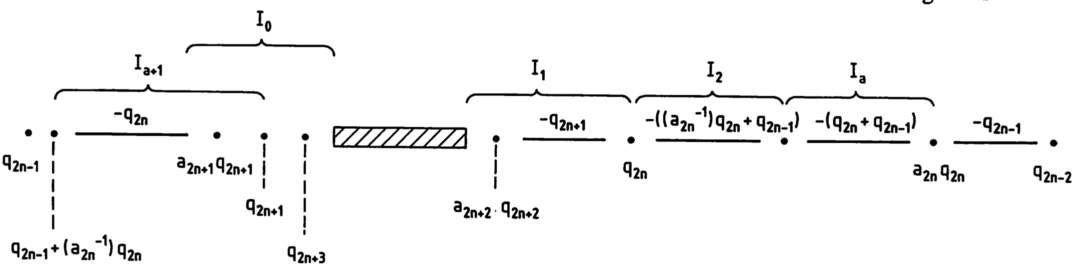

Fig. 4.1. The configuration for irrational number of the forward and backward orbit of the flat spot

Proof. The situation for even n is drawn in Fig. 4.1. We prove it for n even only. The proof for n is odd is similar.

The three points in the lemma are images of $(i+1)q_{2n}$, iq_{2n}, $(i-1)q_{2n}$ under $f^{q_{2n-1}}$. Now

$$f^{q_{2n-1}} = f^{q_{2n-1}-1} \circ S_r,$$

where S_r is the singularity near the right boundary of U. The interval

$$[(i+1)q_{2n}+q_{2n-1}, (i-1)q_{2n}+q_{2n-1}]$$

is contained in $[(a_{2n-1}+1)q_{2n-1}, l(0)]$. Thus according to Lemma 4.1 $f^{-(q_{2n-1}-1)}$ has small distortion. We obtain:

$$\begin{aligned}
&\frac{[q_{2n-1}+(i-1)q_{2n}]-[q_{2n-1}+iq_{2n}]}{[q_{2n-1}+iq_{2n}]-[q_{2n-1}+(i+1)q_{2n}]} \\
&\approx \frac{[1+(i-1)q_{2n}]-[1+iq_{2n}]}{[1+iq_{2n}]-[1+(i+1)q_{2n}]} \approx \frac{[(i-1)q_{2n}]^{\nu}-[iq_{2n}]^{\nu}}{[iq_{2n}]^{\nu}-[(i+1)q_{2n}+1]^{\nu}} \\
&\approx \left[\frac{iq_{2n}}{(i+1)q_{2n}}\right]^{\nu} \approx [\sigma_i(2n)]^{\nu}.
\end{aligned}$$

We have used that A_r has small distortion and that the scalings $\sigma_i(2n)$ are small. □

Lemma 4.3. $\alpha_i(n) \to 0$.

Proof. Consider the scaling $\alpha_i(2n)$. Upon iterating N times under f, where $N=(a_{2n}-i+1)q_{2n}+q_{2n-1}$, the three points that define the scaling map to q_{2n+1}, $l(0)$, and $q_{2n+1}+q_{2n}$, respectively. Consider f^{-N}. This map extends as a diffeomorphism to the interval $[(a_{2n}-i+1)q_{2n}+q_{2n-1}, r(0)]$. Applying the one-sided Koebe principle (Proposition 3.2), we obtain:

$$\alpha_i(2n) < 4\left|\frac{q_{2n+1}-l(0)}{q_{2n+1}-q_{2n-1}}\right|.$$

By Lemma 4.2 ($q_{2n+1}=q_{2n-1}+a_{2n}q_{2n}$) we obtain that the denominator is of order $|q_{2n-1}-q_{2n+1}|$. Since the scalings $\sigma_i(2n+1)$ are small, the $\alpha_i(2n)$ must also be small. □

Lemma 4.4. $\beta_i(n) \to 0$.

Proof. Consider $\beta_i(2n)$ as indicated in Fig. 2.1. We map the triple of points defining this scaling by f^N, where $N=(a_{2n}-i)q_{2n}+q_{2n-1}$. The points get mapped to:

$$(q_{2n-1}+(a_{2n}-1)q_{2n}), \quad \iota(q_{2n+1}), \quad \text{and} \quad l(0),$$

respectively. The ratio defined by these points will be denoted by R and is (asymptotically) equal to $\sigma_{a_{2n+1}}(2n+1)$ by Lemmas 4.2 and 4.3.

We will now study the effect of f^N on the ratio $\beta_i(2n)$. After one application of $f^{(q_{2n}-1)} \circ S_r$, we now have a ratio R_1 which satisfies:

$$R_1 \approx S_r\{\beta_i(2n)\}.$$

So, after repeating this $a_{2n}-i$ times, using an argument like that in the proof of Lemma 4.2, one obtains the ratio R_i:

$$R_i \approx S_r^{a_{2n}-i}\{\beta_i(2n)\}.$$

(Note that the last time we did this, we had to use Lemma 4.3.) To map this to R, we now apply $f^{(q_{2n-1}-1)} \circ S_r$, in keeping with the above observations. The result is:

$$R \approx \sigma_{a_{2n+1}}(2n+1) \approx S_l \circ S_r^{a_{2n}-i}\{\beta_i(2n)\}.$$

In effect, we have applied $(a_{2n}-i+1)$ times a power law to a ratio. Since $a_{2n}-i$ is uniformly bounded by $\max\{a_n\}$, if $\sigma_{a_{2n+1}}(2n+1)$ goes to zero, then so does $\beta_i(2n)$. □

The above lemmas prove that the pre-images of the flat spot take up ever more space. The sizes of the left-overs are dictated by $\max\limits_n \{a_n\}$ and by $\sigma_i(n)$. Note that only in the last lemma it is important that the rotation number is of bounded type. We can now formulate and prove the estimates for the long-term derivative along the orbit of the critical value.

From now on, we will not distinguish S_l from S_r in our notation. It is always clear from the context, since it depends only on the argument of S.

Proposition 4.5. i) $D(iq_n) \approx D((i-1)q_n)D(q_n)\dfrac{S'(iq_n)}{S'(q_n)}$ *(for $i \leqq a_n$).*

$$\text{ii)} \quad D(q_{n+1}) \approx [D(q_n)]^{a_n} D(q_{n-1}) \frac{S'(q_{n+1})}{S'(q_{n-1})} \prod_{2}^{a_n} \frac{S'(iq_n)}{S'(q_n)}.$$

(Here, $S = S_r$ if n is even, and $S = S_l$ if n is odd.)

Proof. First write for $i \leqq a_n$,

$$D(iq_n) = D((i-1)q_n) Df^{q_n}((i-1)q_n+1).$$

Since f^{q_n-1} has small distortion,

$$Df^{q_n}((i-1)q_n+1) \approx Df^{q_n-1}(1).$$

The first equality now immediately follows.

To obtain ii), observe that by the same reasoning

$$D(a_nq_n+q_{n-1}) \approx D(a_nq_n)D(q_{n-1}) \frac{S'(a_nq_n+q_{n-1})}{S'(q_{n-1})}.$$

Applying (i) repeatedly now yields (ii). □

By a different argument we can express derivatives in terms of scalings.

Theorem 4.6. *The derivative* $D(q_{2n+1})$ *satisfies:*

$$D(q_{2n+1})\approx\frac{v_r^{a_{2n}}v_l}{\sigma_{a_{2n+1}}(2n+1)}\cdot\prod_{i=1}^{a_{2n+1}-1}[\sigma_i(2n+1)]^{v_l-1}.$$

(Similar for 2n.)

Proof. To prove this, consider the intervals

$$I_0=[\imath(q_{2n}),l(0)],$$

$$I_i=[(i-1)q_{2n},iq_{2n}]\quad\text{for}\quad 1\leqq i\leqq a_{2n},$$

and

$$I_{a_{2n}+1}=[q_{2n-1}+(a_{2n}-1)q_{2n},q_{2n+1}]$$

in Fig. 4.1. According to Lemma 4.1, we have that $f^{q_{2n}-1}$ is linear on $S(I_i)$ if $0\leqq i\leqq a_{2n}-1$ and $f^{q_{2n-1}-1}$ is linear on $S(I_{a_{2n}})$. By composing these linear maps with the singularities as explained before, one finds that the derivative satisfies:

$$D(q_{2n+1})\approx\frac{I_1}{S(I_0)}S'(I_1)\frac{I_2}{S(I_1)}S'(I_2)\frac{I_{a_{2n}}}{S(I_{a_{2n}-1})}\cdots$$

$$\times S'(I_{a_{2n}})\frac{I_{a_{2n}+1}}{S(I_{a_{2n}})}S'(q_{2n+1}),$$

where the derivatives of S are taken in the outermost endpoint of I_i. We have the equality:

$$\frac{I_i}{S(I_i)}S'(I_i)\approx\frac{i\dot{q}_{2n}}{S(iq_{2n})}S'(iq_{2n})\approx v_r(1<i\leqq a_{2n}),$$

and

$$\frac{I_i}{S(I_i)}S'(I_i)\approx v_l\quad(i=a_{2n}+1).$$

By Lemma 4.4, we may replace I_0 by $[a_{2n+1}q_{2n+1},l(0)]$. Moreover $I_{a_{2n+1}}\approx[q_{2n-1},l(0)]$. Thus,

$$D(q_{2n+1})\approx v_r^{a_{2n}}\cdot\frac{I_{a_{2n}+1}}{S(I_0)}S'(q_{2n+1})\approx v_r^{a_{2n}}\cdot\frac{q_{2n-1}S'(q_{2n+1})}{S([a_{2n+1}q_{2n+1}])}$$

$$=v_r^{a_{2n}}\cdot\frac{q_{2n-1}}{[a_{2n+1}q_{2n+1}]}\cdot\frac{v_lS'(q_{2n+1})}{S'([a_{2n+1}q_{2n+1}])}$$

$$\approx\frac{v_r^{a_{2n}}v_l}{\sigma_{a_{2n+1}}(2n+1)}\cdot\prod_{i=1}^{a_{2n+1}-1}[\sigma_i(2n+1)]^{v_l-1}.\quad\square$$

Combining Proposition 4.5 and Theorem 4.6 yields recursion relations between the scalings, $\{\sigma_i(m)\}$, $m\in\{2n-1,2n,2n+1\}$. We will consider an example of this in Sect. 6. For completeness, we list the relations below.

Corollary 4.7. *The scalings satisfy:*

$$\frac{[\sigma_{a_{2n+1}}(2n+1)]^{v_l}}{\sigma_{a_{2n-1}}(2n-1)[\sigma_{a_{2n}}(2n)]^{a_{2n}}}\prod_{i=1}^{a_{2n-1}-1}\sigma_i(2n-1)\prod_{i=1}^{a_{2n+1}-1}[\sigma_i(2n+1)]^{v_l-2}$$

$$\times\prod_{i=1}^{a_{2n}-1}[\sigma_i(2n)]^{a_{2n}-(a_{2n}-i)(v_r-1)}\approx v_l^{-a_{2n}a_{2n-1}}v_r^{-a_{2n-2}}.$$

Proof. Substitute Theorem 4.6 in Proposition 4.5. □

This corollary still contains $a+1$ scalings in one equation. The next lemma shows how for fixed n the scalings $\sigma_i(n)$ are related.

Lemma 4.8. i) $D(q_n)\sigma_1(n)\approx v$,
ii) $\sigma_i(n)\approx\sigma_{i-1}(n)^v$ *for* $2\leqq i\leqq a_n$.
Here v *equals* v_r *if* n *is even and* v_l *if odd.*

Proof.

(i) $$\sigma_1(n)=\frac{q_n}{[2q_n]-[q_n]}\approx\frac{q_n}{[q_n]^v D(q_n-1)}=\frac{v}{q_n S'(q_n)D(q_n-1)}=\frac{v}{D(q_n)}.$$

(ii) $$\frac{I_{i+1}}{S(I_i)}\approx D(q_n-1)\quad\text{for}\quad 0\leqq i\leqq a_n.$$

Therefore

$$I_{i+1}\approx D(q_n-1)S(I_i),\qquad I_i\approx D(q_n-1)S(I_{i-1}).$$

We obtain

$$\sigma_i(n)=\frac{I_i}{I_{i+1}}\approx\frac{S(I_{i-1})}{S(I_i)}=[\sigma_{i-1}(n)]^v.\quad\square$$

V. The Truncated Family

Let f be a map as defined in Sect. 2, but with $v_l=1$. Such maps arise as the lower bound of the non-decreasing truncations of non-invertible bimodal circle maps. Assume that the rotation number of f is irrational of bounded type. We prove in this section that the scalings $\sigma_i(n)$ go to zero as n tends to infinity. As is clear from the previous section, this is sufficient to insure that all scalings (also the ones of type α and β) go to zero.

Since $v_l=1$, we can find a half neighborhood N of $r(0)$ such that:
- $f''(x)>0$ on N, and
- $nf(x)$ is bounded on the complement of U and N.

Lemma 5.1. *Let* $I=[a,b]$ *be an interval such that its first iterates* $\{I_i\}$ *are disjoint and contained in* S^1-U. *There exists a constant* C, *independent of* n, I, *such that:*

$$\textit{for all}\quad x\in(a,b):\frac{f^n(b)-f^n(x)}{f^n(x)-f^n(a)}>C\frac{b-x}{x-a}.$$

Proof. Since the intervals I_i are disjoint and contained in S^1-U, all except possibly one are contained either in N or in the complement of U and N. If I_i is contained in N then ($f''>0$)

$$\frac{f(f^i(b))-f(f^i(x))}{f(f^i(x))-f(f^i(a))}>\frac{f^i(b)-f^i(x)}{f^i(x)-f^i(a)}.$$

If I_i is contained in the complement of U and N, then

$$\frac{f(f^i(b))-f(f^i(x))}{f(f^i(x))-f(f^i(a))}>e^{-\sup|nf|\cdot|I_i|}\frac{f^i(b)-f^i(x)}{f^i(x)-f^i(a)}.$$

In case I_i intersects both N and its complement, we get two distortion estimates. Combining both yields a constant C' for the distortion in I_i. We obtain:

$$\frac{f^n(b)-f^n(x)}{f^n(x)-f^n(a)} > C' e^{-C\Sigma|I_i|} \frac{b-x}{x-a}. \quad \square$$

Lemma 5.2. *If n is odd, then $\sigma_i(n) \to 0$ as $n \to \infty$.*

Proof. We have (see Fig. 4.1):

$$U(-q_{2n}) \quad \text{in} \quad [q_{2n-1}, a_{2n+1}q_{2n+1}]$$

and $U(-[q_{2n}+iq_{2n+1}])$ in $[(a_{2n+1}-i+1)q_{2n+1}, (a_{2n+1}-i)q_{2n+1}]$, $1 \leqq i < a_{2n+1}$.
Furthermore, the scaling $\sigma_{a_{2n+1}-i}(2n+1)$ is formed by the points

$$(a_{2n+1}-i+1)q_{2n+1}, \quad (a_{2n+1}-i)q_{2n+1} \quad \text{and} \quad (a_{2n+1}-i-1)q_{2n+1}.$$

We map this configuration forward by f^q, where

$$q = q_{2n} + iq_{2n+1}, \quad 0 \leqq i < a_{2n+1}.$$

Then $U(-[q_{2n}+iq_{2n+1}])$ lands on U, and the three points map to

$$q_{2n+1}+q_{2n+2}, \quad q_{2n+2} \quad \text{and} \quad q_{2n+2}-q_{2n+1}.$$

Since the ratio $\dfrac{[q_{2n+1}+q_{2n+2}]-[q_{2n+2}]}{[q_{2n+2}]-[q_{2n+2}-q_{2n+1}]}$ is small (large denominator) we can now apply Lemma 5.1. Thus $\sigma_{a_{2n+1}-i}(2n+1)$ is small. $\square$

Theorem 5.3. *If $v_l = 1$ and the rotation number has bounded type, then $\sigma_i(n) \to 0$ as $n \to \infty$.*

Proof. In view of Lemma 5.1, it is sufficient to prove this for even n only.
Consider the triple of points (Fig. 4.1):

$$q_{2n-1}+(a_{2n}-1)q_{2n}, \quad q_{2n+1}, \quad x' \in U.$$

Here x' is a point to be determined from the one-sided Koebe principle. The map $g = f^{-(q_{2n-1}-1)}$ extends as a diffeomorphism to $[2q_{2n-1}, r(0)]$. Now choose x such that

(i) $\dfrac{[q_{2n+1}+(a_{2n}-1)q_{2n}]-[q_{2n+1}]}{|q_{2n+1}-x|}$ is small, and

(ii) g has small distortion on $[q_{2n-1}+(a_{2n}-1)q_{2n}, x]$.
Then

$$\frac{[1+(a_{2n}-1)q_{2n}]-[1+a_{2n}q_{2n+1}]}{[1+a_{2n}q_{2n}]-g(x)}$$

is small.
Since S has a power law singularity, this implies that

$$\frac{|[(a_{2n}-1)q_{2n}]-[a_{2n}q_{2n}]|}{|[a_{2n}q_{2n}]-f^{-q_{2n-1}}(x)|}$$

is small. Note that $\sigma_{a_{2n}}(2n)$ is still smaller.

Next we transport this control by $f^{-q_{2n}}$. The procedure is analogous. Choose $x \in U(-q_{2n-1})$ such that

(i) $$\frac{|[(a_{2n}-1)q_{2n}]-[a_{2n}q_{2n}]|}{|[a_{2n}q_{2n}]-x|} \text{ is small, and}$$

(ii) $g' = f^{-q_{2n-1}}$ has small distortion on $[l(0), (a_{2n}+1)q_{2n}]$.
Using the same argument we obtain that $\sigma_{a_{2n}-1}(2n)$ is small. Because a_{2n} is bounded, we can continue to obtain that $\sigma_i(2n)$ are small for all i.

Notice that the best one can expect is (Lemma 4.8.)

$$\sigma_{i-1}(2n) \approx \sigma_i(2n)^{1/\nu}.$$

That is: if we repeat the procedure outlined above, the successive $\sigma_i(2n)$'s increase rapidly. One therefore should expect that if $\{a_i\}$ is not a bounded sequence, the scalings $\sigma_1(2n)$ need not tend to zero. □

This establishes that for a map with one "linear" singularity the scalings approach zero (for irrational rotation number of bounded type). We will argue, though not prove, in the next section that this is not the only case.

VI. Recursion on the Scalings

In Sect. 4 we proved that, essentially, scalings correspond to inverse derivatives. A natural recursion on the derivatives (reflecting the number theoretic properties of the rotation number and the nature of the singularities) then gave rise to relations between the scalings. In this section we work out some of the consequences of these relations. In particular, for the cases that are of interest to us here (see below), we solve for the rates at which the scalings decrease and therefore the rate at which the derivatives grow.

Having established in previous sections that the treatment of rotation numbers of bounded type is analogous to that of the golden mean (denoted by g), we will limit ourselves to that case. This is done in order to not needlessly complicate the equations.

In the continuation of this section, there are two cases that will occupy us. The first case is the one discussed in the previous section, namely $\nu_l = 1$. The second case is where both singularities are equal to ν. We note here that these cases overlap when both singularities are linear. The latter case has been studied in detail by Veerman [1987]. Again, in principle all cases can be investigated by the same methods, but we do not wish to bury the arguments in details.

Note that in dealing only with golden mean scalings, the subscript of $\sigma_i(n)$ may be omitted. The recursion for the scalings described in Corollary 4.7, then runs as follows:

Lemma 6.1.

$$\frac{\sigma(2n+1)^{\nu_l}}{\sigma(2n)\sigma(2n-1)} = \frac{(1+\varepsilon(2n+1))}{\nu_l\nu_r}, \tag{6.1a}$$

$$\frac{\sigma(2n+2)^{\nu_r}}{\sigma(2n+1)\sigma(2n)} = \frac{(1+\varepsilon(2n+2))}{\nu_l\nu_r}, \tag{6.1b}$$

and

$$\lim_n \varepsilon(n) = 0 .$$

Proof. Combine Proposition 4.5 and Proposition 4.6. □

In the remainder of this work we use a slightly weaker version of $\approx$:

$$a(n) \cong b(n)$$

means

$$|\ln|a(n)/b(n)|| \leqq \text{const} .$$

Theorem 6.2. *If* $v_l = 1$, *then*

$$D(q_{2n-1}) \cong \frac{1}{\sigma(2n-1)} \cong e^{C \cdot \lambda^n},$$

$$D(q_{2n}) \cong \frac{1}{\sigma(2n)} \cong e^{C \cdot (\lambda - 1) \cdot \lambda^n},$$

where

$$\lambda = \lambda(v) = \tfrac{1}{2}\{1 + 2/v + \sqrt{(1 + 4/v^2)}\} .$$

Proof. Rewrite Eq. (6.1) with $v = v_r$, $v_l = 1$, and logarithmic variables:

$$\xi(n) = -\ln \sigma(2n-1), \qquad \eta(n) = -\ln \sigma(2n), \qquad k(n) = \ln v - \ln(1 + \varepsilon(n)) .$$

One obtains

$$\begin{aligned} \xi(n+1) &= \xi(n) + \eta(n) + k(2n+1), \\ \eta(n+1) &= 1/v\{\eta(n) + \xi(n+1) + k(2n+2)\} . \end{aligned}$$

To study the second iterate of these equations, substitute the first in the last. This yields a simple iterative scheme of the following form:

$$\begin{pmatrix} \xi(n+1) \\ \eta(n+1) \end{pmatrix} \approx \begin{pmatrix} \xi(n) + \eta(n) + k(2n+1) \\ 1/v\{\xi(n) + 2\eta(n) + k(2n+2) + k(2n+1)\} \end{pmatrix} = T_n L \begin{pmatrix} \xi(n) \\ \eta(n) \end{pmatrix}, \quad (6.2)$$

where T_n is a translation and L is the matrix $\begin{pmatrix} 1 & 1 \\ 1/v & 2/v \end{pmatrix}$.

We are interested in the asymptotic behavior of $(\xi(n), \eta(n))$. According to Theorem 5.3, both $\xi(n)$ and $\eta(n)$ tend to infinity. The eigenvalues of L satisfy:

$$\lambda_{+/-} = \frac{1}{2}\left\{1 + \frac{2}{v} +/- \left(1 + \frac{4}{v^2}\right)^{1/2}\right\} .$$

These eigenvalues are distinct and one of them (denoted by λ) is greater than 1. The behavior of $(\xi(n), \eta(n))$ will thus be dominated by the largest eigenvalue λ of L:

$$(TL)^n \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 1 \\ \lambda - 1 \end{pmatrix} \{\lambda^n C_0 + \lambda^{n-1} q(1) + \ldots \lambda q(n-1) + q(n)\} + O(1),$$

where $\begin{pmatrix} 1 \\ \lambda - 1 \end{pmatrix}$ is the unstable eigenvector of L and $q(n)$ the unstable component of $\begin{pmatrix} k(2n+1) \\ 1/v\{k(2n+1) + k(2n+2)\} \end{pmatrix}$. Note that

$$q(\infty) = \lim q(n) \neq 0 .$$

The term written as $O(1)$ contains the contributions in the stable direction. So,

$$\lambda^{-n}\begin{pmatrix}\xi(n)\\ \eta(n)\end{pmatrix}=\begin{pmatrix}1\\ \lambda-1\end{pmatrix}\left\{C_0+q(\infty)\frac{1-\lambda^{-n}}{\lambda-1}+\ldots\sum_{i=0}^{n-1}\frac{q(i)-q(\infty)}{\lambda^i}\right\}+O(\lambda^{-n}).$$

But by Lemma 6.1,

$$\lim_i|q(i)-q(\infty)|<\text{const}\cdot\lim_i a(i)=0,$$

so that the summation converges geometrically in λ. The expression between brackets then converges to C with a $O(\lambda^{-n})$ error. □

In the case that both singularities are linear ($v=1$), one easily checks that

$$\lambda(1)=g^{-2},\quad\text{where}\quad g=\frac{1+\sqrt{5}}{2}.$$

So,

$$D(q_n)=1/\sigma(n)\approx e^{C\cdot g^n}=e^{C\cdot q_n}.$$

With some care, this is also an estimate for Df^{q_n} on the whole of the non-wandering set which is therefore uniformly hyperbolic (as was proved by an altogether different method in Veerman (1989)).

Now let

$$\tau=\ln C,$$

and

$$\gamma(v,g)=\ln\lambda(v)/\ln g^2\quad(\gamma(2,g)=0.55568\ldots).$$

The following corollary is a reformulation of the above theorem. It introduces the "universal exponent" γ which depends exclusively on the nature of the singularity and the rotation number. This exponent describes the asymptotic behavior of scalings and derivatives.

We remark here that from the proof of the theorem one concludes that the constant C depends on the size of the initial $\sigma(n)$ (n small). That is: the constant C is sensitive to the details of the map and is not universal. In the case that $\gamma=1$, this constant is the Lyapunov exponent.

Corollary 6.3. $D(q_{2n-1})\cong\tau^{q_{2n-1}^{\gamma}}$; $D(q_{2n-1})\cong v_r\tau^{(\lambda-1)\cdot q_{2n}^{\gamma}}$, *where* $0<\gamma\leqq 1$, *and equality holds only if* $v=1$.

Proof. Use the above definitions of τ and γ to rewrite the theorem. The estimate in the proposition for γ follows from

$$1<\lambda(v)\leqq g^{-2}.\quad\Box$$

We will now briefly describe the analysis for the case that *both* singularities are of order v. In this case, we do not know how to generalize the reasoning of Sect. 5. Instead, we have the conjecture:

Conjecture 6.4. If both singularities are of order v, then for $v<2$ all scalings approach zero, and

$$1/\sigma(n)\approx e^{C\lambda(v)^n}.$$

For $v>2$, all scalings are bounded away from zero and one.

The evidence for this conjecture is embodied in the last two results of this section.

Proposition 6.6. *For $v<2$, there is an ε such that if there is a k with $\sigma(k)<\varepsilon$ and $\sigma(k+1)<\varepsilon$ then $1/\sigma(n)\cong e^{C\lambda^n}$ as $n\to\infty$.*

Proof. If $\sigma(k)$ and $\sigma(k+1)$ are small enough, then Eq. (6.1) is correct (according to Eq. 3.5, distortions obtained by the Koebe principle are comparable to scalings). In that case, as in Theorem 6.2, the behavior of the scalings is governed by the largest eigenvalue if it is larger than one.

Because both singularities are identical, it is sufficient to analyze the first iterate of Eq. (6.1). In logarithmic variables, one obtains

$$\begin{pmatrix}\xi'\\ \eta'\end{pmatrix}\approx\begin{pmatrix}\eta\\ 1/v\{\xi+\eta+k\}\end{pmatrix}=TL\begin{pmatrix}\xi\\ \eta\end{pmatrix}, \tag{6.3}$$

where

$$k=2\ln v .$$

The eigenvalues of L satisfy

$$\lambda_{+/-}=\frac{1}{2}\left\{\frac{1}{v}+/-\left(\frac{4}{v^2}+\frac{1}{v}\right)^{1/2}\right\}.$$

Thus

$$0<\lambda\leqq 1\Leftrightarrow v\geqq 2,$$

and

$$|\lambda_-|<1. \quad \square$$

Proposition 6.7. *For $v>2$, there is an ε such that*

$$\sigma(k)<\varepsilon \quad \textit{and} \quad \sigma(k-1)<\varepsilon\Rightarrow\sigma(k+1)>\min\{\sigma(k),\sigma(k-1)\}.$$

Proof. As before, if $\sigma(k-1)$ and $\sigma(k)$ are sufficiently small Eq. (6.3) holds. Therefore (take $\sigma(k-1)$ and $\sigma(k)$ even greater if necessary),

$$\eta'=\frac{1}{2}(\xi+\eta)+\left(\frac{1}{v}-\frac{1}{2}\right)(\xi+\eta+k)+\frac{k}{2}<\frac{1}{2}(\xi+\eta). \quad \square$$

Numerical experiments (for the golden mean rotation number) indicate that for $v>2$ the scalings are bounded away from zero. This suggests that for certain flat spot singularities, the renormalization theory is analogous to that of a single critical point.

VII. The Non-Wandering Set

Consider a circle map with irrational rotation number of bounded type and with one linear singularity. We prove that the non-wandering set is equal to the complement of $\bigcup_{i=0}^{\infty} U(-i)$ and that it has Hausdorff dimension zero. For the definitions related to the Hausdorff dimension we refer to Falconer (1985).

Denote by Ω_n the set $S^1-\bigcup_{i=0}^{q_n} U(-i)$. This set consists of q_n+1 closed intervals which will be denoted by A_i^n. We remark that a closed covering $\{A_i^n\}$ of Ω_n also defines an open covering of Ω_n (just add arbitrarily small pieces to the boundaries of A_i^n).

Theorem 7.1. *Let f be a flat spot map with one linear singularity and irrational rotation number of bounded type. Then its non-wandering set has Hausdorff dimension zero.*

Proof. This is a consequence of the fact that scalings around the critical point tend to zero and that by bounded distortions this property can be propagated along the orbit. The proof uses the techniques explained in Sects. 3 and 4. Below we give an outline.

We define $\varepsilon(n)$ to be a sequence whose terms converge to zero sufficiently slowly so that the following is true (recall that the scalings converge to zero). Since

$$|\imath(q_{2n})-l(0)|/|\imath(q_{2n-2})-l(0)|$$

is comparable to a scaling, we have that either

$$|\imath(q_{2n-1})-r(0)|/|\imath(q_{2n-2})-l(0)|<\varepsilon(2n)$$

or else:

$$|\imath(q_{2n})-l(0)|/|\imath(q_{2n-1})-r(0)|<\varepsilon(2n+1).$$

Suppose (without loss of generality) the first is true.

Cover Ω_{2n-2} with $\{A_i^{2n-2}\}$ and set

$$H_{2n-2}^s=\sum_{i=0}^{q_{2n-2}}|A_i^{2n-2}|^s.$$

At the next level of construction $\{A_i^{2n}\}$ covers Ω_{2n}.

Choose $x_0'^{2n}\in U$, and

$$x_i'^{2n}\in U(-[(a_{2n-1}-i)q_{2n-1}+q_{2n-2}]) \quad \text{for} \quad i\in\{1,\dots,a_{2n-1}\},$$

such that one can use the one-sided Koebe principle to see that the maps indicated by solid lines in Fig. 7.1 have the following properties:

i) They are almost linear,

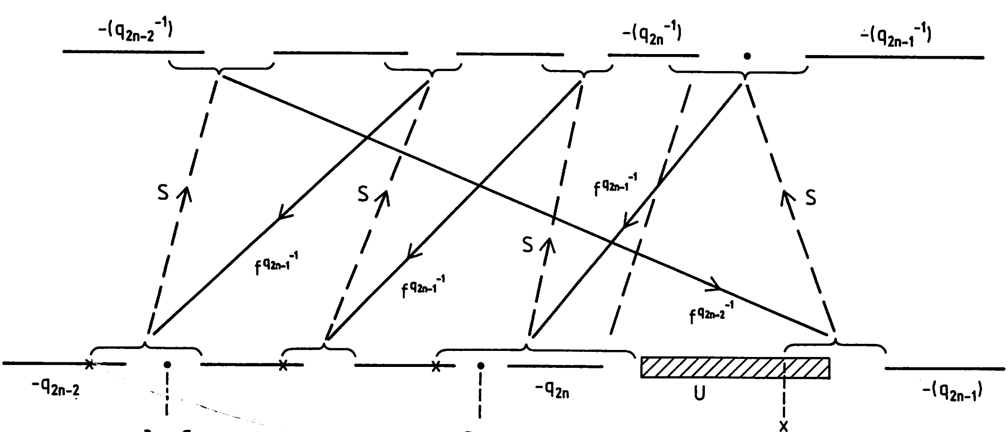

Fig. 7.1. Propagation of the scalings along an orbit

ii) The pieces intersecting Ω_{2n} make up a fraction less than $\varepsilon(n)$ of the pieces represented by the braces.
This can be achieved by noting that all relevant scalings tend to zero and applying the one-sided Koebe principle. It is easy to check that the intersection with Ω_{2n} with the domains of the above maps together with their $q_{2n-1}-1$ forward images under f (except for the leftmost where one applies at most $q_{2n-2}-1$ iterates of f) form a covering of Ω_{2n} with the following properties:
i) Each interval A_i^{2n-2} of Ω_{2n-2} is split up in at least two intervals A_j^{2n} whose length satisfies $|A_j^{2n}|<\varepsilon(2n)\,|A_i^{2n-2}|$.
ii) Each interval A_i^{2n-2} is split up in at most $1+2\max\{a_k\}$ sub-intervals.
Consequently,

$$H_{2n}^s=\sum_{i=0}^{q_{2n}}|A_i^{2n}|^s\leqq(1+2\max\{a_k\})\,\varepsilon(2n)^s H_{2n-2}^s\,.$$

Since all relevant scalings tend to zero, we can easily pick a sequence $\varepsilon(n)$ such that all the above holds. Since also the maximum diameter of the covering $\{A_i^n\}$ of Ω_n goes to zero as n increases, the Hausdorff s-measure of Ω must be zero for each positive x. This implies the theorem. □

VIII. Concluding Remarks

We suspect that Theorem 4.6 for the orbit of the critical point is much more general than the current content. A relation of the type

$$D(q_n)\sigma(n)\approx K$$

clearly also holds for renormalizable maps: the relevant case here is that of folding maps with period doubling kneading sequence and (critical) circle maps with golden mean rotation number. There is no simple proof of such a relation, however. What one can prove for the unimodal maps (period doubling) is that (Sullivan 1987, 1988) the scalings are bounded away from one and therefore that there is a K with

$$\frac{1}{K}<D(q_n)\sigma(n)<K\,.$$

We also conjecture in the context of Sect. 7 that the Hausdorff dimension of the set of parameters such that the rotation number of f_t is irrational has Hausdorff dimension zero. This is eminently reasonable, because scalings in that set are (asymptotically) equal to zero.

Finally we wish to emphasize again that the exponent γ solely depends on the criticality. It might be a good tool to probe higher dimensional systems that can be characterized by rotation intervals, for example Birkhoff attractors in a two dimensional dissipative twist map. To such a Birkhoff attractor one can assign a rotation interval.

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