

Conformal Dynamics Problem List

Edited by Ben Bielefeld

The following list of unsolved problems was given at the Conformal Dynamics Conference which was held at SUNY Stony Brook in November 1989. Problems were contributed by Ben Bielefeld, Adrien Douady, Curt McMullen, Jack Milnor, Misuhiro Shishikura, Folkert Tangerman, and Peter Veerman.

§1. Local connectivity of Julia sets

Let $f_\lambda(z) = \lambda z + z^2$ where $\lambda = \exp(2\pi i\theta)$. Call such a polynomial *parabolic* if θ is rational. If θ is irrational, and f_λ is analytically conjugate to a rotation near 0 we say f_λ is a *Siegel polynomial*. Otherwise we call f_λ a *Cremer polynomial*. Douady and Sullivan [Sul] have shown that the Julia set of a Cremer polynomial is never locally connected. In the generic case, Douady (unpublished) has described specific examples of external rays which do not land, but rather have an entire continuum of limit points in the Julia set.

Question 1. Is there an arc joining 0 to $-\lambda$, in the Julia set of a Cremer polynomial? ($-\lambda$ is the preimage of the fixed point.)

Question 2. Give a plausible topological model for the Julia set of a Cremer polynomial.

Question 3. Make a good computer picture of the Julia set of some Cremer polynomial.

Question 4. Are there any rays landing at 0 for a Cremer polynomial?

Question 5. For which Cremer polynomials is the critical point accessible?

Question 6. If we remove the fixed point from the Julia set of a Cremer polynomial, how many connected components are there in the resulting set $J(f_\lambda) - \{0\}$, ie., is the number of components countably infinite?

Question 7. Are there any Siegel polynomials whose Siegel disk has a boundary which is not a Jordan curve?

Let $P_c(z) = z^2 + c$ where c ranges over values for which the Julia set of P_c is connected.

Question 8. For which c is the Julia set of P_c locally connected? [Reportedly Yoccoz has recently proved local connectivity except at points on boundaries of hyperbolic components and infinitely renormalizable points.]

Question 9. Is the Julia set of P_c locally connected when c is real?

Question 10. If c is the Feigenbaum point, is the Julia set of P_c locally connected?

Question 11. If one could show that topological conjugacy of P_{c_1} and P_{c_2} implies that P_{c_1} is quasiconformally conjugate to P_{c_2} , would this imply local connectivity of the quadratic connectedness locus (ie., the Mandelbrot set, consisting of all c for which the Julia set of P_c is connected)?

Let $\tau = p/q$, and let M_τ be the limb of the connectedness locus with interior angle p/q .

Question 12. Is the diameter of M_τ less than K/q^2 for some constant K independent of τ ? If not, is it at least less than $K \log(q)/q^2$? (Remark: the still unpublished Yoccoz inequality implies that the diameter is bounded by a constant over q . Compare [P].)

Question 13. If P_c is nonrecurrent does this imply the existence of a conformal metric (a metric of the form $\rho(z)|dz|$ with integrable singularities) for which P_c is expanding on its Julia set? [Yoccoz has proved local connectivity in the nonrecurrent case.]

References:

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[DH1] A. Douady and J.H. Hubbard, *Systèmes Dynamiques Holomorphes I,II: Itération des Polynômes Complexes*, Publ. Math. Orsay **84.02 and 85.04**.

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[Sul] D. Sullivan., *Conformal dynamical systems*, Geometric Dynamics, 725-752, Springer-Verlag Lecture Notes No. 1007, 1983.

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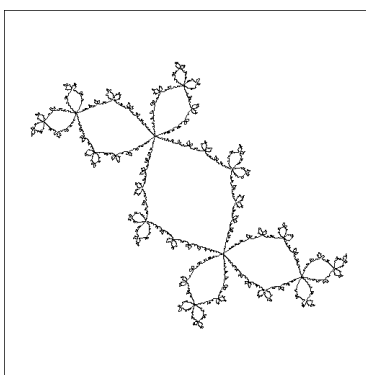
§2. Quasiconformal Surgery (Douady, Bielefeld, Shishikura)

It is possible to investigate rational functions using the technique of quasiconformal surgery as developed in [DH2], [BD] and [S]. There are various methods of gluing together polynomials via quasiconformal surgery to make new polynomials or rational functions. The idea of quasiconformal surgery is to cut and paste the dynamical spaces for two polynomials so as to end up with a branched map whose dynamics combines the dynamics of the two polynomials. One then tries to

find a conformal structure that is preserved under this branched map of the sphere to itself, so that using the Ahlfors-Bers theorem the map is conjugate to a rational function. There are several topological surgeries which experimentally seem to exist, but for which no one has yet been able to find a preserved complex structure.

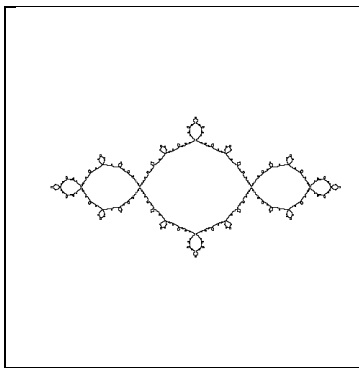
The first such kind of topological surgery is **mating** of two monic polynomials with the same degree. (Compare [TL].) The first step is to think of each polynomial as a map on a closed disk by thinking of infinity as a circle worth of points, one point for each angular direction. The obvious extension of the polynomial at the circle at infinity is $\theta \mapsto d\theta$ where d is the degree of the polynomial. Now glue two such polynomials together at the circles at infinity by mapping the θ of the first polynomial to $-\theta$ in the second. Finally, we must shrink each of the external rays for the two polynomials to a single point. The result should be conjugate to a rational map of degree d . (Surprisingly this construction sometimes seems to make sense even when the filled Julia sets for both polynomials have vacuous interior.)

For instance we can take the rabbit to be the first polynomial, that is $z^2 + c$ where the critical point is periodic of period 3 ($c \sim -.122561 + .744862i$). The Julia set appears in the following picture.



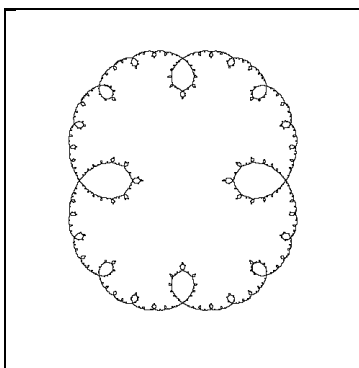
The rabbit

Then for the second polynomial we could take the basilica, that is $z^2 - 1$ (it is named after the Basilica San Marco in Venice. One can see the basilica on top and its reflection in the water below). The Julia set for the basilica appears in the following figure.



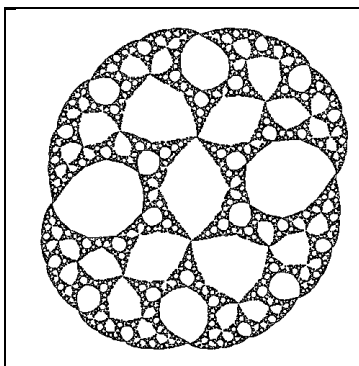
The basilica

Next we show the basilica inside-out ($\frac{z^2}{z^2-1}$) which is what we will glue to the rabbit.



The inside-out basilica

And finally we have the Julia set for the mating ($\frac{z^2+c}{z^2-1}$ where $c = \frac{1+\sqrt{-3}}{2}$).



The basilica mated with the rabbit

Question 1. Which matings correspond to rational functions? There are some known obstructions. For example, Tan Lei has shown that matings between quadratic polynomials can exist only if they do not belong to complex conjugate limbs of the Mandelbrot set.

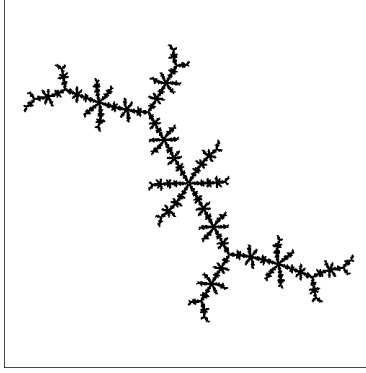
Question 2. Can matings be constructed with quasiconformal surgery?

Question 3. If one polynomial is held fixed and the other is varied continuously, does the resulting rational function vary continuously? Is mating a continuous function of two variables?

The second type of topological surgery is **tuning**. First take a polynomial P_1 with a periodic critical point ω of period k , and assume that no other critical points are in the entire basin of this superattractive cycle. Let P_2 be a polynomial with one critical point whose degree is the same as the degree of ω . We also assume that the Julia sets of P_1 and P_2 are connected. We give two descriptions. For the first description we assume the closure \bar{B} of the immediate basin of ω is homeomorphic to the closed unit disk \bar{D} , and that the Julia set for P_2 is locally connected. Now, P_1^k maps \bar{B} to itself by a map which is conjugate to the map $z \mapsto z^d$ of \bar{D} , where d is the degree of the critical point. (In fact, if $d > 2$, then there are $d - 1$ possible choices for the conjugating homeomorphism, and we must choose one of them.) Intuitively the idea is now the following. Replace the basin B by a copy of the dynamical plane for P_2 , gluing the “circle at infinity” for this plane onto the boundary of B so that external angles for P_2 correspond to internal angles in \bar{B} . Now shrink each external ray for P_2 to a point. Also, make an analogous modification at each pre-image of B . The map from the modified B to its image will be given by P_2 , and the map on all other inverse images of the modified B will be the identity. The result, P_3 , called P_1 tuned with P_2 at ω , should be conjugate to a polynomial having the same degree as P_1 . Conversely P_2 is said to be obtained from P_3 by renormalization.

In the case of quadratic polynomials, the tunings can be made also in the case when P_2 is not locally connected. Also it may be that the case where there is a critical point on the boundary of the basin is different.

As an example we can take P_1 to be the rabbit polynomial. Then we can take $P_2(z) = z^2 - 2$ which has the closed segment from -2 to 2 as its Julia set. The following figure shows the resulting quadratic Julia set tuning the rabbit with the segment ($z^2 + c$ where $c \sim -.101096 + .956287i$).



The rabbit tuned with the segment

In the picture we see each ear of the rabbit replaced with a segment.

Question 4. Does the tuning construction always give a result which is conjugate to a polynomial? This is true when P_1 and P_2 are quadratic.

Question 5. Can tunings be constructed with quasiconformal surgery?

Question 6. Does the resulting polynomial vary continuously with P_2 ? This is true when P_1 and P_2 are quadratic [DH2].

Question 7. Does the resulting tuning vary continuously with P_1 ? (here we consider only P_1 with a superstable orbit.)

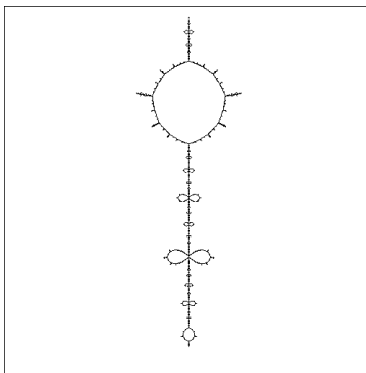
Question 8. Given a sequence $P_{1,k}$ of tending to some limit, do the tunings of $P_{1,k}$ with P_2 tend to a limit which is independent of P_2 ?

The third kind of surgery is **intertwining surgery**.

Let P_1 be a monic polynomial with connected Julia set having a repelling fixed point x_0 which has ray landing on it with rotation number p/q . Look at a cycle of q rays which are the forward images of the first. Cut along these rays and we get q disjoint wedges. Now let P_2 be a monic polynomial with a ray of the same rotation number landing on a repelling periodic point of some period dividing q (such as 1 or q). Slit this dynamical plane along the same rays making holes for the wedges. Fill the holes in by the corresponding wedges above making a new sphere. The new map will given by P_1 and P_2 except on a neighborhood of the inverse images of the cut rays where it will have to be adjusted to make it continuous. This construction should be possible to do quasiconformally using the methods in [BD] together with Shishikura's new (unpublished) method of presurgery in the case where the rays in the P_2 space land at a repelling orbit. This construction doesn't seem to work when the rays land at a parabolic orbit.

For instance we can take $P_1(z) = z^2$ and $P_2(z) = z^2 - 2$. The Julia set for P_1 is the unit circle with repelling fixed point at 1 and the ray at angle 0 lands on it with rotation number 0. The Julia set for P_2 is the

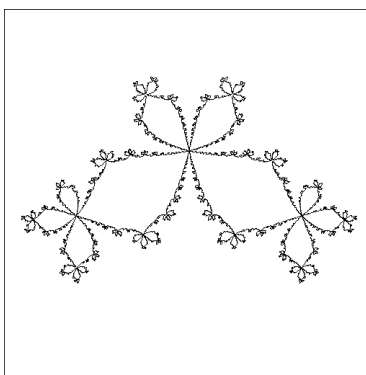
closed segment from -2 to 2 with repelling fixed point 2 and the ray at angle 0 lands on it with rotation number 0. We cut along the 0 ray in both cases. Opening the cut in the first dynamical space gives us one wedge. The space created by opening the cut in the second space is the hole into which we put the wedge. The resulting cubic Julia set is shown in the following picture (the polynomial is $z^3 + az$ where $a \sim 2.55799i$).



A circle intertwined with a segment

We see in the picture the circle and the segment, and at the inverse image of the fixed point on the segment we see another circle. At the other inverse of the fixed point on the circle we see a segment attached. All the other decorations come from taking various inverses of the main circle and segment.

As a second example we can intertwine the basilica with itself. The ray $1/3$ lands at a fixed point and has rotation number $1/2$. The following is the Julia set for the basilica intertwined with itself (the polynomial here is $z^3 - \frac{3}{4}z + \frac{\sqrt{-7}}{4}$).



A basilica intertwined with itself

Question 9. When does an intertwining construction give something which is conjugate to a polynomial?

Question 10. Can intertwinings be constructed with quasiconformal surgery?

Question 11. Does the resulting polynomial vary continuously in P_2 ?

Here is a different kind of continuity question. Consider the space of all monic polynomials

$$z \mapsto z^n + a_{n-1}z^{n-1} + \cdots + a_1z$$

with $|a_1| \geq 1$, so that there is an unattractive fixed point at the origin. Here we do not require that the Julia set be connected. If at least one external ray lands at the origin, then there is a well defined “rotation number” of these external rays under the map $\theta \mapsto n\theta \pmod{1}$.

Question 12. Does this rotation number extend uniquely to a continuous map from our space of polynomials to \mathbb{R}/\mathbb{Z} ? (When $a_1 = e^{2\pi i\theta}$, this rotation number map must take the value θ .)

References:

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§3. Thurston’s algorithm for real functions (Bielefeld, Tangerman, Veerman, Milnor)

Work in the space $P = P(n, \mu)$ of piecewise monotone maps f of the interval $I = [0, 1]$ which have n laps, and which map the boundary $\{0, 1\}$ into itself by some specified map μ . Starting with any map f_0 in P , let

p_0 be the unique degree n polynomial in P which has all critical points in $[0,1]$ and which has the same critical values, encountered in the same order, as does f_0 . Then there is a unique homeomorphism h_0 of $[0,1]$ which takes the critical points of f_0 to the critical points of p_0 and which satisfies $p_0 \circ h_0 = f_0$. It follows that the map $f_1 = h_0 \circ p_0 = h_0 \circ f_0 \circ h_0^{-1}$ is topologically conjugate to f_0 . Now continue inductively, constructing maps $f_{k+1} = h_k \circ f_k \circ h_k^{-1}$.

$$\begin{array}{ccc}
 I & \xrightarrow{f_2} & I \\
 \uparrow h_1 & \searrow p_1 & \uparrow h_1 \\
 I & \xrightarrow{f_1} & I \\
 \uparrow h_0 & \searrow p_0 & \uparrow h_0 \\
 I & \xrightarrow{f_0} & I
 \end{array}$$

If f_0 is post-critically finite, and if there is no Thurston obstruction, then it follows from Thurston's argument that the resulting sequence of polynomials p_k converges to a polynomial p_∞ with the same kneading sequence as f_0 . Computer experiments suggest that in many cases the sequence of maps f_k converges to this same limit. Furthermore, this seems to be true even if f_0 is not post-critically finite.

Question 1. Formulate and prove some precise statement in this direction.

Now we can consider the same problem except that instead of lifting by polynomials we lift by some other family in P . For instance computer experiments suggest that we can choose the p_k 's from a family of the form $p(x) = -|x|^\alpha + c$ where $\alpha > 1$. (To be more precise, we must first change coordinates with an affine map so that the boundary $0,1$ maps to 0 . This yields the family of maps $x \mapsto k - k|2x - 1|^\alpha$ from the unit interval to itself where $0 < k \leq 1$.)

Question 2. Given f_0 with a preperiodic or periodic kneading sequence does Thurston's algorithm converge for the specific family above. There is a proof only for α an even integer, which is the polynomial case.

Question 3. Give a general property for the lifting family which will guarantee convergence of Thurston's algorithm for preperiodic or periodic kneading sequences.

Question 4. Give a general property for the lifting family which will guarantee convergence of Thurston's algorithm for arbitrary kneading sequences.

References:

[MT] J. Milnor and W. Thurston, Iterated maps of the interval, pp. 465-563 of “Dynamical Systems (Maryland 1986-87)”, edit. J.C.Alexander, Lect. Notes Math. **1342**, Springer 1988 (cf. §13.4).

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§4. Stable regions for complex Hénon type maps (Milnor)

Let f be a polynomial diffeomorphism of \mathbb{C}^2 with Jacobian determinant δ . Suppose that the set $K^+(f)$ of points with bounded forward orbit, has a non-empty interior. Let U be some connected component of this interior. According to Montel, the set of iterates of f restricted to U possesses a convergent subsequence, which converges say to $g : U \rightarrow \mathbb{C}^2$. Evidently the rank r of g is zero or one if $|\delta| < 1$, and is two if $|\delta| = 1$.

Question 1. Can U be a wandering component? If so, we must have $|\delta| < 1$. Can the rank of g be either zero or one? Can U be either bounded, or unbounded of finite volume, or of infinite volume?

If U is not a wandering component, then it is strictly periodic under f , and, after replacing f by some finite iterate, we may assume that $f(U) = U$. Note then that f commutes with g . One useful family of examples is provided by the Hénon maps. Given any two non-zero complex numbers λ and μ , there is an essentially unique (quadratic) Hénon map $H_{\lambda,\mu}$ which has a fixed point with eigenvalues λ and μ .

Rank Zero Case. If $g(U) = \mathbf{x}_0 \in U$, then \mathbf{x}_0 is an attracting fixed point. Examples are provided by the Hénon maps $H_{\lambda,\mu}$ where the eigenvalues λ and μ can be any two numbers in the punctured open disk $D - 0$. If $\mathbf{x}_0 \in \partial U$, then it is conjectured that one eigenvalue must be equal to 1. Evidently the other eigenvalue must be in $D - 0$. Here $H_{1,\mu}$ provides an example.

Rank One Case. If g has a fixed point in U , then g must be a retraction of U onto a Siegel disk. There are examples of the form $H_{\lambda,\mu}$ with $1 = |\lambda| > |\mu|$. (Compare Zehnder.) Can g be a retraction onto a Herman ring (or onto a punctured Siegel disk)?

Rank Two Case. If g has a fixed point, then U is a “Siegel bi-disk”. There are examples of the form $H_{\lambda,\mu}$ with $|\lambda| = |\mu| = 1$. (Again see Zehnder.) Can U also be the product of a Herman ring with a Herman ring, or the product of a Herman ring with a Siegel disk?

References:

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[FM] Friedland and J. Milnor, Dynamical properties of plane polynomial automorphisms, Erg. Th. & Dy. Sy. **9** (1989), 67-99.

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[BS] E. Bedford and J. Smillie, Polynomial diffeomorphisms of \mathbb{C}^2 : currents, equilibrium measure and hyperbolicity, to appear.

§5. Geometrically finite maps and Kleinian groups (McMullen)

Geometrically finite rational maps

Let $f(z)$ be a rational map, C its set of critical points, $P = \cup_1^\infty f^n(C)$ its post-critical set and J its Julia set. The map f is *expanding* if $\overline{P} \cap J = \emptyset$. It is well-known that f is expanding iff some fixed iterate of f uniformly expands the spherical metric on the Julia set; these maps are also called *hyperbolic* or *Axiom A*.

Let the space Rat_d (respectively Poly_d) of rational (polynomial) maps of degree d be equipped with the topology of uniform convergence. A well-known and fundamental problem is to resolve the following:

Conjecture. The expanding maps form a dense subset of Rat_d and Poly_d .

Cf. [MSS] where this is related to the problem of invariant measurable line fields supported on the Julia set. (It is known that the set of expanding maps is open). This is not even known in the case of quadratic polynomials. The corresponding problems for real maps are also open.

In many ways an expanding rational map is well-behaved (cf. [Sul]); it is like a Kleinian group with compact convex core in \mathbb{H}^3 .

More generally, let us say a rational map is *geometrically finite* if $\overline{P} \cap J$ is a finite set. Equivalently, every critical point in the Julia set is pre-periodic. (In this case rationally indifferent cycles are allowed). These maps should be compared to geometrically finite Kleinian groups.

For a geometrically finite rational map f :

Problem 1. Show that either the Julia set J is the whole sphere and the action of f on J is ergodic, or the Hausdorff dimension δ of J is less than 2. In the latter case, what can be said about the δ -dimensional measure of J and the dynamics with respect to this measure class?

Problem 2. Show every component of J is locally connected.

Problem 3. Develop for f an analogue of the Haken decomposition for 3-manifolds. For example, if J is disconnected, can f be constructed by surgery from rational maps with connected Julia sets?

Problem 4. Extend Thurston’s combinatorial theory of critical finite rational maps (those for which $|P| < \infty$) to all geometrically finite maps. That is, describe f up to combinatorial equivalence $\text{rel } \overline{P}$ by a finite amount of topological data, and characterize those combinatorial types which arise as rational maps.

Convex hyperbolic 3-manifolds

Let N be a complete hyperbolic 3-manifold presented as the quotient of \mathbb{H}^3 by the action of a Kleinian group Γ . The *convex core* of N is the quotient of the convex hull of the limit set. All closed geodesics in N are contained in the convex core.

Question. Suppose $\pi_1(N)$ is generated by n elements. Is there an upper bound R_n to the radius of an embedded ball entirely contained in the convex core of N ? (Here R_n should depend only on n).

The question has a positive answer when N is a quasifuchsian group. By results of Thurston [**Th Ch. 13**] there is a pleated surface near every point in the convex core, and this provides an upper bound on the injectivity radius.

The question also has an (easy) positive solution for hyperbolic 2-manifolds, and we know of no counterexample for hyperbolic manifolds of any dimension.

Critically finite rational maps on \mathbb{P}^n

A basic tool aiding the study of critically finite rational maps on the Riemann sphere is the Poincaré metric on the complement of the post-critical set P (assuming $|P| > 2$). This metric is expanded by f . One should be able to apply the same sort of arguments to critically finite rational maps $f : \mathbb{P}^n \rightarrow \mathbb{P}^n$, $n > 1$, such that the complement of the post-critical set is Kobayashi hyperbolic.

More precisely, say f is *critical finite* if there exist (possibly reducible) hypersurfaces $V \subset W \subset \mathbb{P}^n$ such that

$$f : (\mathbb{P}^n - W) \rightarrow (\mathbb{P}^n - V)$$

is a covering map.

Problem. Are there nontrivial examples of critically finite maps with $\mathbb{P}^n - V$ Kobayashi hyperbolic? How do they behave dynamically?

References:

- [**MSS**] R. Mañé, P. Sad, and D. Sullivan., *On the dynamics of rational maps*, *Ann. Sci. Éc. Norm. Sup.* 16 (1983), 193–217.
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- [**Th**] W. P. Thurston., *Geometry and Topology of Three-Manifolds*, Princeton lecture notes, 1979.