

# The Critical Locus for Complex Hénon Maps.

Tanya Firsova

## Abstract

We give a topological model of the critical locus for complex Hénon maps that are perturbations of the quadratic polynomial with disconnected Julia set.

## 1 Introduction. Foliations.

The family of Hénon mappings is a basic example of nonlinear dynamics. Both real and complex versions of these maps were extensively studied, but still there is a great deal of unknown about them.

In this article we study complex quadratic Hénon mappings. These are maps of the form:

$$f_a \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + c - ay \\ x \end{pmatrix}.$$

Note that these maps are biholomorphisms of  $\mathbb{C}^2$ . They have constant Jacobian, which equals to the parameter  $a$ . As  $a \rightarrow 0$ , Hénon mappings degenerate to quadratic polynomial maps  $x \mapsto x^2 + c$ , which act on the parabola  $x = y^2 + c$ . The Hénon mappings that we study are perturbations of quadratic polynomials with disconnected Julia set.

In analogy with one-dimensional dynamics, Hubbard and Oberste-Vorth [HOV94] introduced the functions  $G_a^+$  and  $G_a^-$  that measure the growth rate of the forward and backwards iterations of the orbit. These functions are pluriharmonic on the set of points  $U_a^+$ ,  $U_a^-$ , whose orbits tend to infinity under forward and backwards iterates. Their level sets are foliated by Riemann surfaces. These natural foliations  $\mathcal{F}_a^+$  and  $\mathcal{F}_a^-$  were introduced and extensively studied in [HOV94].

The foliations  $\mathcal{F}_a^+$ ,  $\mathcal{F}_a^-$  can also be characterized in terms of Böttcher coordinates. There are maps  $\phi_{a,+}$  and  $\phi_{a,-}$  that semiconjugate the map  $f_a$  and the map  $f_a^{-1}$  to  $z \mapsto z^2$  and  $z \mapsto z^2/a$  correspondingly. These functions were constructed in [HOV94]. We recall the definitions of these functions and list their properties in Section 4. The level sets of  $\phi_{a,+}$  and  $\phi_{a,-}$  define  $\mathcal{F}_a^+$  and  $\mathcal{F}_a^-$  in some domains near the infinity and can be propagated by dynamics to all of  $U_a^+$  and  $U_a^-$  correspondingly.

The dynamical description of these foliations is the following:

**Lemma 1.1** ([BS98a]). *The leaves of  $\mathcal{F}_a^+$  are “super-stable manifolds of  $\infty$ ”, i.e. if  $z_1, z_2$  belong to the same leaf, then  $d(f_a^n(z_1), f_a^n(z_2)) \rightarrow 0$  super-exponentially, where  $d$  is Euclidean distance in  $\mathbb{C}^2$ . If  $z_1, z_2 \in U_a^+$  do not belong to the same leaf, then  $d(f_a^n(z_1), f_a^n(z_2)) \not\rightarrow 0$ .*

The leaves of  $\mathcal{F}_a^-$  are “super-unstable manifolds of  $\infty$ ”.

One would like to think of these foliations as coordinates in  $U_a^+ \cap U_a^-$ . However, for all Hénon mappings there is a codimension one subvariety of tangencies between  $\mathcal{F}_a^+$  and  $\mathcal{F}_a^-$  [BS98a].

**Definition 1.1.** *The critical locus  $\mathcal{C}_a$  is the set of tangencies between foliations  $\mathcal{F}_a^+$  and  $\mathcal{F}_a^-$ .*

Thus, the critical locus is the set of “heteroclinic tangencies” between the “super-stable” and “super-unstable” manifolds.

We give an explicit description of the critical locus for Hénon mappings

$$(x, y) \mapsto (x^2 + c - ay, x),$$

where  $x^2 + c$  has disconnected Julia set, and  $a$  is sufficiently small. The topological model of the critical locus for such Hénon maps was conjectured by John Hubbard. We justify his picture.

Lyubich and Robertson ([LR]) gave the description of the critical locus for Hénon mappings

$$(x, y) \mapsto (p(x) - ay, x),$$

where  $p(x)$  is a hyperbolic polynomial with connected Julia set,  $a$  is sufficiently small. They showed that for each critical point  $c$  of  $p$  there is a component of the critical locus that is asymptotic to the line  $y = c$ . Each component of the critical locus is an iterate of these ones, and each is a punctured disk.

They used critical locus to show that a pair of quadratic Hénon maps of the studied type, taken along with the natural foliations, gives a rigid object. This means that if a conjugacy between two Hénon maps sends the natural foliations of the first map to the natural foliations of the second map then the two Hénon maps are conjugated by a holomorphic or antiholomorphic affine map.

Since the work of Lyubich and Robertson [LR] is unpublished, we include the proof of all the results that we are using.

Some of the sources of the fundamental results about Hénon mappings are [BLS93], [BS91a], [BS91b], [BS92], [BS98a], [BS98b], [BS99], [HOV94], [HOV95], [FM89], [FS92].

## 1.1 Acknowledgements.

The author is grateful to Mikhail Lyubich for statement of the problem, numerous discussions and useful suggestions. We are thankful to John Hubbard for showing the conjectural picture of the critical locus.

## 2 Topological model for the critical locus.

In our case truncated spheres serve as building blocks for the critical locus. Consider a sphere  $S$  and a pair of disjoint Cantor sets  $\Sigma, \Omega \subset S$ . The elements of  $\Sigma, \Omega$  can be parametrized by one-sided infinite sequences of 0, 1’s. Denote by  $\sigma_\alpha \in \Sigma, \omega_\alpha \in \Omega$  elements parametrized by a sequence  $\alpha$ .

Let  $\alpha_n$  be a  $n$ -string of 0’s and 1’s. For each  $\alpha_n, n \in \mathbb{N} \cup \{0\}$ , take a disk  $V_{\alpha_n} \subset S \setminus (\Sigma \cup \Omega)$  with the smooth boundary. (Let  $V$  denote the disk that corresponds to an empty sequence.) We require that  $V_{\alpha_n}$  are disjoint. Moreover,  $V_{\alpha_n}$  converge to  $\sigma_\alpha$ , where  $\alpha_n$  is the string of the first  $n$  elements of  $\alpha$ .

$U_{\alpha_n}$  play the same role for  $\Omega$  as  $V_{\alpha_n}$  for  $\Sigma$ .

We assume that there is a fixed homeomorphism  $\tilde{h}$  between the boundaries of  $V_{\alpha_n}$  and  $U_{\alpha_n}$ .

Let  $p \in S$  be a point.

We say that  $S \setminus (\Sigma \cup \Omega \cup \sum_{\alpha_n} [U_{\alpha_n} \cup V_{\alpha_n}] \cup p)$  is a truncated sphere.

Note that all truncated spheres are homeomorphic one to another.

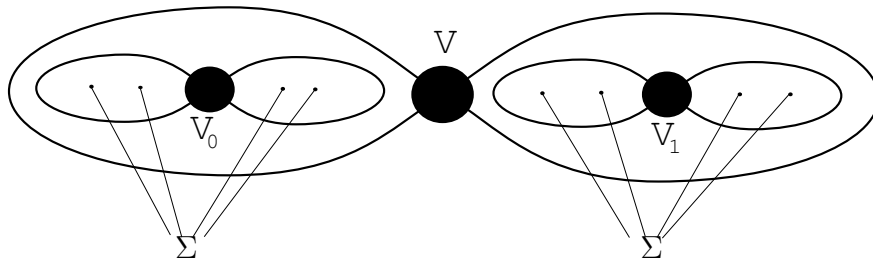


Figure 1: The geometry of a truncated sphere

**Theorem 2.1.** *Suppose  $x^2 + c$  has disconnected Julia set. There exists  $\delta$ , such that  $\forall |a| < \delta$  the critical locus of the map*

$$f_a \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + c - ay \\ x \end{pmatrix}$$

*is smooth, and has the following topological model: take countably many truncated spheres  $S_n$ ,  $n \in \mathbb{Z}$ , and glue the boundary of  $V_{\alpha_n}$  on  $S_k$  to the boundary of  $U_{\alpha_n}$  of  $S_{n+k}$  using the homeomorphism  $h$ . Map  $f_a$  acts on the critical locus by sending  $S_n$  to  $S_{n+1}$ .*

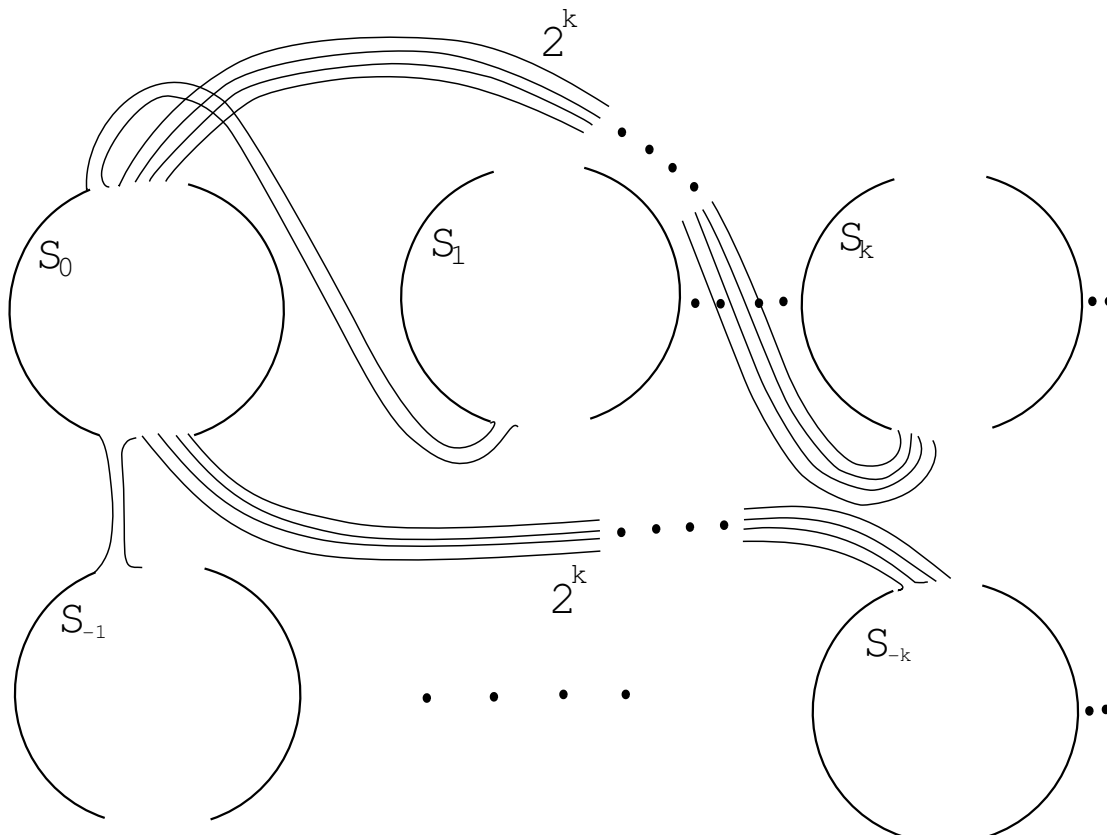


Figure 2: The topological model.

**Note 2.1.** *It is easy to see that the topological model of the critical locus is well-defined up to a homeomorphism.*

### 3 Strategy of description.

When  $a = 0$ , the Hénon mapping reduces to

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^2 + c \\ x \end{pmatrix}$$

which is a map  $x \mapsto p(x)$  acting on the curve  $x = p(y)$ .

As  $a \rightarrow 0$  the map degenerates, but the foliations and the Green functions persist and become easy to analyze.

In Section 4 we study  $\phi_{a,+}$  and  $\phi_{a,-}$ , paying extra attention to the degeneration as  $a \rightarrow 0$ .

Section 5 is devoted to Green's functions. We prove that  $G_a^+$  and  $G_a^-$  depend continuously on  $x, y$  and the parameter  $a$ .

In Section 6 we describe the foliations  $\mathcal{F}_a^+$  and  $\mathcal{F}_a^-$  and the critical locus  $\mathcal{C}$  in terms of  $\phi_{a,+}$  and  $\phi_{a,-}$ . We also calculate the critical locus in the degenerate case. As  $a$  deviates from zero, we carefully describe the perturbation. The latter is done in several steps:

First, we choose appropriate values  $r$  and  $\alpha$  and describe the critical locus in the domain

$$\Omega_a = \{G_a^+ \leq r\} \cap \{|y| \leq \alpha\} \cap \{|p(y) - x| \geq |a|\alpha\}.$$

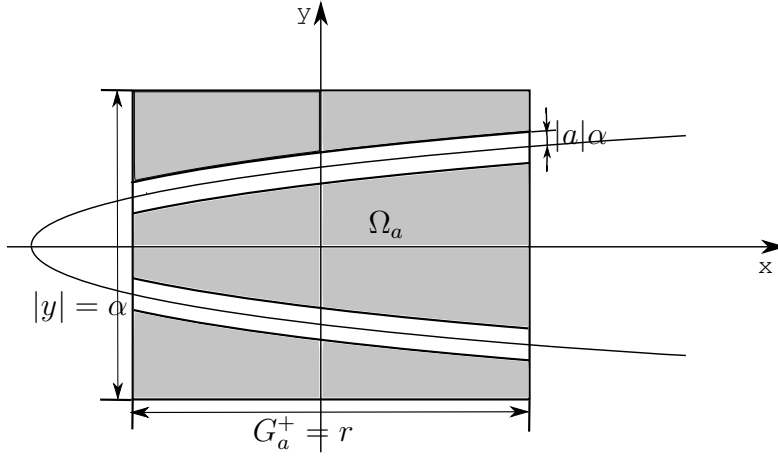


Figure 3: Domain  $\Omega_a$

We choose  $r$  that lies between the critical point level and critical value level:

$$G_p(0) < r < G_p(c),$$

where  $G_p$  is the Green function of polynomial  $p$ .

We choose  $\alpha$  such that  $G_a^+|_{\{|x|>\alpha, |x|>|y|\}} > r$ . Moreover, in Section 9 we choose  $\alpha$  such that leaves of foliation  $\mathcal{F}_a^-$  in  $\Omega_a$  form a family horizontal of parabolas.

In Section 10 we give a description of the foliation  $\mathcal{F}_a^+$  in

$$\{G_a^+ \leq r\} \cap \{|y| \leq \alpha\}.$$

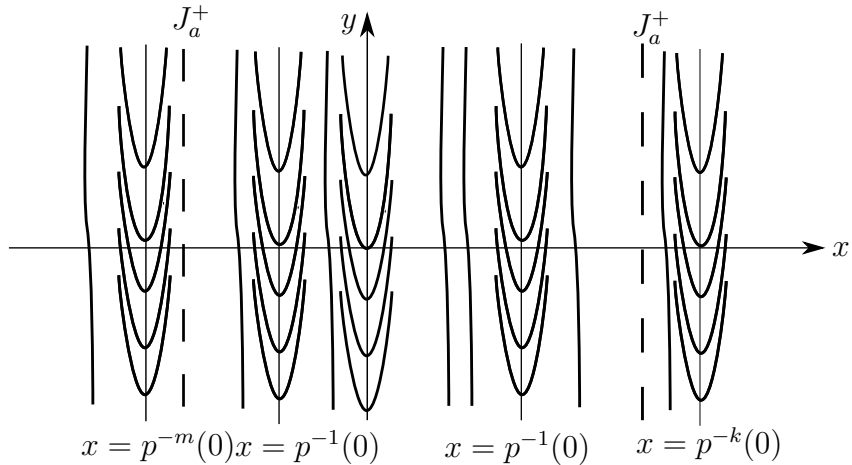


Figure 4: The foliation  $\mathcal{F}_a^+$

We show that the local leaves of  $\mathcal{F}_a^+$  are either vertical-like or vertical parabolas. There is exactly one family of vertical parabolas (see Fig. 4) corresponding to each line  $x = \xi_k$ , where  $p^{\circ k}(\xi_k) = 0$ .

The critical locus in  $\Omega_a$  is the set of tangencies of a family of horizontal parabolas with vertical-like leaves and vertical parabolic leaves. In Section 11 we show that for each family of parabolas there is exactly one handle.

In Section 7 we show that foliations  $\mathcal{F}_a^+$  and  $\mathcal{F}_a^-$  extend holomorphically to  $x = \infty$ . We show that a point  $(0, \infty)$  belongs to the extension and calculate the tangent line to the critical locus at this point. This gives us a description of the critical locus in

$$\{|y| < \epsilon\} \cap \{|x| > \alpha\}.$$

In Section 12 we show that the critical locus can be extended up to a  $a|\alpha|$ -neighborhood of parabola  $p(y) - x = \text{const}$  along  $y = 0$ .

In Section 13 we combine the results from the previous sections to get the description of the fundamental domain of the certain component of the critical locus. We rule out ghost components. We also do a dynamical regluing of the fundamental domain of the critical locus to obtain a description in terms of truncated spheres.

## 4 Functions $\phi_{a,+}$ and $\phi_{a,-}$

In this section we construct functions  $\phi_{a,+}$  and  $\phi_{a,-}$ . In their description we follow [HOV94]. In description of the degeneration of  $\phi_{a,+}$  and  $\phi_{a,-}$  as  $a \rightarrow 0$ , we follow [LR].

### 4.1 Large scale behavior of the Hénon map

The study of Hénon mappings usually begins with introduction of domains  $V_+$  and  $V_-$  which are invariant under the action of Hénon map:

$$f_a(V_+) \subset V_+, \quad f_a^{-1}(V_-) \subset V_-.$$

Moreover, one requires that every point that has unbounded forward orbit eventually enters  $V_+$  and every point that has unbounded backward orbit enters  $V_-$ .

Fix  $R$ ,  $0 < r < 1$ , and choose  $\alpha$  so that

$$\frac{|c|}{|y^2|} + \frac{R+1}{|y|} < r \quad (4.1)$$

$$|p(y)| > (2R+1)|y| \quad (4.2)$$

for all  $|y| \geq \alpha$ .

Consider the following partition of  $\mathbb{C}^2$  :

$$V_+ = \{(x, y) : |x| > |y|, |x| > |\alpha|\};$$

$$V_- = \{(x, y) : |y| > |x|, |y| > \alpha\};$$

$$W = \{(x, y) : |x| \leq \alpha, |y| \leq \alpha\}.$$

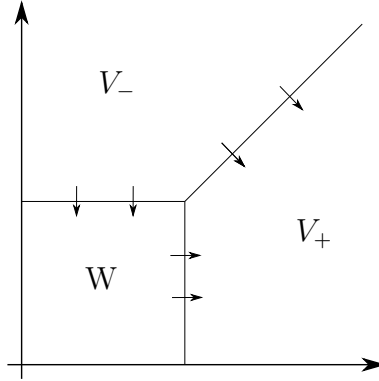


Figure 5: The crude picture of the dynamics of  $f_a$

**Lemma 4.1.** For  $a \in D_R$ ,  $f_a(V_+) \subset V_+$ .

*Proof.*  $|p(x) - ay| \geq |p(x)| + |a||y| \geq (2R+1-R)|y| \geq |y|$   
 $|p(x) - ay| \geq (R+1)|y| \geq \alpha$  □

Let

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = f_a^n \begin{pmatrix} x \\ y \end{pmatrix}.$$

**Lemma 4.2.**  $U_a^+ = \bigcup_n f_a^{-n}(V_+)$ ;

*Proof.* Let  $f_a^n(x) \rightarrow \infty$ . For some  $n$ ,  $f_a^n(x) \in V_+$  or  $f_a^n(x) \in V_-$ .

Suppose that for all  $n \geq n_0$   $(x_n, y_n) \in V_-$ . Note that  $|y_{n+1}| = |x_n| \leq |y_n|$ . Sequence  $\{|y_n|\}$  is decreasing. Contradiction. □

**Lemma 4.3.** For  $a \in D_R$   $f_a^{-1}(V_-) \subset V_-$ .

*Proof.*  $\frac{|p(y)-x|}{|a|} \geq \frac{|2R+1||y|-|x|}{R} \geq |y|$   
 $\frac{|p(y)-x|}{|a|} \geq |y| \geq |\alpha|$  □

**Lemma 4.4.**  $U_a^- = \cup_n f_a^n(V_-)$ .

*Proof.* The proof is the same as in the previous lemma. □

## 4.2 Function $\phi_{a,+}$

The function  $\phi_{a,+}$  is constructed as the limit

$$\phi_{a,+}(x, y) = \lim_{n \rightarrow \infty} |x_n|^{\frac{1}{2^n}}.$$

with the appropriate choice of the branch of root. We are particularly interested in this function in  $V_+$ . Sense is made of the above limit using the telescopic formula

$$\phi_{a,+} = \lim_{n \rightarrow \infty} x \exp \left( \frac{1}{2} \log \frac{x_1}{x^2} + \dots + \frac{1}{2^n} \log \frac{x_n}{x_{n-1}^2} + \dots \right). \quad (1)$$

**Lemma 4.5.** The function  $\phi_{a,+}$ , defined by formula (1), is well-defined and holomorphic for all  $(x, y) \in V_+$  and all  $a \in D_R$ . Moreover, there exists  $B$  so that

$$B^{-1} < \left| \frac{\phi_{a,+}}{x} \right| < B.$$

*Proof.* Let  $s_n^+ = \frac{x_n}{x_{n-1}^2} - 1$ . By property (4.1),  $|s_n^+| < r$ .

$\log \frac{x_n}{x_{n-1}^2} = \log(1 + s_n^+)$  is calculated using the principle branch of log. Therefore, the series

$$\frac{1}{2} \log \frac{x_1}{x^2} + \dots + \frac{1}{2^n} \log \frac{x_n}{x_{n-1}^2} + \dots \quad (2)$$

converges absolutely and uniformly. Since  $|\log \frac{x_n}{x_{n-1}^2}| < -\frac{\log(1-r)}{2^n}$ , the infinite sum (2) is no bigger than  $-\log B = -\log(1-r)$ .

The final claim follows immediately from the expression (1) and the bound derived for the series (2). □

We show that  $\phi_{a,+} \sim x$  as  $x \rightarrow \infty$  in Section 7.

Let  $\mathcal{D}_{n,+} = \{(x, y, a) \mid f_a^n(x, y, a) \in V_+, a \in D_R\}$

**Lemma 4.6.** The function  $\phi_{a,+}^{2^n}$  extends to a holomorphic function on  $\mathcal{D}_{n,+}$  given by  $\phi_{a,+}^{2^n} = \phi_{a,+} \circ f_a^n$ . Moreover,  $\phi_{0,+}^{2^n}(x, y) = b_p^{2^n}(x)$ .

*Proof.* The function  $\phi_{a,+}^{2^n}$  is holomorphic by definition.

As  $a \rightarrow 0$  the Hénon mappings degenerate to a one-dimensional map  $x \rightarrow p(x)$ , acting on  $y = p(x)$ . Therefore,  $\phi_{0,+}(x, y) = b_p(x)$  on  $V_+$ , and  $\phi_{0,+}^{2^n}(x, y) = b_p^{2^n}(x)$  if  $f_0^n(x, y) \in V_+$ . □

Note that

$$\begin{aligned} K_0^+ &= J_0^+ = J_p \times \mathbb{C}, \\ U_0^+ &= U_p \times \mathbb{C}, \end{aligned}$$

where  $J_p$  is the Julia set for the one-dimensional map  $x \mapsto p(x)$ ;  $U_p$  is the set of points, whose orbits escape to  $\infty$  under the map  $x \mapsto p(x)$ .

### 4.3 Function $\phi_{a,-}$

We start by working the leading terms of  $y_{-n}$  as a polynomial in  $y$  and as a polynomial in  $\frac{1}{a}$ . The following notation simplifies the statement of the result:

$$\sigma_k = 1 + 2 + \dots + 2^{k-1} \text{ for } k \geq 1 \text{ and } \sigma_k = 0 \text{ for } k \leq 0.$$

By an easy induction we get:

**Lemma 4.7.** *The leading term of  $y_{-n}(x, y, a)$  considered as a polynomial in  $y$  is  $y^{2^n}/a^{\sigma_n}$ . The leading term of  $y_{-n}$  considered as a polynomial in  $\frac{1}{a}$  is  $\frac{1}{a^{\sigma_n}}(p(y) - x)^{2^{n-1}}$ .*

We define the function  $\phi_{a,-}$  as a limit

$$\phi_{a,-} = \lim_{n \rightarrow \infty} (y_{-n} \circ a^{\sigma_n})^{\frac{1}{2^n}} \quad (3)$$

with an appropriate choice of branch of root. Note that the factor  $a^{\sigma_n}$  is chosen, so that the leading term of  $y_n \cdot a^{\sigma_n}$ , as a polynomial in  $y$ , is  $y^{2^n}$ .

Let  $D_R^* = D_R \setminus \{0\}$ .

Sense is made of the limit in  $V_- \times D_R^*$  using the telescopic formula:

$$\phi_{a,-}(x, y) = \lim_{n \rightarrow \infty} \exp \left( \frac{1}{2} \log \frac{ay_{-1}}{y^2} + \frac{1}{2^2} \log \frac{ay_{-2}}{y_{-1}^2} + \dots \right) \quad (4)$$

**Lemma 4.8.** *The function  $\phi_{a,-}$ , defined by formula (4) is well-defined and holomorphic on  $V_- \times D_R^*$ . Moreover, there exists  $B$  so that*

$$B^{-1} < \left| \frac{\phi_{a,-}}{y} \right| < B.$$

*Proof.* Let  $s_n^- = \frac{c - x_{-(n-1)}}{y_{-(n-1)}^2}$

Note that  $|s_n^-| < r$  for  $(x, y) \in V_-$ ,  $a \in D_R^*$ .

We evaluate  $\frac{ay_{-n}}{y_{-(n-1)}^2} = \log(1 + s_n^-)$  using the principle branch of  $\log$ .

Since  $\log(1 + s_n^-) \leq -\log(1 - r)$ , the series

$$\frac{1}{2} \log \frac{ay_{-1}}{y^2} + \frac{1}{2^2} \log \frac{ay_{-2}}{y_{-1}^2} + \dots \quad (5)$$

converges uniformly and absolutely to a holomorphic function bounded by  $-\log(1 - r)$ . Letting  $B = (r - 1)^{-1}$ , the last statement of the Lemma follows.  $\square$

We show that  $\phi_{a,-} \sim y$  as  $y \rightarrow \infty$  in Section 7.

The next lemma states that one can extend  $\phi_{a,-}$  to a holomorphic function on  $V_- \times D_R$ .



**Lemma 4.9.**  $\phi_{a,-} = (p(y) - x)^{\frac{1}{2}} + ah(x, y, a)$  for some holomorphic function  $h$  on  $V_- \times D_R$ .

*Proof.* By Lemma 4.7 the leading term of  $y_{-n}$  as a polynomial in  $\frac{1}{a}$  is  $\frac{1}{a^{\sigma_n}}(p(y) - x)^{2^{n-1}}$ . Since  $x_{-n} = y_{-(n-1)}$ . The leading term of  $x_{-n}$  as a polynomial in  $\frac{1}{a}$  is the leading term of  $y_{-(n-1)}$ .

Recall  $s_n^- = \frac{c-x_{-(n-1)}}{y_{-(n-1)}^2}$ . It follows that  $s_n^-$  is a polynomial in  $a$  and it vanishes in  $a$  to the order  $2\sigma_{n-1} - \sigma_{n-2}$ . Hence the series (5) takes the form

$$\frac{1}{2} \log \frac{ay_{-1}}{y^2} + ag(x, y, a),$$

where  $g(x, y, a)$  is a holomorphic function on  $V_+ \times D_R$ .

By (4),  $\phi_{a,-}(x, y) = y \exp\left(\frac{1}{2} \log \frac{ay_1}{y^2}\right) \exp(ag(x, y, a))$ . The conclusion follows.  $\square$

Let

$$C(p) = \{p(y) - x = 0\}$$

Domain  $f_a(V_-)$  swells to  $\mathbb{C}^2 \setminus C(p)$  as  $a \rightarrow 0$ .

**Lemma 4.10.** For  $(x, y) \in \mathbb{C}^2 \setminus C(p)$  we have  $(x, y) \in f_a(V_-)$  for all sufficiently small  $a$ . Moreover, for all  $K \Subset \mathbb{C}^2 \setminus C(p) \Rightarrow K \Subset f_a(V_-)$  for all small enough  $a$ .

*Proof.* Both statements follow from

$$f_a(V_-) = \{|p(y) - x| \geq |a|\alpha, |p(y) - x| \geq |a||y|\}.$$

$\square$

$$\mathcal{D}_{-,n} = \{(x, y, a) \mid (x, y) \in f_a^n(V_-), a \neq 0 \text{ or } (x, y) \in \mathbb{C}^2 \setminus C(p), a = 0\}.$$

**Lemma 4.11.** The function  $\phi_{a,-}^{2^n}(x, y)$  extends from a holomorphic function on  $V_- \times D_R$  to  $\mathcal{D}_{-,n}$ , by letting

1.  $\phi_{a,-}^{2^n} = a^{\sigma_n} \phi_{a,-} \circ f_a^{-n}$  for  $a \neq 0$ ;
2.  $\phi_{0,-}^{2^n}(x, y) = (p(y) - x)^{2^{n-1}}$ .

*Proof.* If  $a \neq 0$ , then we can extend the function  $\phi_{a,-}$  to be holomorphic by defining  $\phi_{a,-}^{2^n} = a^{\sigma_n} \phi_{a,-} \circ f_a^{-n}$ . This agrees with  $\phi_{a,-}^{2^n}$  on  $V_-$ .

By Lemma 4.9

$$\phi_{a,-}^{2^n} = a^{\sigma_n} \phi_{a,-} \circ f_a^{-n} = a^{\sigma_n} (p(y_{-n}) - x_{-n})^{\frac{1}{2}} + a^{\sigma_n+1} h(x_{-n}, y_{-n}, a) \quad (6)$$

By Lemma 4.7 the leading term of  $y_{-n}$ , as a polynomial in  $\frac{1}{a}$ , is  $\frac{(p(y)-x)^{2^n}}{a^\sigma}$ . Since  $x_{-n} = y_{-(n-1)}$ , the leading term of  $x_{-n}$  as a polynomial in  $\frac{1}{a}$  is  $\frac{(p(y)-x)^{2^{n-2}}}{a^{\sigma_{n-1}}}$ . Thus  $a^{\sigma_n} (p(y_{-n}) - x_{-n})^{\frac{1}{2}}$  is a polynomial in  $x, y, a$  and the only term free in  $a$  is  $(p(y) - x)^{2^{n-1}}$ .

By the same argument the function  $a^{\sigma_n} h(x_{-n}, y_{-n}, a)$  is holomorphic in  $(x, y, a)$  on  $V_+ \times D_R$ . Therefore,  $a^{\sigma_n+1} h(x_{-n}, y_{-n}, a)$  vanishes in  $a$ .

Thus,  $\phi_{a,-}^{2^n} = (p(y) - x)^{2^{n-1}} + ah_1(x, y, a)$ .  $\square$

We denote  $K_0^-$  and  $J_0^-$  to be  $C(p)$ . We set  $U_0^- = \mathbb{C}^2 \setminus J_0^-$ . The previous two statements justify these notations.

## 5 Green's functions

The next two lemmas are from [HOV94]. We provide the proof of Lemma 5.1 to make the paper self-contained.

**Lemma 5.1.** *The function*

$$G_a^+(x, y) = \lim_{n \rightarrow \infty} \log^+ |f_a^n(x, y)| \quad (7)$$

*is well-defined in  $\mathbb{C}^2 \times D_R$ . It satisfies the functional equation*

$$G_a^+(f_a) = 2G_a^+. \quad (8)$$

*Moreover, it is pluriharmonic in  $U_a^+$ .*

*Proof.* We define, the function  $G_a^+ = \log |\phi_{a,+}|$  on  $V_+$ . We use the functional equation (8) to extend it to  $U_a^+$ . We also set  $G_a^+(x, y) = 0$  for  $(x, y) \in J_a^+$ .  $G_a^+$ , defined this way, is pluriharmonic in  $V_+$ , since it is a logarithm of a holomorphic non-zero function  $\phi_{a,+}$ . It is pluriharmonic in  $U_a^+$ , since for each  $n$ , on  $f_a^{-n}(V_+)$ ,  $G_a^+$  is a pull-back of a pluriharmonic function by a holomorphic change of coordinates.

Moreover, notice that the function defined this way satisfies (7) and there is a unique function that satisfies (7).  $\square$

**Lemma 5.2.** *The function*

$$G_a^-(x, y) = \lim_{n \rightarrow \infty} |f_a^{-n}(x, y)| + \log |a| \quad (9)$$

*is well-defined in  $\mathbb{C}^2 \times D_R^*$ . It satisfies the functional equation*

$$G_a^-(x, y) \circ f_a^- = 2G_a^- - \log |a|. \quad (10)$$

*Moreover, it is pluriharmonic on  $U_a^-$ .*

Equation (10) is sometimes more conveniently written

$$(G_a^- - \log |a|) \circ f_a^{-1} = 2(G_a^- - \log |a|).$$

We set

$$G_0^-(x, y) = \log |\phi_{0,-}(x, y)| = \begin{cases} \frac{1}{2} \log |p(y) - x| & \text{for } (x, y) \notin C(p); \\ -\infty & \text{for } (x, y) \in C(p). \end{cases} \quad (11)$$

Hubbard & Oberste-Vorth proved that the Green's functions are continuous when  $f_a$  is non-degenerate and the same argument gives continuity in  $x, y$  and  $a$  for  $G_a^+$  when  $a = 0$ . Lybich-Robertson [LR] extend this to  $G_a^-$  when  $a = 0$ .

**Lemma 5.3** ([LR]). *The functions  $G_a^+(x, y)$  and  $G_a^-(x, y)$  are continuous in  $x, y$  and  $a$  for  $a \in D_R$ .*

*Proof.* It follows by the same argument as is used in [HOV94] except in the case of  $G_a^-$  when  $a = 0$ . For  $(x', y') \notin C(p)$  the continuity of  $G_a^-$  at  $(x', y')$  and  $a = 0$  follows from Lemma 4.11. For  $(x', y') \in C(p)$  more work is required. If we restrict  $G_a^-$  to the slice  $a = 0$ , then we have shown continuity, so we will assume for most of the rest of this proof that  $a \neq 0$  (so  $f_a^{-1}$  is defined).

Let us fix  $M > 0$ , and find a neighborhood  $U$  of  $(x', y')$  so that for all  $(x, y) \in U$ ,  $G_a^-(x, y) < -M$

If  $(x, y) \in J_a^-$ , then it is enough to require that  $|a| < e^{-M}$ .

If  $(x, y) \in U_a^-$ , then  $f_a^{-n}(x, y) \in V_-$  for all  $n > n_0$ .

If  $f_a^{-n}(x, y) \in V_-$ , then by Lemma 4.8  $B^{-1} < \left| \frac{\phi_{a,-}(x_{-n}, y_{-n})}{y_{-n}} \right| < B$ .

Therefore,  $G_a^-(x, y) < \frac{1}{2} \log B + \frac{1}{2^n} \log |a^{\sigma_n} y_{-n}|$ .

$$y_{-n} = \frac{1}{a} (p(y_{-(n-1)}) - y_{-(n-2)}) \text{ for } n \geq 2. \quad (12)$$

$$y_0 = y, \quad y_{-1} = \frac{p(y) - x}{a}$$

We wish to estimate  $a^{\sigma_n} y_{-n}$  in terms of  $y_0$  and  $y_{-n}$ .

It is convenient to introduce a new variable  $z_{-n} = a^{\sigma_n} y_{-n}$  and a notation  $p(x, y) = y^2 p\left(\frac{x}{y}\right)$ .

In these new notations the recurrence relation (12) takes form

$$z_{-n} = p(z_{-(n-1)}, a^{\sigma_{n-1}}) - a^{\sigma_n - \sigma_{n-2} - 1} z_{n-2}, \quad (13)$$

where  $z_0 = y$ ,  $z_{-1} = p(y) - x$ .

$$|z_{-n}| \leq 2 \max(|p(z_{-(n-1)})|, |a|^{\sigma_{n-1}}, |a|^{2^n + 2^{n-1} - 1} |z_{n-2}|)$$

Using the estimate  $p(x, y) \leq C \max(x^2, a^2)$ , where  $C$  is a constant, we get

$$|z_{-n}| \leq 2 \max(|z_{-(n-1)}^2|, |a|^{2\sigma_{n-1}}, |a|^{2^n + 2^{n-1} - 1} |z_{n-2}|)$$

Consider a neighborhood

$$U(\epsilon) = \{(x, y, a) \mid |p(y) - x| < \epsilon^2, |a| < \min(2C\epsilon^2, \frac{1}{2}), |a||y| < C\epsilon^2\} \quad (14)$$

By induction one can show that if  $(x, y, a) \in U(\epsilon)$ , then  $z_{-n} < (2C)^{\sigma_n} \epsilon^{2^n}$ .

Therefore, for  $(x, y, a) \in U(\epsilon)$

$$G_a^-(x, y, a) < \log(|2C|\epsilon) \quad (15)$$

Therefore, for small enough  $\epsilon$ ,  $G_a^-(x, y, a) < -M$ .

Intersecting  $U(\epsilon)$  with the  $e^{-M}$ -neighborhood in  $a$ -variable, we get the desired neighborhood.  $\square$

## 6 Description of the foliations and the critical locus in terms of $\phi_{a,+}$ and $\phi_{a,-}$

Note that

$$G_a^+(x, y) = \log |\phi_{a,+}(x, y)|;$$

Therefore, the foliation  $\phi_{a,+} = \text{const}$  on  $V_+$  is exactly  $\mathcal{F}_a^+$ . It can be propagated by dynamics to the rest of  $U_a^+$ .

The same way,

$$G_a^-(x, y) = \log |\phi_{a,-}(x, y)|.$$

The foliation  $\phi_{a,-} = \text{const}$  on  $V_-$  is  $\mathcal{F}_a^-$  and it propagates by dynamics to  $U_a^-$

$$\mathcal{U}^\pm = \{(x, y, a) \mid a \in D_R, (x, y) \in U_a^\pm\}$$

Below we show that the critical locus is defined by a global holomorphic form on  $\mathcal{U}^+ \cap \mathcal{U}^-$ . Therefore, it is a proper analytic subset of  $\mathcal{U}^+ \cap \mathcal{U}^-$ .

The forms  $d \log \phi_{a,+}$  and  $d \log \phi_{a,-}$  are well-defined and holomorphic in  $\mathcal{U}^+$  and  $\mathcal{U}^-$  correspondingly.

Critical locus  $\mathcal{C}$  is given by the zeroes of the form

$$w(x, y, a) dx \wedge dy \wedge da = d \log \phi_{a,+} \wedge d \log \phi_{a,-} \wedge da$$

that is holomorphic in  $\mathcal{U}^+ \cap \mathcal{U}^-$ . When we prefer to think of  $w(x, y, a)$  as a function of two variables we use the notation  $w_a(x, y) = w(x, y, a)$ .

Let  $\mathcal{C}_{a_0} = \mathcal{C} \cap \{a = a_0\}$ .

We recall a definition of a proper analytic subset:

**Definition 6.1.** *A is a proper analytic subset of a complex manifold M, if for every point  $x \in M$ , there exist a neighborhood U and a set of functions  $f_1, \dots, f_n$  so that*

$$f_1 = \dots = f_n = 0$$

define A in U.

**Lemma 6.1.**  *$\mathcal{C}$  is a proper analytic subset of  $\mathcal{U}^+ \cap \mathcal{U}^-$ .*

*Proof.*  $\mathcal{C}$  is defined by the zeroes of the form that is holomorphic in  $\mathcal{U}^+ \cap \mathcal{U}^-$ . □

## 6.1 Degenerate critical locus.

As  $a \rightarrow 0$  the Hénon mapping degenerates to  $(x, y) \mapsto (p(x), x)$ . The foliation  $\mathcal{F}_a^+$  is defined by the form  $d\phi_{a,+}$  on  $U_a^+$ . Note that  $d\phi_{a,+} \neq 0$  on  $U_a^+$  for  $a \in D_R^*$ . Therefore, the foliation  $\mathcal{F}_a^+$  for  $a \in D_R^*$  is nondegenerate. The limiting foliation  $\mathcal{F}_0^+$  is degenerate. It has double leaves.

Below we explain what it means for the foliation to have a double leaf.

Recall that  $U_a^+ = \sum_n f_a^{-n}(V_+)$ , on each  $f_a^{-n}(V_+)$  the foliation  $\mathcal{F}_a^+$  is defined by the level sets of the function  $\phi_{a,+}^{2^n}$ .

Suppose that a foliation  $\mathcal{F}$  in  $\Omega \subset \mathbb{C}^2$  is defined by the level sets of a function  $\phi$ . Suppose that in a neighborhood of each point  $(x, y)$  one can choose local coordinates  $(u, t)$ , so that  $\phi = u^n$ .

**Definition 6.2.** We say that a leaf  $L$  of the foliation  $\mathcal{F}$  is double if in a neighborhood of a point  $(x, y) \in L$ ,  $\phi = u^2$ .

**Note 6.1.** The definition depends on the defining function  $\phi$  and does not depend on the choice of the local parameter  $u$ , nor on a point  $(x, y) \in L$ .

**Lemma 6.2.** The foliation  $\mathcal{F}_0^+$  in  $U_0^+$  is a vertical foliation  $x = \text{const}$  with leaves  $x = p^{-k}(0)$  being double for all  $k \geq 0$ .

*Proof.* For each point  $x_0$  there exists  $n$ , such that in a neighborhood of the line  $x = x_0$  the foliation  $\mathcal{F}_0^+$  is determined by the level sets of the function  $\phi_{0,+}^{2^n}$ .

Since  $\phi_{0,+}^{2^n}(x, y) = b_p^{2^n}(x)$ , the foliation is vertical.

The multiple leaves appear when  $(b_p^{2^n})'(x) = 0$ .  $(b_p^{2^n})'(x) = 0$  if and only if  $p^n(x) = 0$ . Moreover, all zeros of  $(b_p^{2^n})'$  are non-degenerate. Therefore, all the leaves  $x = p^{-n}(0)$  are double.  $\square$

The foliation  $\mathcal{F}_a^-$  in  $U^-a$  for  $a \in D_R^*$  is defined by the form  $d\phi_{a,-}^{2^n}$ . By Lemma 4.10 as  $a \rightarrow 0$ ,  $U_a^-$  swell to  $\mathbb{C}^2 \setminus C(p)$ . The form  $d\phi_{a,-}$  extends to  $\mathbb{C}^2 \setminus C(p)$  as a holomorphic form. Let  $\mathcal{F}_0^-$  denote the limiting foliation defined by the form  $d\phi_{0,-}$ .

**Lemma 6.3.** The foliation  $\mathcal{F}_0^-$  in  $U_0^-$  consists of the leaves  $\{p(y) - x = \text{const}\}$ .

*Proof.* By Lemma 4.9  $\phi_{0,-} = p(y) - x$  and is defined in  $U_0^-$ . The statement of the Lemma immediately follows.  $\square$

**Corollary 6.1.**  $\mathcal{C}_0 = [\{y = 0\} \cup_k \{x = p^{-k}(0)\}] \cap U_0^+ \cap U_0^-$

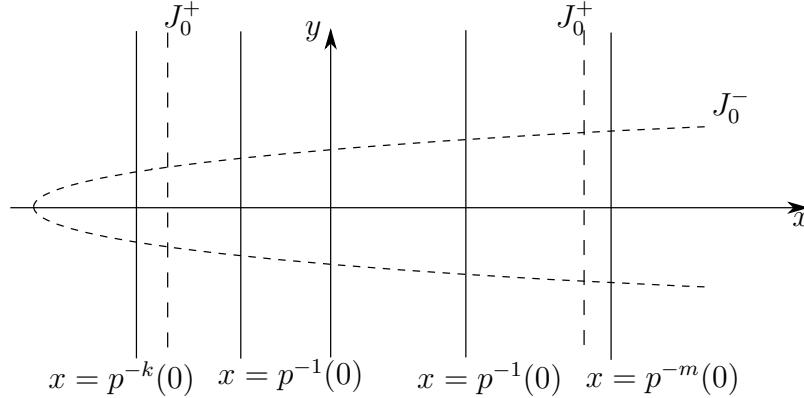


Figure 6: The degenerate Critical Locus

## 7 Critical locus near infinity.

The goal of this section is to calculate the critical locus at a neighborhood of  $x = \infty$  in the compactification of  $\mathbb{C}^2$  given by  $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ .

First, we extend the foliation  $\mathcal{F}_a^+$  to the line  $x = \infty$  in  $\mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$  compactification of  $\mathbb{C}^2$ . Let

$$\hat{V}_+ = \{(x, y) \in \mathbb{CP}^1 \times \mathbb{CP}^1 \mid (x, y) \in V_+ \text{ or } \{x = \infty, y \neq \infty\}\}$$

Let  $t = \frac{1}{x}$ ,  $v = \frac{1}{y}$ .

**Lemma 7.1.** *The function  $\frac{\phi_{a,+}}{x}$  extends as a holomorphic function to  $\hat{V}_+ \times D_R$ . Moreover,  $t\phi_{a,+} = 1 + th_1(t, y, a)$ ,  $\frac{\partial h_1}{\partial y} = u\tilde{h}_1(t, y, a)$ .*

*Proof.* By the Riemann Extension Theorem,  $\frac{\phi_{a,+}}{x}$  can be extended to  $x = \infty$ .

Note that the functions  $s_n^+$  and  $\frac{\partial s_n^+}{\partial y}$  vanish in  $t$  for all  $k$ . Thus, in the sum (2) every term vanishes in  $u$ . Therefore, the infinite sum (2) vanishes in  $u$  as well. And  $\frac{\phi_{a,+}}{x} = 1 + th_1(t, y, a)$ , where  $h_1$  is a holomorphic function in  $t, y, a$ . The same way one proves,  $\frac{\partial h_1}{\partial y} = t\tilde{h}_1(t, y, a)$ .  $\square$

$$\hat{V}_- = \{(x, y) \in \mathbb{CP}^1 \times \mathbb{CP}^1 \mid (x, y) \in V_- \text{ or } \{y = \infty, x \neq \infty\}\}$$

**Lemma 7.2** ([LR]). *The function  $\frac{\phi_{a,-}}{y}$  extends holomorphically to  $\hat{V}_- \times D_R$ . Moreover,  $v\phi_{a,-} = 1 + vh_2(x, v, a)$ ,  $\frac{\partial h_2}{\partial x} = v\tilde{h}_2(x, v, a)$ .*

*Proof.* The proof is the same as in the previous lemma.  $\square$

In  $V_+ \cap \mathcal{D}_{1,-}$  the critical locus is given by the zeroes of the form

$$\omega = d\phi_{a,+} \wedge d\phi_{a,-}^2 \wedge da$$

For the next lemma we fix  $(t, y)$ -coordinates in  $\hat{V}_+$ .

**Lemma 7.3.** *The critical locus extends holomorphically to  $(\hat{V}_+ \times D_R) \cap \mathcal{D}_{1,-}$ . The point  $(0, y, a) \in \mathcal{C}_a \cap \hat{V}_+ \cap \mathcal{D}_{1,-}$  iff  $y = 0$ . The tangent line to the critical locus at  $(0, 0, a)$  is given by  $2dy + Cdt = 0$  with  $C$  depending on  $a$ .*

*Proof.*

$$d\phi_{a,+} = -\frac{dt}{t^2} + \frac{\partial h_1}{\partial t} dt + \frac{\partial h_1}{\partial y} dy + \frac{\partial h_1}{\partial a} da = \frac{1}{t^2} \left( -1 + t^2 \frac{\partial h_1}{\partial t} dt + t^3 \tilde{h}_1 dy + t^2 \frac{\partial h_1}{\partial a} da \right)$$

$$\phi_{a,-}^2(x, y) = a\phi_{a,-} \circ f_a^{-1}(x, y) = a\phi_{a,-} \left( y, \frac{p(y) - x}{a} \right) = \frac{tp(y) - 1}{t} + ah_2\left(y, \frac{at}{tp(y) - 1}, a\right)$$

$$\begin{aligned} d\phi_{a,-} &= \left( p'(y) + a \frac{\partial h_2}{\partial y} \left( y, \frac{at}{tp(y) - 1}, a \right) + \frac{a^2 t^2 p'(y)}{(tp(y) - 1)^2} \frac{\partial h_2}{\partial v} \left( y, \frac{at}{tp(y) - 1}, a \right) \right) dy \\ &\quad + \left( \frac{1}{t^2} + \left[ \frac{a}{tp(y) - 1} + \frac{atp(y)}{(tp(y) - 1)^2} \right] \frac{\partial h_2}{\partial v} \right) dt + h_3(t, y, a) da, \end{aligned}$$

where  $h_3(t, y, a)$  is some holomorphic function in  $t, y, a$ .

By Lemma 7.2,  $\frac{\partial h_2}{\partial x} = v\tilde{h}_2$ . Hence  $\frac{\partial h_2}{\partial x}(y, \frac{at}{tp(y)-1}, a) = \frac{at}{tp(y)-1} \frac{\partial h_2}{\partial x}(y, \frac{at}{tp(y)-1})$ . Thus, there are holomorphic functions  $h_4, h_5$  on  $\mathcal{D}_1^-$  so that

$$d\phi_{a,-} = (p'(y) + th_4(t, y, a))dy + (\frac{1}{t^2}h_5(t, y, a))dt + h_3(t, y, a)da$$

Therefore, there exists a holomorphic function  $h_6$  on  $(\hat{V}_+ \times D_R) \cap \mathcal{D}_{1,-}$  so that

$$\omega t^2 = (p'(y) + th_6) dy \wedge dt \wedge da$$

Conclusion follows. □

**Corollary 7.1.** *Fix  $\epsilon$ . There exists  $\delta$  so that for  $|a| < \delta$  the critical locus in*

$$\{|y| \leq \epsilon\} \cap \{|x| \geq \alpha\}$$

*is the graph of a function  $y(x)$ .*

## 8 Horizontal and vertical invariant cones.

### 8.1 Horizontal cones.

Fix a domain  $B$ .

**Definition 8.1.** *A family of cones  $C_x$  in the tangent bundle to  $B$  is  $f_a$ -invariant iff for any point  $x \in B$ , such that  $f_a(x) \in B$ , we have  $df_a(C_x) \cup C_{f(x)}$ .*

**Lemma 8.1.** *Fix  $r' > r$  and  $\beta$ . For every  $C < \min\{x : G_p(x) \leq \frac{r'}{2}\}$  there exists  $\delta$  such that for all  $|a| < \delta$  the family of horizontal cones  $|\xi| > C|\eta|$  is  $f_a$ -invariant in  $\{G_a^+(x, y) \leq r'\} \cap \{|y| \leq \beta\}$  (where  $(\xi, \eta) \in T_{(x,y)}\mathbb{C}^2$ ).*

**Note 8.1.** *Note that we chose the box  $\{G_a^+(x, y) \leq r'\} \cap \{|y| \leq \beta\}$  so that the tip of the parabola does not belong to the box. This allows us to have an invariant horizontal family of cones.*

*Proof.*  $Df_a(x, y) = \begin{pmatrix} 2x & a \\ 1 & 0 \end{pmatrix}$

$$\text{Let } \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} = \begin{pmatrix} 2x & a \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

We want to find  $C$  such that

$$|\xi| > C|\eta| \Rightarrow |2x\xi + a\eta| > C|\xi|.$$

Take  $C = \min(2x) - \epsilon$ , where  $b_p(x) \leq \frac{r'}{2}$  and  $\epsilon$  is any number. Then  $\delta_2 = \epsilon \max(2x)$ . □

### 8.2 Vertical cones.

**Lemma 8.2.** *Fix  $r' \geq r$ ,  $\alpha$ ,  $C$ . There exists  $\delta$  such that for all  $|a| < \delta$  the family of cones  $|\xi| < C|a||\eta|$  is  $f_a^{-1}$ -invariant in  $\{G_a^+(x, y) \leq r'\} \cap \{|y| \leq \alpha\}$ ,  $(\xi, \eta) \in T_{(x,y)}\mathbb{C}^2$ .*

*Proof.*  $Df_a^{-1} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{a} & \frac{2y}{a} \end{pmatrix}$

$$\text{Let } \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{a} & \frac{2y}{a} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

$$\left| \frac{\xi_1}{\eta_1} \right| = \frac{|a|}{|-\frac{\xi}{a} + 2y|}$$

Suppose  $(x, y) = f_a^{-1}(u, v)$ , where  $\{(u, v) \mid G_a^+(u, v) \leq r', |v| \leq |\alpha|\}$ , then  $|y| > C_1$ .

Therefore, when  $|a| < \delta$ ,  $|2y - \frac{\xi}{a}| \geq |C_1|$  □

## 9 Description of $\mathcal{F}_a^-$ .

In this section we give a description of  $\mathcal{F}_a^-$  in

$$W \cap \{|p(y) - x| \geq |a|\alpha\}.$$

We choose  $\alpha$  in the definition of  $W$  such that the description of  $\mathcal{F}_a^-$  is especially nice.

The function  $\phi_{a,-}^2$  is well-defined in this region, so it is natural to expect that the foliation

$$\mathcal{F}_a^- = \{\phi_{a,-}^2 = \text{const}\}$$

is close to the foliation

$$\mathcal{F}_0^- = \{\phi_{0,-}^2 = p(y) - x = \text{const}\}.$$

The only region where it really needs to be checked is when we approach  $a|\alpha|$ -neighborhood of  $C_p$ .

We also prove that the leaves of  $\mathcal{F}_a^-$  that intersect the boundaries

$$\{G_a^+ \leq r\} \cap \{|p(y) - x| = a|\alpha|\}$$

$$\{G_a^+ \leq r\} \cap \{|y| \leq |\alpha|\}$$

are horizontal-like. In order to guarantee this we will need to choose appropriate  $\alpha$ .

We start by fixing preliminary  $\tilde{\alpha}$  such that conditions (4.1) and (4.2) are satisfied.

Notice that  $\phi_{a,-}^2$  is well-defined in

$$f_a(V_-) = \{|p(y) - x| \geq |a|\tilde{\alpha}, |p(y) - x| \geq |a||y|\}$$

Therefore, the domain of definition of  $\phi_{a,-}^2$  in  $W$  is

$$f_a(V_-) \cap W = \{|p(y) - x| \geq |a|\tilde{\alpha}\}.$$

**Lemma 9.1.** *There exists  $\alpha$  such for all  $a \in D_R$*

$$\min\{|\phi_{a,-}^2(x, y)| : (x, y) \in W, |p(y) - x| = \alpha|a|\} >$$

$$\max\{|\phi_{a,-}^2(x, y)| : (x, y) \in W, |p(y) - x| = \tilde{\alpha}|a|\}.$$



*Proof.*  $\phi_{a,-}^2(x, y) = a\phi_{a,-}(y, \frac{p(y)-x}{a})$ .  
 $\phi_{a,-}(x, y) \sim y$  as  $y \rightarrow \infty$ , Therefore,

$$\min\{|\phi_{a,-}(x, y)| : |x| < \theta, |y| = \alpha\} > \max\{|\phi_{a,-}(x, y)| : |x| < \theta, |y| = \tilde{\alpha}\}$$

for big enough  $\alpha$ . Take  $\theta = \max(p^{-1}(D_{|x|+|a|\alpha}))$ . □

**Corollary 9.1.** *For a point  $q \in W \cap \{|p(y) - x| \geq |a|\alpha\}$  a connected component of a leaf  $L_q$  of the foliation  $\mathcal{F}_a^-$ , passing through a point  $q$ , stays outside of  $\tilde{\alpha}|a|$ -neighborhood of  $C_p$ .*

**Lemma 9.2.** *For all  $a \in D_R$ ,  $\frac{\partial\phi_{a,-}^2/\partial y}{\partial\phi_{a,-}^2/\partial x}$  is  $a$ -close to  $\frac{\partial\phi_{0,-}^2/\partial y}{\partial\phi_{0,-}^2/\partial x} = -p'(y)$  in  $\Omega$ .*

**Note 9.1.** *Note that  $\frac{\partial\phi_{a,-}^2}{\partial y}, \frac{\partial\phi_{a,-}^2}{\partial x}$  do not stay bounded in  $\Omega$  as  $a \rightarrow 0$ , but their ratio does.*

*Proof.* We chose  $\alpha$  so that the function  $\phi_{a,-}^2$  is well-defined in  $\{G_a^+ \leq r\} \cap \{|a|\tilde{\alpha} \leq |p(y) - x| \leq \kappa\}$  and that the leaves of  $\mathcal{F}_a^-$  we consider do not leave this neighborhood.

Let us show that in this region  $\frac{\partial\phi_{a,-}^2/\partial y}{\partial\phi_{a,-}^2/\partial x}$  is  $a$ -close to  $\frac{\partial\phi_{0,-}^2/\partial y}{\partial\phi_{0,-}^2/\partial x}$ .

$$\begin{aligned} \frac{\partial\phi_{a,-}^2}{\partial y}(x, y) &= \frac{\partial}{\partial y} \left( a\phi_{a,-}(y, \frac{p(y)-x}{a}) \right) = a \frac{\partial\phi_{a,-}}{\partial x} \left( y, \frac{p(y)-x}{a} \right) + \\ &\quad p'(y) \frac{\partial\phi_{a,-}}{\partial y} \left( y, \frac{p(y)-x}{a} \right) \\ \frac{\partial\phi_{a,-}^2}{\partial x}(x, y) &= \frac{\partial}{\partial x} \left( a\phi_{a,-}(y, \frac{p(y)-x}{a}) \right) = -\frac{\partial\phi_{a,-}}{\partial y} \end{aligned}$$

Let us do a change of coordinates  $(u, y, v) = (p(y) - x, y, \frac{a}{p(y)-x})$ . First, we introduce the  $u$ -coordinate that measures the distance to parabola. Then we do a blow-up in each line  $y = \text{const}$  that blows-up a cone in  $u, a$ -coordinates, which corresponds to the compliment of  $|a|\alpha$  neighborhood, to a polydisk in  $u, v$  coordinates.

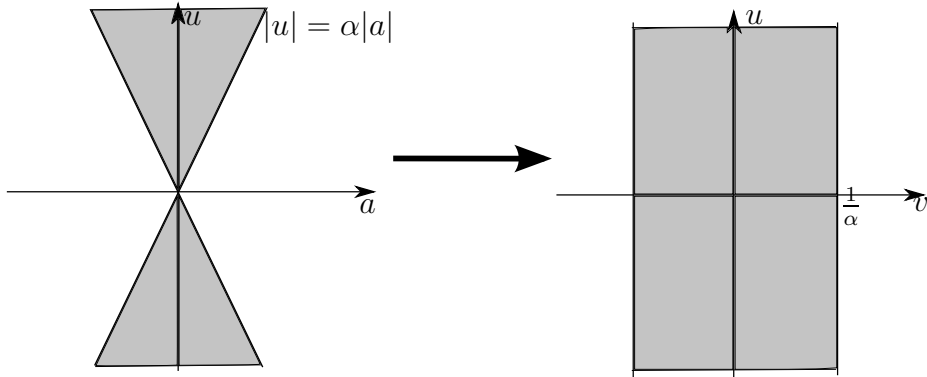


Figure 7: The blow-up

Denote by  $\tilde{\phi}_-^2(u, y, v) = \phi_{uv,-}^2(p(y) - u, y)$ . We show that a function  $\frac{\partial\tilde{\phi}_-^2/\partial y}{\partial\tilde{\phi}_-^2/\partial x}$  lifts to a holomorphic function on the blown-up space  $\{|y| < \alpha, |u| < \epsilon, |v| < \frac{1}{\alpha}\}$ .

$$\begin{aligned}
a \frac{\partial \phi_{a,-}}{\partial x} \left( y, \frac{p(y) - x}{a} \right) &= uv \frac{\partial \phi_{uv,-}}{\partial x} \left( y, \frac{1}{v} \right) \\
\lim_{u \rightarrow 0} uv \frac{\partial \phi_{uv,-}}{\partial x} \left( y, \frac{1}{v} \right) &= 0 \\
\lim_{u \rightarrow 0} \frac{\partial \tilde{\phi}_-^2 / \partial y}{\partial \tilde{\phi}_-^2 / \partial x} &= -p'(y).
\end{aligned}$$

Thus,  $\frac{\partial \phi_{a,-}^2 / \partial y}{\partial \phi_{a,-}^2 / \partial x}$  is  $a$ -close to  $\frac{\partial \phi_{0,-}^2 / \partial y}{\partial \phi_{0,-}^2 / \partial x}$  in  $\Omega \cap \{|p(y) - x| \geq |a|\alpha\}$ .  $\square$

Let  $L_q$  be a leaf of foliation  $\mathcal{F}_a^-$  that passes through a point  $q$ .

**Corollary 9.2.** *There exist  $\kappa$  and  $\delta$  such that for all  $|a| < \delta$  and all*

$$q \in \{G_a^+ \leq r\} \cap \{|p(y) - x| \geq |a|\alpha\} \cap \{|y| \geq \kappa\}$$

*a connected component of  $L_q$  is horizontal-like.*

*Proof.* One takes  $\kappa$  such that every leaf of  $\mathcal{F}_0^-$  that intersects  $|y| = \kappa$  is horizontal like.  $\square$

**Corollary 9.3.** *There exists  $\delta$  so that for all  $|a| < \delta$  and all  $(x, y) \in \Omega$  the tangent plane to the foliation  $\mathcal{F}_a^-$  is not horizontal.*

## 10 Description of $\mathcal{F}_a^+$ .

**Definition 10.1.** *We say that a curve  $C$  in a domain  $B$  is horizontal-like iff there exists a family of  $f_a$ -invariant horizontal cones in  $B$ , such that the tangent lines to  $C$  belong to this family.*

The function  $G_0^+(x, y) = G_p(x)$  is self-similar:

$$G_p(p(x)) = 2G_p(x).$$

Recall that we chose  $r$  so that

$$G_p(0) < r < G_p(p(0)).$$

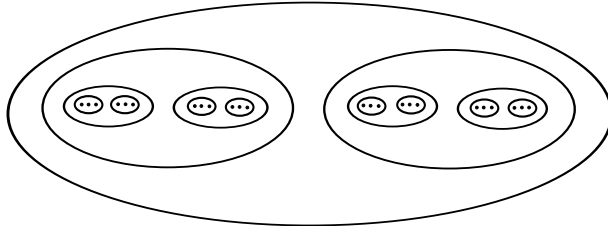


Figure 8: Level sets  $G_p = \frac{r}{2^n}$ .

The picture of the level sets of  $G_p$  inside  $\{G_p < r\}$  is self-similar. Inside each connected component of  $\{G_p = \frac{r}{2^n}\}$  there are exactly two connected components  $\{G_p = \frac{r}{2^{n+1}}\}$ .

There is exactly one critical level in each connected component

$$\frac{r}{2^{n+1}} \leq G_p \leq \frac{r}{2^n}.$$

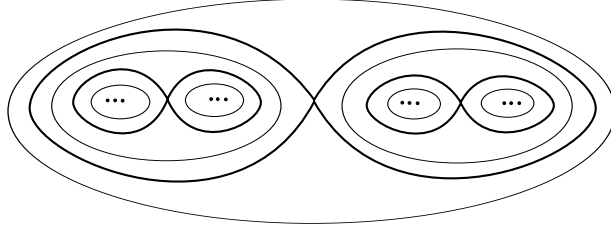


Figure 9: Level sets of  $G_p$ .

In this section we show that the picture persists for the level sets of  $G_a^+$  for small enough  $a$  on each horizontal-like curve in a box

$$\{G_a^+ \leq r\} \cap \{|y| \leq \beta\}.$$

We choose  $\beta = 2 \max\{p(x) \mid G_p(x) \leq r\}$  to apply this construction to the leaves of  $\mathcal{F}_a^-$  that intersect  $\{|y| = \alpha\}$  in  $\{G_a^+ \leq r\}$ .

**Lemma 10.1.** *There exists  $\delta$  such that for  $|a| < \delta$   $\{G_a^+ = r\}$ ,  $\{G_a^+ = \frac{r}{2}\}$  are non-critical on each horizontal-like curve inside  $|y| < \beta$ . Moreover,  $\{G_a^+ = r\}$  has one connected component,  $\{G_a^+ = \frac{r}{2}\}$  has two connected components.*

*Proof.*  $G_a^+$  is a function that depends analytically on  $a$  on  $\mathcal{U}_+$  for all  $a \in D$ . Therefore, since the level sets  $\{G_0^+ = r\}$  and  $\{G_0^+ = \frac{r}{2}\}$  on  $y = b$  are non-degenerate, they remain non-degenerate for small enough  $a$ . The same is true for the number of components.  $\square$

**Lemma 10.2.** *There exists  $\delta$  so that for all  $|a| < \delta$ , there is exactly one critical level of  $G_a^+$  between  $\frac{r}{2}$  and  $r$  on each horizontal-like curve in  $|y| < \beta$ . The corresponding critical point is non-degenerate.*

*Proof.* Let  $C$  be a horizontal-like curve. Then the domain  $\{G_a^+ \leq r\}$  inside  $C$  is parametrized by a planar domain. Therefore, the index of  $\text{grad}G_0^+$  along the boundary of  $\{\frac{r}{2} \leq G_0^+ \leq r\}$  is well-defined and is equal to one. The function  $G_a^+$  depends holomorphically on  $a$ . Thus, the index of  $\text{grad}G_a^+$  along the boundary of  $\{\frac{r}{2} \leq G_a^+ \leq r\}$  is one as well for small  $a$ . Therefore, there is only one critical point inside and it is non-degenerate.  $\square$

**Lemma 10.3.** *There exists  $\delta$  so that for all  $|a| < \delta$   $T_r = \{G_a^+ = r\} \cap W$  is a solid torus,  $\{G_a^+ = \frac{r}{2}\} \cap W$  consists of two solid tori  $T_{r/2}^1, T_{r/2}^2$  (the core coordinate can be chosen to be real-analytic, the disk coordinate holomorphic).*

*Proof.*  $\{G_a^+ = r\} = \{(\phi_{a,+}, y), |\phi_{a,+}| = r, |y| \leq \alpha\}$ .

For  $\{G_a^+ = \frac{r}{2}\} \cap W$  the proof goes the same way.  $\square$

Take any horizontal-like curve. We want to prove by induction that inside each component  $\{G_a^+ = \frac{r}{2^n}\}$  there are exactly two components  $\{G_a^+ = \frac{r}{2^{n+1}}\}$ , and they are non-critical. Therefore, there is exactly one critical level in between, and the corresponding critical point is non-degenerate.

**Lemma 10.4.** *There exists  $\delta$  so that for all  $|a| < \delta$ , the level set  $\{G_a^+ = \frac{r}{2^n}\}$  on each horizontal-like curve in  $|y| < \beta$  is non-critical.*

*Proof.*  $f_a^n(\{G_a^+ \leq \frac{r}{2^n}\})$  is horizontal-like, since it is an image of a horizontal-like curve and it belongs to the box with  $f_a$ -invariant horizontal cones.

$f_a^n(G_a^+ = \frac{r}{2^n}) \in T_r$  and it projects one-to-one to  $x$ -axis. Therefore, it is non-critical.  $\square$

**Lemma 10.5.** *For  $|a| < \delta$  on each horizontal-like curve for every  $n$  there are exactly two level sets  $\{G_a^+ = \frac{r}{2^{n+1}}\}$  inside  $\{G_a^+ = \frac{r}{2^n}\}$  and they are non-critical.*

*Proof.* For every  $n$ ,  $f_a^n(\{G_a^+ \leq \frac{r}{2^n}\})$  is a disk that projects one-to-one to  $x$ -axis with the boundary on  $T_r$ . It intersects  $T_{r/2}^1, T_{r/2}^2$  by a circle each. On the intersection  $\{G_a^+ = \frac{r}{2}\}$ . This proves the lemma.  $\square$

**Corollary 10.1.** *For  $|a| < \delta$  on each horizontal-like curve there is only one critical level  $\{G_a^+ = r'\}$ , where  $\frac{r}{2^{n+1}} < r' < \frac{r}{2^n}$  for each connected component  $\{G_a^+ = \frac{r}{2^n}\}$ .*

The next lemma states that the foliation  $\mathcal{F}_a^+$  is not only vertical-like (projects one-to-one to  $y$ -axis), but is uniformly close to vertical on some thickening of  $\{G_a^+ = \frac{r}{2^n}\}$ .

**Lemma 10.6.** *Fix a small  $\lambda$ . There exists  $\delta$  s.t.  $\forall |a| < \delta$*

$$\left| \frac{\partial \phi_{a,+}^{2^n} / \partial y}{\partial \phi_{a,+}^{2^n} / \partial x} \right| < C|a|$$

on  $\{\frac{r-\lambda}{2^n} \leq G_a^+ \leq \frac{r+\lambda}{2^n}\}$  with  $C$  independent on  $n$ .

*Proof.* On  $\{r - \lambda \leq G_a^+ \leq r + \lambda\}$  the inequality follows from the fact that  $\phi_{a,+}$  is a holomorphic function in  $a$ .

The leaves of foliation  $\mathcal{F}_a^+$  in  $\{\frac{r-\lambda}{2^{n+1}} \leq G_a^+ \leq \frac{r+\lambda}{2^{n+1}}\} \cap W$  are preimages under  $f_a^{-1}$  of the leaves of  $\mathcal{F}_a^+$  in  $\{\frac{r-\lambda}{2^n} \leq G_a^+ \leq \frac{r+\lambda}{2^n}\} \cap W$ . Therefore, by induction we check that they belong to  $f_a^{-1}$ -invariant vertical cones.  $\square$

## 11 Critical Locus in $\Omega$ .

Recall that

$$\Omega_a = \{G_a^+ \leq r\} \cap \{|y| \leq \alpha\} \cap \{|p(y) - x| \geq |a|\alpha\}.$$

See Figure 3. We will omit the subscript  $a$ , when it is clear from the context which Hénon mapping is under consideration.

In this section we subdivide  $\Omega_a$  into countably many regions. Note that the level sets  $\{G_a^+ = \frac{r}{2^n}\}$  on each horizontal line  $y = k$ ,  $|k| < \alpha$  depend continuously with respect to parameter  $a$ . Therefore, the partition of

$$\left\{\frac{r}{2^{n+1}} \leq G_a^+ \leq \frac{r}{2^n}\right\} \cap \Omega_a \quad (16)$$

into connected components depends continuously on  $a$ .

Therefore, it is enough to enumerate the connected components of  $\Omega_0$ . We call  $\Omega_0^{\xi_n}$  the connected component of

$$\left\{\frac{r}{2^{n+1}} \leq G_0^+ \leq \frac{r}{2^n}\right\} \cap \Omega_0,$$

which contains the critical line  $x = \xi_n$ . We call  $\Omega_a^{\xi_n}$  the connected component of (16) that is the continuation of  $\Omega_0^{\xi_n}$ .

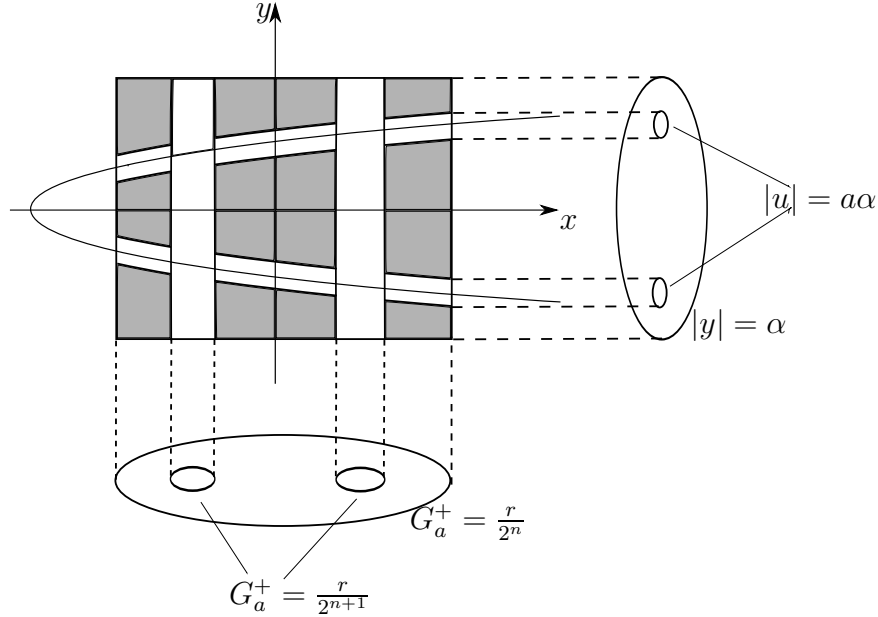


Figure 10: Domain  $\Omega_a^{\xi_n}$

**Lemma 11.1.** *Fix a small  $\lambda$ . There exists  $\delta$  (independent on  $n$ ) so that for all  $|a| < \delta$  the critical locus in each connected component of*

$$\left\{\frac{r-\lambda}{2^n} \leq G_a^+ \leq \frac{r+\lambda}{2^n}\right\} \cap \{|y| \leq \alpha\}$$

*is an annulus which is a graph of function  $y(\phi_{a,+})$ .*

*Proof.* Fix  $\gamma$  so that the set  $\{G_0^+ \leq r\} \cap \{|y| \leq \gamma\}$  is disjoint from  $C_p$ . Then for all  $|a| < \delta'$  the set  $\{G_a^+ \leq r\} \cap \{|y| \leq \gamma\}$  is disjoint from  $C_p$ .

The inverse function theorem implies that  $\forall n \exists \delta_n$  s.t.  $|a| < \delta_n$  the critical locus in  $\{\frac{r-\lambda}{2^n} \leq G_a^+ \leq \frac{r+\lambda}{2^n}\} \cap \{|y| \leq \gamma\}$  is an annulus on the component that is a perturbation of  $y = 0$ .

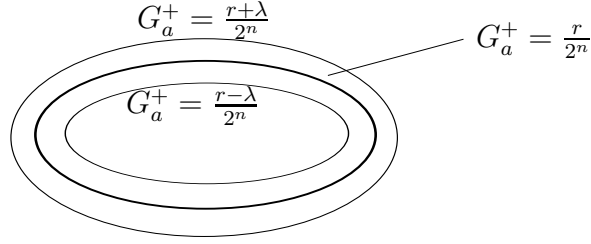


Figure 11: The thickening of  $G_a^+ = \frac{r}{2^n}$

Since in the region

$$\left\{ \frac{r-\lambda}{2^n} \leq G_a^+ \leq \frac{r+\lambda}{2^n} \right\} \cap \{ \gamma < |y| < \alpha \} \cap \{ |p(y) - x| \geq a|\alpha| \}$$

the foliation  $\mathcal{F}_a^-$  is  $a$ -close to  $\mathcal{F}_0^-$  and by Lemma 10.6  $\mathcal{F}_a^+$  is almost vertical, for  $|a| < \delta''$  there are no points of the critical locus in this region.

Take some  $n$  and fix some connected component

$$\left\{ \frac{r-\lambda}{2^n} \leq G_a^+ \leq \frac{r+\lambda}{2^n} \right\} \cap \{ |y| \leq \gamma \}.$$

Let us show that the critical annuli in this connected component persists for all  $|a| \leq \delta''$ .  $\mathcal{C}_a$  is a analytic set in the region

$$\left\{ (x, y, a) \mid \frac{r-\lambda}{2^n} \leq G_a^+(x, y) \leq \frac{r+\lambda}{2^n}, |y| < \gamma, |a| < \delta'' \right\}$$

There are no zeroes of  $\text{grad}G_a^-$  on the leaves of  $\mathcal{F}_a^+$  on curve  $|y| = \gamma$ . Therefore, index of  $\text{grad}G_a^-$  is constant. At  $a = 0$  it is equal to 1. Thus, the critical annulus in

$$\left\{ (x, y, a) \mid \frac{r-\lambda}{2^n} \leq G_a^+(x, y) \leq \frac{r+\lambda}{2^n}, |y| < \gamma, |a| < \delta'' \right\}$$

persists. □

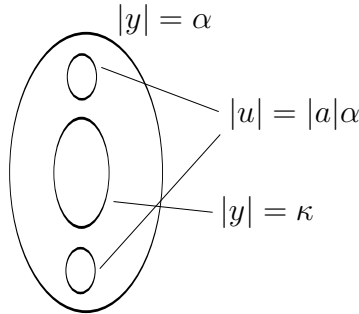


Figure 12: The critical locus in  $\{ |a|\alpha \leq |p(y) - x| \} \cap \{ \frac{r}{2^{n+1}} \leq G_a^+ \leq \frac{r}{2^n} \} \cap \{ |y| \geq \kappa \}$

**Lemma 11.2.** *There exist  $\kappa$  and  $\delta$  such that for all  $|a| < \delta$  the critical locus in each connected component  $\{ |a|\alpha \leq |p(y) - x| \} \cap \{ \frac{r}{2^{n+1}} \leq G_a^+ \leq \frac{r}{2^n} \} \cap \{ |y| \geq \kappa \}$  is an annulus with two holes and is a graph of function  $y(\phi_a, -)$ .*

*Proof.* It follows from Corollary 9.2 □

We review the notion of the Milnor number of a singularity that we will need for the proof of the next lemma.

Let  $z_0$  be an isolated singular point of a holomorphic function  $f : \mathbb{C}^n \rightarrow \mathbb{C}$ . Let  $B_\rho$  be a disk of radius  $\rho$  around  $z_0$ . One can take  $\rho$  and  $\epsilon$  small enough, so that the non-singular level sets  $\{f = \epsilon\} \cap B_\rho$  have homotopy type of a bouquet of spheres of dimension  $(n - 1)$  ([AGV87], Section 2.1).

**Definition 11.1.** *The number of spheres in  $\{f = \epsilon\} \cap B_\rho$  is called the Milnor number of the singular point  $z_0$ .*

**Lemma 11.3** ([AGV87], Section 2.1). *The Milnor number of the singular point  $z_0$  is equal to  $\mathcal{O}_{z_0} / \langle f_1, \dots, f_n \rangle$ , where  $\mathcal{O}_{z_0}$  are functions regular in a neighborhood of  $z_0$  and  $\langle f_1, \dots, f_n \rangle$  is the ideal in  $\mathcal{O}_{z_0}$ , generated by the functions  $f_1, \dots, f_n$ .*

**Lemma 11.4.** *For  $|a| < \delta$  the critical locus in  $\Omega_a$  is a smooth curve. In  $\Omega_a^{\xi_n}$  it is a connected sum of two disks  $D_1$  and  $D_2$  with two holes. The boundary of  $D_1$  belongs to  $\{|y| = \alpha\}$ , and the holes of  $D_1$  have boundaries on  $\{|u| = |a|\alpha\}$ . The boundary of  $D_2$  belongs to  $\{G_a^+ = \frac{r}{2^n}\}$  and the holes to  $\{G_a^+ = \frac{r}{2^{n+1}}\}$ .*

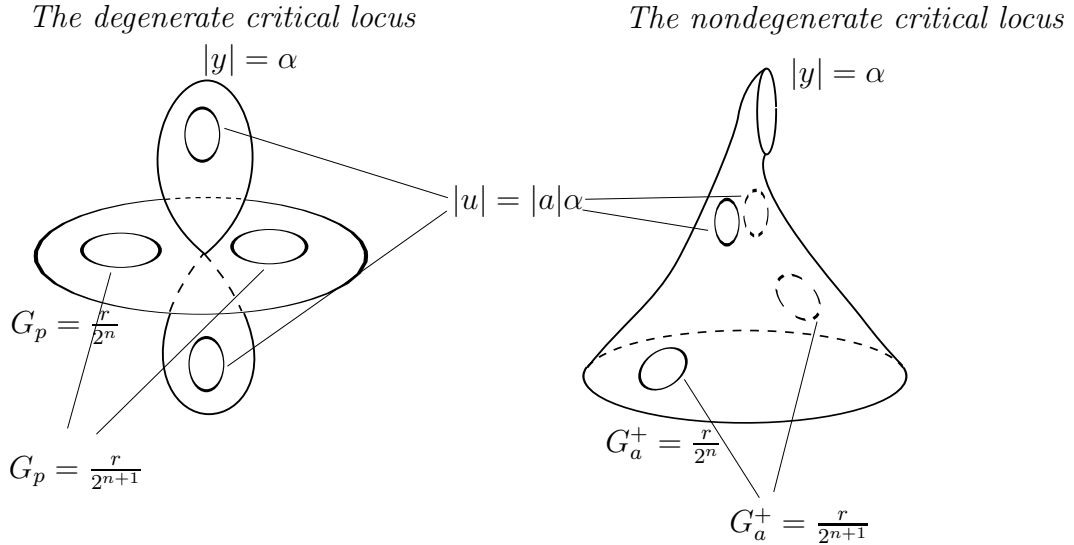


Figure 13: The critical locus in  $\Omega_a^{\xi_n}$

**Note 11.1.** *As  $a \rightarrow 0$ , the curve degenerates to  $(x - \xi_n)y = 0$ . The holes of  $D_1$  degenerate to points  $(0, \xi_{n+1}), (0, \xi'_{n+1})$ , where  $p(\xi'_{n+1}) = p(\xi_{n+1}) = \xi_n$ . The holes of  $D_2$  tend to circles  $\{(x, 0) \mid G_p(x) = \frac{r}{2^{n+1}}\}$ .*

*Proof.* Fix  $\xi_n$ . The critical locus in  $\Omega_0^{\xi_n}$  is a singular curve. It is a union of two intersecting lines  $y = 0$  and  $x = \xi_n$ . By Lemma 11.3 the Milnor number of this singularity is 1. The critical locus  $\mathcal{C}_a$  is the zero level set of  $w_a(x, y)$ . By Lemmas 11.1, 11.2 there exists  $\delta$  so that the critical locus in  $\Omega_a^{\xi_n}$  for  $|a| < \delta$  is transverse to the boundary. Thus, there is  $\epsilon$  so that the

level sets of  $w_a(x, y) = c$ ,  $|c| < \epsilon$  are transverse to the boundary of  $\Omega_a^{\xi_n}$  for all  $|a| < \delta$ . Let us consider  $(w_a(x, y), a) : \Omega_a \times D_\delta \rightarrow \mathbb{C} \times D_\delta$ . By Ehresmann's Fibration Theorem the level sets of this function form a locally trivial fibration. Thus, they are homeomorphic one to the other. By Lemmas 11.1 and 11.2, the critical locus intersects the boundary exactly as prescribed. Therefore, it is enough to show that the critical locus  $\mathcal{C}_a \Omega_a^{\xi_n}$  for  $|a| < \delta$  is non-degenerate.

Suppose there are critical level sets of the function  $w_a(x, y)$ . Then the Milnor number of the corresponding singularity is 1. Therefore, the singularity is locally an intersection of two disks. It is easy to see that one of this curves should be a 'horizontal' curve, i.e. project one-to-one to  $y$ -axis. Its boundary belongs to  $\{|y| = \alpha\}$ . This curve necessarily intersects the 'vertical' curve on which the tangent line to  $\mathcal{F}_a^+$  are horizontal. But then on the point of intersection the tangent line to the foliation  $\mathcal{F}_a^-$  is horizontal. That is impossible by Lemma 9.3.

Therefore, the critical locus in  $\Omega_a^{\xi_n}$  is noncritical for  $a \neq 0$ . The conclusion of the lemma follows.  $\square$

## 12 Extension of the critical locus up to $|a|\alpha$ -neighborhood of parabola.

**Lemma 12.1.** *There exists  $\delta$  so that for all  $|a| < \delta$  the component of the critical locus, that is a perturbation of  $y = 0$  can be extended up to  $|a|\alpha$ -neighborhood of parabola as a graph of function  $y(x)$ .*

*Proof.* The domain of definition of  $\phi_{a,-}^2$  is  $f_a(V_-) = \{(x, y) \mid |p(y) - x| \geq |a|\alpha, |p(y) - x| \geq |a||y|\}$ .

Therefore, in  $W$  the domain of definition of  $\phi_{a,-}^2$  is  $\{|p(y) - x| \geq |a|\alpha\}$ .

Denote  $u = p(y) - x$ .

Consider new variables  $(u, y, v) = (u, y, \frac{a}{u})$ . Denote by  $\pi$  the projection

$$\pi : (u, y, v) \mapsto (u, y, uv).$$

Notice that  $\pi^{-1}$  blows up a point  $u = 0$  on each line  $y = y_0$ .

Let us prove that one can extend the critical locus to

$$S = \{|u| < \beta, |y| < \epsilon, |v| < \frac{1}{\alpha}\}.$$

Note that  $\phi_{a,+}$  can be extended to  $S$  since it's a well-defined holomorphic function in  $\pi(S)$ .  $\phi_{a,-}^2(x, y) = uv\phi_{uv,-}(y, 1/v)$ .

By [LR]  $\frac{\phi_{a,-}(x,y)}{y}$  extends to be a holomorphic function to a neighborhood of  $y = \infty$ .

Therefore  $\phi_{a,-}^2$  extends to  $S$ . Moreover, notice that on blown-up lines  $\phi_{uv,-}^2 = 0$ .

The critical locus is given by the zeroes of the function

$$w = \frac{d\phi_{a,+} \wedge d\phi_{a,-}^2 \wedge da}{dx \wedge dy \wedge da}.$$

Let

$$\tilde{w} = -\frac{d\phi_{uv,+} \wedge d\phi_{uv,-}^2 \wedge dv}{udu \wedge dy \wedge dv}.$$



Notice that  $\tilde{w} = w \circ \pi$ .

$$\tilde{w} = -uv \frac{\partial \phi_{uv,+}}{\partial x}(p(y) - u, y) \frac{\partial \phi_{uv,-}}{\partial x}(y, \frac{1}{v}) +$$

$$\left( \frac{\partial \phi_{uv,+}}{\partial x}(p(y) - u, y) p'(y) + \frac{\partial \phi_{uv,+}}{\partial y}(p(y) - u, y) \right) \frac{\partial \phi_{uv,-}}{\partial y}(y, \frac{1}{v})$$

Note that  $\frac{\phi_{a,-}}{y} = 1 + aH(x, \frac{1}{y}, a)$ , where  $H$  is a holomorphic function in some neighborhood of  $(x, 0, 0)$ .

$$\lim_{u \rightarrow 0} v \frac{\partial \phi_{uv,-}}{\partial x} = 0$$

$$\lim_{u \rightarrow 0} \frac{\partial \phi_{uv,-}}{\partial y} = 1$$

$$\tilde{w}(0, y, v) = b'_p(p(y))p'(y)$$

Note that for all  $|v| < \frac{1}{\alpha} |y| < \alpha$   $\tilde{w}(0, y, v) = 0$  only when  $y = 0$  and the zero is not multiple. Therefore, by Weierstrass theorem for  $v$ ,  $y = g(u, v)$ .  $\square$

### 13 Description of the critical locus.

Fix some  $\epsilon$ . Denote

$$\hat{\Omega}_1 = \{(x, y) \in \mathbb{C}P^2 \mid |y| < \epsilon, |p(y) - x| \geq |a|\alpha, G_a^+(x, y) \geq r\}.$$

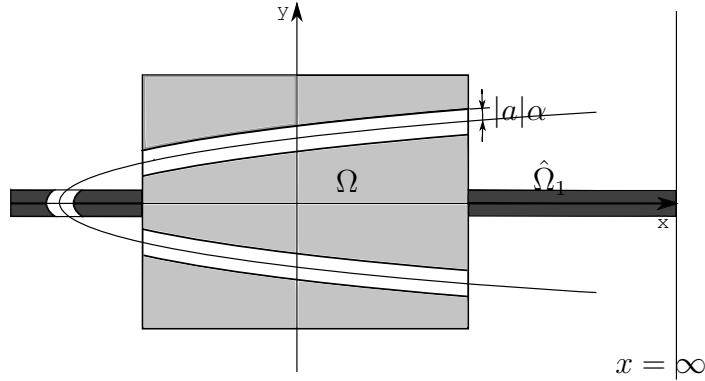


Figure 14: Domain  $\hat{\Omega}_1$

**Lemma 13.1.** *There exists  $\delta$  such that  $\forall |a| < \delta$  the critical locus  $\mathcal{C}_a$  in  $\hat{\Omega}_1$  is a punctured disk, with a hole removed. The puncture is at the point  $(\infty, 0)$ , the boundary of the hole belongs to  $\{|p(y) - x| = |a|\alpha\}$ .*

*Proof.* By Lemma 6.1 the degenerate critical locus  $\mathcal{C}_0$  in  $\hat{\Omega}_1$  is  $y = 0$ , with point  $x = c$  removed. By Lemma 7.3 it persists in some neighborhood of  $x = \infty$ . By the inverse function theorem it can be extended as a graph of function  $y(x)$  to  $\{G_a^+(x, y) \geq r\}$  excluding  $\epsilon$ -neighborhood of  $\mathcal{C}_p$ . By Lemma 12.1 it can be extended up to  $|a|\alpha$ -neighborhood of  $\mathcal{C}_p$ .  $\square$

In the following 3 lemmas we show that every component of the critical locus intersects  $\Omega$ . It follows from Lemma 11.4 that it consists of one component.

**Lemma 13.2.** *Let  $C_a$  be a component of the critical locus  $\mathcal{C}_a$ . Then there exists a point on  $\partial C_a$  that belongs to  $J_a^+ \cup J_a^-$ .*

*Proof.* Consider  $(G_a^+ + G_a^-)$ . This function is pluriharmonic and strictly positive in  $U_a^+ \cap U_a^-$ . Therefore,  $\inf(G_a^+ + G_a^-)$  cannot be attained at the interior point.  $\square$

**Lemma 13.3.** *There exists  $\delta$  such that for all  $|a| < \delta$   $J_a^+ \cap \Omega$  is a fundamental domain for  $J_a^+ \setminus J_a$ .*

*Proof.* [HOV95] There it is prove for Hénon mappings that are perturbations of hyperbolic polynomials with connected Julia set. In the connected case the proof is the same.  $\square$

**Lemma 13.4.** *Suppose  $|a| < \delta$  and  $C_a$  is a component of the critical locus. Then there exists an iterate of  $f_a^n(C_a)$  that intersects  $\Omega$ .*

*Proof.* Lemma 13.2 states that there exists  $z \in (J_a^+ \cup J_a^-) \cap (\partial C_a)$ . Suppose  $z \in J_a^-$ . Take a sequence of points  $z_n \in C_a$ ,  $z_n \rightarrow z$ .

$G_a^-(z_n) \rightarrow 0$  as  $n \rightarrow \infty$ ;  $G_a^+$  is bounded.

For every  $n$  there exists  $k_n$  such that  $1 < G_a^-(f_a^{-k_n}(z_n)) \leq 2$ . Then  $k_n \rightarrow \infty$ .

$$G_a^+(f_a^{-k_n}(z_n)) \rightarrow 0.$$

Taking a subsequence one may assume  $f^{-k_n}(z_n) \rightarrow z$ .  $z \in J_a^+ \setminus J_a$ . Since by Lemma 13.3  $f_a^m(z) \in J_a^+ \cap \Omega$ , there exists an iterate of  $C_a$  that intersects  $\Omega$ .  $\square$

Below we will prove that  $\mathcal{C}_a \cap (\Omega \cup \hat{\Omega}_1)$  is a fundamental domain of the critical locus.

The next lemma is a variation of the Classical Poincaré's Polyhedron Theorem on the fundamental domain under the group action ([M88], IV.H).

**Lemma 13.5.** *Let  $C$  be a Riemann surface. Let  $f : C \rightarrow C$  be an automorphism. Let  $D \subset C$  be an open domain. Suppose*

1.  $f^n(D) \cap D = \emptyset$ ;
2.  $\partial D \subset C$  consists of countably many smooth curves  $\gamma_i$ ;
3.  $\gamma_i$  are being paired by the map  $f$ ;
4. for any sequence  $\{z_n\}$ ,  $z_n \in D$ ,  $f^n(z_n)$  does not have an accumulation point in  $S$

*Then  $D$  is a fundamental domain of  $S$  for the map  $f_a$ .*

*Proof.* Denote by  $S$  a Riemann surface obtained by gluing the images  $f^n(D)$  to  $D$ . The natural map  $i : S \rightarrow C$  is injective by (1). Let us prove that it is proper. If it is not, then there exists a sequence of points  $z_n \in S$  so that  $z_n \rightarrow \partial S$ ,  $i(z_n) \rightarrow z \in C$ . Take  $z'_n \in D$  so that  $z_n = f^{k_n}(z'_n)$ . If there are infinitely many same  $k_n$ 's. Then the sequence  $\{z'_n\}$  has an accumulation point in  $D$ . That contradicts (2) and (3).

Therefore, taking a subsequence one can assume  $\{k_n\}$  increase. This contradicts (4).  $\square$

**Lemma 13.6.** *The images of  $(W \setminus \{|p(y) - x| \geq a|\alpha\}) \cup \{|y| < \epsilon\}$  under the map  $f_a$  are disjoint.*

*Proof.* 1.

$$f_a^n(W \setminus \{|p(y) - x| \leq |a|\alpha\}) \cap [W \setminus \{|p(y) - x| \leq |a|\alpha\}] = \emptyset.$$

$f_a(W) \subset W \cup V_+$ ,  $f_a(V_+) \subset V_+$ , therefore,  $(f_a^n(W) \cap W) \subset (f_a(W) \cap W) \subset \{|p(y) - x| \leq |a|\alpha\}$

2.

$$f_a^n(W) \cap \{|y| < \epsilon\} \cap \{|x| \geq \alpha\} = \emptyset;$$

Let  $(x, y) \in W$ . Suppose  $f_a^n(x, y) \in V_+$  and  $n$  is the smallest number such that this happens. Then  $|p(x_n) - y_n| \leq |a|\alpha$ . Thus,  $|y_n| > \epsilon$ . For points in  $V_+$ ,  $|y_{n+1}| > |y_n|$ .

3.

$$f_a^n(\{|y| < \epsilon\} \cap \{|x| > \alpha\}) \cap \{|y| < \epsilon\} \cap \{|x| > \alpha\} = \emptyset.$$

Suppose  $(x, y) \in \{|y| < \epsilon\} \cap \{|x| > \alpha\}$ . Since  $|y| < \epsilon$ ,  $f_a(x, y)$  belongs to  $|a|\epsilon$ -neighborhood of parabola. Since  $(x, y) \in V_+$ ,  $f_a(x, y) \in V_+$ . Therefore,  $|y_1| > \epsilon$ .  $|y_{n+1}| > |y_n| > \epsilon$ , since  $(x_n, y_n) \in V_+$ .

4.

$$f_a^n(\{|y| < \epsilon\} \cap \{|x| > \alpha\}) \cap W = \emptyset.$$

This is true, since  $\{|y| < \epsilon\} \cap \{|x| > \alpha\} \subset V_+$ , and  $f_a(V_+) \subset V_+$ .

□

**Lemma 13.7.** *There exists  $\delta$  such that for all  $|a| < \delta$  the critical locus in  $\Omega \cup \hat{\Omega}_1$  forms a fundamental domain of  $\mathcal{C}_a$ .*

*Proof.* Denote by  $D = \mathcal{C}_a \cap (\Omega \cup \hat{\Omega}_1)$ . Let us check that the conditions of the Lemma 13.5 are satisfied. Since  $\Omega \cup \hat{\Omega} \subset (W \setminus \{|p(y) - x| \leq |a|\alpha\}) \cup \{|y| \leq \epsilon\}$ . All the forward images of  $D$  are disjoint from it. Therefore, (1) is satisfied.

The boundary of  $D$  in each  $\Omega_a^{\xi_k}$  consists of vertical circles:  $|y| = \alpha$  and  $|p(y) - x| = |a|\alpha$ . The circles  $|p(y) - x| = |a|\alpha$  are parametrized by  $\xi'_{k+1}$ ,  $\xi''_{k+1}$ ,  $(p(\xi'_{k+1}) = p(\xi''_{k+1}) = \xi_k)$ , which stands for the approximate value of  $y$  on the circle.

There is also one horizontal circle  $|p(y) - x| = |a|\alpha$  on a perturbation of  $y = 0$ .

$f_a$  maps  $|y| = \alpha$  in  $\Omega_a^{\xi_k}$  to  $|p(y) - x| = |a|\alpha$  in  $\Omega_a^{p(\xi_k)}$ , parametrized by  $\xi_k$

$f_a$  maps  $|y| = \alpha$  on perturbation of  $x = 0$  to a horizontal circle  $|p(y) - x| = |a|\alpha$ .

Therefore, boundary components of  $D$  are being paired by  $f_a$ . And condition (2) and (3) of Lemma 13.5 are satisfied.

Suppose there exists a sequence of  $z_n \in D$  so that  $f_a^n(z_n) \rightarrow \partial S$ ,  $z_n \rightarrow z_* \in C$ .

If  $\{z_n\}$  has an accumulation point  $z$  in  $D$ . Then  $f_a^n(z) \rightarrow z^*$ . Which is impossible, since  $f_a^n(z) \rightarrow \infty$  in  $\mathbb{C}^2$ .

If  $\{z_n\}$  does not have an accumulation point  $z$  in  $D$ . Then  $z_n$  accumulate to  $z \in J_a^+$ . Therefore,  $f_a^{kn}(z') \rightarrow z \in J_a$ . That contradicts to  $z \in C$ .

So condition (4) is satisfied as well. Therefore,  $D$  is a fundamental domain of the critical locus  $\mathcal{C}_a$ . □

*Proof of theorem 2.1.* To obtain a description in terms of truncated spheres we do a dynamical regluing. We fix some small  $\epsilon$  and cut the fundamental domain of  $\mathcal{C}_a$  along the hypersurface  $|y| = \epsilon$ . We call the connected component of the perturbation of  $y = 0$  main component. The rest of components we call handles:  $H_{\xi_k}$  is a component in  $\Omega_{\xi_k}$

The boundary of  $H_{\xi_k}$  consists of four circles:

$|y| = \alpha$ ,  $|y| = \epsilon$  and two connected components of  $|p(y) - x| = |a|\alpha$ , parametrized as previously by  $\xi'_{k+1}$ ,  $\xi''_{k+1}$ , where  $p(\xi'_{k+1}) = p(\xi''_{k+1}) = \xi_k$ .

We glue  $H_{\xi_k}$  to the main component by the map  $f_a^k$ . Under this procedure the boundary  $|y| = \alpha$  of  $H_{\xi_k}$  is being glued to  $|p(y) - x| = |a|\alpha$ -boundary of  $H_{p(\xi_k)}$ , parametrized by  $\xi_k$ . By “generalized uniformization theorem”, it can be straighten to be a sphere.

The fundamental domain of the critical locus  $\mathcal{C}_a$ , obtained after regluing, is a truncated sphere.

The preimages of 0 under the map  $p^k$  are parametrized by  $k$ -strings of 0 and 1's.

Let  $\alpha_k$  be a  $k$ -string that parametrizes  $\xi_k$ .  $V_{\alpha_k}$  is the interior of  $|y| = \epsilon$  obtained by cutting  $H_{\xi_k}$  from the main component.  $U_{\alpha_k}$  is the interior of  $f_a^k(|y| = \epsilon)$  on  $f_a^k(H_{\xi_k})$ . The rest of the boundary corresponds to  $G_a^+ = 0$  and  $G_a^- = 0$ . Therefore, it is parametrized by two Cantor sets  $\Sigma, \Omega$ .

We show these are true Cantor sets by moduli counting.

**Lemma 13.8.** *Let  $U_1 \supset U_2 \supset \dots U_n \supset \dots$  be a sequence of open domains, such that  $U_i \setminus U_{i-1}$  is an annulus with moduli  $M_i \geq M$ . Then  $\bigcap \bar{U}_i$  is a point.*

Take a point  $\sigma \in \Sigma$ . Let  $M_1$  be a modulus of the annulus  $\{r - \lambda \leq G_a^+ \leq r\}$  on  $y = 0$ . Then  $M_1$  is a modulus of the annulus  $\{\frac{r-\lambda}{2^n} \leq G_a^+ \leq \frac{r}{2^n}\}$ . The main component project one-to-one to each of this annulus. There is a sequence of connected component  $\{\frac{r-\lambda}{2^n} \leq G_a^+ \leq \frac{r}{2^n}\}$  that bound a hole parametrized by  $\sigma$ . Therefore,  $\sigma$  is a point.  $\Sigma$  is a Cantor set.

Fix some small  $\epsilon$ . Let  $M_1$  be a modulus of the annulus  $\{\alpha - \epsilon \leq |y| \leq \alpha\}$ . All handles  $H_{\xi_k}$  project one-to-one to this annulus. So by the same argument we get that  $\Omega$  is a Cantor set.  $\square$

## 14 List of standard notations

We provide a list of notations here as a reference.

| Notation                           | Section | Meaning   |
|------------------------------------|---------|---|
| $f_a$                              | 1       | Hénon mapping under consideration   |
| $a$                                | 1       | Jacobian of Hénon mapping under consideration   |
| $p(x)$                             | 1       | polynomial used in the definition of Hénon mapping  |
| $U_a^+, U_a^-$                     | 1       | set of points whose orbits under forward (backward) iteration of $f_a$ escape to infinity   |
| $K_a^+, K_a^-$                     | 1       | set of points whose orbits under forward (backward) iteration of $f_a$ remain bounded   |
| $J_a^+, J_a^-$                     | 1       | boundaries of $K_a^+, K_a^-$  |
| $J_a$                              | 1       | $J_a^+ \cap J_a^-$  |
| $G_a^+, G_a^-$                     | 1       | pluriharmonic functions that measure the rate of escape to infinity under forward (backward) iterates of $f_a$  |
| $\alpha$                           | 4       | parameter used in the definition of $V_+, V_-$  |
| $V_+, V_-$                         | 4       | $\{ x  > \alpha,  x  >  y \}, \{ y  > \alpha,  y  >  x \}$ – regions which describe the large scale behavior of the Hénon map                               |
| $W$                                | 4       | $\{ x  \leq \alpha,  y  \leq \alpha\}$  |
| $D_R$                              | 4       | the disk of radius $R$ in the parameter space, used to define $V_+$ and $V_-$   |
| $\phi_{a,+}, \phi_{a,-}$           | 4       | holomorphic functions that semiconjugate dynamics in $V_+, V_-$ to $z \rightarrow z^2, z \rightarrow z^2/a$   |
| $s_k^+, s_k^-$                     | 4       | auxillary functions, used to study $\phi_{a,+}$ and $\phi_{a,-}$  |
| $C(p)$                             | 4       | the curve $y = p(x)$ , this is $J_0^-$  |
| $G_p$                              | 3       | Green function for the map $x \rightarrow p(x)$   |
| $b_p$                              | 4       | Böttcher coordinate for the map $x \rightarrow p(x)$  |
| $\mathcal{F}_a^+, \mathcal{F}_a^-$ | 1       | foliations of $U_a^+, U_a^-$  |
| $\mathcal{C}_a$                    | 1       | the critical locus, the set of tangencies between foliations $\mathcal{F}_a^+$ and $\mathcal{F}_a^-$  |
| $(x_n, y_n)$                       | 4       | $(x_n, y_n) = f_a(x, y)$  |
| $\mathcal{D}_{k,+}$                | 4       | the domain of definition of $\phi_{a,+}^k$  |
| $\mathcal{D}_{k,-}$                | 4       | the domain of definition of $\phi_{a,-}^k$  |
| $u$                                |         | $u = p(y) - x$ , measures the distance from a point $(x, y)$ to $C(p)$  |
| $\Omega_a$                         | 3       | $\{G_a^+ \leq r\} \cap \{ y  \leq \alpha\} \cap \{ p(y) - x  < \alpha a \}$ , the domain that does not intersect with its images under $f_a$ and $f_a^{-1}$ |
| $\hat{\Omega}_1$                   | 13      | $\{(x, y) \in \mathbb{C}\mathbb{P}^2 \mid  y  < \epsilon,  x  \geq \alpha\}$  |

## References

- [AGV87] Arnol'd, V.I.; Gusein-Zade, S.M.; Varchenko, A.N. Singularities of differentialbe maps. Vol.II. Monodromy and asymptotics of integrals, under the editorship of V.I. Arnold, translated from Russian by Hugh Porteous. Monographs in Mathematics, 83. Birhauser Boston, Inc., Boston, MA, 1987
- [BLS93] Eric Bedford, Mikhail Lyubich, and John Smillie. Polynomial diffeomorphisms of  $\mathbb{C}^2$ . IV. The measure of maximal entropy and laminar currents. *Invent. Math.*, 112(1):77-125, 1993.
- [BS91a] Eric Bedford and John Smillie. Polynomial diffeomorphisms of  $\mathbb{C}^2$  : currents, equilibrium measure and hyperbolicity. *Invent. Math.*, 103(1):69-99, 1991
- [BS91b] Eric Bedford and John Smillie. Polynomial diffeomorphisms of  $\mathbb{C}^2$ . II, Stable manifolds and recurrence. *J. Amer. Math. Soc.*, 4(4):657-679, 1991.
- [BS92] Eric Bedford and John Smillie. Polynomial diffeomorphisms of  $\mathbb{C}^2$ . III. Ergodicity, exponents and entropy of the equilibrium measure. *Math. Ann.*, 294(3): 395-420, 1992
- [BS98a] Eric Bedford and John Smillie. Polynomial diffeomorphism of  $\mathbb{C}^2$ . V. Critical points and Lyapunov exponents. *J. Geom. Anal.*, 8(3):349-383, 1998.
- [BS98b] Eric Bedford and John Smillie. Polynomial diffeomorphisms of  $\mathbb{C}^2$ . VI. Connectivity of *J. Ann. of Math. (2)*, 148(2): 695-735, 1998
- [BS99] Eric Bedford and John Smillie. Polynomial diffeomorphisms of  $\mathbb{C}^2$ . VII. Hyperbolicity and external rays. *Ann. Sci. Ecole Norm. Sup.(4)*, 32(4):455-497, 1999
- [FM89] Shmuel Friedland and John Milnor. Dynamical properties of plane polynomial automorphisms. *Ergodic Theory Dynam. Systems*, 9(1):67-99, 1989.
- [FS92] John Erik Fornæss and Nessim Sibony. Complex Hénon mappings in  $\mathbb{C}^2$  and Fatou-Bieberbach domains. *Duke Mathematical Journal*, 65(2):345-380, 1992
- [HOV94] John H. Hubbard and Ralph W. Oberste-Vorth. Hénon mappings in the complex domain. I. The global topology of dynamical space. *Inst. Hautes Études Sci. Publ. Math.*, (79):5-46, 1994.
- [HOV95] John H. Hubbard and Ralph W. Oberste-Vorth. Hénon mappings in the complex domain. II. Projective and inductive limits of polynomials. *Real and Complex dynamical systems (Hillerod, 1993)*, volume 464 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pp. 89-132, Kluwer Acad. Publ., Dordrecht, 1995.
- [M88] B. Maskit, *Kleinian Groups*. Springer-Verlag, 1988
- [LR] Mikhail Lyubich, John Robertson. The critical locus and rigidity of foliations of complex Hénon maps. Unpublished.