A NOTE ON HYPERBOLIC LEAVES AND WILD LAMINATIONS OF RATIONAL FUNCTIONS

JEREMY KAHN, MIKHAIL LYUBICH, AND LASSE REMPE

Abstract. We study the affine orbifold laminations that were constructed in [LM]. An important question left open in [LM] is whether these laminations are always locally compact. We show that this is not the case.

The counterexample we construct has the property that the *regular leaf space* contains (many) hyperbolic leaves that intersect the Julia set; whether this can happen is itself a question raised in [LM].

1. Introduction

Providing a new line in the "Sullivan dictionary" between rational maps and Kleinian groups, the article [LM] associated an "Affine Orbifold Lamination" $\mathcal{A}_f$ to any rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. (See Section 2 for an overview of the definitions.)

This lamination is particularly useful when it is locally compact; e.g. this condition allows the construction of transverse conformal measures and invariant measures on the lamination [KL]. Local compactness is satisfied in certain important cases, including geometrically finite rational maps and Feigenbaum-like quadratic polynomials.[1] The question whether the lamination $\mathcal{A}_f$ is always locally compact was raised in [LM, §10, Question 9].

1.1. Theorem (Failure of local compactness)**.**
There exists a quadratic polynomial f whose affine orbifold lamination $\mathcal{A}_f$ is not locally compact.

Our proof of Theorem 1.1 is related to another question from [LM]. As we review in Section 2, the *regular leaf space* $\mathcal{R}_f$ consists of those backward orbits under f along which some disk can be pulled back with a bounded amount of branching. The path-connected components (*leaves*) of $\mathcal{R}_f$ have a natural Riemann surface structure; in many cases all such leaves are parabolic planes. It was asked in [LM, §10, Question 2] whether rational maps can have leaves that are hyperbolic but do not arise from Siegel disks or Herman rings. Hubbard (personal communication) was the first to suggest an example with this property (see the remark at the end of Section 3). The hyperbolic leaves of this example lie over a single Fatou component, so the question remained whether hyperbolic leaves can intersect the Julia set. We give a positive answer.

The second author has been partially supported by the NSF and NSERC. The third author has been partially supported by a postdoctoral fellowship of the German Academic Exchange Service (DAAD), and later by EPSRC fellowship EP/E052851/1.

[1] We should note that there is a different construction of laminations for rational maps, due to Meiyu Su [S]. These laminations are never locally compact.

Stony Brook IMS Preprint #2008/5
October 2008

1.2. Theorem (Hyperbolic leaves intersecting the Julia set)**.**
There exists a quadratic polynomial whose regular leaf space contains a hyperbolic leaf that intersects the Julia set.

Remark. Rivera-Letelier (personal communication) has announced a proof of the stronger result that this polynomial can be chosen to satisfy the topological Collet-Eckmann condition.

We then use the following result to deduce Theorem 1.1 from Theorem 1.2.

1.3. Theorem (Hyperbolic leaves and local compactness)**.**
Let f be a rational function whose regular leaf space contains a hyperbolic leaf that intersects the Julia set. Then the affine orbifold lamination $\mathcal{A}_f$ is not locally compact.

Acknowledgements. We thank Carlos Cabrera and Juan Rivera-Letelier for useful discussions. The third author would like to thank the Institute for Mathematical Sciences at Stony Brook and the Simons endowment for its continued support and hospitality.

2. Preliminaries

In this section, we introduce basic notations and give an account of the construction of $\mathcal{A}_f$ that is sufficient to provide a self-contained proof of our results. This account will necessarily be kept concise; for more details, we refer the reader to [LM].

Basic definitions. The complex plane and Riemann sphere are denoted $\mathbb{C}$ and $\hat{\mathbb{C}}$, as usual. A (spherical) disk of radius ε around $z \in \hat{\mathbb{C}}$ is denoted $D_\varepsilon(z)$. If V and U are open sets such that $\overline{V}$ is a compact subset of U, then we say that V is *compactly contained* in U and write $V \Subset U$.

Throughout this article, $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ will be a rational endomorphism of the Riemann sphere. As usual, $F(f)$ and $J(f)$ denote its Fatou and Julia sets. The set of critical points of f is denoted $\operatorname{Crit}(f) = (f')^{-1}(0)$, and the postcritical set is

$$\mathcal{P}(f) := \overline{\bigcup_{j\geq 1} f^j\big(\operatorname{Crit}(f)\big)}.$$

A point $z \in \hat{\mathbb{C}}$ is *exceptional* if $\bigcup_n f^{-n}(z)$ is finite; there are at most two such points.

Natural extension and regular leaf space. We denote by $\mathcal{N}_f$ the space of backward orbits of f. The function f induces an invertible map $\hat{f} : \mathcal{N}_f \to \mathcal{N}_f$ (whose inverse is the shift map); this map is called the *natural extension* of f. Whenever $A \subset \hat{\mathbb{C}}$ satisfies $f(A) \supset A$, its *invariant lift* $\hat{A} \subset \mathcal{N}_f$ is the set of all backward orbits that remain in A. Abusing notation slightly, we will refer to the invariant lift of the Julia set $J(f)$ also simply as "the Julia set".

If $\hat{z} = (z_0 \hookleftarrow z_{-1} \hookleftarrow z_{-2} \dots)$ is a backward orbit and U_0 is a connected open neighborhood of z_0, then the *pullback* of U_0 along $\hat{z}$ is the sequence $U_0 \leftarrow U_{-1} \leftarrow \dots$, where U_{-k} is the component of $f^{-k}(U_0)$ that contains z_{-k}. This pullback is called *univalent* if $f : U_{-(k+1)} \to U_{-k}$ is univalent for all k, and *regular* if this is true for all sufficiently large

k. If the pullback is univalent, then we also say that U_0 is *univalent* along $\hat{z}$. We will denote the set of all backward orbits of z_0 along which U_0 is univalent by $\mathrm{Univ}(z_0, U_0)$.

The point $\hat{z}$ is called *unbranched* if it does not pass through any critical points; it is called *regular* if there is some open connected neighborhood of z_0 whose pullback along $\hat{z}$ is regular. Note that, if $\hat{z}$ is unbranched and regular, then there exists a neighborhood of z_0 that is univalent along $\hat{z}$.

The set of all regular backward orbits is called the *regular leaf space* and denoted $\mathcal{R}_f$. The topology of $\mathcal{R}_f$ as a subset of the infinite product $\hat{\mathbb{C}}^{\mathbb{N}}$ is called the *natural topology* of $\mathcal{R}_f$. If $\hat{z} \in \mathcal{R}_f$, then the path-connected component $L(\hat{z})$ of $\mathcal{R}_f$ containing $\hat{z}$ is called the *leaf* of $\hat{z}$. Every leaf can be turned into a Riemann surface by using the projections $\pi_{-k} : U \to \hat{\mathbb{C}}, \hat{\zeta} \mapsto \zeta_{-k}$ as charts (for sufficiently large k).

We shall call a backward orbit $\hat{z}$ *parabolic* or *hyperbolic* depending on whether $L(\hat{z})$ is a parabolic or hyperbolic Riemann surface. The set of all parabolic backward orbits is called the *affine leaf space* and denoted by $\mathcal{A}_f^n$. We will use the following facts about regular points and their leaves; compare [LM].

- If $z_{-n} \notin \mathcal{P}(f)$ for large n, then $\hat{z}$ is regular.
- Invariant lifts of Cremer, attracting and parabolic cycles and of boundaries of rotation domains are never regular.
- Every leaf that is not the invariant lift of a Herman ring is a hyperbolic or parabolic plane.
- The periodic leaves associated to repelling periodic points are always parabolic; in fact, they are the Riemann surface of the classical Kœnigs linearization coordinate.
- Similarly, for every repelling petal based at a parabolic periodic point, there is an associated parabolic leaf, uniformized by the Fatou coordinate. Every backward orbit converging to a parabolic orbit belongs to such a leaf.
- Any parabolic leaf intersects the Julia set by Picard's theorem.

The Lamination $\mathcal{A}_f$. We will now describe how the affine orbifold lamination $\mathcal{A}_f$ is obtained from the affine part of the regular leaf space. We note that this lamination will not be used until the end of Section 4. Even there, the main fact that is utilized is Proposition 2.1 below, which can be understood without the exact details of the construction of $\mathcal{A}_f$.

The group of linear transformations $z \mapsto az$, $a \neq 0$, acts on the space $\mathcal{U}$ of nonconstant meromorphic functions $\psi : \mathbb{C} \to \hat{\mathbb{C}}$ by precomposition. Let $\mathcal{U}^a$ denote the quotient of $\mathcal{U}$ by this action. f acts on $\mathcal{U}^a$ by postcomposition, and we can form the inverse limit space $\widehat{\mathcal{U}^a}$ of sequences $\hat{\psi} = (\psi_0 \leftarrow \psi_{-1} \leftarrow \psi_{-2} \leftarrow \dots)$ with $\psi_i = f \circ \psi_{i-1}$.

Now if $\hat{z} \in \mathcal{A}_f^n$, then $L(\hat{z})$ is a parabolic plane, so there exists a conformal isomorphism $\varphi : \mathbb{C} \to L(\hat{z})$ with $\varphi(0) = \hat{z}$. Thus $\psi_k := \pi_{-k} \circ \varphi$ defines an element of $\widehat{\mathcal{U}^a}$; note that ψ_k depends only on $\hat{z}$ since ψ is unique up to precomposition with a linear transformation.

The orbifold lamination $\mathcal{A}_f$ is now defined as the closure in $\widehat{\mathcal{U}^a}$ of all such sequences. In a slight abuse of notation, we will denote the sequence ψ_k associated to $\hat{z}$ also by $\hat{z}$ and thus not differentiate between $\mathcal{A}_f^n$ and its copy inside $\mathcal{A}_f$. As suggested by its name,

$\mathcal{A}_f$ is again a lamination; its leaves are the (parabolic) one-dimensional orbifolds

$$L^{\mathrm{lam}}(\hat{\psi}) := \{\hat{\psi} \circ T_a : a \in \mathbb{C}\},$$

where $T_a(z) = z + a$ and $\hat{\psi} \circ T_a$ is the sequence with entries given by $\psi_{-j} \circ T_a$.

Note that there are now two topologies defined on $\mathcal{A}_f^n \subset \mathcal{R}_f$: the original (natural) topology and that induced from $\mathcal{A}_f$, called the "laminar" topology; the latter topological space will be denoted by $\mathcal{A}_f^\ell$. Rather than working directly with the above definition of $\mathcal{A}_f$, we can use a criterion from [LM] that describes the topology of $\mathcal{A}_f^\ell$ simply in terms of the natural extension. If V and W are two simply connected domains, let us say that V is *well inside* W if $\mathrm{mod}\,(W \setminus V) \geq 2$.

2.1. Proposition (Laminar topology [LM, Proposition 7.5])**.**
A sequence of points $\hat{z}^k \in \mathcal{A}_f^\ell$ converges to $\hat{\zeta} \in \mathcal{A}_f^\ell$ in the laminar topology if and only if

(a) *$\hat{z}^k \to \hat{\zeta}$ in the natural topology and*
(b) *for any $N > 0$, if V and W are simply connected neighborhoods of ζ_{-N} such that $\hat{f}^{-N}(\hat{\zeta}) \in \mathrm{Univ}(\zeta_{-N}, W)$ and V is well inside W, then $\hat{f}^{-N}(\hat{z}^k) \in \mathrm{Univ}(z_{-N}, V)$ for large enough k.* $\square$

Remark. Condition (b) is formally weaker than that given in [LM]; however, the proof remains the same.

Auxiliary results. We will occasionally use the following classical fact, which is a weak version of the *Shrinking Lemma* (see e.g. [LM, Appendix 2]).

2.2. Lemma (Univalent shrinking lemma)**.**
Suppose that U is a domain univalent along some backward orbit $\hat{z}$ that does not lie in the invariant lift of a rotation domain. Let $V_0 \Subset U$ and denote by $V_0 \leftarrow V_{-1} \leftarrow \dots$ the pullback of V_0 along $\hat{z}$. Then $\mathrm{diam}\, V_{-j} \to 0$ (where diam denotes spherical diameter). $\square$

Also, we will be concerned with the existence of unbranched backward orbits of a point $z \in \hat{\mathbb{C}}$ under f. Let us say that z is a *branch exceptional point* if z has at most finitely many unbranched backward orbits, and denote the set of such points by E_B.

2.3. Lemma (Branch exceptional points)**.**
E_B contains at most four points. If $z_0 \notin E_B$, then for every $z \in J(f)$, there is some w arbitrarily close to z such that $f^n(w) = z_0$ and $(f^n)'(w) \neq 0$ for some n.

Proof. Every preimage of a point in E_B either belongs to E_B or is a critical point of f. Setting $d = \deg(f)$, it follows that

$$d \cdot (\#E_B) - \sum_c (\deg(c) - 1) = \#f^{-1}(E_B) \leq \#E_B + \#(\mathrm{Crit}(f) \cap f^{-1}(E_B)),$$

where the sum is taken over $c \in \mathrm{Crit}(f) \cap f^{-1}(E_B)$. Since f has exactly $2d-2$ critical points, counting with multiplicities, it follows that

$$(d-1)\#E_B \leq \sum_c \deg(c) \leq 4d - 4,$$

and hence $\#E_B \leq 4$.

If $z \notin E_B$, then z has a non-periodic unbranched backward orbit $z \hookleftarrow z_{-1} \hookleftarrow z_{-2} \dots$. If n is large enough, then z_{-n} is not on a critical orbit, and any backward orbit of z_{-n} is unbranched. Since iterated preimages of z_{-n} are dense in the Julia set, the claim follows. ■

Remark. For an alternative proof which applies also to transcendental meromorphic functions (using Nevanlinna's theorem on completely branched values), see [RvS, Lemma 5.2].

Branch exceptional points have a relation to the existence of isolated leaves in the affine orbifold lamination $\mathcal{A}_f$. More precisely, if p is a repelling periodic point in E_B, then the periodic leaf $L(\hat{p})$ associated to the periodic backward orbit $\hat{p}$ of p is isolated in $\mathcal{A}_f$. (This is an immediate consequence of Proposition 2.1.)[2] Proposition 4.5 shows that these are the only examples of isolated leaves. In particular, every isolated leaf is periodic.

The famous *Lattès* and *Chebyshev* examples have branch exceptional repelling fixed points. In fact, for polynomials, we can give a description of maps with branch-exceptional periodic points. Indeed, suppose that p is such a polynomial; then a calculation analogous to the proof of the previous lemma shows that p has at most two branch-exceptional points apart from the superattracting fixed point at ∞. Furthermore, if $E_B(p) = \{\zeta_1, \zeta_2\}$ with $\zeta_1 \neq \zeta_2$, then the set of critical points of p coincides exactly with $p^{-1}(E_B(p)) \setminus E_B(p)$. It is well-known (see e.g. [DH2, Proposition 9.2]) that this implies $p = T$ or $p = -T$, where T is a Chebyshev polynomial.

On the other hand, if $E_B(p)$ consists of a single fixed point z_0, then every preimage of z_0 is a critical point, and hence p is conjugate to a polynomial of the form

$$p(z) = z^n \cdot (z - a_1)^{k_1} \cdot \dots \cdot (z - a_m)^{k_m},$$

where $n \geq 1$, $a_j \in \mathbb{C} \setminus \{0\}$ and $k_j \geq 2$.

For rational maps, there are many more possible combinatorics for branch-exceptional periodic points. Even among Lattès maps, one can find functions with periodic points in E_B which have periods 2, 3 and 4, with a number of different combinatorial configurations. (Compare [M3].)

Other examples are given e.g. by the family

$$f_c(z) := z \cdot \frac{z-1}{z^2 - z - \frac{1}{\lambda}}, \quad \lambda \in \mathbb{C} \setminus \{0\}.$$

Indeed, ∞ is a critical point of f_c, with

$$\infty \mapsto 1 \mapsto 0 \mapsto 0.$$

Since f_c is a quadratic rational map, it follows that every non-periodic backward orbit of 0 passes through the critical point ∞, and hence $0 \in E_B$. Note that the fixed point 0 has multiplier λ; in particular 0 may be attracting, parabolic or irrationally indifferent.

---

[2] In [LM, Proposition 7.6], it is stated (incorrectly) that such isolated leaves can only occur for Lattès and Chebyshev polynomials. Proposition 4.5 below provides a corrected version of this assertion.

Quadratic-like maps and renormalization. We quickly review the concepts regarding renormalization of quadratic polynomials relevant for Section 3; compare e.g. [DH1] for details. A *quadratic-like map* is a proper map $\varphi : U \to V$ of degree 2, where U and V are Jordan domains with $U \Subset V$. The *filled Julia set* of φ is

$$K(\varphi) := \{z \in U : \varphi^n(z) \in U \text{ for all } n\}.$$

By Douady and Hubbard's *Straightening Theorem*, for every quadratic-like map φ there exists a quadratic polynomial f (the *straightening* of φ) that, restricted to a neighborhood of its filled Julia set, is (quasiconformally) conjugate to φ.

A quadratic polynomial g is called *renormalizable* if there exists $n \geq 2$ and $U \subset \mathbb{C}$ such that $\varphi := g^n|_U$ is a quadratic-like map with $K(\varphi)$ connected. If f is the straightening of φ, then g is also called a *tuning* of f.

It is well-known that every quadratic polynomial f has (infinitely many) tunings; compare e.g. [M2, Section 3] for a precise statement.

3. Existence of Hyperbolic Leaves

Our proof of Theorem 1.2 begins with a result that establishes the existence of rational functions with many hyperbolic leaves that do *not* intersect the Julia set. Recall that a subset of a Baire space is called *generic* if it is the countable intersection of open dense sets.

3.1. Theorem (Maps with large postcritical set)**.**
Let f be a rational map and suppose that $J(f) \subset \mathcal{P}(f)$. Let $z_0 \in F(f)$ be a non-exceptional point (i.e., a point that has infinitely many backward orbits).

Then, for a generic backward orbit $\hat{z}$ of z_0, the leaf $L(\hat{z})$ does not intersect the Julia set. (In particular, $\hat{z}$ is hyperbolic.)

Proof. Let D be the Fatou component containing z_0. We first prove the theorem under the assumption that D is not a rotation domain and that $z_0 \notin \mathcal{P}(f)$. Below, we indicate how this implies the general case.

Denote the space of all backward orbits of z_0 by $\mathcal{Z} \subset \mathcal{N}_f$; note that by assumption these are all unbranched and regular. Note also that $\mathcal{P}(f) \cap D$ is countable and has at most one accumulation point in D, which is then necessarily an attracting periodic point.

Suppose that $\gamma : [0,1] \to D \setminus \mathcal{P}(f)$ is a curve with $\gamma(0) = z_0$. If $\hat{z} \in \mathcal{Z}$, let us denote by $\gamma(\hat{z})$ the endpoint of the corresponding lift of γ to $\mathcal{N}_f$. Note that the holonomy

$$\hat{z} \mapsto \gamma(\hat{z})$$

is a homeomorphism between the space of backward orbits of z_0 and the space of backward orbits of $\gamma(1)$.

Let $U \subset \hat{\mathbb{C}}$ be any connected open set with $U \cap \partial D \neq \emptyset$ and let γ be a curve as above that satisfies $\gamma(1) \in U$. Consider the set $A_{U,\gamma,n}$ of all backward orbits $\hat{z} \in \mathcal{Z}$ for which the pullback of U along $\gamma(\hat{z})$ passes through a critical point at least n times. Clearly the set $A_{U,\gamma,n}$ is open; by the density of $\mathcal{P}(f)$ in the Julia set, it is also dense.

It follows that the set $A_{U,\gamma} := \bigcap_n A_{U,\gamma,n}$, which consists of all $\hat{z} \in \mathcal{Z}$ for which the pullback of U along $\gamma(\hat{z})$ is not regular, is generic.

Now note that the pullback $\gamma(\hat{z})$ depends only on the homotopy class of γ in $D \setminus \mathcal{P}(f)$. Since the fundamental group of $D \setminus \mathcal{P}(f)$ is countable, and since $D \cap U$ has only countably many components, the set $A_U := \bigcap_\gamma A_{U,\gamma}$ is also generic.

Finally, let U_j be a countable collection of open sets such that $\{U_j \cap \partial D\}$ is a base for the topology of ∂D. Then

$$\mathcal{A} := \bigcap_j A_{U_j}$$

is generic. We claim that $\pi_0(L(\hat{z})) \subset D$ for all $\hat{z} \in \mathcal{A}$ (that is, $L(\hat{z})$ does not intersect the Julia set).

Indeed, otherwise there would be a curve $\hat{\gamma} : [0,1] \to L(\hat{z})$ such that $\hat{\gamma}(0) = \hat{z}$; $\pi_0(\hat{\gamma}(t)) \in D$ for $t \neq 1$ and $\pi_0(\hat{\gamma}(1)) \in \partial D$. Let $\gamma := \pi_0 \circ \hat{\gamma}$; the curve $\hat{\gamma}$ can be easily chosen so that $\gamma\big([0,1)\big) \cap \mathcal{P}(f) = \emptyset$. Since $\hat{\gamma}(1)$ is regular, there exists some small neighborhood U of $\gamma(1)$ whose pullback along $\gamma(\hat{z})$ is regular. This contradicts the construction of $\mathcal{A}$.

To conclude, consider the case where z_0 lies in a rotation domain or in the postcritical set. Suppose that $z_{-n} \in f^{-n}(z_0)$ is a preimage that does not lie in a rotation domain or in the postcritical set. Then, for a generic point in the set $\mathcal{Z}(z_{-n})$ of all backward orbits of z_{-n}, the corresponding leaf does not intersect the Julia set. Therefore the same is true of a generic point in $\hat{f}^n(\mathcal{Z}(z_{-n}))$. There is at most one backward orbit $\hat{z}^0$ of z_0 that belongs to the invariant lift of a rotation domain or of the postsingular set (recall that $z_0 \in F(f)$). Furthermore, $\mathcal{Z} \setminus \{\hat{z}^0\}$ can be written as the disjoint union of (countably many) sets of the form $\hat{f}^n(\mathcal{Z}(z_{-n}))$. The claim follows. ∎

The hyperbolic leaves produced by the preceding theorem do not intersect the Julia set. It seems plausible that under the same hypotheses, there also exist some hyperbolic leaves that do intersect the Julia set.

Instead, we will use Theorem 3.1 and the notion of tuning to prove Theorem 1.2. (This idea is due to Rivera-Letelier.)

3.2. Proposition (Hyperbolic leaves over the Julia set)**.**
Let f be a quadratic polynomial whose regular leaf space contains a hyperbolic leaf over the basin of infinity. Then any tuning g of f has a hyperbolic leaf that intersects the Julia set.

We will use the following fact.

3.3. Lemma (Almost every radial line lifts)**.**
Let f be a polynomial and let $z_0 \in \mathbb{C}$. If $\hat{z}$ is an unbranched backward orbit of z_0 that belongs to a parabolic leaf, then for almost every $\vartheta \in \mathbb{R}/\mathbb{Z}$, the line

$$R_\vartheta := \{z_0 + re^{2\pi i\vartheta} : r \geq 0\}$$

lifts to a curve in $L(\hat{z})$ starting at $\hat{z}$.

Proof. Let $\varphi : \mathbb{C} \to L(\hat{z})$ be a conformal isomorphism with $\varphi(0) = \hat{z}$. Consider the entire function $\psi := \pi_0 \circ \varphi$. By the Gross star theorem [N, Page 292], the branch α of ψ^{-1} that carries z_0 to 0 can be continued along almost every radial ray R_ϑ. The curve $\varphi(\alpha(R_\vartheta)) \subset L(\hat{z})$ is then the required lift of R_ϑ. ∎

Proof of Proposition 3.2. By assumption, there exists a domain U such that $\varphi := g^n : U \to g(U)$ is quadratic-like and conjugate to f for some $n \geq 2$. Let us denote the straightening conjugacy by $h : U \to \mathbb{C}$.

Note that every leaf that intersects the basin of infinity must also intersect the Julia set, since every backward orbit that does not belong to the invariant lift of

$$\bigcup_{j=0}^{n-1} g^j(K(\varphi)) \quad (\supset \mathcal{P}(f))$$

is regular.

Since $K(\varphi)$ is connected, we can find some point $z_0 \in U \setminus K(\varphi)$ such that the set

$$T := \{\vartheta : \exists r_0 > 0 : z_0 + r_0 e^{2\pi i\vartheta} \in K(\varphi) \text{ and } \\ z_0 + re^{2\pi i\vartheta} \in U \setminus K(\varphi) \text{ for } 0 \leq r < r_0\}$$

contains a nondegenerate interval.

By assumption, $h(z_0)$ has a backward orbit whose leaf does not extend to $J(f)$ at all. Let $\hat{z}$ be the corresponding backward orbit under g. Then, for $\vartheta \in T$, the radial ray at angle ϑ starting in z_0 does not lift to $L(\hat{z})$. By Lemma 3.3, this implies that $L(\hat{z})$ is hyperbolic. ∎

Remark. As noted in the introduction, the idea for an example of a hyperbolic leaf that does not arise from a rotation domain was first suggested by Hubbard. He proposed constructing a cubic polynomial with a superattracting fixed point at 0 and a recurrent critical point in the boundary of the basin of attraction of 0, carefully chosen to make sure that some leaf does not extend beyond this basin of attraction.

4. Failure of Local Compactness

To prove Theorem 1.3, let us begin with the following statement, which roughly asserts that, given the presence of a hyperbolic leaf intersecting the Julia set, we can find hyperbolic (and parabolic) leaves close to any backward orbit.

4.1. Proposition (Hyperbolic leaves near unbranched orbits)**.**
Let f be a rational function and let L be a leaf of $\mathcal{R}_f$ that intersects the Julia set. Let $\hat{z} \in \mathcal{R}_f$ be any unbranched backward orbit of f that does not lie in the invariant lift of a rotation domain of f, and let V be an open simply connected neighborhood of z_0 that is univalent along $\hat{z}$. Assume furthermore that $\hat{z}$ does not belong to the (isolated) periodic leaf of a branch-exceptional repelling periodic point.

Then, for every domain $V_0 \Subset V$ with $z_0 \in V_0$, and for any neighborhood N of $\hat{z}$ in the natural topology, there is $m \in \mathbb{N}$ such that $N \cap \operatorname{Univ}(z_0, V_0) \cap \hat{f}^m(L) \neq \emptyset$.

Remark. The condition that L intersects the Julia set is clearly necessary, since V itself may intersect the Julia set.

Proof. Let $\hat{W}$ be an open subset of L such that $W := \pi_0(\hat{W})$ is a simply connected domain intersecting $J(f)$ and such that $\hat{W}$ is a univalent pullback of W (i.e., every backward orbit in $\hat{W}$ is unbranched).

Under the hypotheses of the proposition, let V_{-n} be the component of $f^{-n}(V_0)$ containing z_{-n}. We will show that there are infinitely many n for which there is a univalent branch of f^{-j} (for some $j \in \mathbb{N}$) that takes V_{-n} to a subset of W. This will complete the proof, as we can then continue pulling this subset back along the (univalent) pullback $\hat{W}$ of W.

Let A be the limit set of z_{-n} as $n \to \infty$; note that $A \subset J(f)$. We distinguish two cases.

First case: A is contained in the branch exceptional set E_B. Since E_B is finite by Lemma 2.3, A then consists of a single periodic orbit; let $p \in A$ be a point of this orbit. By assumption, p is not repelling. Clearly p cannot be attracting, and by a result of Perez-Marco [P-M], p is not an irrationally indifferent orbit. Hence p must be parabolic, and the point $\hat{z}$ belongs to the periodic leaf associated to some repelling petal based at this orbit. For simplicity, let us assume that p is fixed (the periodic case is analogous). Note that $p \notin V$ — indeed, the only unbranched backward orbit of p is its invariant lift $\hat{p}$, and an invariant lift of a parabolic periodic orbit is never regular.

Since the backward orbit of p is dense in $J(f)$, we can find some $w \in W$ such that $f^j(w) = p$ for some j. If $\varepsilon > 0$ is sufficiently small, then w has a neighborhood $U \subset W$ such that $f^j : U \to D_\varepsilon(p)$ has no critical points except w. For sufficiently large n, V_{-n} is contained in a repelling petal P that is itself a simply connected subset of $D_\varepsilon(p)$. Hence there is a branch of f^{-j} defined on P, and hence on V_{-n}, that takes values in $U \subset W$. This completes the proof in this case.

(We recall that the family of rational functions f_C given in Section 2 contains maps with branch exceptional parabolic points, so this case may indeed occur.)

Second case: There is some $a \in A \setminus E_B$. Then by Lemma 2.3, there is some $j \in \mathbb{N}$ and $w \in W$ such that $f^j(w) = a$ and such that w is not a critical point of f^j. Let $\varepsilon > 0$ be sufficiently small such that the component U of $f^{-j}(D_\varepsilon(a))$ is contained in W and $f : U \to D_\varepsilon(a)$ is a conformal isomorphism.

If z_{-n_k} is a subsequence of $\hat{z}$ with $z_{-n_k} \to a$, then by Lemma 2.2, $\operatorname{diam} V_{-n_k} \to \infty$ as $n \to \infty$, and hence $V_{-n_k} \subset D_\varepsilon(a)$ for sufficiently large k. Again, we see that there is a branch of f^{-j} defined on V_{-n_k} that takes values in W, and are done. ■

The reason that the presence of hyperbolic backward orbits leads to failure of local compactness is that parabolic leaves accumulating at such an orbit cannot converge, even in the weaker topology of $\mathcal{A}_f$:

4.2. Lemma (No convergence to hyperbolic leaves)**.**
Let $\hat{z}^n$ be a sequence of points in $\mathcal{A}_f^\ell$ that converges, in the natural topology of $\mathcal{R}_f$, to a point $\hat{\zeta}$ for which $L(\hat{\zeta})$ is hyperbolic. Then $\hat{z}^n$ does not converge in $\mathcal{A}_f$.

Proof. Suppose that $\hat{\psi}$ is a limit point of $\hat{z}^n$ in $\mathcal{A}_f$. Then $\psi_{-n}(0) = \zeta_{-n}$ for all n. However, this means that the projection

$$p : L^{\mathrm{lam}}(\hat{\psi}) \to L(\hat{\zeta});\ \big(\varphi_0 \leftarrow \varphi_{-1} \leftarrow \dots\big) \mapsto \big(\varphi_0(0) \leftarrow \varphi_{-1}(0) \leftarrow \dots\big)$$

is a nonconstant holomorphic map from the affine orbifold $L^{\mathrm{lam}}(\hat{\psi})$ to the hyperbolic surface $L(\hat{\zeta})$ (cf. [LM, §6.1]). This is impossible. ■

After these preliminaries, we are ready to prove Theorem 1.3.

4.3. Lemma (Non-pre-compact boxes)**.**
Suppose that f has a hyperbolic leaf L_h that intersects the Julia set. Let $\hat{\zeta} \in \mathcal{A}_f^n$ be an unbranched backward orbit that does not belong to an isolated leaf of $\mathcal{A}_f$.

Let V and W be simply connected neighborhoods of ζ_0 such that V is well inside W and W is univalent along $\hat{\zeta}$. Then $\operatorname{Univ}(\zeta_0, V) \cap \mathcal{A}_f^n$ is not pre-compact in $\mathcal{A}_f$.

Proof. Choose some V_0 with $V \Subset V_0 \Subset W$. By Proposition 4.1, there exists a backward orbit $\hat{z} \in \operatorname{Univ}(\zeta_0, V_0)$ such that the leaf $L(\hat{z})$ is hyperbolic. Now we can apply Proposition 4.1 to $\hat{z}$, this time with L being a parabolic leaf. Hence there is a sequence $\hat{z}^k \in \operatorname{Univ}(\zeta_0, V_0) \cap \mathcal{A}_f^n$ that converges to $\hat{z}$ in the natural topology. By Lemma 4.2, this sequence has no convergent subsequence in $\mathcal{A}_f$. ■

4.4. Corollary (Failure of local compactness)**.**
Let f be a rational function and suppose that $\mathcal{R}_f$ contains some hyperbolic leaf L that intersects the Julia set. Then, for all $\hat{\zeta} \in \mathcal{A}_f$ that are not on isolated leaves, $\mathcal{A}_f$ is not locally compact at $\hat{\zeta}$.

Remark. This completes the proof of Theorem 1.3, and thus of Theorem 1.1.

Proof. By definition, $\mathcal{A}_f^\ell$ is dense in $\mathcal{A}_f$. So it is sufficient to restrict to the case of $\hat{\zeta} \in \mathcal{A}_f^\ell$. Also, unbranched backward orbits are dense in $\mathcal{A}_f^\ell$, so we can assume that $\hat{\zeta}$ is unbranched.

By Proposition 2.1, the sets $\hat{f}^n\big(\operatorname{Univ}(\zeta_{-n}, V)\big)$, where $n \geq 0$ and V, W are Jordan neighborhoods of ζ_{-n} such that W is univalent along $\hat{f}^n(\zeta_{-n})$ and V is well inside W, form a neighborhood base of $\hat{\zeta}$ in the laminar topology of $\mathcal{A}_f^\ell$. By Lemma 4.3, none of these sets is pre-compact in $\mathcal{A}_f$. So $\mathcal{A}_f$ is not locally compact at $\hat{\zeta}$. ■

Finally, we remark that Proposition 4.1 also proves minimality of the lamination $\mathcal{A}_f$ (after removing finitely many isolated leaves).

4.5. Proposition (Minimality)**.**
Let L be a leaf of $\mathcal{A}_f$, and let $\hat{z} \in \mathcal{A}_f^\ell$ be a point that does not belong to the isolated leaf associated to a branch-exceptional repelling periodic point. Then L accumulates on $\hat{z}$ in the topology of $\mathcal{A}_f$.

In particular, let $\mathcal{A}_f'$ be obtained from $\mathcal{A}_f$ by removing all (finitely many) leaves associated to branch-exceptional repelling orbits. Then $\mathcal{A}_f'$ is minimal; i.e., every leaf of $\mathcal{A}_f'$ is dense in $\mathcal{A}_f'$.

Remark. By Lemma 2.3, there are at most four branch-exceptional periodic points, so the lamination $\mathcal{A}_f$ contains at most four isolated leaves. This bound is achieved for some Lattès maps.

Proof. (Compare [LM, Proposition 7.6].) If L is an invariant leaf in $\mathcal{A}_f^\ell$, then the claim follows immediately from Propositions 4.1 and 2.1.

Now suppose that $L = L(\hat{\psi})$ is an arbitrary leaf of $\mathcal{A}_f$; by passing to an iterate, we may assume that f has at least five repelling fixed points $\alpha_1, \dots, \alpha_5$ that are not branch-exceptional. We show that L accumulates at the invariant leaf of at least one of these fixed points, which completes the proof.

To do so, let D_k be pairwise disjoint linearizing Jordan neighborhoods of the fixed points α_k. By the Ahlfors Five Islands theorem (see [B]), for every j there is some k_j such that $\psi_{-j} : \mathbb{C} \to \hat{\mathbb{C}}$ has an *island* over D_{k_j}. That is, there is a domain V_j such that $\psi_{-j} : V_j \to D_{k_j}$ is a conformal isomorphism.

There is some k such that $k_j = k$ for infinitely many j. Analogously to the proof of [LM, Proposition 7.5], it follows that that L accumulates on the invariant leaf $L(\hat{\alpha_k})$, as desired. ■

References

[B] Walter Bergweiler, *A new proof of the Ahlfors five islands theorem*, J. Anal. Math. **76** (1998), 337–347.

[DH1] Adrien Douady and John H. Hubbard, *On the dynamics of polynomial-like mappings*, Ann. Sci. École Norm. Sup. (4) **18** (1985), no. 2, 287–343.

[DH2] ———, *A proof of Thurston's topological characterization of rational functions*, Acta Math. **171** (1993), no. 2, 263–297.

[KL] Vadim A. Kaimanovich and Mikhail Lyubich, *Conformal and harmonic measures on laminations associated with rational maps*, Mem. Amer. Math. Soc. **173** (2005), no. 820, vi+119.

[LM] Mikhail Lyubich and Yair Minsky, *Laminations in holomorphic dynamics*, J. Differential Geom. **47** (1997), no. 1, 17–94, arXiv:math.DS/9412233.

[M1] John Milnor, *Dynamics in one complex variable*, third ed., Annals of Mathematics Studies, vol. 160, Princeton University Press, Princeton, NJ, 2006.

[M2] ———, *Local connectivity of Julia sets: expository lectures*, The Mandelbrot set, theme and variations, London Math. Soc. Lecture Note Ser., vol. 274, Cambridge Univ. Press, Cambridge, 2000, pp. 67–116.

[M3] ———, *On Lattès maps*, Dynamics on the Riemann sphere, Eur. Math. Soc., Zürich, 2006, arXiv:math.DS/0402147, pp. 9–43.

[N] Rolf Nevanlinna, *Eindeutige analytische Funktionen*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Bd XLVI, Springer-Verlag, Berlin, 1953, 2te Aufl.

[P-M] Ricardo Pérez Marco, *Sur une question de Dulac et Fatou*, C. R. Acad. Sci. Paris Sér. I Math. **321** (1995), no. 8, 1045–1048.

[RvS] Lasse Rempe and Sebastian van Strien, *Absence of line fields and Mañé's theorem for non-recurrent transcendental functions*, Preprint, 2007, arXiv:0802.0666; submitted for publication.

[S] Meiyu Su, *Measured solenoidal Riemann surfaces and holomorphic dynamics*, J. Differential Geom. **47** (1997), no. 1, 170–195.

Institute for Mathematical Sciences, SUNY Stony Brook, NY 11794-3660, USA
E-mail address: kahn@math.sunysb.edu

Institute for Mathematical Sciences, SUNY Stony Brook, NY 11794-3660, USA
E-mail address: mlyubich@math.sunysb.edu

Dept. of Math. Sciences, University of Liverpool, Liverpool L69 7ZL, UK
E-mail address: l.rempe@liverpool.ac.uk