COMBINATORIAL RIGIDITY FOR SOME INFINITELY RENORMALIZABLE UNICRITICAL POLYNOMIALS

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Abstract. We prove Combinatorial rigidity for *infinitely renormalizable* unicritical polynomials, $f_c : z \mapsto z^d + c$, with *a priori bounds* and some "combinatorial condition". Combining with [KL2], this implies local connectivity of the connectedness locus (the "Mandelbrot set" when $d = 2$) at the corresponding parameter values.

1. Introduction

The *Multibrot set* M_d or the *connectedness locus* is the set of parameter values c in $\mathbb{C}$ for which the *Julia set* of $f_c : z \mapsto z^d + c$ is connected. M_2 is the well-known *Mandelbrot* set.

There is a way of defining graded partitions of Multibrot set into pieces such that dynamics of maps f_c in each piece has some special combinatorial property. Maps in the same piece of partition of a certain level will be called *combinatorially equivalent* up to that level. Conjecturally, combinatorially equivalent (up to all levels) non-hyperbolic maps in this family are *conformally equivalent*. As stated in [DH1] for $d = 2$, this *Rigidity Conjecture* is equivalent to the local connectivity of the Mandelbrot set and it naturally extends to degree d unicritical maps. In the case of quadratic family, this conjecture is formulated as MLC by A. Douady and J.H. Hubbard. They also proved there that MLC implies Density of *hyperbolic maps* among quadratic maps. These discussions had been extended to degree d unicritical maps by D. Schleicher in [Sch].

In 1990's, Yoccoz proved MLC at all non-hyperbolic parameter values which are at most finitely *renormalizable*. He also proved local connectivity of *Julia* set for these parameters, See [H]. Degree 2 assumption was essential in his proof.

In [L1], M. Lyubich proved combinatorial rigidity for a class of infinitely renormalizable quadratic polynomials. These are degree two polynomials satisfying *Secondary limbs condition*, $\mathcal{SL}$, with sufficiently high returns. Proof in this case also depends on degree 2. Local connectivity of Julia sets for unicritical degree d polynomials which are at most finitely renormalizable has been shown in [KL1] which is done by "controlling" geometry of *Modified principal nest*. The same controlling technique is used to settle the rigidity problem for these parameters in [AKLS]. In more recent works the *a priori* bounds (for renormalization levels) established for more parameters. In [K2] it is proved for *primitive* infinitely renormalizable maps of

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bounded type, in [KL2] it is proved for all parameters in *Decorations* and in [KL3] for *Molecules*. Here we prove that *a priori* bounds imply combinatorial rigidity for infinitely renormalizable maps in $\mathcal{SL}$, which includes all parameters for which *a priori* bounds are known to us.

Theorem 1.0.1 (Rigidity Theorem)**.** *Let f_c, for $c \in M_d$, be an infinitely renormalizable degree d unicritical polynomial with a priori bounds in $\mathcal{SL}$. Then f_c is combinatorially rigid.*

The proof of rigidity in [L1], stated for quadratic maps under general assumption of the *a priori* bounds and $\mathcal{SL}$ but it also goes through for degree d unicritical polynomials. The main difference between our proof and the one in [L1] is that the construction of *Thurston conjugacy* in [L1] was done along all the principal nest but the modified principal nest introduced in [KL1] and the beautiful idea in the proof of rigidity for *non-renormalizable* maps in [AKLS] helps us to pass over principal nest much easier which makes the whole construction simpler.

Combining the above theorem with [K2] and [KL2] we have the following:

Corollary 1.0.2. *Let f and $\tilde{f}$ be two infinitely renormalizable unicritical degree d polynomial-like maps satisfying molecule condition. If f and $\tilde{f}$ are combinatorially equivalent then they are hybrid equivalent.*

The following infinitely renormalizable parameter values are known to enjoy the *a priori bounds*: real infinitely renormalizable unicritical polynomials (see [LV] and [LY]), for decorations and for molecules which includes infinitely *primitive* renormalizable parameter values of *bounded* type. We will see that these parameters are in $\mathcal{SL}$, so combining with the theorem mentioned above, this will imply combinatorial rigidity for these parameter values.

The structure of the paper is as follows. In §2 we introduce some background in holomorphic dynamics required for our work. In §3, Yoccoz puzzle pieces are defined, the modified principal nest is constructed and finally combinatorics of unicritical polynomials is discussed. In §4, we reduce the rigidity problem to existence of a *Thurston conjugacy* and construct it.

1.1. **Acknowledgment.** I am indebted to M. Lyubich because this paper would have never been finished without his great patience and he also suggested the problem. Further thanks is due to R.Pérez for our very useful discussions on the combinatorics of the Mandelbrot set.

2. Polynomials and Multibrot sets

2.1. **External rays and Equipotentials.** The general references for the following material are [M1] and [B].

Let $f : \mathbb{C} \to \mathbb{C}$ be a monic polynomial of degree d, $f(z) = z^d + a_1 z^{d-1} + ... + a_d, \infty$ is a super attracting fixed point of f and its *basin of attraction* is defined as

$$D_f(\infty) = \{z \in \mathbb{C} : f^n(z) \to \infty\}$$

Its complement is called the *filled Julia* set : $K(f) = \mathbb{C} \setminus D_f(\infty)$. The *Julia set*, $J(f)$, is the common boundary of $K(f)$ and $D_f(\infty)$. It is well-known that the Julia set and the filled Julia set are connected if and only if all critical points stay bounded under iteration of f.

With f as above, there exists a conformal change of coordinate, *Böttcher coordinate*, B_f which conjugates f to the *dth* power map $z \mapsto z^d$ throughout some neighborhood of infinity U_f, that is,

$$B_f : U_f \to \{z : |z| > r_f \geq 1\}$$

such that $B_f(f(z)) = (B_f(z))^d$ and $B_f(z) \sim z$ as $z \to \infty$.

In particular, if the filled Julia set is connected, B_f coincides with the Riemann mapping of the whole basin D_f of infinity onto the complement of the closed unit disk.

The *External rays* $R^\theta = R^\theta_f$ of angel θ and *equipotentials* $E(r) = E^r_f$ of radius r are defined as the B_f-preimages of the straight rays $\{re^{i\theta} : r_f < r < \infty\}$ and the round circles $\{re^{i\theta} : 0 \leq \theta \leq 2\pi\}$. It follows from equivariance relation that $f(R^\theta) = R^{d\theta}$.

A ray R^θ is called periodic ray of period p if $f^p(R^\theta) = R^\theta$. It is easy to see that a ray is fixed ($p = 1$) if and only if θ is a rational number of the form $2j\pi/(d-1)$. By definition, a ray R^θ lands at a well defined point z of $J(f)$ if the limiting value of the ray R^θ exists and equals to z. Such a point $z \in J(f)$ is called the landing point of the ray. The following theorem characterizes the landing points of the periodic rays. See [DH1] for further discussions.

Theorem 2.1.1. *Let f be a polynomial of degree $d \geq 2$ with connected Julia set. Every periodic ray lands at a well defined periodic point which is either repelling or parabolic. Vice versa, every repelling or parabolic periodic point is the landing point of at least one and at most finitely many periodic rays with the same ray period.*

In particular, this theorem implies that the external rays landing at a periodic point a are organized in several cycles. Suppose $\overline{a} = \{a_k\}_{k=0}^{p-1}$ is a repelling or parabolic cycle of f and denote by $\Re(a_k)$ the union of closed external rays landing at a_k. The configuration

$$\Re(\overline{a}) = \bigcup_{k=0}^{p-1}(\Re(a_k))$$

with the rays labeled by their external angels, is called the *periodic point portrait* of f associated to the cycle $\overline{a}$.

2.2. **Unicritical family.** Any degree d polynomial with only one critical point is affinely conjugate to $P_c(z) = z^d + c$ for some complex number c.

A case of especial interest is the following fixed point portrait. The $d-1$ fixed rays $R^{2\pi j/(d-1)}$ land at $d-1$ fixed points called β_j and moreover these are the only rays landing at β_j's. These fixed points are *non-dividing* which means $K(F) \setminus \beta_j$ is connected for any j. If the other fixed point called α is also repelling, then there are at least 2 rays landing at it, so it is *dividing* and by above result these rays are permuted under dynamics. The following statement has been shown in [M2] for quadratic polynomials. The same ideas apply to show it for degree d polynomials.

Proposition 2.2.1. *If at least 2 rays land at one of the fixed points α of f, we have*

- *The component of $\mathbb{C} \setminus \Re(\alpha)$ containing the critical value is a sector bounded by two external rays.*
- *The component of $\mathbb{C} \setminus f^{-1}(\Re(\alpha))$ containing the critical point is a region bounded by $2d$ external rays landing in pairs at the points $e^{2\pi k/d}\alpha$ for $k = 0, 1, \ldots, d-1$.*

The *Multibrot* set M_d of degree d is defined as the set of parameters c in $\mathbb{C}$ for which $J(P_c)$ is connected or equivalently the critical point does not escape to infinity under iteration of P_c. In particular, M_2 is the well-known *Mandelbrot* set, See figure 1 and 2. A well-known result due to Douady and Hubbard, see [DH1], shows that the Multibrot set, M_d, is connected. The proof is by constructing an explicit conformal isomorphism

$$B_{M_d} : \mathbb{C} \setminus M_d \to \{z : |z| > 1\}$$

which is given by $B_{M_d}(c) = B_c(c)$, where B_c is the Böttcher coordinate for P_c.

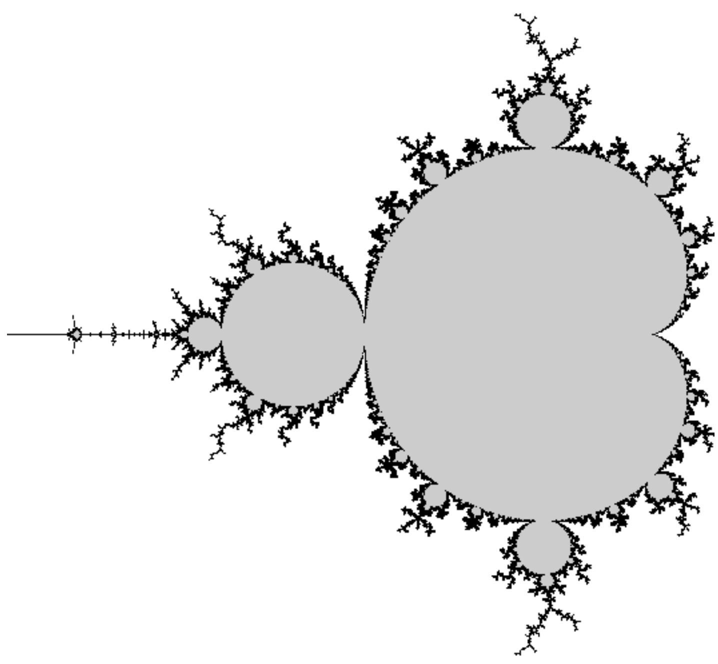


FIGURE 1. The Mandelbrot set

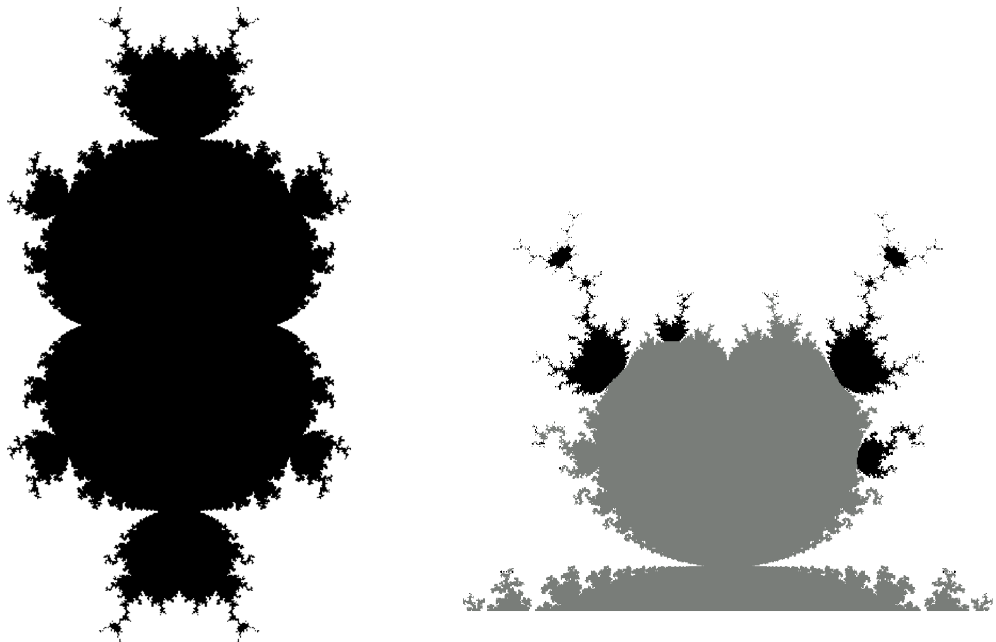


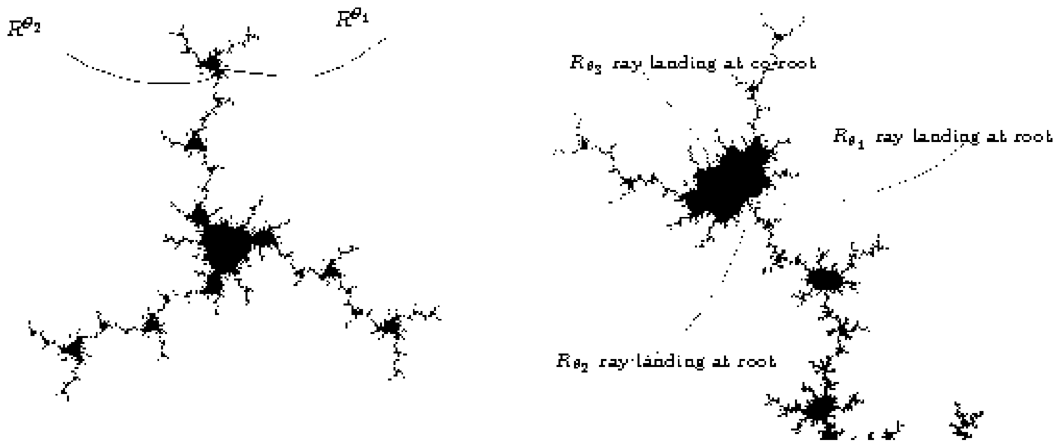
FIGURE 2. Figure on the left shows the Multibrot set M_3. The figure on the right is an enlargement of the $1/2$ limb in M_3. The dark regions show some of the secondary limbs

By means of this conformal isomorphism, B_{M_d}, the parameter external rays and equipotentials are defined as the B_{M_d}-preimages of the straight rays going to infinity and round circles around 0. This provides us with two orthogonal foliations of the complement of the Multibrot set.

A polynomial P_c (and the corresponding parameter c) is called *hyperbolic* if P_c has an attracting periodic point. The set of hyperbolic parameters in M_d is the union of some components of $intM_d$ which are called *hyperbolic components*.

The main hyperbolic component is defined as the set of parameter values c for which P_c has an attracting fixed point. Outside of the closure of this set all the fixed points become repelling. Now, consider a hyperbolic component $H \subset \text{int } M_d$ and suppose $\bar{b}_c$ is the corresponding attracting cycle of period k. On the boundary of H this cycle becomes neutral and there are $d-1$ points b_i where P_{b_i} has a parabolic orbit. The one, $d \in \partial H$, which divides the Multibrot set into two piece is called the *root* of H, (See [DH1] for degree 2 case). Indeed, any hyperbolic component has one root and $d-2$ *co-roots*. The root is the landing point of two parameter rays, while every co-root is the landing point of a single parameter ray, See figure 3. More details for combinatorics of degree d unicritical maps can be found in [Sch].

If c is in a hyperbolic component H which is not the main component of the Multibrot set, the basin of attraction, A_c , is defined as the set of points z such that $P_c(z)$ converges to the cycle $\bar{b}_c$. The boundary of its component containing b_c, D_c, is a Jordan curve and moreover P_c^k on D_c is topologically conjugate to θ to $d\theta$ on the unit circle, so there are $d-1$ fixed points of P_c^k which are repelling periodic points (of P_c) of period dividing period of b_c (its period can be strictly less than period of b_c). Among all the rays landing at these periodic points, let θ_1 and θ_2 be the angels of the external rays bounding the sector containing the critical value of P_c. The following theorem makes connection between external rays R^{θ_1}, R^{θ_2} and the corresponding parameter external rays R_{θ_1}, R_{θ_2}. See [DH1] or [Sch] for the proof.



(a) dynamical space

(b) parameter space

FIGURE 3. The picture on the left shows a primitive renormalizable Julia set and the external rays R^{θ_1} and R^{θ_2} landing at the corresponding periodic point. The figure on the right is the corresponding primitive little multibrot copy. It also shows the parameter external rays R_{θ_1} and R_{θ_2} landing at the root point.

Theorem 2.2.2. *The parameter external rays R_{θ_1} and R_{θ_2} land at the root d of H and these are the only rays that land at d.*

Two parameter external rays R_{θ_1} and R_{θ_2} cut the plane into two components. The one containing the component H, with the root point d attached to it is called the *wake* W_d. So a wake is an open set with a root point attached to its boundary. For a wake W_d and an equipotential of radius η, $E(\eta)$, the *truncated wake* $W(\eta) = W_d(\eta)$ is the bounded component of $W_d \setminus E(\eta)$. Part of the Multibrot set contained in the wake W_d with the root point d attached to it is called *Limb* L_d of the Multibrot set *originated* at H. By definition, every Limb is a closed set.

The wakes attached to the main cardioid are called *primary wakes* and a limb associated to such a primary wake will be called *primary limb*. If H is a hyperbolic component attached to the main cardioid, all the wakes attached to such a component H are called *secondary wakes* and similarly, a limb associated to a secondary wake will be called *secondary limb*. A *truncated limb* is obtained from a limb by removing a neighborhood of its root. Some secondary limbs are shown on figure 2.

Given a parameter c in H, we have the attracting cycle b_c as above and the associated repelling cycle $\overline{a}_c$ which contains the landing point of the external rays R^{θ_1} and R^{θ_2}. The following result gives the dynamical meaning of the parameter values in the wake W_d (See [Sch] for the proof).

Theorem 2.2.3. *For parameter c in W_d, the repelling cycle $\overline{a}_c$ stays repelling and the isotopic type of rays portrait $\Re(\overline{a}_c)$ is fixed throughout W_d .*

2.3. **Polynomial-like maps.** A holomorphic branched covering $f : U' \to U$ such that U' is compactly contained in U is called a *polynomial-like map*. Reader can consult [DH2] for the following material about polynomial-like maps. Every polynomial can be viewed as a polynomial-like map after restricting it onto an appropriate neighborhood of the filled Julia set. In what follows we will only consider polynomial-like maps with one branched point of degree d (which is assumed to be at zero after normalization) and refer to them as *unicritical polynomial-like* maps.

The filled Julia set $K(f)$ is naturally defined as

$$K(f) = \{z : f^n(z) \in U', \quad n = 0, 1, 2, ...\}.$$

The Julia set $J(f)$ is defined as the boundary of $K(f)$. They are connected if and only if $K(f)$ contains the critical point.

Given a polynomial-like map $f : U' \to U$, we can consider the *fundamental annulus* $A = U \setminus U'$. It is not canonic because any choice of $V' \Subset V$ such that $f : V' \to V$ is a polynomial-like map with the same Julia set will give a different annulus but we can associate a real number, *modulus of* f, to any polynomial-like map f as follows:

$$\operatorname{mod}(f) = \sup \operatorname{mod}(A)$$

where the sup is taken over all possible fundamental annuli A of f.

Two polynomial-like maps f and g are called *topologically (quasi-conformally, conformally, affinely) conjugate* if there is a choice of domains $f : U' \to U$ and $g : V' \to V$ and a homeomorphism $h : (U, U') \to (V, V')$ (quasi conformal, conformal or affine isomorphism correspondingly) such that $h \circ f|U' = g \circ h|U'$.

Two polynomial like maps f and g are *hybrid or internally equivalent* if there is a qc conjugacy (quasi-conformal conjugacy for short) h between f and g such that $\bar{\partial} h = 0$ on $K(f)$. The following theorem due to Douady and Hubbard makes the connection between polynomial-like maps and polynomials (See [DH2] for more details).

Theorem 2.3.1 (Straightening Theorem)**.** *Every polynomial-like map f is hybrid equivalent to (suitable restriction of) a polynomial P of the same degree. Moreover, P is unique up to affine conjugacy when $K(f)$ is connected.*

In particular, any unicritical polynomial-like map with connected Julia set corresponds to a unique (up to affine conjugacy) unicritical polynomial $z \mapsto z^d + c$ with c in the Multibrot set M_d. Note that $z^d + c$ and $z^d + c/\lambda$ are conjugate via $z \mapsto \lambda z$ for every $d-1$th root of unity λ.

The *Teichmüller distance* between two hybrid equivalent polynomial-like maps f and g is defined as

$$\mathrm{dist}_T(f, g) = \inf \log Dil(h)$$

where h runs over all hybrid conjugacies between f and g, and $\mathrm{Dil}(h)$ denotes the qc dilatation of the map h .

It can be seen from the construction of the straightening that the Teichmüller distance between f and the corresponding polynomial $P_{c(f)} : z \mapsto z^d + c$ is controlled by modulus of f.

Proposition 2.3.2. *If $mod(f) \geq \mu > 0$ then $dist_T(f, P_{c(f)}) \leq C$ where C only depends on μ and moreover $C(\mu) \to 0$ as $\mu \to \infty$.*

3. Modified principal nest

3.1. **Yoccoz puzzle pieces.** Recall that for a parameter $c \in M_d$ outside of the main component of the Multibrot set, $f = P_c$ has a unique dividing fixed point α_c. The $q \geq 2$ external rays $\Re(\alpha_c)$ landing at this fixed point together with an equipotential E, cut the domain inside E into q closed topological disks $Y_j^0, j = 0, 1, \ldots, q-1$, called *puzzle pieces of level zero*. The main property of this partition is that $P_c(\partial Y_j^0)$ does not intersect interior of any piece Y_i^0.

Now the *puzzle pieces Y_j^n of level or depth n* are defined as the closures of the connected components of $f^{-n}(\mathrm{int}(Y_j^0))$. They partition the neighborhood of the filled Julia set bounded by the equipotential $f^{-n}(E)$ into finite number of closed disks and moreover, they are bounded by piecewise analytic curves. The *label* of each puzzle piece is the set of the angels of external rays bounding that puzzle piece. If the critical point does not land on the fixed point α_c there is a unique puzzle piece $Y^{(n)} = Y_0^n$ of level n containing the critical point.

Let $\mathcal{Y}_f$ denote the family of all puzzle pieces of f of all levels. It has the following *Markov property*:

- Puzzle pieces are disjoint or nested. In the latter case, the puzzle piece of higher level is contain in the puzzle piece of lower level.
- The image of any puzzle piece of level $n \geqslant 1$ is a puzzle piece of level $n-1$ and in addition, $f : Y_j^n \to Y_k^{n-1}$ is d-to-1 branched covering or univalent depending on whether Y_j^n contains the critical point or not.

On the first level there are $d(q-1)+1$ puzzle pieces. One critical piece Y_0^1, $q-1$ ones, Y_i^1, attached to the fixed point α_c, and the $(d-1)(q-1)$ symmetric ones

Z_i^1 attached to $\omega\alpha_c$ where ω's are dth roots of unity (except for $\omega = 1$). Moreover $f|Y_0^1$ d-to-1 covers Y_1^1, $f|Y_i^1$ univalently covers $Y_{i+1}^1, i = 1, ..., q-2$ and $f|Y_{q-1}^1$ univalently covers, $Y_0^1 \cup \bigcup_{i=1}^{(d-1)(q-1)} Z_i^1$. So $f^q(Y_1^0)$ truncated by $f^{-1}(E)$ is the union of Y_1^1 and Z_i^1's.

We will assume after this that $f^n(0) \neq \alpha$-fixed point for all n, So that the critical puzzle pieces of all levels are well defined. As it will be apparent in a moment, this condition is always the case for renormalizable maps.

3.2. **The complex bounds in the favorite nest and renormalization.** For a puzzle piece $V \ni 0$, let $R_V : \mathrm{Dom}R_V \subseteq V \to V$ denote *the first return map* to V. It is defined for the points z in V for which there exists a positive t such that $P_c^t(z) \in \mathrm{int}V$. Markov property of puzzle pieces implies that any component of $DomR_V$ is contained in V and the restriction of this map (P_c^t, for some t) to each component U of $DomR_V$ is d-to-1 or 1-to-1 proper map onto V depending on whether U contains the critical point or not. In the former case U is called *central component* of R_V. If the image of critical point under the first return map belongs to the central component, the return is called *central return*.

The *first landing map* L_V to a puzzle piece $V \ni 0$ is also well defined and it univalently maps each component of Dom L_V onto V (L_V is the identity on the component V).

Consider a puzzle piece $Q \ni 0$. The central component $P \subset Q$ of R_Q is the pullback of Q by P_c^p along the orbit of the critical point, where p is the first moment when critical orbit enters int Q. This puzzple piece P is called the *first child* of Q. Recall that $P_c^p : P \to Q$ is a proper map of degree d.

The *favorite child* Q' of Q is constructed as follows, Let $p > 0$ be the first moment when $R_Q^p(0) \in \mathrm{int}\,(Q \setminus P)$ and $q > 0$ be the first moment when $R^{p+q}(0) \in \mathrm{int}P$ ($p+q$ is the moment of the first return back to P after the first escape of the critical point from P under iterates of R_Q). Now Q' is defined as the pullback of Q under R_Q^{p+q} containing the critical point. Markov property implies that the map $P_c^k = R_Q^{p+q}$ (with an appropriate $k > 0$) from Q' to Q is proper of degree d. The main property of the favorite child is that the image of critical point under the map $P_c^k : Q' \to Q$ is in the first child P.

A map $f = P_c$ is called *immediately renormalizable* (or *satellite renormalizable*) if

$$f^{lq}(0) \in Y_0^1, \quad l = 0, 1, 2, \dots.$$

By slight "thickening" of the domain of this map (see [M3]) it can be turned into a unicritical polynomial-like map. Note that above condition implies that the critical point does not scape the domain, so the corresponding little Julia set is connected.

If f is not immediately renormalizable then there is a first moment k such that $f^{kq}(0)$ belongs to some Z_i^1. Define Q^1 as the pullback of Z_i^1 under f^{kq}. By the above construction we form the first child P^1 and the favorite child Q^2 of Q^1. Repeating the above process we get a nest of puzzle pieces

$$Q^1 \supset P^1 \supset Q^2 \supset P^2 \supset .... \supset Q^n \supset P^n \tag{1}$$

where P^i is the first child of Q^i, and Q^{i+1} is the favorite child of Q^i. The above process stops if and only if one of the following happens:

- The map f is combinatorially non-recurrent, that is, the critical point does not return to some critical puzzle piece; or
- The critical point does not escape the first child P^n under iterates of R_{Q^n} for some n, or equivalently, returns to all critical puzzle pieces of level bigger than n are central and we get an infinite cascade of central returns.

In the latter case, $R_{Q^n} = P_c^k : P^n \to Q^n$ (for an appropriate k) is a unicritical polynomial-like map, and the map P is called *primitively renormalizable.* Note that the corresponding little Julia set is connected because all the returns of critical point to Q^n are central by definition (critical point does not escape).

A map $P_c : \mathbb{C} \to \mathbb{C}$ is called *renormalizable* if it is immediately or primitively renormalizable.

Combinatorial rigidity in the critically non-recurrent case has been taken care of in [M3]. To deal with the non-renormalizable polynomials, the following *a priori* bounds have been proved in [AKLS] which is a slightly modified version of a priori bounds that appeared in [KL1] for the first time.

Theorem 3.2.1. *There exists $\delta > 0$ such that for every $\varepsilon > 0$ there exists $n_0 > 0$ with the following property. For the nest of puzzle pieces*

$$Q^1 \supset P^1 \supset Q^2 \supset P^2 \supset .... \supset Q^m \supseteq P^m$$

as above, if mod$(Q^0 \setminus P^0) > \varepsilon$ and $n_0 < n < m$ then mod$(Q^n \setminus P^n) > \delta$.

If a map f is combinatorially recurrent, the critical point does not land at α-fixed point, then puzzle pieces of all levels are well defined. The *combinatorics of f* up to level n is the set of labels of puzzle pieces of level n. Equivalently, the combinatorics up to level $n + t$ determines the puzzle piece Y_j^n of level n containing the critical value $f^t(0)$. Two non-renormalizable maps are called *combinatorially equivalent* if they have the same combinatorics up to an arbitrary level n. Combinatorics of a renormalizable map will be defined in section 3.3.

Two unicritical polynomials f and $\tilde{f}$ with the same combinatorics up to level n are called *pseudo-conjugate (up to level n)* if there is an orientation preserving homeomorphism $H : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, such that $H(Y_j^0) = \widetilde{Y}_j^0$ for all j and $H \circ f = \tilde{f} \circ H$ outside of the critical puzzle piece Y_0^n. A pseudo-conjugacy H, is said to *match the Böttcher marking* if near infinity it becomes identity in the Böttcher coordinates for f and $\tilde{f}$, so it is identity outside of $\cup_j Y_j^n$ by its equivariance property.

Recall that a *holomorphic motion* of a given subset X of $\mathbb{C}$, parameterized by a complex manifold $\mathcal{M}$, is a map $\Phi : \mathcal{M} \times X \to \mathcal{M} \times \mathbb{C}$ of the form $(c, z) \to (c, \phi^c(z))$, which is holomorphic on each slice $\mathcal{M} \times \{z\}$, injective on each slice $\{c\} \times X$ and satisfies $\phi^{c_0}(z) \equiv z$ for some $c_0 \in \mathcal{M}$.

Let q_m and p_m be respectively the levels of puzzle pieces Q^m and P^m, that is, $Q^m = Y^{q_m}$ and $P^m = Y^{p_m}$. The following is the main technical result of [AKLS] which will be used frequently in our construction.

Theorem 3.2.2. *Assume that a nest of puzzle pieces as in (1) is obtained for f. If $\tilde{f}$ is combinatorially equivalent to f, then there exists a K-qc pseudo-conjugacy H (up to level q_m) between f and $\tilde{f}$ which matches the Böttcher marking and moreover $K = K(f, \tilde{f})$ depends only on the hyperbolic distance between c and $\tilde{c}$ in the primary wake truncated by some equipotential of radius η.*

A brief sketch of the proof: Combinatorial equivalence of f and $\tilde{f}$ up to level zero implies that the corresponding parameters c and $\tilde{c}$ are in the same truncate wake $W(\eta)$.

Inside $W(\eta)$ the q external rays $\Re(\alpha)$ and the equipotential of height $E(h)$(for every $h > \eta$) move holomorphically in $\mathbb{C} \setminus 0$, that is, there exists a holomorphic motion Φ of $\Re(\alpha) \cup E(h)$ parameterized by $W(\eta)$ such that $\phi(\tilde{c}, \Re(\alpha) \cup E(h)) = (\tilde{c}, \widetilde{\Re}(\alpha) \cup \widetilde{E}(h))$ This is given by $B_{\tilde{c}}^{-1} \circ B_c$.

Outside equipotential $E(h)$ this holomorphic motion extends to a motion holomorphic in both variables (c, z) which is coming from the Böttcher coordinate near ∞. By [BR] the map $\phi(\tilde{c}, .) \circ \phi(c, .)^{-1}$ extends to a K_0 qc map $H_0 : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, where K_0 only depends on the hyperbolic distance between c and $\tilde{c}$ inside the truncated wake $W(\eta)$. This gives a qc map $H_0 : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ which is a conjugacy outside union of puzzle pieces of level zero.

By adjusting the qc map H_0 inside equipotential $E(h)$ such that it send c to $\tilde{c}$ we get a qc map (not necessarily with the same dilatation) H'_0. By pulling back H'_0 via f and $\tilde{f}$ we get a new qc map H_1. Repeating this process for $i = 1, 2, ..., n$, by adjusting qc map H_i inside the union of puzzle pieces of level $i+1$ so that it sends c to $\tilde{c}$ (still qc but not with the same K) and pulling it back we get a qc map H_{i+1} which is a conjugacy outside union of puzzle pieces of level $i+1$. At the end we will have a qc map H_n which is a conjugacy outside of the equipotential $E(h/d^n)$.

The nest of puzzle pieces $\widetilde{Q}^1 \supset \widetilde{P}^1 \supset \widetilde{Q}^2 \supset \widetilde{P}^2 \supset .... \supset \widetilde{Q}^m \supseteq \widetilde{P}^m$ are defined as the image of the nest of puzzle pieces $Q^1 \supset P^1 \supset Q^2 \supset P^2 \supset .... \supset Q^m \supseteq P^m$ under the map H_n. Combinatorial equivalence of f and $\tilde{f}$ implies that this new nest has the properties required by Theorem 3.2.1, then it also has the *a priori* bounds property. By properties of this nest, one constructs a K-qc map H from the critical puzzle piece Q^n to the corresponding one $\widetilde{Q}^n$ where K only depends on the *a priori* bounds δ and the hyperbolic distance of the parameters c and $\tilde{c}$ in the parabolic wake truncated by some equipotential E^r. The pseudo-conjugacy H_n is obtained by univalent pullbacks of H onto other puzzle pieces.

Remark 3.2.3. *If f is combinatorially recurrent and non renormalizable, the above process repeats to construct an infinite nest of puzzle pieces P^i, Q^i and pseudo-conjugacies H_i. The a priori bounds property 3.2.1 of the nest implies that the critical puzzle pieces shrink to the critical point. Now the qc conjugacy between f and $\tilde{f}$ follows from precompactness of K-qc maps H_i from $\mathbb{C}$ to $\mathbb{C}$ normalized on the postcritical set.*

But if f is renormalizable, the process of constructing modified principal nest stops at some level χ and returns to the critical puzzle pieces are central after level χ. This implies that the critical puzzle pieces do not shrink to 0. In the following section we will deal with this problem under some combinatorial and *a priori* bounds type conditions.

3.3. **Combinatorics of a map.** If $f : z \mapsto z^d + c_0, c_0 \in M_d$, is renormalizable then there is a homeomorphic copy $M_d^0 \ni c_0$ of the Multibrot set with the following properties (see[DH2]): for $c \in (M_d^0 \setminus$ the root point) the polynomial $P_c : z \mapsto z^d + c$ is renormalizable and there is a holomorphic motion of the fixed point α_c and the rays landing at it on a neighborhood of ($M_d^0 \setminus$ the root point) such that the renormalization of P_c is associated to this fixed point and external rays. The copies corresponding to satellite renormalizations are attached to the main component of the Multibrot set.

We will see below that among all renormalizations there is the first one denoted by Rf which corresponds to a maximal (not included in any other copy except M_d itself) copy of the Multibrot set inside the Multibrot set. Let $\mathcal{M}_d$ denote the family of all maximal Multibrot copies.

If Rf is also renormalizable, its renormalization will be denoted by R^2f and f will be called twice renormalizable. So there will be a canonical finite or infinite sequence $f, Rf, R^2f, ...$ associated to f and accordingly, f is classified as at *most finitely or infinitely renormalizable*. Equivalently, there will be a finite or infinite sequence $\tau = \{M_d^1, M_d^2, ...\}$ of maximal Multibrot copies associated to f such that M_d^n corresponds to the renormalization R^nf of $R^{n-1}f$. In the case of infinite renormalizable, τ is called *combinatorics* of f.

Two infinitely renormalizable maps are called combinatorially equivalent if they have the same combinatorics. Two at most finitely renormalizable maps are combinatorially equivalent if they are the same number of times renormalizable with the same sequence of Multibrot copies and their last renormalizations are combinatorially equivalent in the sense of definition following Theorem 3.2.1.

An infinitely renormalizable map f satisfies the *secondary limbs condition* if all the Multibrot copies in the combinatorics, τ, of f belong to finite number of truncated secondary limbs. Let $\mathcal{SL}$ stand for the class of unicritical polynomial like maps satisfying the secondary limbs condition.

All these combinatorial notions extend to unicritical polynomial-like maps by means of straightening.

An infinitely renormalizable map f is said to have *a priori bounds* if there is an $\varepsilon > 0$ such that $mod(R^m f) \geq \varepsilon > 0$ for all renormalizations.

4. The pullback argument

In this section we begin to prove the rigidity theorem stated in the introduction.

4.1. **Reductions.**

Theorem 4.1.1 (Rigidity theorem). *Let f and $\tilde{f}$ be two infinitely renormalizable unicritical polynomial-like maps in $\mathcal{SL}$ with a priori bounds. If f and $\tilde{f}$ are combinatorially equivalent, then they are hybrid equivalent.*

Remark 4.1.2. *If two maps f and $\tilde{f}$ in the above theorem are polynomials, then hybrid equivalence becomes conformal equivalence. This is because Böttcher coordinate conformally conjugates them on the complement of the Julia sets. See Proposition* 6 *in* [DH2]

The proof is divided into following three steps:

- f and $\tilde{f}$ are topologically equivalent
- f and $\tilde{f}$ are qc equivalent
- f and $\tilde{f}$ are hybrid equivalent

It has been shown in [J] that any unbranched infinitely renormalizable map with *a priori* bounds has locally connected Julia set. Here unbranched condition follows from our combinatorial condition and *a priori* bounds (see [L1] lemma 9.3). Then the first step (topological equivalence of two combinatorially equivalent maps) follows from the local connectivity of the Julia sets by the Carathéodory theory. See [M1].

The last step follows from McMullen's Rigidity Theorem [McM1]. He has shown that an infinitely renormalizable degree 2 polynomial-like map with *a priori* bounds (the same proof works for degree d unicritical polynomial-like maps) does not have any nontrivial invariant line field on its Julia set. It follows that any qc conjugacy h between f and $\tilde{f}$ satisfies $\overline{\partial} h = 0$ almost everywhere on the Julia set, so h is a hybrid conjugacy between f and $\tilde{f}$.

If all infinitely renormalizable unicritical maps in a given combinatorial class satisfy *a priori* bounds condition (with a uniform ε for all of them), it is easier to show that qc-conjugacy implies hybrid conjugacy for that class rather than showing that there is no nontrivial invariant line field on the Julia set. Since we are finally going to apply our theorem to combinatorial classes for which *a priori* bounds have been established, we will show this in proposition 4.5.5

So assume f and $\tilde{f}$ are topologically conjugate. We want to show the following:

Theorem 4.1.3. *If two infinitely renormalizable unicritical polynomial-like maps f and $\tilde{f}$ with a priori bounds in $\mathcal{SL}$ are topologically conjugate then they are qc conjugate.*

4.2. **Thurston equivalence.** The proof uses the following notion of Thurston conjugacy. Suppose two unicritical polynomial-like maps $f : U' \to U$ and $\tilde{f} : \widetilde{U}' \to \widetilde{U}$ are topologically conjugate. A qc map $h : (U, U', \mathcal{O}(0)) \to (\widetilde{U}, \widetilde{U}', \widetilde{\mathcal{O}}(0))$ which is homotopic to a topological conjugacy $\psi : (U, U', \mathcal{O}(0)) \to (\widetilde{U}, \widetilde{U}', \widetilde{\mathcal{O}}(0))$ relative $\partial U \cup \partial U' \cup \mathcal{O}(0)$ is called *Thurston conjugacy* where $\mathcal{O}(0)$ denotes the *postcritical* set

$$\mathcal{O}(0) = \overline{\cup_{i=1}^{\infty} P^i(0)}$$

in other words the post critical set is the closure of the orbit of critical point. Two maps f and $\tilde{f}$ are called *Thurston equivalent* if there is such a h for an appropriate choices of domains $U, U', \widetilde{U}, \widetilde{U}'$.

The following result is due to Thurston and Sullivan (see [S]) which originates the "pull-back method" in holomorphic Dynamics.

Lemma 4.2.1. *If two unicritical polynomial-like maps are Thurston equivalent then, they are qc equivalent.*

Proof. Assume $h_1 : (U_1, U_2, \mathcal{O}(0)) \to (\widetilde{U}_1, \widetilde{U}_2, \widetilde{\mathcal{O}}(0))$ is a Thurston conjugacy homotopic to the topological conjugacy $\Psi : (U_1, U_2, \mathcal{O}(0)) \to (\widetilde{U}_1, \widetilde{U}_2, \widetilde{\mathcal{O}}(0))$ relative $\partial U_1 \cup \partial U_2 \cup \mathcal{O}(0)$.

As $f : (U_2 \backslash \{0\}) \to (U_1 \backslash \{f(0)\})$ and $\tilde{f} : (\widetilde{U}_2 \backslash \{0\}) \to (\widetilde{U}_1 \backslash \{\tilde{f}(0)\})$ are covering maps, $h_1 : (U_1 \backslash \{f(0)\}) \to (\widetilde{U}_1 \backslash \{\tilde{f}(0)\})$ can be lifted to a homeomorphism $h_2 : (U_2 \backslash \{0\}) \to (\widetilde{U}_2 \backslash \{0\})$ and since h_1 satisfies the *equivariance relation*, $h_1 \circ f = \tilde{f} \circ h_1$, on the boundary of U_2, h_2 can be extended to $U_1 \backslash U_2$ by h_1. It also extends to the critical point by sending it to the critical point of $\tilde{f}$. Let us denote this new map h_2. For the same reason, every homotopy h_t from Ψ to h_1 can be lifted to a homotopy from Ψ to h_2. As f and $\tilde{f}$ are holomorphic maps, h_2 has the same dilatation as dilatation of h_1. This implies that the new map h_2 is also a Thurston conjugacy with the same dilatation.

By definition, The new map h_2 satisfies the equivariance relation on the annulus $U_2 \backslash (f^{-1}(U_2))$.

This process can be continued to make a sequence of K-qc maps h_n from U_1 to $\widetilde{U}_1$ which satisfies equivariance relation on the annulus $U_2 \setminus f^{-n}(U_2)$. Compactness of the family of K-qc maps normalized at two points (normalized on $\mathcal{O}(0)$ in this case) implies that there is a subsequence h_{n_j} which converges to some K-qc map H. For every z outside of the Julia set, the sequence $h_{n_j}(z)$ stabilizes and by construction eventually $h_{n_j} \circ f(z) = f \circ h_{n_j}(z)$. Taking limit will imply that $H \circ f(z) = f \circ H(z)$ for every such a z. Equivariance for an arbitrary z on the Julia set follows from continuity as the filled Julia set of an infinitely renormalizable unicritical map does not have interior .

□

By the *a priori* bounds assumption in the theorem, there are topological disks $0 \in V_{n,0} \Subset U_{n,0}$ such that $R^n f : V_{n,0} \to U_{n,0}$ is a unicritical degree d polynomial-like map and the moduli of the annuli $U_{n,0} \setminus V_{n,0}$ are at least some positive ε. By slight shrinking the domains we may assume that these domains have smooth boundaries.

We will use the following notations throughout the construction:

$f : V_0 \to U_0, K_{0,0} = K(f),$
$Rf = f^{t_1} : V_{1,0} \to U_{1,0}, K_{1,0} = K(Rf)$
$R^2 f = f^{t_2} : V_{2,0} \to U_{2,0}, K_{2,0} = K(R^2 f)$
$\vdots$
$R^n f = f^{t_n} : V_{n,0} \to U_{n,0}, K_{n,0} = K(R^n f)$
$\vdots$

Define $V_{n,i}$ as the pullback of $V_{n,0}$ by f^{-i} containing the little Julia set $J_{n,i} = f^{t_n - i}(J_{n,0})$ and $U_{n,i}$ the component of $f^{-i}(U_{n,0})$ containing $V_{n,i}$, for $i = 1, 2, ..., t_n - 1$. $W_{n,i}$ will denote the pre-image of $V_{n,i}$ under the map $f^{t_n} : V_{n,i} \to U_{n,i}$.

Accordingly, $K_{n,i}$ is defined as the component of $f^{-i}(K_{n,0})$ inside $V_{n,i}$. Note that $R^n f : V_{n,i} \to U_{n,i}$ is a polynomial-like map with the filled Julia set $K_{n,i}$ which is conjugate to $R^n f : V_{n,0} \to U_{n,0}$ by conformal isomorphism $f^i : U_{n,i} \to U_{n,0}$. It has been proved in [L1] (Lemma 9.2) that there is always definite space in between Julia sets in the primitive case for parameters under our assumption. Compare the proof of Prop 4.4.2. It has been shown in [McM2] that definite space between little Julia sets implies that there exist choice of domains $U_{n,i}$ which are disjoint for different i's and moreover the annuli $U_{n,i} \setminus V_{n,i}$ have definite moduli. So we will assume that on the primitive levels, the domains $V_{n,i}$ are disjoint for different i's.

In all of the above notation, the first lower subscripts denote the level of renormalization and the second lower subscripts run over little filled Julia sets, Julia sets and their neighborhoods accordingly. In what follows all corresponding objects for $\widetilde{f}$ will be marked with a tilde and any notation introduced for f will be automatically transferred to $\tilde{f}$.

To construct a Thurston conjugacy, first we will construct multiply connected domains $\Omega_{n(k),i}$ (and $\widetilde{\Omega}_{n(k),i}$) in $\mathbb{C}$ for an appropriate subsequence $n(k)$ of the renormalization levels and a sequence of K-qc maps

$$h_{n(k),i} : \Omega_{n(k),i} \to \widetilde{\Omega}_{n(k),i}$$

for $k = 0, 1, 2, ...$ and $i = 0, 1, 2, ..., t_{n(k)} - 1$ where $t_{n(k)}$ is the period of the renormalization of level $n(k)$. The domains will satisfy the following properties:

- Each $\Omega_{n(k),j}$ is a topological disk minus some topological disks $D_{n(k),i}$.
- Each $\Omega_{n(k+1),i}$ is well inside the topological disk $D_{n(k),i}$, that is, the modulus of the annulus obtained from $D_{n(k),i} \setminus \Omega_{n(k+1),i}$ is at least $\zeta > 0$ which only depends on *a priori* bounds ε.
- Each set $J_{n,k} \cap \mathcal{O}(0)$ is well inside the topological disk $D_{n,k}$.
- Every $D_{n,i}$ is the pullback of $D_{n,0}$ by f^{-i} containing $J_{n,i} \cap \mathcal{O}(0)$ and every $\Omega_{n(k+1),i}$ is the component of $f^{-i}(\Omega_{n(k+1),0})$ inside $D_{n(k),i}$.

Finally, the Thurston conjugacy will be constructed by an appropriate gluing of these maps on the complement of all these multiply connected domains (which is the union of annuli). See figure 4.

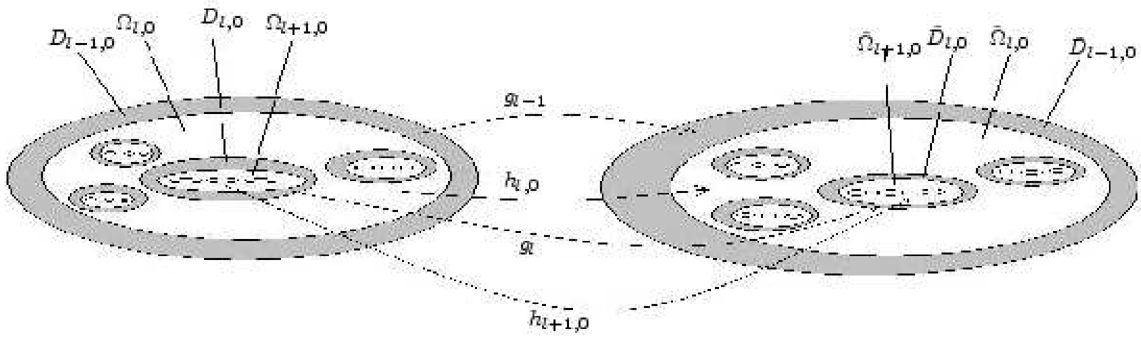


FIGURE 4. The multiply connected domains and the buffers

4.3. **Construction of $\Omega_{n,j}$ and $h_{n,j}$.** Suppose we are on level $n-1$ of the construction. Because of the difference in type of renormalizations, we will consider the following three cases:

(I) $R^{n-1}f$ is primitively renormalizable.
(II) $R^{n-1}f$ is immediately renormalizable and $R^n f$ is primitively renormalizable.
(III) $R^{n-1}f$ is immediately renormalizable and $R^n f$ is also immediately renormalizable.

This will cover all renormalization levels for infinitely renormalizable maps.

Case (I) :

By applying the Straightening Theorem to the polynomial-like map

$$R^{n-1}f : V_{n-1,0} \to U_{n-1,0}$$

we get a $K_1(\varepsilon)$-qc map and a unicritical polynomial $\mathbf{f}_{c_{n-1}}$

$$S_{n-1} : (U_{n-1,0}, V_{n-1,0}, 0) \to (\Upsilon^0_{n-1}, \Upsilon^1_{n-1}, 0) \tag{2}$$
$$\text{such that } S_{n-1} \circ R^{n-1}f = \mathbf{f}_{c_{n-1}} \circ S_{n-1}$$

(See Figure 5).

To make the notations easier to follow, we will drop the second subscript whenever it is zero and it does not create confusion. Also, all the objects on the dynamical planes of $\mathbf{f}_{c_{n-1}}$ and $\mathbf{f}_{\tilde{c}_{n-1}}$ (the ones after straightening) will be denoted by the **bold** versions of the notations used for the objects on the dynamical planes of f and $\tilde{f}$.

We need the following lemma to show that there are equipotentials of sufficiently large radii $\eta(\varepsilon)$ inside $S_{n-1}(W_{n-1,0})$ and $\tilde{S}_{n-1}(\widetilde{W}_{n-1,0})$ in the dynamical planes for the two maps $\mathbf{f}_{c_{n-1}}$ and $\mathbf{f}_{\tilde{c}_{n-1}}$.

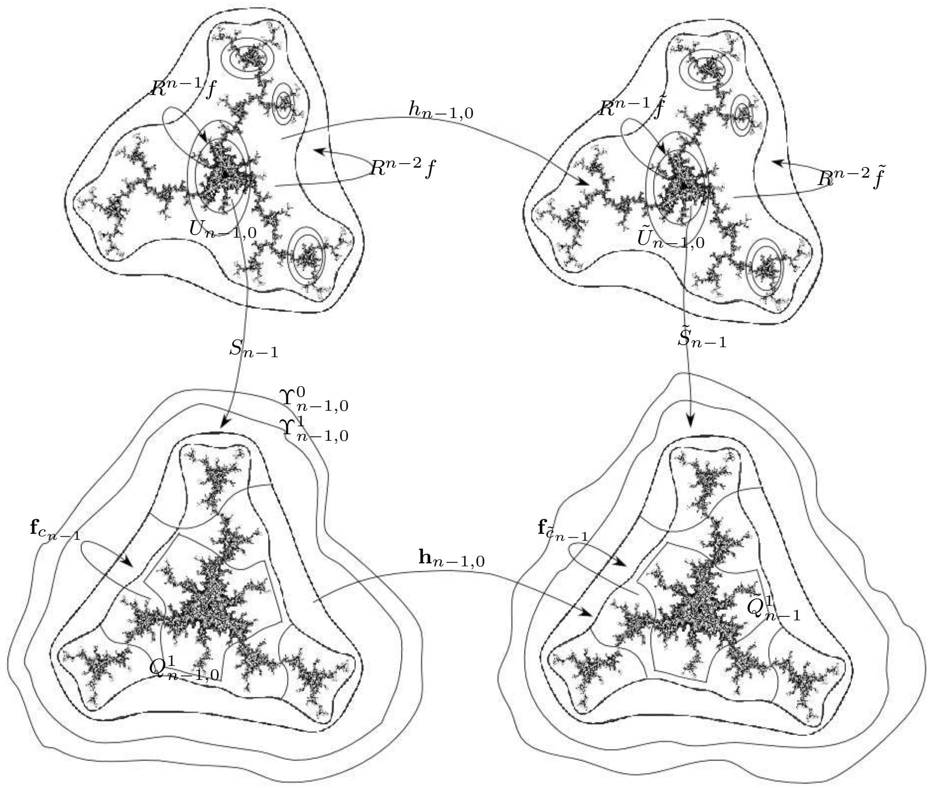


FIGURE 5. Primitive case

Lemma 4.3.1. *Let $P_c : U' \to U$ be a unicritical polynomial with connected Julia set such that mod$(U \setminus U') \geq \varepsilon$. Then U' contains equipotentials of radius less than $\eta(\varepsilon)$.*

Proof. The map P_c on the complement of $K(P_c)$ is conjugate to P_0 on the complement of the closed unit disk by B_c (Böttcher coordinate). Since radii of the equipotentials are preserved and modulus is conformal invariant, it is enough to prove the statement for $P_0 : U' \to U$. As $P_0 : P_0^{-1}(U \setminus U') \to (U \setminus U')$ is a covering of degree d, modulus of the annulus $P_0^{-1}(U \setminus U')$ is ε/d which implies that modulus of $U' \setminus \mathbb{D} \geq \varepsilon/d$. Combining this with Grötzsch Problem in [Ah](see section A in chapter III) we conclude the lemma.

$\square$

Now, we are in the position to apply the Theorem 3.2.2 to $\mathbf{f}_{c_{n-1}}$ and $\mathbf{f}_{\tilde{c}_{n-1}}$. Consider the equipotentials of radius $\eta(\varepsilon)$ (obtained in the previous lemma) and the external rays landing at the dividing fixed points α_{n-1} and $\tilde{\alpha}_{n-1}$ for the two maps $\mathbf{f}_{c_{n-1}}$, $\mathbf{f}_{\tilde{c}_{n-1}}$, and form the favorite nest of puzzle pieces (1) introduced in section 3.2. The hyperbolic distance between c_{n-1} and $\tilde{c}_{n-1}$ in the truncated wake, $W(\eta)$, is bounded by some $M(\varepsilon)$ which only depends on ε. This is because parameters belong to finitely many limbs which is a compact subset of the wake, so dilatation of the pseudo-conjugacy from Theorem 3.2.2 is bounded by some constant $K_2(\varepsilon)$ independent of n.

Consider the last critical puzzle pieces $Y_{n,0}^{q_{\chi_n}} = Q_{n,0}^{\chi_n}$ and $P_{n,0}^{\chi_n}$ obtained in the nest of puzzle pieces (1) and the corresponding $K_2(\varepsilon)$ pseudo-conjugacy $\mathbf{h}_{n-1} = \mathbf{h}_{n-1,0}$. Denote components of $\mathbf{f}_{c_{n-1}}^{-i}(Q_{n,0}^{\chi_n})$ and $\mathbf{f}_{c_{n-1}}^{-i}(P_{n,0}^{\chi_n})$ containing the little Julia sets $J_{n,i}$ by $Q_{n,i}^{\chi_n}$ and $P_{n,i}^{\chi_n}$ for $i = 0, 1, 2, ..., t_n/t_{n-1} - 1$ (recall that t_n/t_{n-1} is the period of the first renormalization of $\mathbf{f}_{c_{n-1}}$).

Two polynomials $\mathbf{f}_{c_{n-1}}$ and $\mathbf{f}_{\tilde{c}_{n-1}}$ also satisfy our combinatorial and *a priori* bounds assumptions, so there is a topological conjugacy between them which we denote it by $\boldsymbol{\psi}_{n-1} = \boldsymbol{\psi}_{n-1,0}$.

Denote the closure of the annulus $Q_n^{\chi_n} \setminus P_n^{\chi_n}$ by A_n^0 and the component of its pullback by $\mathbf{f}_{c_{n-1}}^{-kt_n/t_{n-1}}$ around $J_{n,0}$ by A_n^k for $k = 0, 1, 2, ...$. Lifting $\mathbf{h}_{n-1}$ by $\mathbf{f}_{c_{n-1}}^{t_n/t_{n-1}}$ and $\mathbf{f}_{\tilde{c}_{n-1}}^{t_n/t_{n-1}}$ we obtain a K_2-qc map g from A_n^0 to $\widetilde{A}_n^0$ which is homotopic to $\boldsymbol{\psi}_{n-1}$ relative boundaries of A_n^0. This is because by the external rays connecting $\partial P_n^{\chi_n}$ to $\partial Q_n^{\chi_n}$, the annulus A_n^0 is partitioned into some topological disks and the equivariance relation implies that the two maps coincide on the boundaries of these topological disks.

As $\mathbf{f}_{c_{n-1}}^{kt_n/t_{n-1}} : A_n^k \to A_n^0$ and $\mathbf{f}_{\tilde{c}_{n-1}}^{kt_n/t_{n-1}} : \widetilde{A}_n^k \to \widetilde{A}_n^0$ are holomorphic unbranched coverings, g can be lifted to a K_2-qc map on A_n^k. All these lifted maps are the identity in the Böttcher coordinate on the boundaries of annuli so they match together to give a K_2-qc conjugacy from $Q_n^{\chi_n} \setminus J_n$ to $\widetilde{Q}_n^{\chi_n} \setminus \tilde{J}_n$.

The following lemma guaranties to extend g over the little Julia set J_n.

For a given Polynomial P_c with connected Julia set J, we can define a *rotation* of angle θ, ρ_θ, on the complement of the Julia set as the rotation of angle θ in the Bötcher coordinate, that is $B_c^{-1}(e^{i\theta}.B_c)$. By means of straightening one can define rotations of angle θ on the complement of the Julia set of polynomial-like maps. It is not canonic as it depends on the choice of straightening but it's effect on the landing points of the external rays is canonic.

Proposition 4.3.2. *Let $f : V_2 \to V_1$ be a polynomial-like map with connected Julia set J. If ϕ is a homeomorphism from $V_1 \setminus J$ to $V_1 \setminus J$ which commutes with f, then there exists a rotation of angle $2j\pi/(d-1)$, ρ_j, such that $\rho_j \circ \phi$ extends onto the Julia set J and this extension is the identity map on the Julia set.*

Proof. Consider an external ray R landing at the non-dividing fixed point β_0 of f. As this ray is invariant under f and ϕ commutes with f, we have $f \circ \phi(R) = \phi \circ f(R) = \phi(R)$ which means $\phi(R)$ is also invariant under f. By Theorem 2.1.1 this ray will land at a non-dividing fixed point β_j of f. Now choose ρ_j such that $\rho_j(\phi(R))$ lands at β_0. Denote $\rho_j \circ \phi$ by ψ and $\rho_j(\phi(R))$ by R'. Obviously ψ commutes with f and R' is invariant under f.

The external ray, R, cuts the annulus $V_1 \setminus V_2$ into a quadrilateral $I_{0,1}$. f-preimage of this quadrilateral produces d quadrilaterals denoted by $I_{1,1}, I_{1,2}, \dots, I_{1,d}$ ordered clockwise starting from R and similarly the f^n-preimages of the quadrilateral $I_{0,1}$ produces d^n quadrilateral $I_{n,1}, I_{n,2}, \dots, I_{n,d^n}$ (ordered clockwise). In the same way, the external ray R' produces quadrilaterals denoted by $I'_{n,j}$ ordered clockwise starting from R'. First we will show that the Euclidean diameter of every quadrilateral $I_{n,j}$ (similarly $I'_{n,j}$) goes to zero when n goes to infinity.

Define V_{i+1} as $f^{-i}(V_1)$ and denote by d_{i+1} the hyperbolic distance on this annulus. As $I_{n,j} \subseteq V_{n-1}$ and the intersection of the topological disks V_n is J, the quadrilaterals $I_{n,j}$ get close to the boundary of the set $V_1 \setminus J$ by making n large enough. In order to show that the Euclidean diameter of each quadrilateral goes to zero it is enough to show that their hyperbolic diameters are bounded with respect to the metric d_1 on the annulus $V_1 \setminus J$. The map $f^{n-2} : (V_{n-1} \setminus J, d_{n-1}) \to (V_1 \setminus J, d_1)$ is an unbranched covering of degree d^{n-2} so it is an isometry and also the inclusion map from $(V_n \setminus J, d_n)$ to $(V_1 \setminus J, d_1)$ is a contraction. Since image of $I_{n,j}$ under f^{n-2} is a compact subset of $V_1 \setminus J$ we conclude that $I_{n,j}$ has bounded hyperbolic diameter in $(V_1 \setminus J, d_1)$.

As the map ψ satisfies the equivariance relation, it sends the quadrilateral $I_{n,j}$ to the quadrilateral $I'_{n,j}$. The same argument as the above one implies that the hyperbolic distance between $I_{n,j}$ and $I'_{n,j}$ inside $V_1 \setminus J$ is uniformly bounded.

To conclude the lemma we need to show that $\psi(w)$ converges to w when w converges to J. As w and $\psi(w)$ will belong to $I_{n,j}$ and $I'_{n,j}$ for large n's, the above argument implies that the Euclidean distance between them goes to zero. $\square$

Note that no rotation is needed if $d = 2$, that is, every homeomorphism which commutes with f extends on to the Julia set as the identity map.

Applying the above lemma to $\boldsymbol{\psi}_{n-1}^{-1} \circ g$ by choosing $Q_n^{\chi_n}$, $P_n^{\chi_n}$ and an external ray connecting the boundary of $Q_n^{\chi_n}$ to the little Julia set inside it we conclude that g can be extended onto the little Julia set $J_{n,0}$ and this extension equals to $\boldsymbol{\psi}_{n-1}$ on $J_{n,0}$. Proof of the above lemma also implies that g and $\boldsymbol{\psi}_{n-1}$ are homotopic relative J_n and boundary of $Q_n^{\chi_n}$. This is because the boundaries of the annuli and the curves obtained in the above lemma cut the puzzle piece $Q_n^{\chi_n}$ into (infinite) topological disks such that g and $\boldsymbol{\psi}_{n-1}$ are equal on their boundaries.

In the same way, the map $\mathbf{h}_{n-1}$ can be extended onto the other puzzle pieces $Q_{n,i}^{\chi_n}$ and moreover, the map $\mathbf{h}_{n-1}$ is homotopic to ψ_{n-1} on $Q_{n,i}^{\chi_n}$ relative $J_{n,i}$ and boundary of $Q_{n,i}^{\chi_n}$. We will denote the map obtained by gluing $\mathbf{h}_{n-1}$ and the extensions by the same notation as $\mathbf{h}_{n-1}$.

Lemma 4.3.3. *The map $\mathbf{h}_{n-1}$ can be adjusted on a neighborhood of the little Julia set $J_{n,0}$ such that it sends $\mathbf{V}_{n,i} = S_{n-1}(V_{n,i})$ quasi conformally onto $\widetilde{\mathbf{V}}_{n,i} = \tilde{S}_{n-1}(\widetilde{V}_{n,i})$.*

Proof. First we will show this for $i=0$. Choose a topological disk $\widetilde{\mathbf{X}}$ containing $\widetilde{\mathbf{V}}_n = \widetilde{\mathbf{V}}_{n,0}$ and disjoint from other domains $\widetilde{\mathbf{V}}_{n,i}$ $(i \neq 0)$ such that the annuli $\widetilde{\mathbf{X}} \setminus \mathbf{h}_{n-1}(\mathbf{V}_n)$ and $\widetilde{\mathbf{X}} \setminus \widetilde{\mathbf{V}}_n$ have moduli bounded from below and above (only depending on ε). Bounded from below is because $\tilde{S}_{n-1}(\widetilde{U}_n \setminus \widetilde{V}_n)$ and image of $U_n \setminus V_n$ under the qc map $\mathbf{h}_{n-1} \circ S_{n-1}$ gives some definite space around $\widetilde{\mathbf{V}}_{n,i}$'s and $\mathbf{h}_{n-1}(\mathbf{V}_{n,i})$'s. Bounded from above is easily done by making $\widetilde{\mathbf{X}}$ small enough. Denote $\mathbf{h}_{n-1}^{-1}(\widetilde{\mathbf{X}})$ by $\mathbf{X}$.

Consider a qc homeomorphism Σ from $\widetilde{\mathbf{X}} \setminus J_n$ onto $\mathbb{D}_4 \setminus \mathbb{D}_1$ such that image of $\partial(\mathbf{h}_{n-1}(\mathbf{V}_n))$ under the map Σ is $\partial\mathbb{D}_3$. Build a family of diffeomorphisms ϕ_t of $\mathbb{D}_4 \setminus \mathbb{D}_1$ for $t \in [0,1]$ such that ϕ_0 is identity on $\mathbb{D}_4 \setminus \mathbb{D}_1$ and ϕ_1 is identity on the boundaries of $\mathbb{D}_4 \setminus \mathbb{D}_1$ and it maps the circle of radius 3 to a circle of radius r for some small $r > 1$ which will be determined later. Such an example can be made by sliding the points along rays passing from origin.

The composition map $g_t = \Sigma^{-1} \circ \phi_t \circ \Sigma \circ \mathbf{h}_{n-1}$ is a continuous family of homeomorphisms from $\mathbf{X} \setminus J_n$ to $\widetilde{\mathbf{X}} \setminus \tilde{J}_n$ which is an isotopy relative $\partial\mathbf{X}$ and moreover g_1 sends $\partial\mathbf{V}_n$ to a closed curve close to $\tilde{J}_n$. If r in the construction is small enough depending only on ε, the image of this curve will be definitely contained in $\widetilde{\mathbf{V}}_n$. A lower bound for r can be found in terms of $mod(\widetilde{X} \setminus \mathbf{h}_{n-1}(\mathbf{V}_n))$ which we made it bounded from above and below earlier.

Now consider a qc map Ξ from $\widetilde{\mathbf{X}} \setminus J_n$ onto $\mathbb{D}_4 \setminus \mathbb{D}_1$ such that the image of $\partial(\widetilde{\mathbf{V}}_n)$ is $\partial\mathbb{D}_3$ and image of $\partial(g_1(\mathbf{V}_n))$ is the circle of radius r. Now concatenating the family g_t with the family $\Xi^{-1} \circ \phi_{2-t} \circ \Xi \circ g_1$ for $t \in [1,2]$ gives a deformation of $\mathbf{h}_{n-1}$ which maps $\mathbf{V}_n$ to $\widetilde{\mathbf{V}}_n$. Denote the time 2 map by $\mathbf{h}'_{n-1}$. Upper and lower bounds in the previous paragraph and construction of ϕ_t implies that this map is K_3-qc where K_3 only depends on the *a priori* bounds ε.

By pulling back the above construction with $\mathbf{f}_{c_{n-1}}^{-k}$ for $k = 1, 2, 3, ..., t_n/t_{n-1}$ we adjust $\mathbf{h}_{n-1}$ such that it maps $\mathbf{V}_{n,i}$ to $\widetilde{\mathbf{V}}_{n,i}$. $\square$

Remark 4.3.4. *In the above Lemma, it would be easier to show that such a map exists but we will later use the property that this new map is homotopic to* $\mathbf{h}_{n-1,0}$ *relative little Julia set* $J_{n,0}$.

Now, let $\Delta_{n-1,0}$ be S_{n-1}-preimage of the domain bounded by the equipotential of height $\eta(\varepsilon)$ and let $h_{n-1,0}$ be $\tilde{S}_{n-1}^{-1} \circ \mathbf{h}'_{n-1} \circ S_{n-1}$.

Let $\Omega_{n-1,0}$ be $\Delta_{n-1,0}$ minus topological disks $V_{n,it_{n-1}}$, for $i = 0, 1, ..., t_n/t_{n-1}$. It is clear that $h_{n-1,0}(\Omega_{n-1,0}) = \widetilde{\Omega}_{n-1,0}$ and it is K^3-qc where $K = \max\{K_1, K_3\}$. Pulling back $\Delta_{n-1,0}$ by f^{-i} for $i = 1, 2, ..., t_{n-1}$ along the orbit of critical point we get $\Delta_{n-1,i}$'s. The map $h_{n-1,i}$ from $\Delta_{n-1,i}$ to $\widetilde{\Delta}_{n-1,i}$ is defined as $\tilde{f}^{-i} \circ h_{n-1,0} \circ f^i$. Notation $\Omega_{n-1,i}$ is self explanatory.

Finally, the annulus $V_{n-1,0} \setminus W_{n-1,0}$ which has definite modulus (depending only on ε) is around $\Omega_{n-1,0}$ and contained in the disk $V_{n-1,0}$. Moreover, preimages of this annulus by conformal maps f^{-i} give definite annuli around $\Omega_{n-1,i}$ which are contained in the disks $V_{n-1,i}$. This proves that the domains $\Omega_{n-1,i}$ are well inside the disks $V_{n-1,i}$.

Case (II) :

Given a polynomial-like map $R^{n-1}f : V_{n-1,0} \to U_{n-1,0}$, there is a $k_1(\varepsilon)$-qc map S_{n-1} satisfying equation (2).

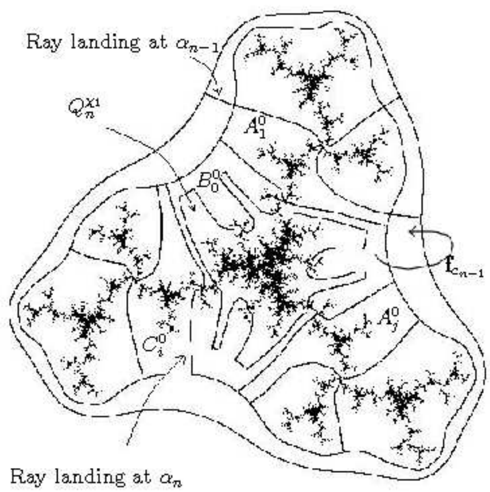


FIGURE 6. Figure of an infinitely renormalizable Julia set. The first renormalization is of satellite type and the second one is of primitive type. The puzzle piece at the center , $Q_n^{\chi_1}$, is the first puzzle piece in the favorite nest.

In this case, $\mathbf{f}_{c_{n-1}}$ is immediately renormalizable and its second renormalization is of primitive type. Consider an equipotential of radius $\eta(\varepsilon)$ (see lemma 4.3.1), the external rays landing at the α_{n-1} fixed point and the external rays landing at the $\mathbf{f}_{c_{n-1}}$-orbit of the fixed point α_n of the renormalization of $\mathbf{f}_{c_{n-1}}$ (See figure 6). These rays and the equipotential move holomorphically inside the secondary wake $W(\eta)$ containing the secondary limb.

Now we need to introduce some new notations for puzzle pieces. Let us denote $\mathbf{f}_{c_{n-1}}^{t_n/t_{n-1}}$ by g (through this case), and let Y_0^0 as before, denote the puzzle piece containing the critical point which is bounded by the equipotential $E(\eta)$, the external rays landing at α_{n-1} and its $\mathbf{f}_{c_{n-1}}$-preimages ($\omega^i\alpha_{n-1}$ where ω is a dth root of unity). External rays landing at α_n and their g-preimages cut the puzzle piece Y_0^0 into some pieces. Let us denote the critical one by B_0^0, the non-critical ones which have a boundary ray landing at α_n fixed point by C_i^0 and the rest of them by A_j^0 (these are the ones which have a boundary external ray landing at $\omega^i\alpha_n$).

Consider the critical puzzle piece Y_0^0, g-preimage of this set along the postcritical set is inside itself and since all processes of making modified principal nest and pseudo-conjugacy are obtained by pullback arguments, the same ideas will be applicable except that we do not have equipotentials for the second renormalization. As we will see in a moment, external rays and part of the equipotential bounding Y_0^0 will play the role of a equipotential for the second renormalization of $\mathbf{f}_{c_{n-1}}$.

As $\mathbf{f}_{c_{n-1}}$ is immediately renormalizable, every $g^n(0)$ belongs to Y_0^0 and since the second renormalization of $\mathbf{f}_{c_{n-1}}$ is of primitive type, there is a first moment t such that $g^t(0)$ is in A_1^0 (rearrange the indices if required). Pulling back A_1^0 by g^{-t} along the critical orbit, we get a critical puzzle piece $Q_n^{\chi_1}$ which is strictly inside B_0^0. This is because B_0^0 is bounded by the external rays landing at α_n and its g-preimages, so if $Q_n^{\chi_1}$ intersects boundary of B_0^0, orbit of this intersection under g^k for $k \geq 2$ will stay on the rays landing at α_n which implies that image of $Q_n^{\chi_1}$ can never be A_1^0. Obviously they can not intersect at equipotentials. Now, consider the first moment $m > t$ when $g^m(0)$ returns back to $Q_n^{\chi_1}$ and pullback $Q_n^{\chi_1}$ by g^{-m} along the critical orbit to get $P_n^{\chi_1}$. The rest of the process to form the favorite nest is the same as in section 3.2

The Dilatation of the K_1-qc map obtained in the Theorem 3.2.2 depends on the hyperbolic distance between c_{n-1} and $\tilde{c}_{n-1}$ inside one of the secondary wakes under our consideration, $W(\eta)$, which only depends on the *a priori* bounds ε and the combinatorial condition. So we obtain a K_2-qc map $\mathbf{h}_{n-1} = \mathbf{h}_{n-1,0}$ from Y_0^0 to $\tilde{Y}_0^0$. Lifting this map by $\mathbf{f}_{c_{n-1}}$ we have a K_2-qc pseudo-conjugacy from the interior of equipotential $E(\eta)$ to the interior of equipotential $\tilde{E}(\eta)$. In other words, it is obtained by lifting $\mathbf{h}_{n-1}$ on the domain of $L_{Y_0^0}$ (first landing map on Y_0^0) and taking the Böttcher coordinate on the complement of Julia set. As the residual set is hyperbolic, this map extends to a qc map with the same dilatation as dilatation of K_2.

Now, we use the same method as Case (I) to adjust this map to get a new K_3-qc map $\mathbf{h}'_{n-1,0}$ which sends $S_{n-1}(V_{n,it_{n-1}})$ to $\tilde{S}_{n-1}(\widetilde{V}_{n,it_{n-1}})$ for $i = 0, 1, 2, ..., t_n/t_{n-1} - 1$. Now, $\Delta_{n-1,0}$ is defined as S_{n-1}-pullback of the domain inside the equipotential $E(\eta)$ and $h_{n-1,0}$ is defined as the lift of $\mathbf{h}'_{n-1,0}$ by S_{n-1} and $\tilde{S}_{n-1}$. The domain $\Omega_{n-1,0}$ is $\Delta_{n-1,0}$ minus topological disks $V_{n-1,it_{n-1}}$ for $i = 0, 1, 2, ..., t_n/t_{n-1} - 1$. The sets $\Delta_{n-1,i}$, $\Omega_{n-1,i}$ and the maps $h_{n-1,i}$ for $i = 1, 2, 3, ...t_{n-1}$ are defined as pullbacks and lifts of $\Delta_{n-1,0}$, $\Omega_{n-1,0}$ and $h_{n-1,0}$ by f^{-i}. By taking $K = \max\{K_1, K_3\}$, $h_{n,i}$'s are K^3-qc. The same reason as the one in Case (I) shows that $\Omega_{n-1,i}$ are well inside the disks $V_{n-1,i}$.

Case (III) :

Consider the K_1-qc straightening (2) for the polynomial-like map $R^{n-1}f$: $V_{n-1,0} \to U_{n-1,0}$ and the corresponding polynomial $\mathbf{f}_{c_{n-1}}$ which is twice immediately renormalizable in this case. The argument in this case relies more on the compactness of the parameters under consideration rather than a dynamical construction.

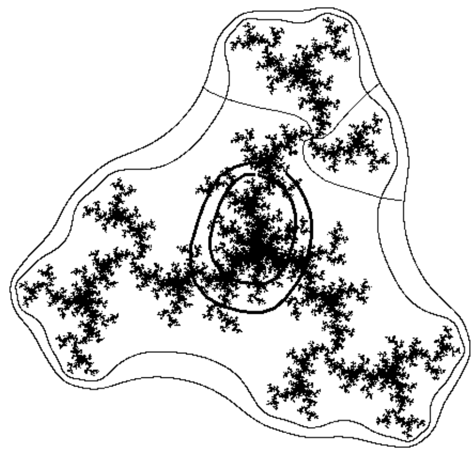


FIGURE 7. A twice satellite renormalizable map. It also shows the domains $L_{n-1,0}$ and $L'_{n-1,0}$.

Little Julia sets of the first renormalization of $\mathbf{f}_{c_{n-1}}$ are joined at the α fixed point of $\mathbf{f}_{c_{n-1}}$. Note that this fixed point is one of the β fixed points of each little Julia set. Lets call the union of this little Julia sets Julia *bouquet* and denote it by $B_{1,0}$. Similarly the little Julia sets $J_{2,i}$, $i = 0, 1, 2, ..., (t_{n+1}/t_{n-1}) - 1$ of the second renormalization of $\mathbf{f}_{c_{n-1}}$ are organized in pairwise disjoint *bouquets*, $B_{2,j}$, of Julia sets touching at the same periodic point, that is, each $B_{2,j}$ consists of t_{n+1}/t_n little Julia sets $J_{2,i}$ touching at one of their β fixed points (as usual $B_{2,0}$ denotes the one containing the critical point).

By an equipotential of radius η depending on ε obtained in lemma 4.3.1 and the external rays landing at the α fixed point of $\mathbf{f}_{c_{n-1}}$ we have the puzzle pieces of level zero. Recall that Y_0^0 denotes the critical one. Now we want to show that the annulus $Y_0^0 \setminus B_{2,0}$ has definite Modulus once we restrict the parameter to the finite number of truncated primary limbs.

The *Hausdorff distance*, $d_H(A, B)$, between two compact subsets A and B of the complex plane $\mathbb{C}$ endowed with the Euclidean metric d is the infimum of all the ε such that the ε neighborhood (with metric d) of A covers B and vise versa. A set valued map $c \mapsto A_c$ is said to be *upper semi-continuous* if $c_n \to c$ implies that the Hausdorff limit of A_{c_n} is contained in A_c.

Let us say that a family of simply connected domains U_λ depends continuously on λ if there exits a continuous choice of uniformizations, ψ_λ, of the domains U_λ.

Lemma 4.3.5. *Let $(P_\lambda : U_\lambda \to V_\lambda, \lambda \in \Lambda)$ be a continuous family of polynomial like maps with connected filled Julia sets K_λ. The map $\lambda \mapsto K_\lambda$ is upper semi-continuous.*

Proof. Let $\lambda_n \mapsto \lambda$ and assume $K_n = K_{\lambda_n}$ and K_λ denote the filled Julia sets of $P_n = P_{\lambda_n}$ and P_λ respectively . To prove $K = \lim K_n$ is contained in K_λ it is enough to show that for every $\varepsilon > 0$, $K_n \subseteq B_\varepsilon(K_\lambda)$ for sufficiently large n's. That is because by definition of the Hausdorff limit, for a given $t \in K$ there exists a sequence of $t_n \in K_n$ which converges to t. If $K_n \subseteq B_\varepsilon(K_\lambda)$ for arbitrary $\varepsilon > 0$ and large n's as K_λ is compact we conclude that $t \in K_\lambda$.

To see that $K_n \subseteq B_\varepsilon(K_\lambda)$ for large n's, assume x is not in $B_\varepsilon(K_\lambda)$ then there exists a finite time s such that $P_\lambda^s(x) \in V_\lambda \setminus U_\lambda$. As P_{λ_n} converges to P_λ and the simply connected domains depend continuously on λ, $P_n^s(x)$ or $P_n^{s+1}(x)$ belongs to $U_{\lambda_n} \setminus V_{\lambda_n}$ which implies that x is not in K_n. $\square$

Actually one can show that the above map is continuous at the parameters c such that $K_c = J_c$ but we don't need this statement here.

For the satellite renormalizable parameter c_{n-1} in a truncated primary limb, the equipotential of radius η moves continuously (indeed holomorphically) and one can easily construct a continuous "thickening" of the domains to obtain a continuous family of polynomial like maps. As the Julia Bouquet is compactly contained in the interior of Y_0^0 for any such a parameter and the set of parameters in our assumption in the Theorem 4.1.1 is compact, applying the above lemma implies that there is a definite space around $B_{2,j}$ inside Y_0^0.

From the above argument we conclude that there are simply connected domains $\mathbf{L}'_{n-1} \subseteq \mathbf{L}_{n-1}$ (and the tilde objects) such that the annuli $Y_0^0 \setminus \mathbf{L}_{n-1}$, $\mathbf{L}_{n-1} \setminus \mathbf{L}'_{n-1}$, $\mathbf{L}'_{n-1} \setminus B_{2,0}$ (and the corresponding tilde objects) have definite (depending only on ε) and bounded above moduli. This implies that there exits a K-qc map $\mathbf{h}_{n-1}$ from $Y_0^0 \setminus \mathbf{L}_{n-1}$ to $\tilde{Y}_0^0 \setminus \widetilde{\mathbf{L}}_{n-1}$ which matches the Böttcher marking on the boundary of Y_0^0.

Now by lifting $\mathbf{h}_{n-1}$ via $\mathbf{f}_{c_{n-1}}^{-l}$ and $\mathbf{f}_{\tilde{c}_{n-1}}^{-l}$ we obtain K-qc maps from $\mathbf{f}_{c_{n-1}}^{-l}(Y_0^0 \setminus \mathbf{L}_{n-1})$ to $\tilde{\mathbf{f}}_{c_{n-1}}^{-l}(\tilde{Y}_0^0 \setminus \tilde{\mathbf{L}}_{n-1})$ for $l = 1, 2, ..., t_n/t_{n-1} - 1$. The domain of each such a map is a puzzle piece Y_j^0 ($j = t_n/t_{n-1} - l$) cut off by the equipotential of radius η/d^l, minus $\mathbf{f}_{c_{n-1}}^{-l}$-preimage of $\mathbf{L}_{n-1}$. As all these maps match the Böttcher marking on the boundaries of their domains, they glue together to give a well defined K-qc map. Finally one can extend this map by Böttcher coordinates on the spaces between equipotential of radius η and equipotentials of radius η/d^l which will be denoted with the same notation $\mathbf{h}_{n-1}$.

Like in the previous Cases, $\Delta_{n-1,0}$ is defined as S_{n-1} pullback of the domain inside equipotential $E(\eta)$ and $\Omega_{n-1,0}$ is defined as $\Delta_{n-1,0}$ minus $L_{n-1,l} = S_{n-1}^{-1}(\mathbf{f}_{c_{n-1}}^{-l}(\mathbf{L}_{n-1}))$ for $l = 0, 1, ..., t_n/t_{n-1} - 1$. The map $h_{n-1,0}$ is defined as lift of $\mathbf{h}_{n-1}$ on this set. The same argument shows that $\Omega_{n-1,i}$ are well inside the disks $V_{n-1,i}$. This completes the construction in case (III).

For a given infinitely renormalizable map f_c, the renormalization on each level is of primitive or satellite type, so we may associate a word $P \dots PS \dots SP \dots = P^{l_1} S^{l_2} P^{l_3} \dots$ (l_j's are non negative integers) where a P or S on the i's place means that the i's renormalization of f_c is of primitive or satellite type. Corresponding to any such a word, there is a word of cases $(I)^{m_1}(II)^{m_2}(III)^{m_3}\dots$ (for non negative m_j's) which is constructed as follows. For a given word of S and P, starting from left, a P will be replaced by (I), SP by (II) and SS by (III)S. By repeating this process, we obtain a word of cases which will be our guide to build the domains.

To finish building the multiply connected domains $\Omega_{n(k),i}$ (and $\widetilde{\Omega}_{n(k),i}$) and the K-qc maps $h_{n(k),i} : \Omega_{n(k),i} \to \widetilde{\Omega}_{n(k),i}$ we follow the latter word constructed above. In cases (I) and (II) we have adjusted the map $\mathbf{h}_{n-1}$ such that it sends boundary of $V_{n-1,0}$ to the boundary of $\tilde{V}_{n-1,0}$ therefore if any of the three cases of construction is following Case (I) or (II), we consider $R^n f : W_{n,0} \to V_{n,0}$ where $W_{n,0}$ is the component of $f^{-t_n}(V_{n,0})$ inside $V_{n,0}$ and straighten it for these choices of the domains (instead of $R^n f : V_{n,0} \to U_{n,0}$). If a case of construction on level n is following the case (III) the set $\mathbf{\Delta}_{n,0}$ constructed on level n (any of the three cases) will be replaced by $\mathbf{\Delta}_{n,0} \cap S_n(L'_{n-1})$ and $\mathbf{h}_{n,0}$ will be restricted to this set and adjusted so that it sends $S_n(L'_{n-1,i})$ to $\tilde{S}_n(\tilde{L}'_{n-1,i})$. The annulus $L_{n-1} \setminus L'_{n-1}$ will provide the definite space between $\Omega_{n,0}$ and $L_{n-1,0}$.

In the following two sections, we will denote the holes of the domains $\Omega_{n,i}$ by $\mathbb{V}_{n+1,j}$, that is, $\mathbb{V}_{n+1,j}$ is $V_{n+1,j}$, if n belongs to the Case (I) or (II) in the construction or $\mathbb{V}_{n+1,j}$ is $S_n^{-1}(L_n)$ if n belongs to the Case (III).

4.4. **Gluing the maps $h_{n,i}$.** In this section we will construct $K'(\varepsilon)$-qc maps $g^i_{n(k)}$ from the annuli $\mathbb{V}_{n(k),i} \setminus \Delta_{n(k+1),i}$ to the corresponding ones for the tilde objects. Any $g^i_{n(k)}$ has to be identical with $h_{n(k),i}$ on the boundary of $\mathbb{V}_{n(k),i}$ and identical with $h_{n(k+1),i}$ on the boundary of $\Delta_{n(k),i}$ (which is outer boundary of $\Omega_{n(k),i}$). Then gluing all these maps $g^i_{n(k)}$ and $h_{n(k),i}$ will give a qc map, H, with dilatation bounded by maximum of $K(\varepsilon)^3$ and $K'(\varepsilon)$. In what follows, we will use the index n instead of $n(k)$ and assume n runs over subsequence $n(k)$, so for all n, means for all $n(k)$'s.

Like constructions in the previous sections, it is enough to construct g^0_n for all n and pull them back by f^{-i} to obtain g^i_n. Construction of the maps $h_{n,i}$ and the domains $\mathbb{V}_{n,i}$ and $\Omega_{n,i}$ implies that these maps glue together on the boundaries of their domain's of definition. For simplicity, let us denote the map g^0_n by g_n. To build a qc map from an annulus to another annulus with given boundary conditions, there is a choice of the number of "twists" one may consider. Moreover, to have a uniform bound on the dilatation of such a map, the two annuli must have proportional moduli and the number of twists has to be uniformly bounded. The number of twists which is an integer number effects on the homotopy class of the Thurston conjugacy H.

In this section, we will show that the corresponding annuli $\mathbb{V}_{n,0} \setminus \Delta_{n,0}$ and $\widetilde{\mathbb{V}}_{n,0} \setminus \widetilde{\Delta}_{n,0}$ (for all n) have proportional moduli with a constant only depending on the *a priori* bounds ε and also we will define the twist and its relation with the dilatation of the maps g_n. In the next section we will specify the number of twists needed for the isotopy class of the Thurston map.

Lemma 4.4.1. *Let $U' \subset U$ and $\widetilde{U}$ be three annuli whose inner boundaries are the unit circle such that $mod(U \setminus U') \geq \varepsilon$ and $mod(U)$ is bounded from above by some constant M. Let ψ be a K-qc map from U onto $\widetilde{U}$. For any r with D_r contained in U' and $\psi(U')$, the moduli of the annuli $U \setminus D_r$ and $\widetilde{U} \setminus D_r$ are proportional with a constant depending only on M, K and ε.*

Proof. By properties of qc maps we have
$\varepsilon \leq mod(U \setminus D_r) \leq M$ and
$\varepsilon/K \leq mod(\widetilde{U} \setminus D_r) \leq KM$
which implies the lemma.

□

Proposition 4.4.2. *For the topological disks $\mathbb{V}_{n,0}$ and $\Delta_{n,0}$ as above, the moduli of the annuli $\mathbb{V}_{n,0} \setminus \Delta_{n,0}$ and $\widetilde{\mathbb{V}}_{n,0} \setminus \widetilde{\Delta}_{n,0}$ are proportional with a constant depending only on ε.*

Proof. If level n follows case (I) or (II) in the construction, Proof is applying the above lemma to the images of the domains $\mathbb{V}_{n,0}$, $f^{-t_n}(\mathbb{V}_{n,0})$ and $\Delta_{n,0}$ under the map $B_{c(R^n f)} \circ S_n$ and the corresponding tilde objects, where $B_{c(R^n f)}$ is the Böttcher coordinate for the map P_{c_n} and S_n is the straightening of the map $R^n f$. Note that $\Delta_{n,0}$ is mapped to the disk of radius r. To have an upper bound M, it is enough to go some levels lower than the fundamental annulus to have bounded modulus.

If level n follows case (III), by definition $\mathbb{V}_{n,0} \setminus \Delta_{n,0}$ is $L_n \setminus L'_n$ and $\tilde{\mathbb{V}}_{n,0} \setminus \tilde{\Delta}_{n,0}$ is $\tilde{L}_n \setminus \tilde{L}'_n$ which were chosen to be proportional.

□

When a primitive renormalization follows an immediately renormalization, the little Julia sets of the primitive one can be arbitrarily close to the β fixed point of the satellite renormalization and there would not be enough space for gluing. That is why we had to consider these two Cases together.

For a given curve ℓ in an annulus U which is image of a continuous map $f : [a, b] \to U$ such that $f(a)$ is on the inner boundary of U (corresponding to the bounded component of the complement of U) and $f(b)$ is on the outer boundary of U (corresponding to the unbounded one), define the *wrapping number* $\omega(\ell)$ as $[\theta(\phi(f(b))) - \theta(\phi(f(a)))]/2\pi$ where ϕ is a uniformization of the annulus U by the round annulus and θ is the polar angel function calculated continuously at the angel $2k\pi$. Basically, $\omega(\ell)$ is the total turning of the curve ℓ in the uniformized coordinate. Note that $\omega(\ell)$ is invariant under the Automorphism group of the annulus, So it is independent of the choice of uniformization and just like winding number, it is constant over the homotopy class of curves with the same boundary points.

Let $U(r)$ denote the round annulus bounded by circles of radii r and 1. We have:

Lemma 4.4.3. *Given constants $K \geq 1$ and $R > 1$. For every K-qc map $\psi : V(R) \to V(R')$, the image of the horizontal line segment $[1, R]$ under ψ is a curve with wrapping number bounded by $-N$ and N, where N depends only on K and R.*

Proof. Consider the curve family L consisting of all ray segments in $U(R)$ obtained from rotating the segment $[1, R]$ about origin, that is the radial lines in $V(R)$.

Let us denote by $\Gamma(F)$ the extremal length of a given curve family F. We have the inequality $\Gamma(L)/K \leq \Gamma(\psi(L)) \leq K\Gamma(L)$. See [Ah] for more details on curve families and extremal length properties.

It is easy to see that if the interval $[1, R]$ is mapped to a curve with wrapping number T then every curve in L will be mapped to a curve with wrapping number between $T + 1$ and $T - 1$.

By definition of extremal length and choosing conformal metric ρ as the Euclidian metric, we obtain

$$K\Gamma(L) \geq \Gamma(\psi(L)) = \sup_{\rho} \frac{\inf \ell_\rho(\psi(\gamma))^2}{S_\rho} \geq \frac{4\pi(T-1)^2}{R'^2 - 1}.$$

and $\Gamma(L) = \log R/2\pi$

By properties of qc maps we have $R' \leq R^K$ which implies

$$T \leq \frac{1}{\pi}\sqrt{\frac{K\log(R)(R^{2K} - 1)}{8}} + 1$$

∎

Let $S(r, 0)$ denote the circle of radius r about origin, then

Lemma 4.4.4. *Fix round annuli $V(r)$, $V(r')$ such that $mod(V(r'))/K_1 \leq mod(V(r)) \leq K_1 mod(V(r'))$ and integer number $k \leq N$. If homeomorphisms $h_1 : S(r, 0) \to S(r', 0)$ and $h_2 : S(1, 0) \to S(1, 0)$ have K_2-qc extension to some neighborhood of these circles then there exists a K-qc map from $V(r)$ to $V(r')$ which matches with h_i's on the boundary circles and sends the segment $[1, r]$ to a curve with wrapping number $(\theta(h_1(r)) - \theta(h_2(1))) + k$. Moreover K only depends on K_1, K_2 and N.*

Proof. Proof is by constructing such maps for every k. More details left to the reader. ∎

Applying the above lemma to the uniformization of the annuli $\mathbb{V}_{n,0} \setminus \Delta_{n,0}$ and $\widetilde{\mathbb{V}}_{n,0} \setminus \widetilde{\Delta}_{n,0}$, the induced maps from $h_{n-1,0}$ and $h_{n,0}$ on their boundaries and a number k_n will give the required K'-qc maps g_n. In the next section we will specify some especial numbers k_n (which are bounded by a constant only depending on ε) in order to make the K-qc map H, obtained after gluing, homotopic to the topological conjugacy relative the postcritical set. Note that in this Case all the constants K_1, K_2 and N depend only on ε, so dilatation of the map H only depends on ε.

Definite modulus in between the annuli $\mathbb{V}_{n,i}\setminus\Delta_{n,i}$ implies that the holes $\mathbb{V}_{n,i}$ shrink to the postcritical set, so the postcritical set is removable and H can be extended to a well defined K-qc map on the postcritical set. See [St] for more details on removability of sets and extensions of quasi-conformal maps onto them.

4.5. **Proof of isotopy.** In this subsection we will use the same notations as we used in the previous two subsections. Recall that S_n (and $\tilde{S}_n$) is the straightening of the renormalization $R^n f$ (respectively $R_n\tilde{f}$). Let $\mathbf{f}_{c_n} = \mathbf{f}_{c(R^n f)}$, $\mathbf{f}_{\tilde{c}_n} = \mathbf{f}_{c(R^n \tilde{f})}$ and denote by $\boldsymbol{\psi}_{n,0}$ the topological conjugacy between $\mathbf{f}_{c_n}$ and $\mathbf{f}_{\tilde{c}_n}$. So, $\psi_{n,0} = \tilde{S}_n^{-1} \circ \boldsymbol{\psi}_{n,0} \circ S_n$ is a topological conjugacy between $R^n f$ and $R^n\tilde{f}$ on a neighborhood of the little Julia set $J_{n,0}$. Note that this neighborhood covers the domain $\Omega_{n,0}$. In the dynamic plane for $\mathbf{f}_{c_n}$, let $U(\eta)$ denote the domain inside the equipotential of radius η.

Lemma 4.5.1. *If level n belongs to the case(I) or (II), the K-qc maps $h_{n,i} : \Delta_{n,i} \to \widetilde{\Delta}_{n,i}$ are homotopic to*

$$\psi_{n,i} = f^i_{\tilde{c}_n} \circ \tilde{S}^{-1}_{n,i} \circ \boldsymbol{\psi}_{n,i} \circ S_{n,i} \circ f^{-i}_{c_n} : \Delta_{n,i} \to \mathbb{C}$$

relative the little Julia sets $J_{n+1,j}$ of level $n+1$ inside $\Delta_{n,i}$.

Note that $\psi_{n,i}(\Delta_{n,i})$ is a neighborhood of the little Julia sets $\tilde{J}_{n+1,i+t_n j}$ which are contained in $\widetilde{\Delta}_{n,i}$.

Proof. First assume level n follows a case (I) or (II). From the definition of the domains $\Delta_{n,i}$, $\mathbb{V}_{n,i}$ and the K-qc maps $h_{n,i}$, it is enough to prove the statement only for $i = 0$ (pull the homotopies back by f^i or construct them in the same way as for $i = 0$).

As $\Delta_{n,0}$, $\psi_{n,0}$ and the K-qc map $h_{n,0}$ are lifts of $\boldsymbol{\Delta}_{n,0}$, $\boldsymbol{\psi}_{n,0}$ and $\mathbf{h}'_{n,0}$ by straightenings, it is enough to make the homotopy on the dynamic planes for $\mathbf{f}_{c_n}$ and $\mathbf{f}_{\tilde{c}_n}$ and then transfer it to the dynamic planes for $R^n f$ and $R^n\tilde{f}$. Recall that in our construction, $\mathbf{h}'_{n,0}$ was adjustment of $\mathbf{h}_{n,0}$ through some homotopy relative little Julia sets, so proof of this lemma reduces to the homotopy of $\mathbf{h}_{n,0}$ and $\boldsymbol{\psi}_{n,0}$ relative the little Julia sets.

Assume we are in the first Case of the construction. The idea of the proof is to divide the domain $\boldsymbol{\Delta}_{n,0}$ (by means of rays and equipotential arcs) into some topological disks and one annulus such that $\boldsymbol{\psi}_{n,0}$ and $\mathbf{h}_{n,0}$ are identical on the boundaries of these domains.

Consider the puzzle piece $Q^{\chi_n}_{n,0}$ ($Q^{\chi_n}_{n,0} = Y_0^{q_{\chi_n}}$). The equipotential $f^{-\chi_n}(E(\eta))$ and the rays of the puzzles $Q^{\chi_n}_{n,i}$ up to equipotential $f^{-\chi_n}(E(\eta))$ cut the domain $\boldsymbol{\Delta}_{n,0}$ into one annulus($\boldsymbol{\Delta}_{n,0} \setminus f^{-\chi_n}(U(\eta))$) and some topological disks. The topological disks (obtained above) which do not intersect the little Julia sets of level n, the puzzle pieces $Q^{\chi_n}_{n,i}$ and the remaining annulus, $E(\eta) \setminus f^{-\chi_n}(E(\eta))$, are the required domains. Construction in the Theorem 3.2.2 implies that the maps $\mathbf{h}_{n,0}$ and $\boldsymbol{\psi}_{n,0}$ are identical (in the Böttcher coordinate) on the boundaries of these domains. Indeed, the topological conjugacy $\boldsymbol{\psi}_{n,0}$ between $\mathbf{f}_{c_n}$ and $\mathbf{f}_{\tilde{c}_n}$ is identity in the Böttcher coordinate and the pseudo-conjugacy $h_{n,0}$ constructed in the Theorem 3.2.2 matches the Böttcher marking. This proves the homotopy of the two maps outside of the puzzle pieces $Q^{\chi_n}_{n,i}$. To show the homotopy inside puzzle pieces $Q^{\chi_n}_{n,0}$, recall that we started with a qc map on $Q^{\chi_n}_{n,0} \setminus P^{\chi_n}_{n,0}$ which was homotopic to $\boldsymbol{\psi}_{n,0}$ relative boundaries. So all the pullbacks of this map on the annuli $A^k \setminus A^{k+1}$ are homotopic to $\boldsymbol{\psi}_{n,0}$ relative boundaries which implies that these two maps are homotopic and moreover, they are identical on the little Julia sets. Proof for the second Case of construction is applying above argument on every puzzle piece of level zero.

The same proof works if level n follows the case (III). The only difference is that the domain of definition of the homeomorphisms are restricted to a smaller set. In this case we may restrict the homotopy to that domain. $\square$

Now, we will introduce the specific numbers k_n required for the gluing maps g_n in the previous section. In the following, let $V(r)$ denote the annulus bounded by circles of radii r and one.

If level n belongs to case (I) or (II) and it follows a case (I) or (II) in our construction, consider the uniformizations $\phi_1 : V(s) \to (\mathbb{V}_{n,0} \setminus K_{n,0})$, $\phi_2 : V(r) \to (\Delta_{n,0} \setminus K_{n,0})$, $\widetilde{\phi}_1 : V(\tilde{s}) \to (\widetilde{\mathbb{V}}_{n,0} \setminus \tilde{K}_{n,0})$ and $\widetilde{\phi}_2 : V(\tilde{r}) \to (\widetilde{\Delta}_{n,0} \setminus \tilde{K}_{n,0})$ by round annuli. The qc maps $h_{n-1,0}$ and $h_{n,0}$ will lift via ϕ_i and $\widetilde{\phi}_i$ to qc maps $\hat{h}_{n-1,0} : V(s) \to V(\tilde{s})$ and $\hat{h}_{n,0} : V(r) \to V(\tilde{r})$ with the same dilatation. By composing the uniformizations with rotations, we may assume that the point one is mapped to the point one by these two maps. By lemma 4.4.3, the image of the segment $[1, s]$ under the qc map $\hat{h}_{n-1}$ has wrapping number ω_{1n} bounded by some N and image of the segment $[1, r]$ under the qc map $\hat{h}_{n,0}$ has wrapping number ω_{2n} bounded by N which depends only on ε. Take our favorite wrapping number k_n as $\omega_{1n} - \omega_{2n}$ and note that gluing $\hat{h}_{n,0}$ and $\hat{h}_{n-1,0}$ in the lemma 4.4.4 by such a choice makes the image of the segment $[1, s]$ under the two maps g_n and $\hat{h}_{n,0}$ glued together homotopic to image of the segment $[1, s]$ under the map $\hat{h}_{n-1,0}$ relative two boundary circles. This homotopy will lift to a homotopy between $h_{n-1,0}$ and the two maps g_n and $h_{n,0}$ glued together.

Before we define the numbers k_n for the other cases, we need to show that the qc map $\mathbf{h}'_{n-1}$ constructed in the third case has qc extension over the topological disk $\mathbf{L}_{n-1}$.

Lemma 4.5.2. *The K-qc map $\mathbf{h}'_{n-1}$ constructed in the case (III) has a qc extension onto the topological disk $\boldsymbol{L}_{n-1}$ with bounded dilatation depending only on ε and moreover this extension is homotopic to the topological conjugacy $\boldsymbol{\psi}_{n-1,0}$ relative the Julia bouquet $B_{2,0}$ inside $\boldsymbol{L}_{n-1}$.*

Proof. Consider the fundamental annuli $S_{n-1}(U_{n,0} \setminus V_{n,0})$ and $\tilde{S}_{n-1}(\tilde{U}_{n,0} \setminus \tilde{V}_{n,0})$ for the first renormalizations of $\mathbf{f}_{c_{n-1}}$ and $\mathbf{f}_{\tilde{c}_{n-1}}$. Let

$$g_n : S_{n-1}(U_{n,0} \setminus V_{n,0}) \to \tilde{S}_{n-1}(\tilde{U}_{n,0} \setminus \tilde{V}_{n,0})$$

be a qc map which satisfies the equivariance relation on the boundaries of these annuli. By lifting the map g_n on the preimages of these annuli we obtain a qc map g_n from complement of the little Julia set $J_{1,0}$ to the complement of the little Julia set $\tilde{J}_{1,0}$ on the dynamic planes of $\mathbf{f}_{c_{n-1}}$ and $\mathbf{f}_{\tilde{c}_{n-1}}$. By lemma 4.3.2, g_n (or some rotation of it) can be extended on to the the Julia set $J_{1,0}$ as $\boldsymbol{\psi}_{n-1,0}$. Moreover these two maps are homotopic relative this little Julia set. Now we can adjust the map g_n so that it sends $\mathbf{L}'_{n-1}$ to $\tilde{\mathbf{L}}'_{n-1}$. We have three annuli $Y_0^0 \setminus \mathbf{L}_{n-1}$, $\mathbf{L}_{n-1} \setminus \mathbf{L}'_{n-1}$ and $\mathbf{L}'_{n-1} \setminus B_{2,0}$ and the corresponding tilde objects. The map $\mathbf{h}_{n-1}$ is from the first annulus to the corresponding tilde one and the map g_n is from the last annulus to the corresponding tilde one. To glue these two maps on the middle annulus we use the above argument to find the right number of twists on this annulus. Consider a curve γ connecting a point, a, on the bouquet $B_{2,0}$ to a point, d, on the boundary of Y_0^0 such that it intersects the boundaries of the domains $\mathbf{L}'_{n-1}$ and $\mathbf{L}_{n-1}$ only at one point denoted by b and c. Lets denote by γ_{ab}, γ_{bc} and γ_{cd} each segment of this curve between these four points. Consider the integer number $\omega(\boldsymbol{\psi}(\gamma)) - \omega(\mathbf{h}_{n-1}(\gamma_{bd})) - \omega(g_n(\gamma_{ab}))$ which is bounded by lemma 4.4.3 depending only on ε. Now if we glue these two maps by such a number of twists (see lemma 4.4.4), the resulting map will be homotopic to the homeomorphism $\boldsymbol{\psi}_{n-1}$ relative the Julia bouquet $B_{2,0}$ and the boundary of Y_0^0. Note that the two maps $\mathbf{h}'_{n-1}$ and $\boldsymbol{\psi}_{n-1}$ are identical on the boundary of Y_0^0. In the same way one can extend this map over the other topological disks $\mathbf{L}_{n-1,i}$. We will denote the final qc map map by $\mathbf{h}'_{n-1}$. □

If a case (III) follows a case (I) or (II) the number of twists, k_n, is similar to the one introduced above and if level $n-1$ is in case (III) and level n is any of the three cases we use the uniformization of the annuli $\mathbb{V}_{n,0} \setminus B_{n,0}$ and $\Delta_{n,0} \setminus B_{n,0}$ and the corresponding tilde ones instead of the above annuli to define the right number of twists.

We will use the following easy lemma in the proof of isotopy. Proof is left to the reader.

Lemma 4.5.3. *Let U and $\widetilde{U}$ be two closed annuli with boundary curves respectively γ_i and $\tilde{\gamma}_i$ for $i = 1, 2$. Let $h_i^t : \gamma_i \to \tilde{\gamma}_i$ for $t \in [0, 1]$ be two continuous family of homeomorphisms and G^0 be a continuous interpolation of h_1^0 and h_2^0 on U, then G^0 can be extended to a continuous family of interpolations between h_1^t and h_2^t for $t \in [0, 1]$.*

Proposition 4.5.4. *The K-qc map H obtained from gluing g_n^i's and $h_{n,k}$'s is homotopic to the topological conjugacy Ψ relative the postcritical set $\mathcal{O}(0)$.*

Proof. Let H_n denote the map obtained from gluing n qc maps $h_{1,0}, h_{2,i}, ..., h_{n,j}$ and $g_1^0, g^k, \ldots, g_{n-1}^l$ for all possible indexes i, j, k and l's. First we will show that H_1 is homotopic to the topological conjugacy Ψ between $\mathbf{f}_c$ and $\mathbf{f}_{\tilde{c}}$ relative the little Julia sets $J_{1,i}$ and each H_{n-1} is homotopic to H_n relative the little Julia sets of level $n+1$. The map H_1 is just $h_{1,1}$ which is homotopic to $\psi_{1,0}$ by lemma 4.5.1 and this map is homotopic to Ψ by lemma 4.3.2. The two maps H_{n-1} and H_n are equal outside of the domains $\mathbb{V}_{n,j}$ by construction. Inside $\Delta_{n,0}$, H_{n-1} and H_n are $h_{n-1,0}$ and $h_{n,0}$ respectively.

The domain $\mathbb{V}_{n,0}$ is divided into annulus $\mathbb{V}_{n,0} \setminus \Delta_{n,0}$ and the topological disk $\Delta_{n,0}$. On $\Delta_{n,0}$, $h_{n,0}$ and $h_{n-1,0}$ are homotopic to $\psi_{n,0}$ relative $J_{n+1,i}$ by lemma 4.5.1 and 4.5.2. Then there is a homotopy h_n^t for t in $[0,1]$ which starts with $h_{n,0}$ and ends with $h_{n-1,0}$ and moreover it sends boundary of $\Delta_{n,0}$ to the boundary of $\widetilde{\Delta}_{n,0}$ for all t in $[0,1]$. At time zero consider the map $h_{n,0}$ on the inner boundary of this annulus, $h_{n-1,0}$ on the outer boundary of this annulus and the interpolation $G_n = g_n^0$ between them. Applying above lemma to the fixed homeomorphism $h_{n-1,0}$ on the outer boundary and h_n^t on the inner boundary we get a continuous family of interpolations G_n^t between them. The map G_n^1 is a homeomorphism from the annulus to itself which is an interpolation of $h_{n-1,0}$ on the boundaries, but this interpolation has to be homotopic to $h_{n-1,0}$ on the annulus. Indeed these two maps send a curve joining the two different boundaries to two curves (clearly joining the two boundaries) which are homotopic relative end points. This comes from our choices of wrapping numbers for gluing the maps.

Let $t_0 = 0 < t_1 < t_2, ... < 1$ be an increasing sequence of times in $[0,1]$. Assume H^t for t in $[t_0, t_1]$ be the homotopy obtained above between Ψ and H_1 relative the Julia set and H^t for t in $[t_n, t_{n+1}]$ be the homotopy between H_n and H_{n+1} relative the little Julia sets of level $n+2$ for n=1, 2, 3, ...

It is clear from our construction that $H^t(z)$ for a fixed z eventually stabilizes and equals to $H(z)$, indeed, *a priori* bounds implies that the diameter of the topological disks $\mathbb{V}_{n,i}$ goes to zero (by $n \mapsto \infty$), so the uniform distance between H^t and H is going to zero (by $t \mapsto 1$). We conclude that H^t for t in $[0,1]$ is the homotopy between the topological conjugacy Ψ and the Thurston conjugacy H relative the postcritical set. □

Proposition 4.5.5. *Suppose all infinitely renormalizable unicritical polynomials in a given combinatorial class $\tau = \{M_1, M_2, M_3....\}$ in $\mathcal{SL}$ enjoy a priori bound. Then qc conjugacy implies hybrid conjugacy in this class.*

Proof. If this is not true, there are two polynomials P_1 and P_2 which are qc equivalent but not hybrid equivalent. Form the set

$$\Omega = \{c \in \mathbb{C} |\ P_c \text{ is qc equivalent to } P_1\} = \{c \in \mathbb{C} |\ P_c \text{ is qc equivalent to } P_2\}.$$

We will show that Ω is both open and closed subset of $\mathbb{C}$ which is not possible.

Theorem 4.1.1 implies that qc conjugacy is equivalent to combinatorial conjugacy for this class and since the combinatorial class τ is intersection of closed sets, Ω is closed.

Consider a point P in Ω, P is not hybrid equivalent to both of P_1 and P_2 by assumption. Let us assume it is not hybrid equivalent to P_1 (for the other Case just change P_1 to P_2). Let $\phi_1 : \mathbb{C} \to \mathbb{C}$ be a K-qc conjugacy, $\phi_1 \circ P = P_1 \circ \phi_1$ and let μ_0 denote the standard complex structure on $\mathbb{C}$. By pulling back this complex structure via ϕ_1 we get a complex structure μ on $\mathbb{C}$ with dilatation $\frac{K-1}{K+1}$. Consider the complex structures $\mu_\lambda = \lambda\mu$ for λ in the disk of radius $\frac{K+1}{K-1}$ around origin. By applying measurable Riemann mapping Theorem (see [Ah]), There are qc maps ϕ_λ which map complex structure μ_λ to μ_0 and fix the origin and infinity (post compose with a möbius map if required). $P_\lambda = \phi_\lambda^{-1} \circ P_1 \circ \phi_\lambda$ are holomorphic maps of the same degree as degree of P_1 and sends infinity to infinity with the same degree as the degree of P_1, so they are polynomials. For $\lambda = 1$ we will get the polynomial P and for $\lambda = 0$ we get P_1 and By analytic dependence of the solution of the measurable Riemann mapping theorem on complex structure, P_λ will cover a neighborhood of P in Ω. This shows that P is an interior point in Ω and as P was an arbitrary point of Ω, we conclude that Ω is open.

□

It has been shown in [KL1] that infinitely renormalizable parameter values satisfying *decorations* enjoy *a priori* bounds. The dynamical meaning of a parameter c satisfying this condition is that there exists an integer M such that for all renormalizations $f_n = R^n(P_c)$ there exists $t, q < M$ such that $f_n^{kq}(0) \in Y_0^1$ for $k < t$ and $f^{tq}(0) \notin Y_0^1$.

An infinitely renormalizable parameter is said to be of bounded type if the relative return times t_{n+1}/t_n of renormalizations , $R^n(f) = f^{t_n}$, are bounded By some constant M for all renormalization levels. Clearly, the decoration condition includes infinitely primitive renormalizable parameters of bounded type.

To a given infinitely renormalizable unicritical polynomial-like map f, we associated a sequence of maximal Multibrot copies $\tau(f) = \{M^1, M^2, ...\}$ in section 3.3. Let $\pi_n(\tau) = M^n$ for any τ and $n \geq 1$. Define

$$\tau(f, n) = \{c \in M_d | \pi_n(\tau(f)) = \pi_n(\tau(P_c))\}.$$

In other words, the Multibrot copy $\tau(f, n)$ is the set of at least n times renormalizable parameters with the same combinatorics as of f up to level n.

For a given infinitely renormalizable map f as above and an increasing subsequence of renormalization levels $\{n_i\}$, we define a sequence of relative Multibrot copies of M_d in M_d as follows:

$$(\tilde{\tau}(f), \{n_i\}) = \{\tilde{M}^{n_1}, \tilde{M}^{n_2}, \dots, \tilde{M}^{n_k}, \dots\}$$

where $\tilde{M}^{n_k} = \tau(R^{n_{k-1}} f, n_k - n_{k-1})$.

One can see that there is a one to one correspondence between these two sequences thus one may define the latter one to be the combinatorics of f.

Consider the main hyperbolic component of the Multibrot set M_d. There are infinitely many primary hyperbolic components attached to it. Similarly, there are infinitely many hyperbolic components, secondary ones, attached to these components and so on. Consider the set of all hyperbolic components obtained this way, that is, the ones which can be connected to the main hyperbolic component by a chain of hyperbolic components bifurcating one from another. Take the closure of this set and fill it in (i.e. adding the bounded components of its complement[1]). We obtain a set which will be called the *molecule* $\mathcal{M}_d$ of M_d.

An infinitely renormalizable map f satisfies the molecule condition if there exists a positive number $\eta > 0$ and an increasing subsequence $\{n_i\}$ of renormalization levels such that $R^{n_i} f$ is primitive renormalization of $R^{n_i - 1} f$ and the distance between the Multibrot copy $\tilde{M}_d^{n_i}$ and the molecule $\mathcal{M}_d$ is at least η for all i. Note that for a map satisfying this condition, there may be several satellite renormalizable maps once in a while in the sequence $\{R^n f\}$. The condition requires that there are infinite number of primitive levels with the corresponding Multibrot copies uniformly away from the molecule. It is obvious that the parameters in decoration condition satisfy the molecule condition.

For any given $\varepsilon \geq 0$ and hyperbolic component of the Multibrot set, There are at most finite number of limbs with diameter bigger that ε attached to this hyperbolic component, (see [H]). This implies that all the secondary limbs except finite number of them are included in the η neighborhood of the molecule for any $\eta \geq 0$. This implies that the parameters satisfying the molecule condition are in $\mathcal{SL}$ so we have the following.

Corollary 4.5.6. *Let f and $\tilde{f}$ be two infinitely renormalizable unicritical degree d polynomial-like maps satisfying molecule condition. If f and $\tilde{f}$ are combinatorially equivalent then they are hybrid equivalent.*

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