THE EXTERNAL BOUNDARY OF THE BIFURCATION LOCUS M_2

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Abstract. Consider a quadratic rational self-map of the Riemann sphere such that one critical point is periodic of period 2, and the other critical point lies on the boundary of its immediate basin of attraction. We will give explicit topological models for all such maps. We also discuss the corresponding parameter picture.

1. Introduction

1.1. **The family V_2.** Consider the set V_2 of holomorphic conjugacy classes of quadratic rational maps that have a super-attracting periodic cycle of period 2 (we follow the notation of Mary Rees). The complement in V_2 to the class of the single map $z \mapsto 1/z^2$ is denoted by $V_{2,0}$. The set $V_{2,0}$ is parameterized by a single complex number. Indeed, for any map f in $V_{2,0}$, the critical point of period two can be mapped to ∞, its f-image to 0, and the other critical point to -1. Then we obtain a map of the form

$$f_a(z) = \frac{a}{z^2 + 2z}, \quad a \neq 0$$

holomorphically conjugate to f. Thus the set $V_{2,0}$ is identified with $\mathbb{C} - 0$.

The family V_2 is just the second term in the sequence $V_1, V_2, V_3, \dots$, where, by definition, V_n consists of holomorphic conjugacy classes of quadratic rational maps with a periodic critical orbit of period n. Such maps have one "free" critical point, hence each family V_n has complex dimension 1. Note that V_1 is the family of quadratic polynomials, i.e., holomorphic endomorphisms of the Riemann sphere of degree 2 with a fixed critical point at ∞. Any quadratic polynomial is holomorphically conjugate to a map $z \mapsto z^2 + c$. Thus V_1 can be identified with the complex c-plane. For a map $z \mapsto z^2 + c$, the "free" critical point is 0. The family V_1 is the most studied family in complex dynamics. The main object describing the structure of V_1 is the *Mandelbrot set* M defined as the set of all parameter values c such that the orbit of the critical point 0 under $z \mapsto z^2 + c$ is bounded.

Similarly to the case of quadratic polynomials, we can define the set M_2 (an analog of the Mandelbrot set for V_2) as the set of all parameter values a such that the orbit of -1 under f_a is bounded. A conjectural description of the topology of M_2 is given in [27]. In this paper, we deal with maps on the *external boundary* of M_2, i.e. the boundary of the only unbounded component of $\mathbb{C} - M_2$.

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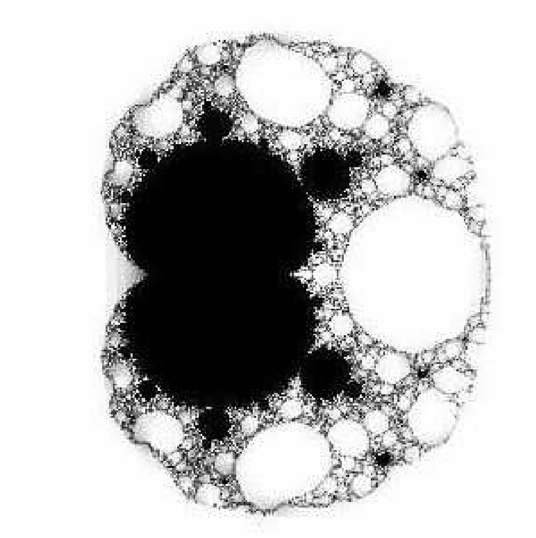

FIGURE 1. The set M_2

In [18], M. Rees studies the parameter plane of V_3, which turns out to be much more complicated than V_2.

1.2. **Invariant laminations.** Invariant laminations were introduced by Thurston [25] to describe quadratic polynomials with locally connected Julia sets. A set L of hyperbolic geodesics in the open unit disk is a *geodesic lamination* if any two different geodesics in L do not intersect, and the union of L is closed with respect to the induced topology on the unit disk. For any pair of points z, w on the unit circle, the geodesic with endpoints z and w will be written as zw. Any geodesic lamination L defines an equivalence relation $\sim_L$ on the unit circle S^1. Namely, two different points on S^1 are equivalent if they are connected by a leaf of L or by a broken line consisting of leaves. For many quadratic polynomials, the Julia set is homeomorphic to the quotient of the unit circle by an equivalence relation $\sim_L$.

We say that a geodesic lamination L on the unit circle is *invariant under the map* $z \mapsto z^2$ if the following conditions hold:

- if $z_1z_2 \in L$, then $z_1^2z_2^2 \in L$ or $z_1^2 = z_2^2$,
- if $z_1z_2 \in L$, then $(-z_1)(-z_2) \in L$,
- if $z_1^2z_2^2 \in L$, then $z_1z_2 \in L$ or $z_1(-z_2) \in L$.

Such laminations are also known as *quadratic invariant laminations.* Any quadratic polynomial p defines a quadratic invariant lamination. In many cases, the quotient of the unit circle by the corresponding equivalence relation is homeomorphic to the Julia set J, and the projection of S^1 onto J semi-conjugates the map $z \mapsto z^2$ with the restriction of p to J.

A *gap* of a geodesic lamination is any component of the complement to all leaves in the unit disk. Let L be a quadratic invariant lamination. The map $z \mapsto z^2$

admits a natural extension over all leaves and gaps of L. This extension is called the *lamination map* of L and is denoted by s_L. The image of any leaf under s_L is a leaf or a single point. The image of any gap is a gap, or a leaf, or a single point. Suppose that L is *clean*, i.e. any two adjacent leaves of L are sides of a common finite-sided gap. Then we can also extend the equivalence relation $\sim_L$ to $\mathbb{C}$. The equivalence classes of $\sim_L$ are defined as finite-sided gaps, leaves, or points.

If L is clean, then the quotient $\mathbb{C}/\sim_L$ is homeomorphic to $\mathbb{C}$. The lamination map s_L defines a continuous self-map $[s_L]$ of this quotient. We say that the lamination L *models* a quadratic polynomial p if the quotient $\mathbb{C}/\sim_L$ is homeomorphic to $\mathbb{C}$, and the map $[s_L]$ is topologically conjugate to p. E.g. any critically finite quadratic polynomial is modeled by the corresponding quadratic invariant lamination. The same is true for many quadratic polynomials with Siegel disks, but not for quadratic polynomials with Cremer points.

Let y_0 be a real number between 0 and 1. Denote by l_0 the diameter connecting the points $e^{\pi i y_0}$ and $-e^{\pi i y_0}$ on the unit circle. Consider all geodesics $z_1 z_2$ in the unit disk such that, for every k, the geodesic $z_1^{2^k} z_2^{2^k}$ does not intersect l_0 or coincides with l_0. This set of geodesics is an invariant lamination, which we denote by $L(y_0)$. If a quadratic polynomial p is modeled by $L(y_0)$, then p belongs to the boundary of the Mandelbrot set. There is a natural *parameter equivalence* relation on the unit circle. Points $e^{2\pi i y_0}$ and $e^{2\pi i y_0'}$ are parameter equivalent if the laminations $L(y_0)$ and $L(y_0')$ correspond to the same quadratic polynomial in a certain well-defined sense, although they may not model this polynomial (e.g. $L(0)$ corresponds to the parabolic map $z \mapsto z^2 + 1/4$, but the equivalence relation $\sim_{L(0)}$ identifies all binary rational points on the unit circle). It turns out that the parameter equivalence relation also corresponds to a geodesic lamination in the unit disk. This lamination is called the *parameter lamination*, or the *quadratic minor lamination*. Thurston [25] gave a description of the parameter lamination using his "minor leaf theory". Conjecturally, the boundary of the Mandelbrot set is homeomorphic to the quotient of the unit circle by the parameter equivalence relation. This conjecture is equivalent to the MLC conjecture (stating that the Mandelbrot set is locally connected).

1.3. **Two-sided laminations.** In the theory of quadratic invariant laminations, the single quadratic polynomial $z \mapsto z^2$ is used to build models for the dynamics of many other quadratic polynomials. The Julia set of $z \mapsto z^2$ is the unit circle, and the unit disk is preserved. A similar idea can be used to build models for rational maps of class V_2. To this end, one can use the rational map $z \mapsto 1/z^2$. This is the only map in V_2 not conjugate to a map of the form f_a. Its Julia set is also the unit circle. However, the map $z \mapsto 1/z^2$ interchanges the inside and the outside of the unit disk.

Let us define an analog of quadratic invariant laminations for the map $z \mapsto 1/z^2$. A *two-sided geodesic lamination* is a set of geodesics that live both inside and outside of the unit disk. Note that the outside of the unit disk is also a topological disk in $\overline{\mathbb{C}}$. Geodesics are in the sense of the Poincaré metric (on the inside or on the outside of the unit disk). We will sometimes use $2L$ to denote a two-sided lamination, but this notation does not assume any multiplication by 2 (in other words, $2L$ is to be thought of as a single piece of notation). A two-sided lamination $2L$ gives rise to a pair of laminations L_0 and L_∞, where the leaves of L_0 are inside of the unit circle, and the leaves of L_∞ are outside. The two-sided lamination $2L = L_0 \cup L_\infty$ is called *invariant* under $z \mapsto 1/z^2$ if the following conditions hold:

- if $z_1 z_2 \in L_0$, then $(1/z_1^2)(1/z_2^2) \in L_\infty$ or $z_1^2 = z_2^2$,
- if $z_1 z_2 \in L_0$, then $(-z_1)(-z_2) \in L_0$,
- if $z_1^2 z_2^2 \in L_0$, then $z_1 z_2 \in L_\infty$ or $z_1(-z_2) \in L_\infty$,

and the same conditions with L_0 and L_∞ interchanged. Let $\sim_0$ and $\sim_\infty$ denote the equivalence relations on the unit circle corresponding to the laminations L_0 and L_∞, respectively.

Two-sided laminations were first considered by D. Ahmadi [2]. He used a different language ("laminations on two disks"). In [2], a classification of two-sided laminations is given, similar to the "minor leaf theory" of Thurston [25].

Gaps of two-sided laminations and the corresponding lamination maps are defined in the same way as for invariant laminations of the unit disk. For a two-sided lamination $2L$, extend the equivalence relations $\sim_0$ and $\sim_\infty$ to the unit disk and to the outside of the unit disk, respectively, in the same way as for invariant quadratic laminations. Define $\sim_{2L}$ to be the smallest equivalence relation containing both $\sim_0$ and $\sim_\infty$. We say that $2L$ *models* a quadratic rational map f if the quotient $\overline{\mathbb{C}}/\sim_{2L}$ is homeomorphic to the sphere, and the map $[s_{2L}]$ is topologically conjugate to f.

We will now define a particular family of two-sided laminations invariant under $z \mapsto 1/z^2$. Let x_0 be a real number strictly between 0 and 1. Consider the arc σ_0 of the unit circle bounded by the points $e^{2\pi i x_0}$ and $-e^{2\pi i x_0}$ and not containing the point 1. Let σ be any component of the full n-fold preimage of σ_0 under $z \mapsto 1/z^2$. Connect the endpoints of σ by a geodesic in the complement to the unit circle. This geodesic should be inside the unit circle if n is even, and outside if n is odd. For certain values of x_0 (which we will describe explicitly later), the set of geodesics thus constructed is a two-sided lamination. We denote this lamination by $2L(x_0)$. If $2L(x_0)$ exists, then it is clearly invariant under the map $z \mapsto 1/z^2$.

1.4. **Statement of the main theorems.** For a map $f_a \in V_2$, denote by Ω the immediate basin of attraction of the critical cycle $\{0, \infty\}$.

Theorem A. *Suppose that $-1 \in \partial\Omega$. Then the Julia set of f_a is locally connected.*

Let Ω_0 and Ω_∞ denote the components of Ω containing 0 and ∞, respectively. As we will see, the critical point -1 cannot be on the boundary of Ω_∞. Thus, under the assumptions of Theorem A, we can only have $-1 \in \partial\Omega_0$. We will prove in this case that $\overline{\Omega}_0$ is a closed topological disk. Moreover, there is a homeomorphism H of the closed unit disk to $\overline{\Omega}_0$ that conjugates the map $z \mapsto z^2$ with the map $f_a^{\circ 2}$. We say that a point in $\overline{\Omega}_0$ *has angle* θ if this point coincides with $H(re^{2\pi i\theta})$ for some $0 \le r \le 1$.

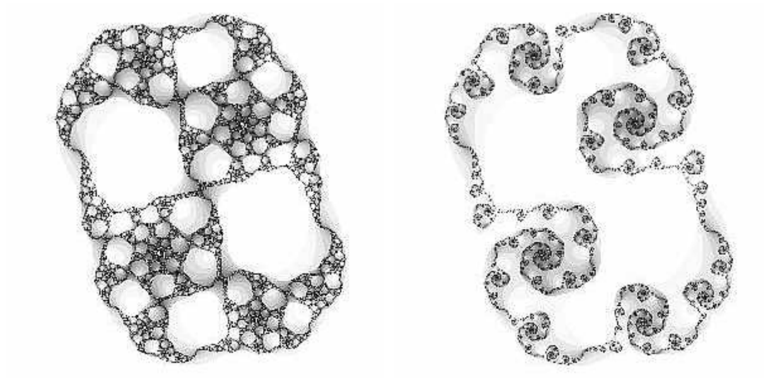

FIGURE 2. The Julia set of $f_a \in V_2$ with $-1 \in \partial\Omega_0$ and of nearby $f_{a'} \in V_2$ with $-1 \in \Omega_0$

Theorem B. *Suppose that the critical point -1 belongs to $\partial\Omega_0$ and has angle θ_0. Then, for*

$$x_0 = \sum_{m=1}^{\infty} \frac{[(2^m - 1)\theta_0] + 1}{2^{2m+1}},$$

the two-sided lamination $2L(x_0)$ exists and models the map f_a.

The maps f_a from Theorems A and B, together with countably many parabolic maps, form the external boundary of M_2 (the boundary of the unbounded component of $\mathbb{C} - M_2$). A more detailed statement will be given below.

1.5. **Matings and anti-matings.** Consider two quadratic invariant laminations L_1 and L_2. Consider the images l^{-1} of all leaves $l \in L_2$ under the transformation $z \mapsto 1/z$. If we straighten all such curves to geodesics in $\{|z| > 1\}$, then we obtain a lamination L_2^{-1} outside the unit disk. We can form the two-sided lamination $L_1 \cup L_2^{-1}$. The lamination $L_1 \cup L_2^{-1}$ is invariant under the map $z \mapsto z^2$ (rather than $z \mapsto 1/z^2$). This lamination is called the *mating* of the laminations L_1 and L_2. If the quadratic invariant laminations L_1 and L_2 correspond to quadratic polynomials p_1 and p_2, and if the lamination $L_1 \cup L_2^{-1}$ models a rational map f, then we say that f is a mating of p_1 and p_2. We write $f = p_1 \sqcup p_2$ in this case.

This definition of mating is equivalent to the following more standard definition. Compactify the complex plane by the circle at infinity. The resulting space is homeomorphic to the closed disk. Let a polynomial p_1 act on one copy of this disk, called D_1, and p_2 act on another copy, called D_2. Denote by $\gamma_i(t)$ the point on the boundary of D_i of angle t. Identify the boundaries of D_1 and D_2 by the formula $\gamma_1(t) = \gamma_2(-t)$. Then the union $D_1 \cup D_2$ is homeomorphic to the sphere. If p_1 and p_2 have the same degree, then the actions of both polynomials match on $\partial D_1 = \partial D_2$. Introduce the minimal equivalence relation $\sim$ on the sphere $D_1 \cup D_2$ such that for any point $z \in \partial D_1 = \partial D_2$ that is a common landing point of two rays, one in D_1 and another in D_2, the union of these two rays and the point z belongs to a single

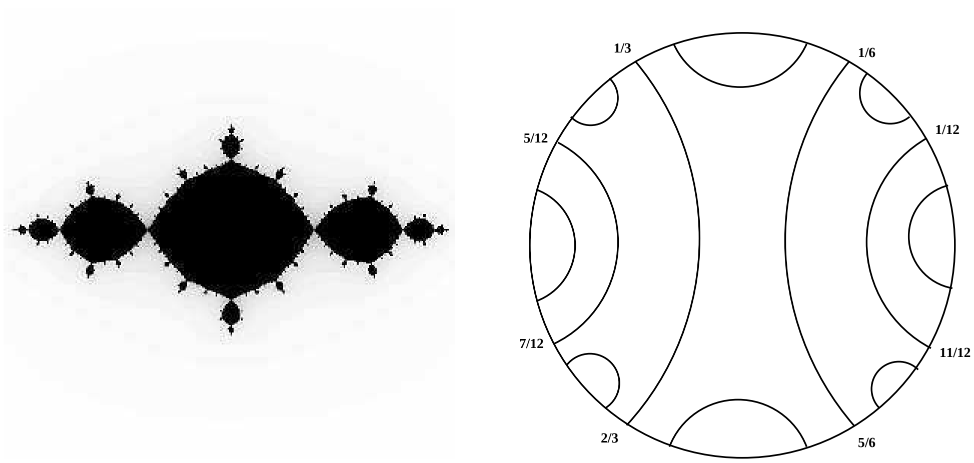

FIGURE 3. The basilica (the Julia set of $z \mapsto z^2 - 1$) and the basilica lamination

equivalence class. If the quotient $D_1 \cup D_2 / \sim$ is homeomorphic to the sphere, and the map $p_1 \cup p_2 / \sim$ is topologically conjugate to a rational map, then this rational map is called the *mating* of p_1 and p_2.

Many maps in V_2 can be described as matings with the quadratic polynomial $z \mapsto z^2 - 1$. The Julia set of this polynomial is called the *basilica*. The dynamics of $z \mapsto z^2 - 1$ can be described by a certain quadratic invariant lamination, which we call the *basilica lamination*. The critical point 0 of the polynomial $z \mapsto z^2 - 1$ is periodic of period two: $f(0) = -1$ and $f(-1) = 0$. Thus $z \mapsto z^2 - 1$ belongs to V_2. Actually, this is the only polynomial of class V_2.

Theorem B*. *Suppose that the critical point -1 of f_a belongs to $\partial\Omega_0$ and has angle θ_0. Let $\theta_0[m]$ denote the m-th binary digit of θ_0. Then, for*

$$y_0 = \frac{1}{3}\left(1 + 3\sum_{m=1}^{\infty} \frac{\theta_0[m]}{4^m}\right),$$

the mating of the basilica lamination with the lamination $L(-2y_0)$ models the map f_a. Moreover, the lamination $L(-2y_0)$ itself models a well-defined quadratic polynomial, so that f_a is a mating of $z \mapsto z^2 - 1$ with another quadratic polynomial.

The formula for y_0 has a simple meaning. Namely, consider the point in the basilica belonging to the boundary of the Fatou component of -1 and having the (internal) angle θ_0 on this boundary. Then y_0 is the external angle of the same point. In the terminology we used, internal angles parameterize dynamical rays emanating from -1, whereas external angles parameterize dynamical rays emanating from ∞.

Theorem B* can be deduced from Theorem B. Actually, the model with a two-sided lamination invariant under $z \mapsto 1/z^2$ is combinatorially equivalent to the mating model. However, the model with a two-sided lamination is simpler in some respects. It can be restated in terms of *anti-matings*. The notion of anti-mating was also introduced by Douady and Hubbard [4]. Consider two closed disks D_1 and D_2 as above, together with the actions of quadratic polynomials p_1 and p_2, respectively. We can glue a topological sphere $D_1 \cup D_2$ out of the disks D_1 and D_2 in the same way as for matings. However, we define a self-map of this topological sphere differently. Namely, a point with coordinate z in D_1 is mapped to the point with coordinate $p_1(z)$ in D_2. Similarly, a point with coordinate w in D_2 is mapped to the point with coordinate $p_2(w)$ in D_1. Thus the disks D_1 and D_2 are interchanged. Suppose that the quotient of $D_1 \cup D_2$ by the equivalence relation $\sim$ is a topological sphere, and the quotient of the map introduced above is topologically conjugate to a rational function. This rational function is then called the *anti-mating* of p_1 and p_2.

Theorem B can be restated as follows:

Theorem B**. *Any map f_a such that $-1 \in \partial\Omega_0$ is an anti-mating of $z \mapsto z^2$ and another quadratic polynomial.*

From this theorem it follows, in particular, that the second iteration $f_a^{\circ 2}$ is the mating of two quartic polynomials. For both of these polynomials, all critical points are either periodic or on the boundaries of immediate super-attracting basins.

For the case, where the critical point -1 is pre-periodic, Theorem A is known, and the proofs of Theorems B and B* are much simpler (they basically follow from the mating criterion given in [24]). In this paper, we will concentrate on the case, where -1 is not pre-periodic. As we will see, the angle θ_0 is irrational in this case (e.g. this follows from Theorem A), however, we do not assume this a priori.

The results of Theorems A, B and B* complement recent results by Aspenberg and Yampolsky [3]. They prove that any non-renormalizable quadratic polynomial, not in the 1/2-limb and with all cycles repelling, is mateable with the basilica. From Theorem B* it follows, in particular, that any map f_a with $-1 \in \partial\Omega$ is a mating with a non-renormalizable polynomial, and, therefore, belongs to the class considered in [3]. The main technical tool of both this paper and [3] are bubble puzzles suggested by Luo [7]. Luo claimed the main result of [3], and gave a sketch of a proof, but many important details were missing. In other contexts, similar constructions were used in [28, 19].

The first version of this paper was written before preprint [3] appeared. It contained a proof of Theorem B based on a direct construction of the puzzle specific to our situation. No analytic continuation was used, but the condition $-1 \in \partial\Omega$ was essential. The technique developed in [3] permits to build the puzzle just for some simple rational maps, and then continue it analytically. We adopt this approach.

1.6. **The exterior hyperbolic component.** All theorems we stated so far are about maps on the external boundary of M_2. It is natural to attempt studying topology and dynamics of such maps by approaching them from the *exterior component* $\mathcal{E}$ — the only unbounded component of the complement to M_2. There is a simple dynamical description of the set $\mathcal{E}$: a map $f_a \in V_2$ belongs to $\mathcal{E}$ if and only if the free critical point -1 belongs to the immediate basin of the critical cycle $\{0, \infty\}$. Then we must have $-1 \in \Omega_0$, as we will see.

The Julia set of any map f_a in $\mathcal{E}$ is a quasi-circle, and the restriction of f_a to the Julia set is conjugate to the map $z \mapsto 1/z^2$. This follows from a more general theorem of Sullivan [23]. Thus the topology and the dynamics of the Julia set is the simplest possible. However, a non-trivial combinatorics and a non-trivial dynamics show up when we consider rays for the second iteration $f_a^{\circ 2}$, and the way they crash into pre-critical points; more details will come soon.

We give topological models for all maps f_a in $\mathcal{E}$ in terms of Blaschke products. The methods used to build these models are not new (cf. Sullivan and McMullen [11]). The second iteration $f_a^{\circ 2}$ of the map f_a preserves both components of the complement to the Julia set. Pick one particular component. This is an open topological disk. Consider a holomorphic uniformization of this topological disk by the round unit disk. The map corresponding to $f_a^{\circ 2}$ under this uniformization takes the unit disk to itself. Therefore, it is a quartic Blaschke product. It is not hard to see that this Blaschke product must actually be the square of a quadratic Blaschke product

$$B : z \mapsto z\frac{z+b}{\bar{b}z+1},$$

where b belongs to the open unit disk. This gives an idea of how to construct a topological model for f_a.

The unit circle divides the Riemann sphere into two disks — the *inside* and the *outside* of the unit circle. Consider the map $1/B$ that takes the inside to the outside, and the map $1/z^2$ that takes the outside to the inside. We would like to glue these maps together but, unfortunately, they do not match on the boundary. Fortunately, there is a quasi-conformal automorphism Q of the outside of the unit circle such that the maps $Q \circ 1/B$ and $1/z^2 \circ Q^{-1}$ do match on the boundary. They define a global topological ramified self-covering g of the Riemann sphere of degree two. Moreover, there is a natural quasi-conformal structure invariant under g. By the Measurable Riemann Mapping theorem of Ahlfors–Bers [1], the ramified self-covering g is topologically conjugate to a quadratic rational map. Clearly, this quadratic rational map must belong to $\mathcal{E}$. Conversely, any map in $\mathcal{E}$ can be obtained by this quasi-conformal surgery.

1.7. **Dynamical rays and external parameter rays.** Let f_a be a map in V_2. The second iteration $f_a^{\circ 2}$ has two super-attracting fixed points 0 and ∞. The other four critical points are -1, the two preimages of -1 under f_a, and -2, which is a preimage of ∞ under f_a.

Consider the Green function G for the map $f_a^{\circ 2}$ that is defined by the usual formula

$$G(z) = \lim_{n\to\infty} \frac{\log |f_a^{\circ 2n}(z)|}{2^n}.$$

This function is negative near 0 and positive near ∞. The gradient of G restricted to the open set $\{G \neq 0\}$ is a smooth vector field that has singularities at all *pre-critical points* (iterated preimages of critical points). Recall that a *ray* is any trajectory of this vector field.

The α-limit set of any ray is a single pre-critical point, more precisely, an iterated preimage of ∞ or an iterated preimage of -1. The ω-limit set is either a pre-critical point or a subset of the Julia set (which is also a single point in a locally connected situation). If the ω-limit set is a pre-critical point, then this point is necessarily an iterated preimage of -1 (because it can not be an iterated preimage of ∞). Consider any iterated preimage z of -1, and assume that $G(z) \neq 0$. The point z is a saddle point of the Green function. Thus there are only two rays emanating from z and only two rays crashing into z. The union of the two rays emanating from z, together with the point z itself, is called the *ray leaf centered at* z. Thus the ray leaves are in one-to-one correspondence with iterated preimages z of -1 such that $G(z) \neq 0$.

Suppose that a belongs to the exterior component $\mathcal{E}$. Then the critical point -1 of f_a belongs to Ω_0. Rays emanating from 0 are parameterized by the *angle*. In a small neighborhood of 0, the map $f_a^{\circ 2}$ is holomorphically conjugate to the map $z \mapsto z^2$. Under this local conjugacy, the point 0 is mapped to 0, and germs of rays are mapped to germs of radial segments. By definition, the angle of a ray is defined as the angle the corresponding radial segment makes with the real axis. We measure angles in radians$/2\pi$. Thus the measure of the full angle is 1. Let $R_0(\theta)$ denote the ray of angle θ emanating from 0. It is not hard to see that there exists a unique ray $R_0(\theta_0)$ that emanates from 0 and crashes into the critical point -1.

Fix an angle θ_0. Consider the set of all parameter values a, for which the ray $R_0(\theta_0)$ crashes into the critical point -1. This set is called the *external parameter ray of angle* θ_0. We call an external parameter ray *periodic* or *non-periodic* according to whether its angle is periodic or non-periodic under the doubling map modulo 1.

M. Rees [17] proved that periodic external parameter rays (except for the zero ray) land at parabolic parameter values.

Theorem C. *All external parameter rays land. Consider the rational map $f_a \in V_2$ corresponding to the landing point a of a non-periodic external parameter ray of angle θ_0. For this map, $-1 \in \partial\Omega_0$. Moreover, the critical point -1 is the point on $\partial\Omega_0$ of angle θ_0, thus the topological dynamics of f_a is described by Theorem B.*

When the first version of this paper was written, I had in mind to deduce this theorem from Theorem B by showing that $-1 \in \partial\Omega$ for all parameter values on the external boundary, except for countably many parabolic points. My argument was overly complicated, and I am grateful to M. Lyubich for suggesting a simpler approach, not using the puzzle. However, in this paper, Theorem C is proved using the parameter puzzle, a version of that in [3]. This approach has the advantage that the same combinatorial constructions are used for both Theorems A and C.

FIGURE 4. Ray leaves for some map in the exterior component of V_2

1.8. **Ray laminations.** Consider a quadratic rational map f_a in the exterior component $\mathcal{E}$. Assume that f_a does not lie on a periodic parameter ray (it can still lie on a strictly pre-periodic parameter ray). Then each ray leaf of f_a is a curve that is closed in the complement to the Julia set. The closure of this curve in the Riemann sphere intersects the Julia set in two points — the *endpoints* of the ray leaf.

Straighten the Julia set to the unit circle, and each ray leaf to a geodesic in the complement to the unit circle. Then we obtain a two-sided geodesic lamination. Since the restriction of the map f_a to the Julia set is conjugate to the map $z \mapsto 1/z^2$, this two-sided lamination is invariant under $z \mapsto 1/z^2$. We will call this lamination the *ray lamination*. Ray laminations can be described explicitly.

Theorem D. *Let $f_a \in V_2$ be a map in the exterior component. Suppose that f_a lies on a non-periodic external parameter ray of angle θ_0. Then the ray lamination for f_a coincides with the two-sided lamination $2L(x_0)$, where*

$$x_0 = \sum_{m=1}^{\infty} \frac{[(2^m - 1)\theta_0] + 1}{2^{2m+1}}.$$

We will see that all maps in the same parameter ray give rise to the same ray lamination. On the other hand, ray laminations corresponding to maps from different parameter rays can never be the same.

What happens if we approach the external boundary along a non-periodic parameter ray? The corresponding ray lamination stays the same, but all leaves become shorter and shorter. In the limit, all leaves of the ray lamination collapse to points. Thus the same two-sided lamination serves both as a ray lamination for a map in the exterior component and as a lamination modeling a map on the external boundary. This picture was the initial motivation for Theorem B stated above. However, the formal proof goes differently. The collapsing of ray leaves can be proved a posteriori, using Theorems B and C.

1.9. **Hyperbolic components of V_2.** From Theorem C it follows that the boundary of the exterior component $\mathcal{E}$ is a topological circle. However, it is not a quasi-circle because it has cusps at all parabolic points. The hyperbolic component $\mathcal{E}$ is special because it is the only type II component in V_2. Recall that, according to the terminology of M. Rees [17], a hyperbolic component in a space of quadratic rational maps is of type II if both critical points belong to the same cycle. A hyperbolic component is of type III if one critical point is strictly pre-periodic and eventually enters the cycle of the other critical point, and of type IV if both critical points are periodic, with disjoint cycles. In V_2, all type III components are capture components, and all type IV components are mating components.

Note that the boundaries of type IV components are real analytic curves. From [3] it follows that the boundaries of type III components are topological circles, and it is very likely that they are quasi-circles. Maps on the boundary of a type III component are never critically recurrent. Thus they exhibit much simpler dynamical behavior, compared with the maps on the external boundary. On the other hand, maps on the boundary of a type IV component can be much more complicated, as complicated as quadratic polynomials can be. In particular, they can have Siegel or Cremer points.

1.10. **A blow-up of $z \mapsto z^2$.** The explicit formula for x_0 in terms of θ_0 used in Theorems B and D may look mysterious. We will now explain this formula by describing a simple topological construction it comes from.

Let z_0 be any point on the unit circle. There is a unique probability measure μ on the unit circle with the following properties:

- The measure μ is supported on countably many points, namely, on all iterated preimages of z_0 under the map $z \mapsto z^2$ (the point z_0 itself is also regarded as an iterated preimage of z_0).
- For any point z on the unit circle different from z_0, we have $\mu\{z^2\} = 4\mu\{z\}$.

The measure μ can be given by the following formula

$$\mu\{z\} = \sum_{m:\ z^{2^m}=z_0} \frac{1}{2 \cdot 4^m}.$$

The summation is over all nonnegative integers m such that $z^{2^m} = z_0$. In particular, if the point z_0 is not periodic under the map $z \mapsto z^2$, then there is at most one summand. The definition of μ can be made simple in the non-periodic case: any preimage of z_0 under the map $z \mapsto z^{2^m}$ has measure $\frac{1}{2\cdot 4^m}$.

It is classically known that there is a unique continuous map $h : S^1 \to S^1$ with the following properties:

- $h(1) = 1$, and 1 is in the center of $h^{-1}(1)$.
- the push-forward of the uniform probability measure under the map h is the measure μ,
- the map h has topological degree 1.

The map h blows up all iterated preimages of the point z_0 under $z \mapsto z^2$ in the following sense. For any point z such that $z^{2^m} = z_0$, the full preimage of z under h is an arc of length $\mu\{z\}$. In particular, the full preimage $h^{-1}(z_0)$ is a half-circle. The following proposition is verified by a simple direct computation:

Proposition 1.1. *If $z_0 = e^{2\pi i\theta_0}$ is not periodic under the squaring map $z \mapsto z^2$, then the half-circle $h^{-1}(z_0)$ is bounded by $e^{2\pi i x_0}$ and $-e^{2\pi i x_0}$, where x_0 is expressed in terms of θ_0 by the formula from Theorems B and D.*

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2. Two-sided laminations $2L(x_0)$

In this section, we will give details on the explicit construction of two-sided laminations that appear in Theorems B and D. Actually, the construction will be slightly more general, including the two-sided laminations for parabolic maps, not considered in this paper.

2.1. **Formulas for x_0.** Recall that, for a real number θ_0 between 0 and 1 that is not an odd denominator rational number, we defined the corresponding real number x_0 by the formula

$$x_0 = \sum_{m=1}^{\infty} \frac{[(2^m - 1)\theta_0] + 1}{2 \cdot 4^m}.$$

In this subsection, we will find the binary expansion of x_0. Define the functions ν_m on real numbers between 0 and 1 as follows:

$$\nu_m(\theta) = \begin{cases} 0, & \{2^m\theta\} < \theta \\ 1, & \{2^m\theta\} \geq \theta \end{cases}$$

Proposition 2.1. *For any real number θ between 0 and 1, we have*

$$1 + [(2^m - 1)\theta] = [2^m\theta] + \nu_m(\theta).$$

Proof. There are two cases: $[2^m\theta] = [(2^m - 1)\theta]$ and $[2^m\theta] = [(2^m - 1)\theta] + 1$. In the first case, subtracting θ from $2^m\theta$ does not change the integer part, therefore, $\{2^m\theta\} > \theta$, and $\nu_m(\theta) = 1$. In the second case, subtracting θ from $2^m\theta$ changes the integer part, therefore, $\{2^m\theta\} < \theta$, and $\nu_m(\theta) = 0$. □

We can now rewrite the formula for x_0 as follows:

$$x_0 = \sum_{m=1}^{\infty} \frac{[2^m\theta_0]}{2^{2m+1}} + \sum_{m=1}^{\infty} \frac{\nu_m(\theta_0)}{2^{2m+1}}.$$

Let us compute the first sum:

Proposition 2.2. *Let $\theta_0[m]$ denote the m-th digit in the binary expansion of θ_0. Then*
$$\sum_{m=1}^{\infty} \frac{[2^m\theta_0]}{2^{2m+1}} = \sum_{m=1}^{\infty} \frac{\theta_0[m]}{2^{2m}}.$$

Proof. Denote by X the left hand side of this equality. Note that the m-th binary digit of a real number θ is equal to $[2^m\theta] - 2[2^{m-1}\theta]$ for $m \geq 1$. Therefore, the right hand side is
$$\sum_{m=1}^{\infty} \frac{[2^m\theta_0] - 2[2^{m-1}\theta_0]}{2^{2m}} = 2X - X = X. \qquad \square$$

We have proved that
$$x_0 = \sum_{m=1}^{\infty} \frac{\theta_0[m]}{2^{2m}} + \sum_{m=1}^{\infty} \frac{\nu_m(\theta_0)}{2^{2m+1}}.$$
This series represents the binary expansion of x_0. Therefore, we have

Proposition 2.3. *Let $x_0[m]$ denote the m-th binary digit of x_0. Then*
$$x_0[2m] = \theta_0[m], \quad x_0[2m+1] = \nu_m(\theta_0)$$

2.2. **A forward invariant lamination.** Fix a point $z_0 = e^{2\pi i\theta_0}$ on the unit circle. Define a lamination L_0 as follows. We first define a probability measure μ on the unit circle. It is given by the following formula:
$$\mu\{z\} = \sum_{m:\ z^{2^m}=z_0} \frac{1}{2\cdot 4^m}.$$
Next, we consider the map h with the following properties:

- $h(1) = 1$, and 1 is in the center of $h^{-1}(1)$.
- the push-forward of the uniform probability measure under the map h is the measure μ,
- the map h has topological degree 1.

It blows up all iterated preimages of z_0. We connect two points on the unit circle by a geodesic if these two points bound the full preimage of a single point under h. The lamination L_0 is the set of all such geodesics. As we will prove shortly, this lamination is *forward invariant under $x \mapsto x^4$*: for any leaf xy of L_0, either $x^4 = y^4$, or the geodesic x^4y^4 is also a leaf of L_0.

Note that in the definition of the lamination L_0, each leaf $l \in L_0$ comes together with a specific arc subtended by l. Namely, for a leaf xy, the corresponding arc is the full preimage of the point $h(x) = h(y)$ under the map h. We will call this arc the *shadow of the leaf* l. Shadows of different leaves in L_0 do not intersect. Given an arc σ on the unit circle, define *the bridge over* σ as the geodesic connecting the boundary points of the arc σ. Thus the bridge over the shadow of a leaf $l \in L_0$ is this leaf l itself. Denote by l_0 the leaf, whose shadow σ_0 is $h^{-1}(z_0)$.

The lamination L_0 has a distinguished gap G_0 such that all leaves of L_0 are on the boundary of G_0.

Proposition 2.4. *The lamination L_0 defined above is forward invariant under the map $x \mapsto x^4$. Moreover, the map h semi-conjugates the endomorphism $x \mapsto x^4$ of the unit circle with the endomorphism $z \mapsto z^2$ everywhere except on the arc σ_0. In other words, $h(x^4) = h(x)^2$ for any point x on the unit circle such that $h(x) \neq z_0$.*

Proof. We first define an endomorphism φ of the unit circle such that L is forward invariant under φ, and then prove that φ is the map $x \mapsto x^4$.

Suppose first that a point x on the unit circle does not belong to a shadow of a leaf of L_0. Then the point $h(x)^2$ has a unique preimage under the map h. Define $\varphi(x)$ to be this preimage. The map φ thus defined admits a continuous extension that maps the full h-preimage of any point z on the unit circle to the full h-preimage of the point z^2, except for $z = z_0$. To fix one such extension, we require that on each arc that is the full h-preimage of some point, the map φ act linearly with respect to the arc-length. Then φ is well-defined everywhere except on σ_0, and the restriction of φ to the full h-preimage of any point on the unit circle multiplies all arc lengths by 4. Indeed, the length of the arc $h^{-1}(z^2)$ is four times bigger that the length of the arc $h^{-1}(z)$, provided that $z \neq z_0$. We can also say where φ should map the arc σ_0 in order to be a self-covering of the unit circle.

In the case, where z_0 is not periodic under $z \mapsto z^2$, the arc σ_0 has length $1/2$. It should be wrapped twice around the circle under the endomorphism φ. Both endpoints of σ_0 should be mapped to the h-preimage of z_0^2, which is a single point. Of course, we require that φ act linearly on σ_0.

In the case, where z_0 is periodic with the minimal period p under the map $z \mapsto z^2$, the orbit of the arc σ_0 under the map $z \mapsto z^4$ consists of p arcs of the following lengths:

$$\frac{4}{2(4^p-1)}, \ \frac{4^2}{2(4^p-1)}, \ \dots, \ \frac{4^p}{2(4^p-1)},$$

the biggest length being that of σ_0. We can arrange that σ_0 wraps more than twice but less than three times around the unit disk under the map φ so that the ends of σ_0 map to the ends of the segment of length $4/2(4^p-1)$ (this segment being covered 3 times by parts of σ_0 under the map φ). In all cases, we can arrange that all arc-lengths in σ_0 get 4 times bigger modulo $\mathbb{Z}$ under the map φ.

We defined a continuous self-map φ of the unit circle that is semi-conjugate to $z \mapsto z^2$ on the complement to the arc σ_0. The semi-conjugacy is given by h. It is not hard to see that φ is a self-covering of the unit circle and that $\varphi(1) = 1$. By definition, the lamination L_0 is forward invariant under the map φ.

We will now prove that the map φ just defined multiplies all arc-lengths by 4 modulo $\mathbb{Z}$ (in other words, it multiplies all small arc-lengths exactly by 4). Consider any arc σ on the unit circle, whose length is smaller than $1/4$. We want to show that the length of the arc $\varphi(\sigma)$ is 4 times bigger than the length of the arc σ. Since on each arc of the form $h^{-1}(z)$, the map φ multiplies all arc-lengths by 4, it suffices to assume that σ is the full preimage of the arc $h(\sigma)$ under h. By definition of the measure μ, we have $\mu(h(\sigma)^2) = 4\mu(h(\sigma))$. We also know that $\mu(h(\sigma)^2)$ coincides with the length of the arc $\varphi(\sigma)$. This implies that the length of $\varphi(\sigma)$ is 4 times bigger than the length of σ.

Since the map φ multiplies all arc-lengths by 4 and fixes 1, it must have the form $x \mapsto x^4$. □

2.3. **An invariant lamination.** In this subsection, we extend the lamination L_0 to a lamination L invariant under the map $x \mapsto x^4$ in the sense of Thurston. Recall that a geodesic lamination in the unit disk is said to be *invariant* under the map $x \mapsto x^d$ if

- it is forward invariant,
- it is *backward invariant*: for any leaf xy of the lamination, there exists a collection of d disjoint leaves, each connecting a preimage of x with a preimage of y under the map $x \mapsto x^d$.
- it is *gap invariant*: for any gap G, the convex hull G' of the image of $\overline{G} \cap S^1$ is a gap, or a leaf, or a single point.

By a *pullback* of a connected set under a continuous map, we mean a connected component of an iterated preimage of this set. Recall that the arc σ_0 was defined as the full preimage of the point z_0 under the map h. The arc σ_0 is the shadow of some leaf l_0. It is easy to see that the shadow of any other leaf in L_0 is a certain pullback of σ_0 under the map $x \mapsto x^4$.

Proposition 2.5. *Consider the set A of all pullbacks of the arc σ_0 under the map $x \mapsto x^4$. The bridges over any two arcs in A are disjoint.*

We need the following lemma:

Lemma 2.6. *Consider two different pullbacks σ and σ' of the arc σ_0 different from σ_0. If the bridges over σ and σ' intersect, then so do the bridges over their images under the map $x \mapsto x^4$, unless σ or σ' coincides with σ_0.*

Proof. If the bridges over σ and σ' intersect, then these arcs intersect each other, but none of them contains the other. The union σ'' of the two arcs is also an arc. If we can show that the length of σ'' is less than 1/4, then we would conclude that the map $z \mapsto z^4$ acts homeomorphically on σ'', and hence the images of σ and σ' have intersecting bridges.

By the *depth* of a pullback of σ_0 we mean the minimal number n such that σ_0 is the image of the pullback under $x \mapsto x^{4^n}$. The arcs σ and σ' cannot be pullbacks of σ_0 of the same depth, because different pullbacks of the same depth are disjoint. By our assumption, neither of the arcs σ, σ' coincides with σ_0. Then the length of one arc is at most

$$\frac{1}{2}\left(\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} \cdots\right),$$

while the length of the other arc is at most

$$\frac{1}{2}\left(\frac{1}{4^2}+\frac{1}{4^3}+\dots\right).$$

The length of σ'' is thus at most

$$\frac{1}{8}+\frac{1}{4^2}+\frac{1}{4^3}+\dots<\frac{1}{4}.$$

This proves the lemma. □

Define the set A_0 as the set of all arcs that are shadows of leaves of L_0.

Lemma 2.7. *The union of the set A_0 is backward invariant. In other words, any pullback of any arc in the set A_0 is a subset of some arc in A_0.*

Indeed, this follows from the proof of Proposition 2.4.

Proof of Proposition 2.5. Suppose that there are two arcs from A such that their bridges intersect. Then, applying to this pair of arcs a suitable iterate of the map $x \mapsto x^4$, we can make one of the arcs be σ_0.

Thus we have a pullback σ of the arc σ_0 such that the bridges over σ_0 and σ intersect. But this contradicts Lemma 2.7. □

We can now define a lamination L as the set of bridges over all pullbacks of the arc σ_0. By Proposition 2.5, the leaves of L are disjoint, so that L is indeed a lamination. It is not hard to see that the lamination L does not have any accumulation points inside the unit disk.

Proposition 2.8. *The lamination L is invariant under the self-map $x \mapsto x^4$ of the unit circle.*

Proof. We have already proved the forward and backward invariance. It remains only to prove the gap invariance. Define the *span $P(l)$ of a leaf $l \in L$* as the open topological disk bounded by l and the shadow of l. Any gap of L different from G_0 can be described as the complement in a span $P(l)$ to the closures of all spans that lie in $P(l)$. Denote by $G(l)$ the gap associated with the leaf l in this way.

Suppose that l is a leaf of L different from l_0. Then the image of l under the map $x \mapsto x^4$ is another leaf l', and the gap $G(l)$ maps to the gap $G(l')$ in the following sense: the intersection $\overline{G(l)} \cap S^1$ maps to the intersection $\overline{G(l')} \cap S^1$. Clearly, the gap G_0 maps to itself under the map $x \mapsto x^4$ in this sense. Moreover, G_0 is a *critical gap of degree two*: the quotient space $\partial G_0/l_0$ maps to ∂G_0 as a topological covering of degree two, if we extend the map $x \mapsto x^4$ linearly over leaves.

It remains to consider the gap $G(l_0)$. This gap is mapped to G_0, and this is also a critical gap. To see that, it is enough to understand what happens with the arc σ_0, but this was described in the proof of Proposition 2.4. □

2.4. **A two-sided lamination.** In this subsection, we extend the lamination L to a two-sided lamination $2L$ invariant under the map $x \mapsto 1/x^2$. By Proposition 1.1, it will be clear that $2L = 2L(x_0)$. In particular, the lamination $2L(x_0)$ exists.

Proposition 2.9. *The lamination L is invariant under the antipodal map $x \mapsto -x$.*

Proof. Indeed, if the shadow σ of some leaf $l \in L$ is a pullback of the arc σ_0 under the map $x \mapsto x^4$, then $-\sigma$ is also a pullback of σ_0. Thus leaves of L map to leaves of L under the map $x \mapsto -x$, and, clearly, gaps map to gaps. □

Consider the set L' of geodesics outside of the unit circle connecting pairs of points $1/x^2$ and $1/y^2$, where x and y are endpoints of a leaf in L.

Proposition 2.10. *The set L' is a geodesic lamination outside of the unit circle.*

Indeed, by Proposition 2.9, the images of different leaves in L are either the same or disjoint.

We can now consider the two-sided lamination $2L$ that is the union of the inside lamination L and the outside lamination L'. By Proposition 1.1, we have $2L = 2L(x_0)$.

3. The exterior component

In this section, we describe maps in the exterior component $\mathcal{E}$ in terms of a special quasi-conformal surgery performed on Blaschke products. We also discuss combinatorics of rays.

3.1. **Anti-matings of Blaschke products.** Anti-matings of polynomials were considered by Douady and Hubbard [4]. In this section, we introduce a similar notion for Blaschke products, together with an explicit quasi-conformal surgery making these anti-matings into rational functions. Let Δ_0 denote the inside of the unit circle, and Δ_∞ the outside of the unit circle (i.e. the complement to the closed unit disk in the Riemann sphere). The closures of the open disks Δ_0 and Δ_∞ are denoted by $\overline{\Delta}_0$ and $\overline{\Delta}_\infty$, respectively.

A *(finite) Blaschke product* is a product of any finite number of holomorphic automorphisms of the unit disk. The product here is in the sense of multiplication of complex numbers. Any holomorphic automorphism of the unit disk extends to a holomorphic automorphism of the Riemann sphere. Therefore, Blaschke products are also defined on the whole Riemann sphere.

Consider two Blaschke products B_0 and B_∞ of the same degree d. We will make the following assumption on B_0 and B_1: *the restrictions of these maps to the unit circle are expanding in the usual metric*. In particular, this implies that both maps B_0 and B_1 are hyperbolic. Let α_0 be the restriction of the map $1/B_0$ to the unit circle. This map takes the unit circle to itself. Moreover, this is an orientation-reversing self-covering of the unit circle of degree $-d$ (the negative sign represents the change of orientation). The restriction α_∞ of the map $1/B_\infty$ to the unit circle satisfies the same properties.

From a classical theorem of M. Shub [21] it follows that any expanding endomorphism of the unit circle is topologically conjugate to a map $z \mapsto z^k$; the conjugating homeomorphism is unique (see e.g. [6]). In particular, the maps α_0 and α_∞ are topologically conjugate to the map $z \mapsto z^{-d}$. Since α_0 and α_∞ are C^∞, by [22], the conjugating homeomorphism is quasi-symmetric.

The following statement is classical, but we give a proof for completeness:

Lemma 3.1. *Consider two endomorphisms of the unit circle, one of which is expanding. If these two maps have the same topological degree and if they commute, then they coincide.*

Proof. The expanding map is conjugate to the map $z \mapsto z^k$ for some $k \neq 0, \pm 1$. If we lift this map to the universal cover of the unit circle (i.e. to the real line), then we obtain just the linear map $x \mapsto kx$. Assume that another map of topological degree k commutes with $z \mapsto z^k$. The lift of this map to the universal cover has the form $x \mapsto kx + P(x)$, where P is a periodic function. Since the two maps commute, we have

$$(kx) + P(kx) = k(x + P(x)).$$

Therefore, $kP(x) = P(kx)$, and then $k^n P(x) = P(k^n x)$ for all n. The function P is periodic, hence bounded. It follows that

$$P(x) = \lim_{n\to\infty} \frac{1}{k^n} P(k^n x) = 0$$

for all x. □

Let φ denote the self-homeomorphism of the unit circle that conjugates $\alpha_0 \circ \alpha_\infty$ with $\alpha_\infty \circ \alpha_0$. Then we have

$$\varphi \circ \alpha_0 \circ \alpha_\infty \circ \varphi^{-1} = \alpha_\infty \circ \alpha_0.$$

From this equation it follows that the maps $\varphi \circ \alpha_0$ and $\alpha_\infty \circ \varphi^{-1}$ commute. By Lemma 3.1, this is only possible when

$$\varphi \circ \alpha_0 = \alpha_\infty \circ \varphi^{-1}.$$

This is an important functional equation on φ that we will use.

There is a quasi-conformal self-homeomorphism Q of the disk $\overline{\Delta}_\infty$ that restricts to the map φ on the unit circle. This is because φ is quasi-symmetric: any quasi-symmetric automorphism of the unit circle extends to a quasi-conformal automorphism of the unit disk, see [1].

Define a self-map F of the unit sphere as follows. On the disk $\overline{\Delta}_0$, we set F to be $Q \circ (1/B_0)$. On the disk $\overline{\Delta}_\infty$, we set F to be $(1/B_\infty) \circ Q^{-1}$. These two maps match on the unit circle by the functional equation on φ.

There is a quasi-conformal structure on the Riemann sphere that is invariant under the map F. Indeed, we can define this structure to be the standard conformal structure on the unit disk Δ_0, and the push-forward of the standard conformal structure under Q on the disk Δ_∞.

By the Measurable Riemann Mapping theorem of Ahlfors and Bers (see [1]), there is a self-homeomorphism of the sphere that takes the quasi-conformal structure we defined to the standard conformal structure. Let f be a self-map of the Riemann sphere corresponding to the self-map F under this homeomorphism, and J the image of the unit circle. The map f is a holomorphic self-map of the Riemann sphere with the Julia set J (which is a quasi-circle). It has topological degree d, hence it is a rational function of degree d.

We call the map f the *anti-mating* of the Blaschke products B_0 and B_∞.

3.2. **The exterior component.** In this subsection, we consider one particular example of the general construction introduced above. For the map B_0, we take a quadratic Blaschke product

$$B_0(z) = z\frac{z+b}{\bar{b}z+1}$$

with $|b| < 1$. The origin is a fixed point for this map. The critical points $c_{1,2}$ of B_0 are given by the equation $\bar{b}z^2 + 2z + b = 0$. Since we have $|c_1c_2| = 1$, one of the critical points, say c_1, satisfies $|c_1| \leq 1$, while for the other critical point c_2 we have $|c_2| \geq 1$. The exact formula for $c_{1,2}$ is

$$c_{1,2} = \frac{-1 \pm \sqrt{1-|b|^2}}{\bar{b}}.$$

We see that c_1 lies in Δ_0, whereas c_2 lies in Δ_∞ (since $|b| < 1$, it is clear from this formula that points $c_{1,2}$ cannot both lie on the unit circle).

Proposition 3.2. *The restriction of B_0 to the unit circle is expanding.*

Proof. By a theorem of Tischler [26], a Blaschke product B restricts to an expanding endomorphism of the unit circle if and only if λB has a fixed point in Δ_0 for all λ in the unit circle. Clearly, the map B_0 satisfies this condition. □

For the map B_∞, we just take $z \mapsto z^2$ (the restriction of this map to the unit circle is obviously expanding). Let $f = f_{[b]}$ be the anti-mating of the Blaschke products B_0 and B_∞. This is a quadratic rational map. It depends smoothly (and even real-analytically) on b. However, the dependence is not complex analytic, because the Blaschke product B_0 does not depend complex analytically on b.

Proposition 3.3. *The map f has a super-attracting cycle of period two.*

Proof. Consider the map F from Subsection 3.1. The image of 0 under F is $Q(\infty)$, and the image of $Q(\infty)$ is 0. Thus $\{0, Q(\infty)\}$ is a periodic cycle of period two for the map F. Moreover, $Q(\infty)$ is a critical point of F, hence this cycle is super-attracting. The map f is quasi-conformally conjugate to F. It follows that f also has a super-attracting cycle of period two. □

This proposition means that f is a map in V_2. In particular, it is holomorphically conjugate to some map of the form

$$f_a : z \mapsto \frac{a}{z^2 + 2z}$$

or to the map $z \mapsto 1/z^2$. Thus, for any $b \neq 0$ in the open unit disk, there is a unique complex number a such that f_a is holomorphically conjugate to $f_{[b]}$. Recall that $f_{[b]}$ was originally defined only up to a holomorphic conjugacy. We can fix this degree of freedom by setting $f_{[b]} = f_a$. For $b = 0$, we obtain the map $z \mapsto 1/z^2$. This defines a map from the unit disk $|b| < 1$ to the parameter space V_2. We will call this map the *anti-mating parameterization*. Actually, it is easy to see that each map $f_{[b]}$ belongs to the exterior component $\mathcal{E}$ (this is because all critical points of $f_{[b]}$ are in the immediate basin of attraction of the super-attracting cycle $\{0, \infty\}$).

Proposition 3.4. *The anti-mating parameterization is one-to-one: if maps $f_{[b]}$ and $f_{[b']}$ are holomorphically conjugate, then $b = b'$.*

Proof. Indeed, if $f_{[b]}$ and $f_{[b']}$ are holomorphically conjugate on the Riemann sphere, then the squares of the corresponding quadratic Blaschke products

$$B_0(z) = z\frac{z + b}{\overline{b}z + 1}, \quad \text{and} \quad B'_0(z) = z\frac{z + b'}{\overline{b'}z + 1}$$

are holomorphically conjugate in the unit disk. Since 0 is the only fixed point for each of the maps B_0^2 and ${B'_0}^2$, a conjugating homeomorphism φ must fix 0. Then φ is just the multiplication by some complex number λ such that $|\lambda| = 1$.

The point $-b$ is the only preimage of 0 under B_0. Similarly, the point $-b'$ is the only preimage of 0 under B'_0. Therefore, we must have $b' = \lambda b$. But then the equation $\lambda B_0^2(z) = {B'}_0^2(\lambda z)$ yields $\lambda = 1$, after all cancelations. In particular, $b = b'$. □

We will need the following obvious lemma:

Lemma 3.5. *Let f and g be holomorphic functions defined on some open subsets of $\overline{\mathbb{C}}$. Suppose that f has no multiple critical points, and fix an open set $U \subseteq \overline{\mathbb{C}}$. If for every critical value v of f, the set $g^{-1}(v) \cap U$ consists of simple critical points of g, then the multi-valued analytic function $f^{-1} \circ g$ has no ramification points in U.*

In particular, if U is simply connected, g is defined everywhere on U, and f is a ramified covering over $g(U)$, then $f^{-1} \circ g|_U$ splits into single-valued branches.

Proposition 3.6. *The anti-mating parameterization is onto: any quadratic rational map of class $\mathcal{E}$ is holomorphically conjugate to $f_{[b]}$ for some b.*

Proof. Consider any map $f \in V_2$ in the exterior hyperbolic component $\mathcal{E}$. We may assume that $f = f_a$ for some a. Let Ω_0 and Ω_∞ denote the immediate basins of 0 and ∞, respectively, for the map $f^{\circ 2}$ (both 0 and ∞ are super-attracting fixed points for this map). The proof that f is holomorphically conjugate to (actually, coincides with) some map $f_{[b]}$, consists of several steps:

Step 1. Conjugate $f^{\circ 2}$ by a Riemann map sending Ω_0 to the unit disk and fixing 0. The result is a holomorphic self-covering g of the unit disk of degree 4 such that 0 is a fixed critical point and a preimage $-b \neq 0$ of 0 is also a critical point. In particular, all preimages of 0 have multiplicity 2, which means by Lemma 3.5 that there is a well-defined holomorphic branch of the function $\sqrt{g}$. Denote this branch by B_0.

Step 2. Since $B_0(0) = 0$, we conclude that $z \mapsto B_0(z)/z$ is a holomorphic automorphism of the unit disk that maps $-b$ to 0. Therefore, it must have the form

$$\lambda \frac{z+b}{\bar{b}z+1},$$

where λ is a complex number such that $|\lambda| = 1$. Conjugating g by a suitable rotation around the origin, we can arrange that $\lambda = 1$ (with a different choice of b).

Step 3. The map $f^{\circ 2}$ is holomorphically conjugate to B_0^2, and hence to $f_{[b]}^{\circ 2}$, on the set Ω_0. More precisely, there is a holomorphic embedding $\varphi_0 : \Omega_0 \to \overline{\mathbb{C}}$ such that

$$\varphi_0 \circ f^{\circ 2} = f_{[b]}^{\circ 2} \circ \varphi_0 \quad on \quad \Omega_0. \tag{1}$$

Moreover, we can assume that $\varphi_0'(0) = 1$. In particular, the 0-ray of $f^{\circ 2}$ emanating from 0 is mapped to the 0-ray of $f_{[b]}^{\circ 2}$ emanating from 0. Since the Julia set of f is locally connected, we can extend φ_0 to the closure of Ω_0.

Step 4. All critical values of $f_{[b]}$ are images under φ_0 of the critical values of f. Therefore, by Lemma 3.5, the multi-valued analytic function $f_{[b]}^{-1} \circ \varphi_0 \circ f$ splits into two single-valued branches over Ω_∞. The 0-ray for f emanating from ∞ gets mapped to the 0-ray for $f_{[b]}$ emanating from 0 under $\varphi_0 \circ f$. The two preimages of the latter ray under $f_{[b]}$ are the 0- and 1/2-rays emanating from ∞. Choose the branch φ_∞ of $f_{[b]}^{-1} \circ \varphi_0 \circ f$ that takes the 0-ray emanating from ∞ to the 0-ray for $f_{[b]}$ emanating from ∞.

Step 5. The map φ_∞ is defined on Ω_∞, and satisfies the following relation:

$$f_{[b]} \circ \varphi_\infty = \varphi_0 \circ f \quad on \quad \Omega_\infty. \tag{2}$$

If we substitute this relation into (1), then we obtain the following:

$$f_{[b]} \circ (\varphi_\infty \circ f) = f_{[b]} \circ (f_{[b]} \circ \varphi_0) \quad on \quad \Omega_0.$$

Using the fact that φ_∞ takes a 0-ray to a 0-ray, we conclude that

$$\varphi_\infty \circ f = f_{[b]} \circ \varphi_0 \quad on \quad \Omega_0. \tag{3}$$

From formulas (2) and (3) it also follows that

$$\varphi_\infty \circ f^{\circ 2} = f_{[b]}^{\circ 2} \circ \varphi_\infty \quad on \quad \Omega_\infty. \tag{4}$$

Step 6. The map φ_∞ also extends continuously to the Julia set of f. The restrictions of the maps φ_0 and φ_∞ to the Julia set of f both conjugate the map $f^{\circ 2}$ with $f_{[b]}^{\circ 2}$. Uniformize both Ω_0 and the basin of 0 for $f_{[b]}^{\circ 2}$ by the unit disk. Both maps $f^{\circ 2}$ and $f_{[b]}^{\circ 2}$ correspond to the self-map $z \mapsto z^4$ of the unit circle. Thus the maps corresponding to φ_0 and φ_∞ both conjugate $z \mapsto z^4$ with itself. It follows that these maps differ by a cubic root of unity. However, both φ_0 and φ_∞ take the 0-rays for f emanating from 0 and ∞ to the 0-rays of $f_{[b]}$ emanating from 0 and ∞, respectively. Therefore, the restrictions of φ_0 and φ_∞ to the Julia set of f must coincide.

Step 7. We can now define a continuous map

$$\varphi = \begin{cases} \varphi_0, & on \quad \overline{\Omega}_0 \\ \varphi_\infty, & on \quad \overline{\Omega}_\infty \end{cases}$$

By formulas (2) and (3) and their extensions to the Julia set, the map φ conjugates f with $f_{[b]}$. Moreover, φ is holomorphic on the Fatou set. It follows that φ is holomorphic on the Riemann sphere, i.e. φ is a Möbius transformation. Since it fixes 0, 1 and ∞, the map φ must be the identity. We conclude that $f = f_{[b]}$. □

3.3. **Ray dynamics: non-periodic case.** Let $f = f_a$ be a map in the exterior component. In this subsection, we will study combinatorics of rays for the map $f^{\circ 2}$.

Consider the ray $R_0 = R_0(\theta_0)$ in Ω_0 that emanates from 0 and crashes into -1. Such ray always exists. Indeed, there is at least one ray emanating from 0 that crashes into a pre-critical point (otherwise, the map $f^{\circ 2}$ would be conjugate to the map $z \mapsto z^2$ everywhere on Ω_0). The pre-critical point this ray crashes into must be an iterated preimage of -1. The image of this ray under the corresponding (necessarily even) iteration of f will be the ray emanating from 0 and crashing into -1.

Suppose that the ray R_0 is not periodic under the map $f^{\circ 2}$ (i.e. no iterated image of R_0 is contained in R_0). This means that the angle θ_0 is not periodic under the doubling. There are exactly two rays R_1 and R_2, whose α-limit set is the critical point -1. The images of these rays under the map $f^{\circ 2}$ coincide and lie on the ray $f^{\circ 2}(R_0)$.

Proposition 3.7. *The rays R_1 and R_2 land in the Julia set.*

Proof. It suffices to prove this for one ray, say, for R_1. First, we need to show that the ray R_1 does not crash into pre-critical points. Assume the contrary: the ω-limit set of R_1 is a pre-critical point x. It is an iterated preimage of -1, so that we can write $f^{\circ 2n}(x) = -1$ for some positive integer n.

The set $f^{\circ 2}(R_1)$ lies on the ray containing $f^{\circ 2}(R_0)$. Therefore, the set $f^{\circ 2n}(R_1)$ lies on the ray containing $f^{\circ 2n}(R_0)$. However, the set $f^{\circ 2n}(R_1)$ has the point -1 in its closure, whereas the ray containing $f^{\circ 2n}(R_0)$ does not (because R_0 is not periodic). A contradiction.

We see that R_1 does not crash into pre-critical points. Therefore, its ω-limit set is a connected subset of the Julia set. If this subset contains more than one point, then it contains an arc (i.e. the preimage of an arc under a homeomorphism between the Julia set and the unit circle). In this case, the ω-limit set of a suitable iterated image of R_1 is the whole Julia set. The iterated images of R_1 belong to the rays containing the iterated images of R_0. Thus the ω-limit set of a ray containing a certain iterated image of R_0 is the Julia set.

Consider two strictly pre-periodic rays R' and R'' of different minimal periods emanating from 0. If R_0 is strictly pre-periodic, we assume additionally that the minimal periods of R' and R'' are different from that of R_0. The rays R' and R'' do not crash into pre-critical points, otherwise their suitable iterated images would belong to the ray R_0, which is not pre-periodic or has a different minimal period. The standard argument of Douady and Hubbard [4] now applies to show that R' and R'' land in the Julia set (so that their ω-limits are single well-defined points different from each other). The closures of the rays R' and R'' divide the closed unit disk into two parts, and the closure of any ray emanating from 0 can only belong to one part . This contradicts the statement that the ω-limit set of a certain ray emanating from 0 is the whole Julia set. □

Proposition 3.8. *Any ray for the map $f^{\circ 2}$ either crashes into an iterated preimage of -1 or lands in the Julia set.*

Proof. Consider any ray R. The α-limit set of this ray is an iterated preimage of 0 or an iterated preimage of -1. Thus we can map R to a ray emanating from 0 or from -1 by a suitable iteration of the map $f^{\circ 2}$. In other terms, we can assume without loss of generality that the ray R emanates from 0 or from -1.

Consider the first case: R emanates from 0. Suppose that R does not crash into a an iterated preimage of -1. Then its ω-limit set is contained in the Julia set. The rest of the proof goes exactly as in Proposition 3.7. In the second case, the ray R must coincide with R_1 or R_2. The result now follows from Proposition 3.7. □

Let φ denote the quasi-symmetric homeomorphism between the unit circle and the Julia set of f that conjugates the map $x \mapsto 1/x^2$ with the map f:

$$f(\varphi(x)) = \varphi(1/x^2), \quad x \in S^1$$

Recall that we defined the two-sided *ray lamination* RL associated with f in the following way: $xy \in RL$ if and only if $\varphi(x)$ and $\varphi(y)$ are the landing points of rays emanating from the same iterated f-preimage of -1. The geodesic xy is drawn inside or outside of the unit circle depending on whether this iterated preimage of -1 belongs to Ω_0 or Ω_∞.

3.4. **Proof of Theorem D.** Consider a parameter value a in the exterior hyperbolic component that does not belong to a periodic external parameter ray, and the corresponding rational map $f = f_a$. Let J denote the Julia set of f. We need to prove that the ray lamination RL coincides with some two-sided lamination $2L(x_0)$ corresponding to a point $z_0 = e^{2\pi i\theta_0}$ on the unit circle that is not periodic under the map $z \mapsto z^2$ (here x_0 is expressed through θ_0 as in Theorems B and D). To this end, we recover the map h of Subsection 2.2 in terms of RL. We will use the homeomorphism $\varphi : S^1 \to J$ from the end of the preceding subsection.

For any iterated preimage z of -1, we defined the *ray leaf* $Rl(z)$ as the union of z and the two rays emanating from z. Define a continuous map $\tilde{h} : S^1 \to S^1$ as follows:

- if $\varphi(e^{2\pi i\theta})$ is the landing point of a ray $R_0(\xi)$, then we set $\tilde{h}(e^{2\pi i\theta}) = e^{2\pi i\xi}$;
- otherwise there is a unique ray $R_0(\xi)$ that splits at a precritical point z and such that $Rl(z) \cup J$ separates 0 from $\varphi(e^{2\pi i\theta})$; we set $\tilde{h}(e^{2\pi i\theta}) = e^{2\pi i\xi}$.

Proposition 3.9. *The map $\tilde{h}$ coincides with the map h from Subsection 2.2, with some choice of the point z_0.*

Proof. We will just check that the map $\tilde{h}$ satisfies all properties of the map h. Since $\varphi(1)$ is the landing point of $R_0(0)$, we have $\tilde{h}(1) = 1$. It is also clear that $\tilde{h}$ has topological degree 1. It only remains to verify that the push-forward of the Lebesgue measure under $\tilde{h}$ is the measure μ corresponding to some point z_0 on the unit circle, as it was defined in Subsection 2.2. We denote by $\tilde{\mu}$ the push-forward of the Lebesgue measure under the map $\tilde{h}$.

Consider the ray leaf $Rl(-1) = \{-1\} \cup R_1 \cup R_2$. The landing points of rays R_1 and R_2 divide the Julia set into two arcs. Choose the arc $\varphi(\tilde{\sigma}_0)$ that is separated from 0 by $Rl(-1)$. The arc $\tilde{\sigma}_0$ of the unit circle has length 1/2 (because the boundary points of $\varphi(\tilde{\sigma}_0)$ are mapped to the same point under f, and hence the boundary points of $\tilde{\sigma}_0$ are mapped to the same point under $x \mapsto 1/x^2$). The image of $\tilde{\sigma}_0$ under $\tilde{h}$ is some point z_0 on the unit circle such that $\tilde{\mu}\{z_0\} = 1/2$. Any ray leaf is an iterated preimage of the leaf $Rl(-1)$. Therefore, the images under $\tilde{h} \circ \varphi^{-1}$ of all arcs in J subtended by ray leaves are points on the unit circle that lie in the backward orbit of z_0 under the map $z \mapsto z^2$. Moreover, if $z^{2^m} = z_0$, then we have $\tilde{\mu}\{z\} = \frac{1}{2 \cdot 4^m}$.

We see that the measure $\tilde{\mu}$ coincides with the measure μ corresponding to the point z_0. Then the map $\tilde{h}$ is also the same as the map h. □

Theorem D follows immediately from this proposition.

4. Analytic continuation

In this section, we approach the external boundary of M_2 from the exterior component. We will define fixed point portraits for maps on the external boundary using an analytic continuation argument similar to that in [3].

4.1. The basin of the super-attracting cycle.

Let us first recall the setup. Our main object is the following family of quadratic rational self-maps of the Riemann sphere:

$$f_a(z) = \frac{a}{z^2 + 2z}.$$

Infinity is a periodic critical point of period 2 for all maps in this family. The corresponding orbit is $\{0, \infty\}$. The other critical point is -1.

Denote by Ω the immediate basin of attraction of the super-attracting cycle $\{0, \infty\}$. Let Ω_0 and Ω_∞ be connected components of Ω containing 0 and ∞, respectively. The restriction of f_a to Ω_∞ is a 2-fold branched covering of Ω_0. It follows that $f_a^{-1}(\Omega_0) = \Omega_\infty$. We will write simply f instead of f_a whenever this notation is unambiguous. The Julia set of f will be denoted by J.

Proposition 4.1. *The critical point -1 does not belong to the set Ω_∞.*

Proof. If $-1 \in \Omega_\infty$, then all critical points of f belong to the same Fatou component. It is known (see e.g. [13, 17]) that in this case, the Fatou component containing the critical points must be invariant, and the Julia set must be totally disconnected. A contradiction. □

Proposition 4.2. *Both sets Ω_0 and Ω_∞ are topological disks.*

Proof. Consider a small disk U containing the origin. For any positive integer n, define the open set U_n as the component of $f^{-n}(U)$ containing 0 or infinity depending on whether n is even or odd. Since $-1 \notin \Omega_\infty$, each set U_n contains at most one critical point. By the Riemann–Hurwitz formula, if U_n is a topological disk, then U_{n+1} is also a topological disk. Thus all U_n are simply connected.

The set Ω_0 is the union of U_n for all even n. As the union of a nested sequence of simply connected open sets, this set is also simply connected. Similarly, Ω_∞ is simply connected. □

4.2. **Radial components.** Let x be an iterated preimage of the critical point ∞. It makes sense to talk about *rays* emanating from x, see Subsection 1.7 for more details. Every ray hits the Julia set or a pre-critical point (namely, an iterated preimage of the critical point -1).

Define the *radial component* of x as the union of $\{x\}$ and all rays emanating from x. We will call the point x the *center* of this radial component. Clearly, every radial component is an open topological disk. If the critical point -1 is not attracted by the cycle $\{0, \infty\}$, then each radial component is just a Fatou component. However, the combinatorial structure of radial components is more stable than that of Fatou components.

Let A_0 and A_∞ denote the radial components of 0 and ∞, respectively. Note that $f(A_\infty) = A_0$, the restriction of f to A_∞ being a ramified covering of degree 2. However, in general, the set $f(A_0)$ is strictly contained in A_∞. The ray of angle θ emanating from x will be denoted by $R_x(\theta)$.

The following proposition is essentially due to Luo [7]:

Proposition 4.3. *Suppose that the parameter a is not on the external parameter ray of angle 0. Then the intersection of $\overline{A}_0$ and $\overline{A}_\infty$ contains a fixed point ω of f that is the landing point of both $R_\infty(0)$ and $R_0(0)$.*

Proof. First note that if a is not on the external parameter ray of angle 0, then the rays $R_\infty(0)$ and $R_0(0)$ both land in the Julia set. Consider the landing point ω of the 0-ray in A_∞. This is a point on the boundary of A_∞ that is either a fixed point or a point of period 2. However, the map f has only one orbit of period two, namely, $\{0, \infty\}$. It follows that ω is a fixed point. Since ω belongs to the boundary of A_∞, it is also on the boundary of $A_0 = f(A_\infty)$. □

Note that the fixed point ω must be repelling. Indeed, this fixed point is a univalent function of the parameter defined on $\mathbb{C} - 0$ with the external parameter ray of angle 0 removed. Since it does not bifurcate over this region, it never becomes parabolic. Actually, the ramification point for ω is exactly the puncture $a = 0$, the value of a that does not correspond to any map in V_2.

Let x be an iterated preimage of the critical point ∞, and n the minimal non-negative integer such that $f^{\circ n}(x) = \infty$. The number n is called the *depth* of x and of the corresponding radial component. The following statement can be easily deduced from Proposition 4.3 by applying pull-backs under the iterates of f:

Proposition 4.4. *Suppose that the parameter a is not on the external parameter ray of a binary rational angle. Let A be a radial component, and $r \neq 0$ a binary rational angle. Then the ray of angle r in A lands at a point in the Julia set that is also the landing point of the 0-ray in a unique radial component A', whose depth is bigger than the depth of A.*

The uniqueness follows from the following fact:

Proposition 4.5. *The ray $R_\infty(0)$ is the only ray in A_∞ landing at ω.*

The proof is similar to that of the following classical statement about quadratic polynomials: there is only one external ray landing at the β fixed point.

Proposition 4.6. *If A is a radial component different from A_∞ and A_0, then the fixed point ω is not in the closure of A.*

Proof. Suppose that ω is in the closure of A. Then ω must be the root point of A, i.e. the landing point of the zero ray in A (because some ray in A must land at ω, and this can only be the ray of angle zero). Note that if A has the property $\omega \in \partial A$, then $f(A)$ has the same property. We can now assume that A has the minimal depth among all radial components with this property, different from A_∞ and A_0. In this case, A must map to Ω_∞ under the first iteration of f, and the root point of A must coincide with the landing point of $R_\infty(1/2)$. But this point is different from ω by Proposition 4.5. □

Corollary 4.7. *Suppose that -1 is not an iterated preimage of ω. Then any iterated preimage of ω is on the boundary of exactly two radial components.*

This statement can be easily reduced to the preceding proposition by using iterations of f.

4.3. **Regulated rays.** Let $r_0, r_1, \dots$ be a finite or infinite sequence of nonzero binary rational angles, and x an iterated preimage of ∞. Define the set $\Gamma(x, r_0, r_1, \dots)$ as follows. Let A_0 be the radial component centered at x. Start at x and go in A_0 along the ray of angle r_0 up to the landing point a_0. By Proposition 4.4, the point a_0 is the landing point of the 0-ray in some radial component A_1. Go along the 0-ray of A_1 to the center of A_1. From the center, go along the ray of angle r_1 up to the landing point a_2. Continuing this process (if possible), we obtain a (finite or infinite) sequence of points a_m and radial components A_m such that a_m is the landing point of the ray of angle r_m in A_m, and, at the same time, the landing point of the 0-ray in A_{m+1}. We define $\Gamma(x, r_0, r_1, \dots)$ to be the union of the centers of A_m, the rays of angles r_m in A_m, the points a_m, and the 0-rays in A_{m+1}. We call $\Gamma(x, r_0, r_1, \dots)$ a *regulated ray* starting at x. It is easy to see that there is a continuous embedding $\gamma : [0, \infty) \to \overline{\mathbb{C}}$ such that $\gamma[0, \infty) = \Gamma(x, r_0, r_1, \dots)$ and $\gamma(n + 1/2) = a_n$ for all $n = 0, 1, \dots$. We say that an infinite regulated ray $\Gamma(x, r_0, r_1, \dots)$ *lands* at a point z if the corresponding path $\gamma(t)$ converges to z as $t \to \infty$. Note that a regulated ray is well defined unless it crashes into a pre-critical point. In particular, if the critical point -1 is not attracted by the cycle $\{0, \infty\}$, then all regulated rays are well defined.

Proposition 4.8. *Any iterated preimage of* ∞ *can be connected to* 0 *or* ∞ *by a finite regulated ray.*

Proof. Note that the full preimage of a regulated ray starting at 0 is a pair of regulated rays starting at ∞:

$$f^{-1}(\Gamma(0, r_1, r_2, \dots)) = \Gamma(\infty, r_1/2, r_2, \dots) \cup \Gamma(\infty, (r_1 + 1)/2, r_2, \dots).$$

Consider a regulated ray $\Gamma(\infty, r_1, r_2, \dots)$ starting at ∞. The preimage of this ray is the union of $\Gamma(0, r_1, r_2, \dots)$ and a regulated ray starting at -2. But the latter is a part of $\Gamma(\infty, 1/2, r_1, r_2, \dots)$. We see that the preimage of any regulated ray lies in the union of regulated rays.

Using this statement, it is now easy to prove the proposition by induction. □

Note that the intersection of any two regulated rays is an initial segment of both. The image of a regulated ray starting at 0 is a regulated ray starting at ∞:

$$f(\Gamma(0, r_1, r_2, \dots)) = \Gamma(\infty, r_1, r_2, \dots).$$

The image of a regulated ray starting at ∞ is either a regulated ray starting at 0 or the union of a regulated ray starting at ∞ and the path between 0 and ∞ along the zero rays of A_0 and A_∞. The latter path will be denoted by $\Gamma[0, \infty]$. We have

$$f(\Gamma(\infty, r_1, r_2, \dots)) = \begin{cases} \Gamma(0, 2r_1, r_2, \dots), & r_1 \neq 1/2, \\ \Gamma(\infty, r_2, \dots) \cup \Gamma[0, \infty], & r_1 = 1/2. \end{cases}$$

Let x be the center of some radial component. The end of a finite regulated ray $\Gamma(x, r_1, \dots, r_n)$ is the center of another radial component, which we will denote by $A(x, r_1, \dots, r_n)$. By Proposition 4.8, radial components are in one-to-one correspondence with finite regulated rays starting at ∞ or 0 and such that all angles r_i are nonzero.

Proposition 4.9. *Let* $r_1, r_2, \dots$ *be an infinite sequence of binary rational numbers, and suppose that the parameter* a *is on the external parameter ray of angle* $\theta_0 \neq 2^k r_m$*. Then the regulated ray* $\Gamma(0, r_1, r_2, \dots)$ *is well defined and lands at a point in the Julia set.*

Proof. The condition $\theta_0 \neq 2^k r_m$ guarantees that the regulated ray $\Gamma(0, r_1, r_2, \dots)$ never crashes into a precritical point. Therefore, it is well defined. From the hyperbolicity of f it follows that the diameter of A_m decays exponentially, therefore, the regulated ray lands. □

To emphasize the dependence of a regulated ray on the parameter a, we will sometimes write $\Gamma_a(\infty, r_1, r_2, \dots)$ instead of $\Gamma(\infty, r_1, r_2, \dots)$. In the sequel, we will need the notion of the *angle* of a regulated ray $\Gamma_a(\infty, r_1, r_2, \dots)$. To define the angle, consider the regulated ray $\Gamma_1(\infty, r_1, r_2, \dots)$ for the rational map f_1, which is Möbious conjugate to the quadratic polynomial $p_{-1} : z \mapsto z^2 - 1$. The landing point of this ray corresponds to a point in the basilica that is the landing point of exactly one external ray of angle θ. We call θ the angle of $\Gamma(\infty, r_1, r_2, \dots)$. Clearly, it depends only on the sequence of binary rational numbers $r_1, r_2, \dots$, not on a specific parameter value a. This definition is parallel to that of [7, 3].

4.4. **Fixed point portraits.** For this subsection, the parameter a is in the exterior hyperbolic component, but not on a rational external parameter ray.

Consider the regulated ray $\Gamma^0 = \Gamma(\infty, 1/2, 1/2, \dots)$. Note that this regulated ray is contained in its image under f. Therefore, the landing point of it must be a fixed point of f. Denote this point by β. For the parameter values under consideration, all periodic points are repelling. In particular, β is a repelling fixed point.

Proposition 4.10. *The fixed point β is different from ω.*

Proof. Suppose that $\beta = \omega$. Consider a small topological disk D around ω. We can arrange that the boundary of this disk intersect each ray $R_\infty(0)$ and $R_0(0)$ at a single point. Then the union of these rays and ω divides D into two parts. The path Γ^0 lies in one part and is invariant under f (in the sense that $D \cap f(D \cap \Gamma^0) = D \cap \Gamma^0$). However, the two parts are interchanged under f, because the rays $R_0(0)$ and $R_\infty(0)$ are interchanged. A contradiction. □

The map f has three fixed points, and we already identified two of them. Denote the remaining fixed point by α (the notation α and β for fixed points is meant to suggest a similarity with quadratic polynomials). The α-fixed point is the most interesting one.

Proposition 4.11. *There is a regulated ray landing at the fixed point α.*

Proof. Let I be the closed segment of the ray $R_\infty(\theta_0)$ between the critical value and the landing point (since θ_0 is irrational, the ray $R_\infty(\theta_0)$ lands in the Julia set). The map f^{-1} has two well-defined holomorphic branches on the set $\Omega_\infty - I$.

Since $\alpha \neq \omega$, the α-fixed point cannot be on the boundary of A_∞. Consider a ray leaf Rl on the boundary of A_∞ that separates ∞ from the fixed point α (this means that any curve in Ω_∞ connecting ∞ with α must intersect Rl). Let D be the component of the complement to $Rl \cup J$ lying in Ω_∞ and containing α on its boundary. There is a holomorphic branch g of f^{-n} mapping $\Omega_\infty - I$ into D and

$A_\infty - I$ into a radial component adjacent to Rl. We have $(\Omega_\infty - I) \supset D \supset g(D) \supset g^{\circ 2}(D) \supset \dots$, and the sets $g^{\circ m}(D)$ converge to α in the Hausdorff metric.

Consider a finite regulated ray $\Gamma(\infty, r_1, r_2, \dots, r_k)$ connecting ∞ with the center of the radial component different from A_∞ and adjacent to the ray leaf Rl. Actually, $k = 1$ in our situation (see Proposition 5.20), but this is not important for the time being. Consider the infinite regulated ray

$$\Gamma_1 = \Gamma(\infty, r_1, r_2, \dots, r_k, r_1, r_2, \dots, r_k, \dots),$$

where the sequence of angles is periodic with period $(r_1, r_2, \dots, r_k)$. Clearly, $g(\Gamma_1) \subset \Gamma_1 \cap D$. Therefore, Γ_1 lands at the fixed point α. □

The map $f^{\circ n}$ takes the path Γ_1 to itself (modulo the regulated segment $\Gamma[0, \infty]$). In this sense, Γ_1 is periodic under f. Denote the minimal period by q. However, Γ_1 is not fixed, because otherwise Γ_1 would coincide with the regulated ray $\Gamma(\infty, 1/2, 1/2, \dots)$ landing at β. Consider all images of Γ_1 under iterations of f (regarded as regulated rays starting at ∞ or 0; the segment $\Gamma[0, \infty]$ appearing in the image should be disregarded), and denote them by $\Gamma_1, \dots, \Gamma_q$, where $\Gamma_i = f^{\circ i-1}(\Gamma_1)$. All regulated rays Γ_i land at the fixed point α. The union $\{\alpha\} \cup \Gamma_1 \cup \dots \cup \Gamma_q$ is called the *fixed point portrait* for f.

With a fixed point portrait consisting of regulated rays $\Gamma_1, \dots, \Gamma_q$, we associate the set of angles $\{\theta_1, \dots, \theta_q\}$, where θ_i is the angle of Γ_i.

4.5. **Regulated parameter rays.** Let us start with the following landing property:

Proposition 4.12. *Any external parameter ray of a nonzero binary rational angle lands at a parameter a, for which the critical point -1 is on the boundary of Ω_0 and is eventually mapped to the fixed point ω.*

Proof. Consider an external parameter ray $\mathcal{R}$ of a binary rational angle θ_0. For any parameter a on this ray, the critical value $-a = f_a(-1)$ belongs to the ray $R_\infty(\theta_0)$. Since θ_0 is strictly pre-periodic under the doubling, the ray $R_\infty(\theta_0)$ lands in the Julia set. The landing point z_a must be an iterated preimage of the fixed point ω, because there are no other fixed points on the boundary of $A_0 \cup A_\infty$. The point z_a moves complex analytically (with respect to the parameter a) with finitely many branch points.

Consider any parameter value a_0 in the boundary of $\mathcal{R}$. If a_0 is not a ramification point for z_a, then z_a moves holomorphically over a neighborhood $O(a_0)$ of a_0. Thus the closure of the ray $R_\infty(\theta_0)$ moves holomorphically (hence equicontinuously, see [9]) over $O(a_0)$. It follows that, for the parameter value a_0, we have $-a_0 = z_{a_0}$, hence it maps eventually to ω. Clearly, if a_0 is a ramification point for z_a, then -1 is also mapped eventually to ω.

There are only finitely many parameter values, for which -1 is mapped eventually to ω. It follows that the parameter ray $\mathcal{R}$ lands. For the landing point a_0, we must have $-a_0 = z_{a_0}$, which can be easily proved by induction on the exponent of the denominator of θ_0. The proposition follows. □

Now recall certain facts from [3] that we will use. We use slightly different language; however, the translation should be straightforward. The type III hyperbolic components of V_2 are in one-to-one correspondence with finite sequences $(r_1, \dots, r_n)$ of nonzero binary rational numbers. For each such sequence $(r_1, \dots, r_n)$, the hyperbolic component $H(r_1, \dots, r_n)$ consists of all parameter values a such that the critical value $-a$ belongs to the radial component $A(\infty, r_1, \dots, r_n)$. The dynamical (Böttcher) coordinate of $-a$ in $A(\infty, r_1, \dots, r_n)$ defines the parameter coordinate of a in $H(r_1, \dots, r_n)$. Thus it makes sense to talk about *internal rays* in $H(r_1, \dots, r_n)$: the internal ray of angle θ consists of all parameter values $a \in H(r_1, \dots, r_n)$ such that the critical value $-a$ belongs to the dynamical ray of angle θ in $A(\infty, r_1, \dots, r_n)$, or, equivalently, the ray of angle θ in a preimage of $A(\infty, r_1, \dots, r_n)$ crashes into the critical point -1. The parameter value a lies on the boundary of $H(r_1, \dots, r_n)$ if and only if the corresponding critical value lies on the boundary of $A(\infty, r_1, \dots, r_n)$. In [3], this statement is deduced from the λ-lemma of Mañe–Sud–Sullivan [9].

Let $\mathcal{R}$ be an external parameter ray of a binary rational angle r. Consider the landing point a of $\mathcal{R}$. For the corresponding rational map f, the critical point -1 lies on the boundary of A_0 and A_{-2}, but also on the boundary of $A(\infty, 1/2, r)$ and $A(0, r)$. It follows that $-a$ is on the boundary of $A(\infty, r)$, hence the parameter value a is on the boundary of the type III component $H(r)$.

For a sequence of nonzero binary rational numbers $r_1, r_2, \dots$, define the *regulated parameter ray* $\Delta(\infty, r_1, r_2, \dots)$ as follows. Start at ∞ and go along the external parameter ray of angle r_1. By Proposition 4.12, this external parameter ray lands at some point on the external boundary, which is also a boundary point of $H(r_1)$. Continue along the zero internal ray of $H(r_1)$ up to the center, and then go along the internal ray of angle r_2 up to a boundary point. It is not hard to see that this boundary point of $H(r_1)$ is also a boundary point of $H(r_1, r_2)$. Continue along the zero internal ray in $H(r_1, r_2)$, etc.

The *angle* of a regulated parameter ray $\Delta(\infty, r_1, r_2, \dots)$ is defined as the angle of the corresponding regulated dynamical ray $\Gamma(\infty, r_1, r_2, \dots)$.

4.6. **Analytic continuation of fixed point portraits.** In this subsection, we essentially follow [3]. Consider a fixed point portrait $\{\alpha\} \cup \Gamma_1 \cup \dots \Gamma_q$ with the set of angles $\{\theta_1, \dots, \theta_q\}$. The angles $\theta_1, \dots, \theta_q$ divide the unit circle into several arcs. The shortest complementary arc is called the *characteristic arc*. Suppose that the characteristic arc is bounded by angles θ_- and θ_+, taken in the counterclockwise order. Then it is not hard to see that the critical value $-a$ must lie between the regulated rays Γ_- and Γ_+ of angles θ_- and θ_+, respectively. The following proposition is proved in [3]:

Proposition 4.13. *The regulated parameter rays Δ_- and Δ_+ of angles θ_- and θ_+, respectively, land at a parabolic point not in the closure of the exterior component.*

The following statement is slightly more general than in [3], but with similar proof:

Proposition 4.14. *The fixed point portrait moves holomorphically over the region* (*called a* parameter wake) *bounded by the regulated parameter rays* Δ_- *and* Δ_+.

Proof. For parameter values in the parameter wake, the critical point never enters a regulated ray of the fixed point portrait. Therefore, each regulated ray moves holomorphically. By the λ-lemma, it follows that the critical portrait also moves holomorphically. □

As a corollary, we have a well-defined fixed point portrait at all points on the external boundary. Moreover, for any external parameter ray $\mathcal{R}$, whose angle is not a binary rational number, the fixed point portrait moves continuously over $\overline{\mathcal{R}}$, and even holomorphically over some neighborhood of $\overline{\mathcal{R}}$. Note that Proposition 4.14 fails if we replace regulated rays with bubble rays (a *bubble ray* corresponding to a regulated ray Γ is the union of the closures of all radial components intersecting Γ). Actually, bubbles (the radial components) do not move continuously on the external boundary.

4.7. **Dynamical and parameter pre-puzzle.** The union of the fixed point portrait and $\Gamma[0,\infty]$ divides the parameter plane into several pieces, called *pre-puzzle pieces of depth 0*. We use the term pre-puzzle, because we do not employ equipotentials as we should do to form the actual puzzle pieces. The point of considering pre-puzzle is that its combinatorics will be stable along each external parameter ray. We define *pre-puzzle pieces of depth n* as n-th pull-backs of the pre-puzzle pieces of depth 0. By *combinatorics* of the pre-puzzle, we mean the information about which rays bound which pre-puzzle pieces. The following statement is immediate:

Proposition 4.15. *The combinatorics of the pre-puzzle stays fixed over each external parameter ray, whose angle is not a binary rational number.*

Define a parameter pre-puzzle piece of depth n as the locus of parameter values a such that f_a has a given combinatorics of pre-puzzle pieces of depth $\le n$.

Proposition 4.16. *Every parameter pre-puzzle piece is an open set bounded by several pairs of regulated parameter rays, each pair having a common landing point.*

Proof. The proof is straightforward. Suppose that there is no neighborhood of a parameter value a_0, over which a specified pre-puzzle piece moves holomorphically. Then, for certain parameter values a in any neighborhood of a_0, a certain iterated image $f_a^{\circ m}(-1)$ of -1 enters a regulated ray in the fixed point portrait. Since a fixed point portrait is invariant, we may assume that $m > 0$, and $f_a^{\circ m-1}(-a)$ lies on some regulated ray in the fixed point portrait. We conclude that a_0 is in the union of closures of finitely many regulated parameter rays.

Thus every parameter pre-puzzle piece is bounded by closures of finitely many regulated parameter rays. It is also easy to see that these regulated parameter rays come in pairs, each pair having a common landing point. For regulated parameter rays of periodic angles, this follows from Proposition 4.13. For regulated parameter rays of strictly pre-periodic angles, this follows from the fact that the fixed point portrait moves equicontinuously over open neighborhoods of the closures of such rays. □

As a corollary, we obtain the following

Proposition 4.17. *Let $\mathcal{R}$ be an external parameter ray, whose angle is not binary rational. For any n, there is an open neighborhood U of $\overline{\mathcal{R}}$ such that all pre-puzzle pieces of depth $\le n$ move holomorphically over U. Therefore, for all points in $\overline{\mathcal{R}}$, the corresponding rational maps have the same combinatorics of the pre-puzzle.*

5. Puzzles, cells and local connectivity

In this section, we deal with maps on the external boundary of M_2. We study two different types of combinatorial partitions for such maps: puzzles and cells. We need puzzles to prove local connectivity of Julia sets, and cells to establish topological models.

5.1. **Puzzle.** Throughout this section, $f = f_a$ corresponds to a parameter a on the boundary of some external parameter ray of angle θ_0. We assume that θ_0 is not binary rational. Denote by E_∞ some equipotential curve in A_∞ and by E_0 some equipotential curve in A_0. Let U be the component of the complement to $E_\infty \cup E_0$ containing -1. By choosing appropriate equipotentials E_∞ and E_0, we can arrange that $f^{-1}(U)$ be compactly contained in U. Let $\{\alpha\} \cup \Gamma_1 \cup \cdots \cup \Gamma_q$ be the fixed point portrait for f. *Puzzle pieces of depth zero* are defined as connected components of the complement to the set

$$G = \Gamma[0, \infty] \cup \bigcup_{i=1}^{q} \Gamma_i \cup \{\alpha\} \cup E_\infty \cup E_0,$$

intersecting the Julia set. A *puzzle piece* P_n of any depth n is defined as a connected component of $f^{-n}(P_0)$, where P_0 is a puzzle piece of depth 0. For any point $z \in J$ not on the boundary of a puzzle piece, let $P_n(z)$ denote the puzzle piece of depth n containing z. Puzzle pieces $P_n(-1)$ are called *critical puzzle pieces.*

A slight variation of this construction leads to the bubble puzzle, obtained by replacing the regulated rays Γ_i with the corresponding bubble rays. However, we use regulated rays instead of the corresponding bubble rays because two different bubble rays may touch at iterated preimages of the critical point -1.

5.2. **Rational-like maps.** P. Roesch [20] generalized the Yoccoz puzzle technique (initially developed for quadratic polynomials) to a broader class rational maps. In this subsection, we briefly recall the terminology of [20]. Let U and U' be two open sets in $\overline{\mathbb{C}}$ with smooth boundaries (in particular, both boundaries have finitely many connected components). Suppose that U' is compactly contained in U. Consider a proper holomorphic map $f : U' \to U$ with finitely many critical points that extends to a continuous map from $\overline{U'}$ to $\overline{U}$. Such a map is called a *rational-like map.* A rational-like map is called *simple* if there is exactly one critical point of f in U', and this critical point is simple. The filled-in Julia set for f is defined as $\bigcap_{n\geq 0} f^{-n}(U)$.

A finite connected topological graph G is called *admissible* for a simple rational-like map $f : U' \to U$ if the following conditions hold:

- the graph G contains ∂U and is contained in $\overline{U}$,
- the graph G is *stable* under f, i.e. we have $G \cap U' \subseteq f^{-1}(G)$,
- the forward orbit of the critical point is disjoint from G.

A *puzzle piece* of depth n (associated with (G, f, U', U)) is defined as any connected component of $f^{-n}(U - G)$. The collection of all puzzle pieces is called the *puzzle*. This is a generalization of the Yoccoz definition to the case of rational-like maps. For any point z in $f^{-n}(U - G)$, there is a unique puzzle piece $P_n(z)$ of depth n containing z. If z is a critical point for f, then the puzzle pieces $P_n(z)$ are called the *critical puzzle pieces*.

Example 5.1. Take $f = f_a$, with and U as in the preceding subsection. Set $U' = f^{-1}(U)$. Then $f : U' \to U$ is a simple rational-like map. The topological graph G introduced in the preceding subsection is admissible for f.

Suppose that the forward orbit of a point z avoids Γ. Define the *tableau* $T(z)$ of z as the matrix $T(z)_{i,j} = P_i(f^{\circ j}(z))$ of puzzle pieces, where i and j run through all nonnegative integers (thus the matrix $T(z)$ is infinite down and to the right). If z is a critical point, then $T(z)$ is called the *critical tableau*. A tableau $T(z)$ is said to be *periodic* of period k if $T_{i,j+k}(z) = T_{i,j}(z)$ for all i and j. The critical tableau T is called *recurrent* if the critical point belongs to $T_{i,j}$ with $j > 0$ and i arbitrarily large.

The following theorem is proved in [3]:

Theorem 5.1. *Let a rational-like map $f : U' \to U$ and a topological graph G be as in Subsection 5.1. If the critical tableau is not periodic, then the critical puzzle pieces $P_n(-1)$ converge to -1. Moreover, for any point z, whose forward orbit is disjoint from G, the puzzle pieces $P_n(z)$ converge to z.*

Below (Subsections 5.3 and 5.4), we sketch a proof of this theorem under the assumption that the parameter value a belongs to the boundary of an irrational parameter ray (this is what we actually need for the proof of the main theorems). For such parameter values, the critical tableau is automatically non-periodic:

Proposition 5.2. *For the parameter values on the boundary of the external parameter ray of angle θ_0, the critical tableau is not periodic, provided that θ_0 is not periodic and not binary rational.*

Proof. The argument below is similar to that in [5]. Consider a parameter value a on the boundary of the external parameter ray of angle θ_0, and the corresponding rational map $f = f_a$. By Proposition 4.17, all critical puzzle pieces intersect both A_0 and A_{-2}. The intersection of $P_n(-1)$ with $A_0 = \Omega_0$ is bounded by two rays in A_0 of binary rational angles θ_n^- and θ_n^+. It is easy to see that both θ_n^- and θ_n^+ converge to θ_0. Therefore, the intersections of the critical puzzle pieces with $\overline{\Omega}_0$ converge to the prime end impression of angle θ_0.

From the combinatorics of the puzzle it also follows that the landing points of binary rational rays in Ω_0 separate the boundary of Ω_0. In particular, the prime end impressions are disjoint. If the critical tableau is periodic, then the prime end impression of angle θ_0 for Ω_0 is also periodic. It follows that θ_0 is periodic, a contradiction. □

An important corollary of this proposition is the following:

Proposition 5.3. *If θ_0 is not periodic, and a is on the boundary of the external parameter ray of angle θ_0, then the critical point -1 lies on the boundary of the Fatou component Ω_0.*

5.3. **An example.** Before discussing general combinatorics of puzzles, let us work out one particular example. We use the same set-up as in Subsection 5.1. Suppose that the regulated rays Γ_i, $i = 1, 2, 3$, converging to the fixed point α are

$$\Gamma_1 = \Gamma\left(\infty, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \dots\right), \quad \Gamma_2 = \Gamma\left(\infty, \frac{1}{4}, \frac{1}{4}, \dots\right), \quad \Gamma_3 = \Gamma\left(0, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \dots\right).$$

Consider also preimages of these regulated rays (or, equivalently, regulated rays symmetric to these regulated rays with respect to -1):

$$\Gamma'_1 = \Gamma\left(0, \frac{1}{4}, \frac{1}{4}, \dots\right), \quad \Gamma'_2 = \Gamma\left(\infty, \frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \dots\right), \quad \Gamma'_3 = \Gamma\left(\infty, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \dots\right).$$

The regulated rays Γ'_1, Γ'_2 and Γ'_3 converge to the point α' symmetric to α with respect to -1, i.e. $\alpha' = -2 - \alpha$. The six paths Γ_i, Γ'_j, $i, j = 1, 2, 3$, divide the open set U into 5 pieces (see Picture 5).

We see that no puzzle piece of depth 1 is compactly contained in a puzzle piece of depth 0. Next, we need to look for puzzle pieces of depth 2 compactly contained in puzzle pieces of depth 0. Indeed, there are two puzzle pieces of depth 2 compactly contained in $P^{(0)}(-1)$. They are marked with sign "+".

5.4. **Critical annuli.** Consider a map $f = f_a$, where the parameter value a is in the closure of an exterior parameter ray $\mathcal{R}$ of an irrational angle θ_0. Let an open set U be as in Subsection 5.1. In this subsection, we study the rational like map $f : U' \to U$, where $U' = f^{-1}(U)$, and the puzzle for such map defined in Subsection 5.1.

We define the *critical annulus of depth n* as $R_n(-1) = P_n(-1) - \overline{P}_{n+1}(-1)$. If this set is not a topological annulus, we say that the annulus $R_n(-1)$ is *degenerate*.

Recall that for quadratic polynomials, the existence of a non-degenerate critical annulus was settled by the following statement (see [12, 8]): for a non-renormalizable quadratic polynomial, the critical orbit enters a non-critical puzzle piece of depth 1 incident to the point $-\alpha$ (where α is the α-fixed point). There is an analog of this statement for the maps under consideration:

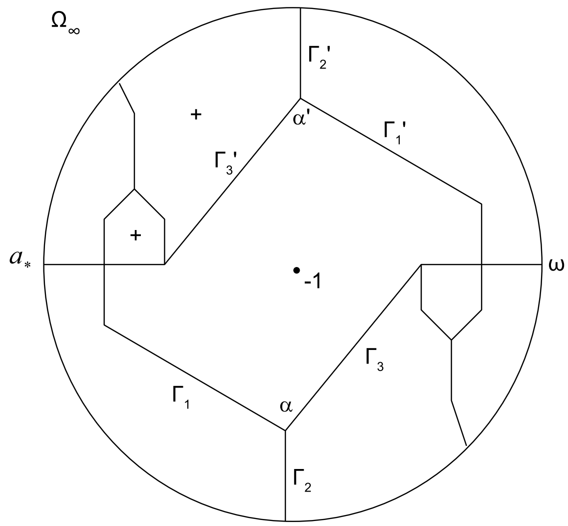

FIGURE 5. An example of the puzzle (this is a very schematic picture not showing equipotentials and rays in Ω_∞)

Proposition 5.4. *Let α' be the preimage of α different from α, i.e. $\alpha' = -2 - \alpha$. The critical orbit enters a puzzle piece of depth 1 incident to α' and not containing the critical point -1.*

Proof. Let Π be the union of pre-puzzle pieces of depth 1 incident to α' and not containing the critical point -1. We will write Π_a to indicate the dependence of Π on the parameter a. We know that the boundary of Π_a moves holomorphically with respect to a over some neighborhood of $\overline{\mathcal{R}}$.

Note that Π_a contains either all rays in A_∞ of angles less than $1/2$ or all rays in A_∞ of angles bigger than $1/2$. Now suppose that $a \in \mathcal{R}$. Then there exists a positive integer n (independent of $a \in \mathcal{R}$!) such that $f_a^{\circ n}(-1) \in \Pi_a$.

Passing to the limit as a approaches the boundary of $\mathcal{R}$, we conclude that $f^{\circ n}(-1) \in \Pi$ for parameter values on the boundary of $\mathcal{R}$. The proposition follows. □

Unfortunately, unlike the case of quadratic polynomials, not all the puzzle pieces of depth 1 from Proposition 5.4 are compactly contained in the critical puzzle piece of depth 0. Note, however, that the set of angles $2^n\theta_0$ is dense in $\mathbb{R}/\mathbb{Z}$. In particular, the critical orbit enters all puzzle pieces of depth 1 intersecting Ω_∞. Let $\Gamma_1 = \Gamma(\infty, r_1, r_2, \dots)$ be a regulated ray landing at α. We can always arrange that $r_1 = 1/2$ by taking forward images of Γ_1 under the iterates of f.

For $r_2 \neq 1/4, 3/4$, there is a puzzle piece of depth 1 that intersects Ω_∞ and is compactly contained in the critical puzzle piece of depth 0. This is because there are regulated rays in the fixed point portrait of α intersecting the boundary of

Ω_∞ at points of angles $2^k r_2$, where $k = 0, 1, 2, \dots$. Since the critical orbit enters this puzzle piece, there is a non-degenerate critical annulus. We can now use the following theorem [20] (see also [14] — it deals with quadratic polynomials only, but the proof can be taken verbatim in our situation):

Theorem 5.5. *Suppose that the critical tableau T is recurrent but not periodic. Also, suppose that there is at least one nondegenerate critical annulus. Then the critical puzzle pieces converge to the critical point.*

It remains to consider the case, where r_2 is $1/4$ or $3/4$ (see Subsection 5.3 above). There are no nondegenerate critical annuli in this case. Note, however, that the following property holds:

Proposition 5.6. *The critical point -1 can only return to the puzzle piece $P_1(-1)$ under an even iteration of f.*

Proof. It suffices to prove the corresponding statement for the critical pre-puzzle piece of depth 1 and for parameter values a in the exterior hyperbolic component. The proposition will follow, if we pass to the limit as a approaches the external boundary of M_2. For the parameter values in the exterior hyperbolic component, the images of -1 under all odd iterates of f belong to A_∞, which is disjoint from the critical pre-puzzle piece of depth 1 (see Picture 5, in which Ω_∞ should be replaced with A_∞). □

Therefore, instead of usual critical annuli, we can consider annuli of the form $P_n(-1) - \overline{P_{n+2}(-1)}$, which we call *double critical annuli*. Nondegenerate double critical annuli exist, because there are puzzle pieces of depth 2 compactly contained in $P_0(-1)$ (see Picture 5). We can apply the tableau technique to the double critical annuli. Namely, the proof of Lemma 1.3 from [14] carries out almost verbatim to double critical annuli. From this lemma and the Grötzsch inequality, it follows that the critical puzzle pieces converge to -1.

From the convergence of critical puzzle pieces and a simple Koebe distortion argument it follows that, for any point z in the Julia set of f but not on the boundary of a puzzle piece, the sequence of puzzle pieces $P_n(z)$ converges to z. The argument goes exactly as for quadratic polynomials. This concludes the proof of theorem 5.1.

5.5. **Cells.** Let $f = f_a$, where a is on the external boundary of M_2. Then, by Proposition 5.3, the critical point -1 is on the boundary of Ω. In particular, the open set $f^{-1}(\Omega_\infty)$ does not contain critical points. By the Riemann–Hurwitz theorem, this set consists of two connected components. One of these components is Ω_0. The other component contains the point -2 (recall that $f(-2) = \infty$). Denote this component by Ω_{-2}. Note that in our case, all radial components are Fatou components, e.g. $A_0 = \Omega_0$, $A_\infty = \Omega_\infty$, and $A_{-2} = \Omega_{-2}$.

Proposition 5.7. *The set $\overline{\mathbb{C}} - \overline{\Omega}$ is connected.*

Proof. Let C_* be the connected component of $\overline{\mathbb{C}} - \overline{\Omega}$ that contains -2. From the existence of the puzzle partition it follows that C_* contains the fixed point α. Indeed, the fixed point portrait contains a regulated ray passing through -2 and landing at α. Note also that there is a regulated ray passing through 0 and landing at α. Therefore, there is a regulated ray passing through -2 and landing at $\alpha' = -2 - \alpha$ (the point α' is characterized by the properties $f(\alpha') = \alpha$ and $\alpha' \neq \alpha$). We see that α' also belongs to C_*.

The full preimage of C_* under f does not contain critical points. Therefore, it consists of two connected components. One of these components contains α, and the other component contains α'. It follows that both components are contained in C_*, i.e. we have $f^{-1}(C_*) \subset C_*$.

Assume that there is a connected component V of $\overline{\mathbb{C}} - \overline{\Omega}$ different from C_*. The forward orbit of V is disjoint from C_*. Therefore, no iterate of V intersects G. It follows that, for any point $x \in V$ and any depth n, we have $V \subset P_n(x)$. This contradicts the convergence of puzzle pieces, see Theorem 5.1. □

The open set $C_* = \overline{\mathbb{C}} - \overline{\Omega}$ is called the *main cell*. Since $-2 \in C_*$, we have $\Omega_{-2} \subseteq C_*$. We define *cells of depth n* as connected components of $f^{-n}(C_*)$. Since no cell contains critical points, there are exactly 2^n cells of depth n. For any cell C of depth n, there is a unique component of $f^{-n}(\Omega_{-2})$ contained in C. This Fatou component is called the *kernel* of the cell. Note that if a cell has depth n, then the depth of its kernel is $n+1$. Conversely, for each radial component A different from Ω_0 and Ω_∞, there is a unique cell containing A as the kernel. The root point of A (i.e. the landing point of the zero ray in A) is also called the *root of the cell*.

The methods of Proposition 5.7 also yield the following important result:

Proposition 5.8. *Any Fatou component of f is eventually mapped to Ω_∞, i.e. it is a radial component.*

Proof. Let V be a Fatou component of f. Suppose that the forward orbit of V is disjoint with Ω. Then, for any point $x \in V$ and any depth n, we have $V \subset P_n(x)$. This contradicts the convergence of puzzle pieces, Theorem 5.1. □

We will use cells to encode the dynamics of f. To this end, the following property is crucial:

Theorem 5.9. *For any infinite nested sequence of cells $C^{(1)} \supset C^{(2)} \supset \dots$, the intersection $\bigcap \overline{C^{(n)}}$ consists of a single point.*

We will prove this theorem in Subsection 6.1. The partition of the Julia set into closures of cells has one major disadvantage: the critical point -1 lies on the boundaries of cells rather than in the interior of a cell. This is the reason why we also need the puzzle partition.

5.6. **Topology of Fatou components.** In this subsection, we study topology of Fatou components, in particular, local connectivity and intersection properties of their boundaries.

Proposition 5.10. *The boundary of Ω_∞ is locally connected.*

Proof. We show that all rays in Ω_∞ land — the proposition will follow. It suffices to consider a ray R of irrational angle θ. Let I be the prime end impression of angle θ and x any point of I. Clearly, the forward orbit of I is disjoint from G. It follows that, for any depth n, we have $I \subset P_n(x)$. By Theorem 5.1, the puzzle pieces $P_n(x)$ converge to x. It follows that $I = \{x\}$, and that R lands at x. □

Proposition 5.11. *Two different rays in Ω_∞ cannot land at the same point.*

Proof. Assume the contrary: there are two rays in Ω_∞ that land at the same point. The union of these rays, the common landing point and ∞ divides the Riemann sphere into two parts. Each part must contain points of the complement to $\overline{\Omega}_\infty$. This contradicts Proposition 5.7. □

Proposition 5.12. *The critical point -1 is the only intersection point of $\overline{\Omega}_0$ and $\overline{\Omega}_{-2}$.*

Proof. By our assumption, the critical point -1 belongs to the boundary of Ω_0. Note that the map $z \mapsto -2 - z$ takes Ω_0 to Ω_{-2}. It follows that -1 is on the boundary of Ω_{-2}, therefore, $-1 \in \overline{\Omega}_0 \cap \overline{\Omega}_{-2}$.

Suppose that $x_0 \neq -1$ is another point in $\overline{\Omega}_0 \cap \overline{\Omega}_{-2}$. Let R_0 and R_{-2} be rays in Ω_0 and Ω_{-2}, respectively, that land at x_0. If R_0 and R_{-2} map to the same ray under f, then x_0 must be a critical point (indeed, f is not injective in any neighborhood of x_0). Suppose that $f(R_0)$ and $f(R_{-2})$ are different rays in Ω_∞. However, they land at the same point $f(x_0)$. Contradiction with Proposition 5.11. □

Proposition 5.13. *The fixed point ω is the only intersection point of $\overline{\Omega}_0$ and $\overline{\Omega}_\infty$.*

Proof. Assume the contrary: x is another point in $\overline{\Omega}_0 \cap \overline{\Omega}_\infty$. The union of $\{0, x, \omega, \infty\}$ and the rays in Ω_0 and Ω_∞ landing at ω and x is a simple closed curve. This curve divides the Riemann sphere into two parts. By Proposition 5.7, only one part can contain points of C_*. Then, in the other part, the boundaries of Ω_0 and Ω_∞ coincide. It is easy to see that, in this case, $\partial\Omega_0 = \partial\Omega_\infty$, a contradiction. □

5.7. **Topology of cells.** There are two cells of depth 1. Denote them by C_0 and C_1. Let a_* be the landing point of the ray $R_\infty(1/2)$. This point belongs to the boundary of both Ω_∞ and Ω_{-2}. The following is a consequence of Propositions 5.12 and 5.13.

Proposition 5.14. *The intersection of $\overline{C}_0$ and $\overline{C}_1$ is $\{\omega, a_*, -1\}$.*

For the following, we need two simple lemmas.

Lemma 5.15. *The kernel of any cell C is the Fatou component of the biggest depth contained in C.*

Proof. Any cell gets eventually mapped to the main cell C_*. For the main cell, the statement is obvious. □

Lemma 5.16. *Let C be a cell. The depth of any Fatou component lying in C is bigger than that of C.*

Proof. Suppose that C has depth n. Then the kernel of C has depth $n+1$. The statement now follows from Lemma 5.15. □

We need to establish convergence of certain nested sequences of puzzle pieces. For any positive integer n, there are two puzzle pieces of depth n containing the fixed point ω on their boundary. One of these puzzle pieces, say, $P_{n,0}(\omega)$, intersects C_0, and the other puzzle piece, $P_{n,1}(\omega)$, intersects C_1. We have

$$P_{n+1,0}(\omega) \subset P_{n,0}(\omega), \quad P_{n+1,1}(\omega) \subset P_{n,1}(\omega).$$

Proposition 5.17. *The nested sequence of closed sets $\overline{P}_{n,0}(\omega)$ converges to ω. Similarly, the nested sequence $\overline{P}_{n,1}(\omega)$ converges to ω.*

Proof. Let x be any point in $\bigcap_{n\geq 1} \overline{P}_{n,0}(\omega)$. If x is different from ω, then it is easy to see that the forward orbit of x is disjoint from G. By Theorem 5.1, it follows that the puzzle pieces $P_n(x)$ converge to x. However, we must have $P_n(x) = P_{n,0}(\omega)$. This is a contradiction, which shows that the sequence $\overline{P}_{n,0}(\omega)$ converges to ω. A proof that $\overline{P}_{n,1}(\omega)$ converges to ω is similar. □

For any positive integer n, there are exactly 2 cells of depth n that contain the critical point ω on the boundary. One of these cells, say, $C_0^{(n)}(\omega)$, is contained in C_0, and the other cell, $C_1^{(n)}(\omega)$ is contained in C_1. It is easy to see that, for $n \geq 1$, the kernel of every cell $C_0^{(n)}(\omega)$ and $C_1^{(n)}(\omega)$ touches both Ω_0 and Ω_∞ (we say that two open sets *touch* if their closures intersect). Moreover, either the point where the kernel of $C_0^{(n)}(\omega)$ touches Ω_0 or the point where the kernel of $C_0^{(n)}(\omega)$ touches Ω_∞ is eventually mapped to ω (the same holds for $C_1^{(n)}(\omega)$). In particular, there is a finite regulated ray $\Gamma(0, r_1)$ or $\Gamma(\infty, r_1)$ passing through Ω_0 or Ω_∞ and the kernel of the cell $C_0^{(n)}(\omega)$.

Proposition 5.18. *The nested sequence of closed sets $\overline{C}_0^{(n)}(\omega)$ converges to ω. Similarly, the nested sequence $\overline{C}_1^{(n)}(\omega)$ converges to ω.*

Proof. We show that every puzzle piece $P_{n,0}(\omega)$ contains some cell $C_0^{(m)}(\omega)$. The proposition will follow then from Proposition 5.17. The puzzle piece $P_{n,0}(\omega)$ is bounded by a finite number of regulated rays and equipotentials. We can choose m such that the kernel A of the cell $C_0^{(m-1)}(\omega)$ touches both Ω_0 and Ω_∞ at interior points of $P_{n,0}(\omega)$. Suppose that, say, $\Gamma(0, r_1)$, is a finite regulated ray passing though Ω_0 and the kernel of the cell $C_0^{(m-1)}(\omega)$ (if it does not exist, then there is a regulated ray $\Gamma(\infty, r_1)$ passing though Ω_∞ and the kernel of the cell $C_0^{(m-1)}(\omega)$). Since the intersection of any pair of regulated rays is an initial segment of both, the regulated ray $\Gamma(0, r_1)$ is disjoint from the boundary of the puzzle piece $P_{n,0}(\omega)$. Therefore, the kernel of the cell $C_0^{(m-1)}(\omega)$ is disjoint from the boundary of the puzzle piece $P_{n,0}(\omega)$, and the cell $C^{(m)}(\omega)$ is contained in $P_{n,0}(\omega)$. □

5.8. **Cells converging to α.** For any point x in the Julia set of f but not on the boundary of a cell, there is a unique cell $C^{(n)}(x)$ of depth n containing x.

Proposition 5.19. *The nested sequence of cells $C^{(n)}(\alpha)$ containing α converges to α, i.e.*

$$\bigcap_{n=1}^{\infty} \overline{C^{(n)}(\alpha)} = \{\alpha\}.$$

Proof. The proof consists of several steps.

Step 1. Let A_n denote the kernel of $C^{(n)}(\alpha)$. If all A_n touch Ω_0 and Ω_∞, then $C^{(n)}(\alpha)$ coincide with $C_0^{(n)}(\omega)$ or with $C_1^{(n)}(\omega)$. However, this contradicts Proposition 5.18.

Step 2. It follows that some A_n does not touch Ω_0 or does not touch Ω_∞. Suppose that n is the minimal index with this property. Then it is easy to see that A_n touches Ω_∞ and Ω_{-2}. It follows that $C^{(n)}(\alpha)$ does not touch Ω_0.

Step 3. Consider the intersection I of $\overline{C}^{(n)}(\alpha)$. This is a compact connected subset of the Julia set for f. By step 2, the set I is disjoint from $\overline{\Omega}_0$. Since I is forward invariant under f, it is also disjoint from $\overline{\Omega}_\infty$. By the same reason, I is disjoint from $\overline{\Omega}_{-2}$.

Step 4. It follows that there is a cell $C^{(n)}(\alpha)$ that does not touch $\Omega_\infty \cup \Omega_0 \cup \Omega_{-2}$. There is a single valued branch of f^{-n+1} that takes the cell $C^{(1)}(\alpha)$ (which is C_0 or C_1) to $C^{(n)}(\alpha)$. Note that $C^{(n)}(\alpha)$ is compactly contained in $C^{(1)}(\alpha)$. The proposition now follows from the Poincaré distance argument. □

Proposition 5.20. *There is a regulated ray of the form*

$$\Gamma(\infty, 1/2, r_2, r_2, \dots, r_2, \dots)$$

converging to α.

Proof. Let a_n be the root point of the cell $C^{(n+1)}(\alpha)$. By Proposition 5.19, not all points a_n are on the boundary of Ω, whereas $a_{-1} = a_*$ is in the boundary of Ω_∞. It follows that there is a nonnegative integer n such that $a_n \in \partial\Omega_{-2}$. Let r_2 be the angle of a_n with respect to Ω_{-2}. Consider the regulated ray $\Gamma_1 = \Gamma(\infty, 1/2, r_2, r_2, \dots)$. This ray is periodic; let q be the minimal period. There is a branch g of f^{-q} that takes $C^{(n+1)}(\alpha)$ to $C^{(n+q+1)}(\alpha) \subset C^{(n+1)}(\alpha)$. Clearly, we have $g(\Gamma_1) \subset \Gamma_1$. The proposition now follows. □

6. Topological models

In this section, we give topological models for rational maps $f = f_a$ satisfying the condition $-1 \in \partial\Omega$. We use the partition of the Julia set into cells to encode the topological dynamics of f.

6.1. **Convergence of cells and a proof of Theorem A.** In this section, we prove Theorem 5.9: all nested sequences of cells converge to singletons.

Proposition 6.1. *Consider any point z in the Julia set of f different from α and such that the forward orbit of z is disjoint from $\{-1, \omega\}$. Then there is a cell $C(z)$ that contains z in its closure and lies in a puzzle piece of depth 0.*

Proof. Since z does not coincide with α, it avoids the closure of a cell $C^{(n)}(\alpha)$ containing α (this follows from Lemma 5.19). Let N denote the maximal depth of a Fatou component intersecting some regulated ray Γ_i but not lying in the cell $C^{(n)}(\alpha)$. It is not hard to see that the cell $C(z) = C^{(N)}(z)$ of depth N lies in some puzzle piece of depth 0, see Lemma 5.16. By definition, z belongs to $\overline{C^{(N)}(z)}$. □

The following statement now follows from the convergence of puzzle pieces.

Proposition 6.2. *Let z be any point in the Julia set of f, whose forward orbit is disjoint from $\{-1, \omega, \alpha\}$. We have*

$$\bigcap_{n=1}^{\infty} \overline{C^{(n)}(z)} = \{z\}.$$

Note that iterated preimages of ω are the only points in the Julia set that lie on the boundaries of puzzle pieces.

Let z be an iterated preimage of -1. Then, for each depth n, there are two cells $C_0^{(n)}(z)$ and $C_1^{(n)}(z)$ having z on the boundary. We can arrange the indexing so that to have

$$C_0^{(n+1)}(z) \subset C_0^{(n)}(z), \quad C_1^{(n+1)}(z) \subset C_1^{(n)}(z).$$

We will also assume that

$$C_0^{(n)}(-1) \subseteq C_0, \quad C_1^{(n)}(-1) \subseteq C_1.$$

Proposition 6.3. *For any iterated preimage z of the critical point -1, we have*

$$\bigcap_{n=1}^{\infty} \overline{C_0^{(n)}(z)} = \bigcap_{n=1}^{\infty} \overline{C_1^{(n)}(z)} = \{z\}.$$

Proof. It suffices to prove this for $z = -1$. Note that $C_0^{(n)}(-1)$ and $C_1^{(n)}(-1)$ are centrally symmetric with respect to -1. If, say, $\alpha \in C_0$, then C_1 is contained in a single puzzle piece of depth 0, namely, in the critical puzzle piece $P_0(-1)$. The critical orbit returns to $\overline{C}_1$, and hence to $P_0(-1)$, infinitely many times. Suppose that $f^{\circ m}(-1) \in \overline{C}_1$. Then, by the pullback argument, $C_0^{(m)}(-1)$ or $C_1^{(m)}(-1)$ is contained in $P_{m-1}(-1)$, which is the pullback of $P_0(-1)$ along the critical orbit. Since m can be made arbitrarily large, the diameters of $C_0^{(n)}(-1)$ and $C_1^{(n)}(-1)$ tend to 0 as $n \to \infty$. □

Proof of Theorem 5.9. Consider a nested sequence of cells $C^{(n)}$. The intersection of all $\overline{C^{(n)}}$ is non-empty. Let z be any point in this intersection. If z is not in the backward orbit of $\{-1, \omega, \alpha\}$, then the convergence follows from Proposition 6.2. If z is an iterated preimage of ω, then the convergence follows from Proposition 5.18. If z is an iterated preimage of -1, then the convergence follows from Proposition 6.3. Finally, if z is an iterated preimage of α, then the convergence follows from Proposition 5.19. □

Note that Theorem A follows from Theorem 5.9, because the intersection of the Julia set with each cell is connected. Indeed, any component of the complement to this intersection is a simply connected Fatou component.

6.2. **Encoding of the Julia set.** In this subsection, we encode all points of the Julia set by binary sequences. Our main tool is Theorem 5.9. Consider a cell C of depth n. The *address of* C is a finite binary sequence $\varepsilon_1 \dots \varepsilon_n$ defined as follows. We set $\varepsilon_k = 0$ or 1 depending on whether $f^{\circ k-1}(C)$ is contained in C_0 or in C_1. We will think of the main cell as having the empty address. For any finite binary sequence $\varepsilon_1 \dots \varepsilon_n$, there is a unique cell $C_{\varepsilon_1 \dots \varepsilon_n}$ with address $\varepsilon_1 \dots \varepsilon_n$. We have $f(C_{\varepsilon_1 \varepsilon_2 \dots \varepsilon_n}) = C_{\varepsilon_2 \dots \varepsilon_n}$.

We can now define a continuous map from all infinite binary sequences to the Julia set of f (the set of infinite binary sequences is considered as a topological space with respect to the direct product topology). Given an infinite binary sequence $\varepsilon_1 \dots \varepsilon_n \dots$, define the point $z_{\varepsilon_1 \dots \varepsilon_n \dots}$ to be the only point in $\bigcap_{n=1}^{\infty} \overline{C}_{\varepsilon_1 \dots \varepsilon_n}$. We have

$$f(z_{\varepsilon_1 \varepsilon_2 \dots \varepsilon_n \dots}) = z_{\varepsilon_2 \dots \varepsilon_n \dots}.$$

The sequence $\varepsilon_1 \dots \varepsilon_n \dots$ is called an *address* of the point $z_{\varepsilon_1 \dots \varepsilon_n \dots}$. Note that the same point can have different addresses.

From now on, we will assume that the cells C_0 and C_1 of depth 1 are indexed so that the landing points of all rays $R_\infty(\theta)$ with $\theta < 1/2$ belong to the closure of C_0. Then the landing points of all rays $R_\infty(\theta)$ with $\theta > 1/2$ belong to the closure of C_1. Clearly, this can be arranged.

Proposition 6.4. *The critical point -1 is encoded by exactly two binary sequences, namely,*

$$-1 = z_{0\varepsilon_1^* \dots \varepsilon_n^* \dots} = z_{1\varepsilon_1^* \dots \varepsilon_n^* \dots}, \quad \varepsilon_{2m}^* = \theta_0[m], \quad \varepsilon_{2m+1}^* = 1 - \nu_m(\theta_0),$$

where $\theta_0[m]$ denotes the m-th digit in the binary expansion of θ_0, and the function ν_m is that introduced in Subsection 2.1.

Proof. The point -1 belongs to the closures of both C_0 and C_1. However, the remaining address of -1 is well-defined: the m-th digit is 0 if $f^{\circ m-1}(-1)$ belongs to $\overline{C}_0$ and 1 if $f^{\circ m-1}(-1)$ belongs to $\overline{C}_1$. We assumed that -1 is not pre-periodic, thus $f^{\circ m-1}(-1)$ cannot belong to the intersection $\overline{C}_0 \cap \overline{C}_1$, and the m-th digit in the address of -1 is well defined. Denote the m-th digit by ε_m^*.

The point $f^{\circ 2m}(-1)$ is on the boundary of Ω_0. This is the landing point of the ray $R_0(2^m\theta_0)$. It belongs to the closure of C_1 or C_0 depending on whether $\{2^m\theta_0\} < \theta_0$ or $\{2^m\theta_0\} > \theta_0$. Therefore, $\varepsilon_{2m+1} = 1 - \nu_m(\theta_0)$. The point $f^{\circ 2m-1}(-1)$ is on the boundary of Ω_∞. This is the landing point of the ray $R_\infty(2^{m-1}\theta_0)$. It belongs to the closure of C_0 or C_1 depending on whether $\{2^{m-1}\theta_0\} < 1/2$ or $\{2^{m-1}\theta_0\} > 1/2$. Therefore, $\varepsilon_{2m} = \theta_0[m]$. □

Define the following equivalence relation $\sim$ on the set of all infinite binary sequences: $x \sim y$ if and only if one of the following formulas holds:

- $x = 010101\dots$, $y = 101010\dots$,
- $x = w0010101\dots$, $y = w1101010\dots$,
- $x = w0\varepsilon_1^*\dots\varepsilon_n^*\dots$, $y = w1\varepsilon_1^*\dots\varepsilon_n^*\dots$,

for some finite binary word w.

Proposition 6.5. *Let x and y be two infinite binary sequences. We have $z_x = z_y$ if and only if $x \sim y$.*

Proof. In one direction, the proposition is obvious: if x and y are as described, then $z_x = z_y$. Suppose now that $z_x = z_y$. Interchanging x and y if necessary, we can write $x = w0x'$ and $y = w1y'$ for some finite binary word w (possibly empty) and infinite binary sequences x' and y'. We have $z_{0x'} = z_{1y'}$. But $z_{0x'}$ belongs to $\overline{C}_0$, whereas $z_{1y'}$ belongs to $\overline{C}_1$. Note that the sets $\overline{C}_0$ and $\overline{C}_1$ intersect at only three points: ω, -1 and a_*. Consider these three cases separately.

Case 1. Suppose first that $z_{0x'} = z_{1y'} = \omega$. In this case, $x' = 101010\dots$ and $y' = 010101\dots$. Indeed, if a cell lies in C_0 and touches the fixed point ω, then the image of this cell lies in C_1, and vice versa.

Case 2. Suppose that $z_{0x'} = z_{1y'} = a_*$. In this case, it is easy to see that $x' = 010101\dots$ and $y' = 101010\dots$. This follows from the fact that $f(a_*) = \omega$.

Case 3. Finally, suppose that $z_{0x'} = z_{1y'} = -1$. Then $x' = y' = x_0$ by Proposition 6.4. □

Corollary 6.6. *The Julia set of f is homeomorphic to the quotient of the space $\{0,1\}^{\mathbb{N}}$ of all infinite binary sequences (equipped with the product topology) by the equivalence relation $\sim$. Moreover, the canonical projection semi-conjugates the Bernoulli shift with the restriction of f to the Julia set.*

6.3. **Proof of Theorem B.** Consider the two-sided lamination $2L(x_0)$, where x_0 is given in terms of θ_0 by the formula from Theorem B. Let us prove that the Julia set of f is homeomorphic to the quotient of the unit circle by the equivalence relation $\sim_{2L(x_0)}$, and that the map f is conjugate to the map $s_{2L(x_0)}/\sim_{2L(x_0)}$.

We can describe the equivalence relation $\sim_{2L(x_0)}$ in terms of binary digits as follows. Identify each point $e^{2\pi i\theta}$ on the unit circle with the binary expansion of θ, in which every second digit is replaced with its opposite. Under this identification, the map $z \mapsto 1/z^2$ identifies with the Bernoulli shift.

The equivalence relation $\sim_{2L(x_0)}$ is given by the following formulas:

- $101010\cdots \sim 010101\ldots,$
- $w001010\cdots \sim w11010\ldots,$
- $w0\varepsilon_1^*\ldots\varepsilon_n^*\cdots \sim w1\varepsilon_1^*\ldots\varepsilon_n^*\ldots.$

Note that the first two formulas represent identifications on the unit circle (due to the fact that the same point on the unit circle can correspond to different binary expansions), and only the last formula represents the equivalence defined by the lamination $2L(x_0)$. The digits ε_m^* are the same as in Proposition 6.4 due to Proposition 2.3.

We see that the equivalence relation on binary sequences corresponding to the relation $\sim_{2L(x_0)}$ is identical with that introduced in Subsection 6.2. Thus both $S^1/\sim_{2L(x_0)}$ and the Julia set of the map f are identified with the quotient of the space of infinite binary sequences by the same equivalence relation. It follows that these two sets are homeomorphic. Moreover, both $s_{2L(x_0)}/\sim_{2L(x_0)}$ and f are represented by the Bernoulli shift on binary sequences. Thus the two maps are topologically conjugate.

It is easy to extend the conjugacy $(S^1/\sim_{2L(x_0)}, s_{2L(x_0)}) \to (J, f)$ over the gaps of the lamination $2L(x_0)$. This finishes the proof of Theorem B.

6.4. **Proof of Theorem B*.** We now sketch a proof of Theorem B*. Consider a map $f \in V_2$ such that $-1 \in \partial\Omega_0$. Let θ_0 be the angle of the ray in Ω_0 landing at the critical point -1. Define the real number y_0 by the formula

$$y_0 = \frac{1}{3}\left(1 + 3\sum_{m=1}^{\infty}\frac{\theta_0[m]}{4^m}\right),$$

where $\theta_0[m]$ is the mth binary digit of θ_0.

Let $L_{\mathcal{B}}$ denote the basilica lamination (i.e. the lamination that models the quadratic polynomial $z \mapsto z^2 - 1$). Recall that $L_{\mathcal{B}}$ is a quadratic invariant lamination containing the leaf $e^{2\pi i(1/3)}e^{2\pi i(2/3)}$ and such that this leaf has the maximal length among all leaves in $L_{\mathcal{B}}$. Consider the mating lamination $L_{\mathcal{B}} \cup L(-2y_0)^{-1}$. It defines the corresponding equivalence relation on the sphere, and the quotient S by this equivalence relation is also a topological sphere (this can be deduced from a theorem of Moore [16] that gives a necessary and sufficient condition for a quotient of the sphere to be homeomorphic to the sphere — this theorem is a standard tool used to define topological matings, see e.g. [15]). Let π denote the canonical projection onto S. The lamination map for $L_{\mathcal{B}} \cup L(-2y_0)^{-1}$ respects this equivalence relation, and, therefore, descends to S. Denote the quotient map by $\pi_*(s)$. We would like to show that $\pi_*(s)$ is topologically conjugate to f.

For the map $\pi_*(s)$, we will arrange a partition into cells with exactly the same combinatorial structure as the partition into cells of f. Let G_∞ be the gap of $L_{\mathcal{B}}$ containing the center of G_0 (which is a critical point for s), and G_0 the gap containing $s(0)$ (recall that $s^{\circ 2}(0) = 0$). The open sets $\pi(G_\infty)$ and $\pi(G_0)$ are topological disks, whose boundaries intersect at exactly one point, which is the image of the leaf $e^{2\pi i(1/3)}e^{2\pi i(2/3)} \in L_{\mathcal{B}}$. Let G_{-2} be the gap $-G_0$.

Let $K = \partial G_0 \cap S^1$. The set K is a Cantor set obtained as follows (we identify K with the corresponding subset of $\mathbb{R}/\mathbb{Z}$): from the segment $[1/3, 2/3]$, we remove two middle quarters, then do the same with the two remaining segments, etc. Thus there is a natural parameterization of K by binary sequences: the point of K corresponding to a binary sequence $\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$ is given by the formula

$$\frac{1}{3}\left(1 + 3\sum_{m=1}^{\infty} \frac{\alpha_m}{4^m}\right).$$

It is easy to see that, under this parameterization, the map $z \mapsto z^4$ (which leaves the set K invariant) acts as the standard Bernoulli shift on binary sequences.

Note that the boundary of $\pi(G_0)$ is exactly $\pi(K)$. The parameterization of K by binary sequences translates into the parameterization of $\partial\pi(G_0)$ by points on the unit circle. The parameterization of $\partial\pi(G_0)$ by points on the unit circle is natural in the sense that the map $\pi_*(s)$ restricted to $\partial\pi(G_0)$ corresponds to the map $z \mapsto z^2$ on the unit circle (this is because on binary sequences, we had the Bernoulli shift). The point on the unit circle parameterizing a given point $z \in \partial\pi(G_0)$ will be called the *angle* of z with respect to $\pi(G_0)$.

Consider the intersection point of $\partial\pi(G_0)$ and $\partial\pi(G_{-2})$. This is a critical point for $\pi_*(s)$ (in the sense that $\pi_*(s)$ is not locally injective near this point). The angle of this point with respect to $\pi(G_0)$ is θ_0 (this can be seen by comparing the formula for y_0 with the formula defining the parameterization of K by binary sequences). It follows that the restriction of $\pi_*(s)$ to $\overline{\pi(G_\infty \cup G_0 \cup G_{-2})}$ is conjugate to the restriction of f to $\overline{\Omega_\infty \cup \Omega_0 \cup \Omega_{-2}}$. Now define the *main cell for* $\pi_*(s)$ as the complement to $\overline{\pi(G_0 \cup G_\infty)}$. The *cells for* $\pi_*(s)$ are defined as the pullbacks of the main cell under $\pi_*(s)$. It is easy to see that the nested sequences of cells converge to points. Thus the conjugacy between $\pi_*(s)$ and f can be extended to the whole Riemann sphere by taking pullbacks and using convergence of cells.

We see that any map f on the external boundary of M_2 is modeled by a certain mating lamination. We need to deduce that f is a mating of the polynomial $z \mapsto z^2 - 1$ with some actual quadratic polynomial. To this end, it suffices to show that the lamination $L(-2y_0)$ models a quadratic polynomial. Note that the lamination $L(-2y_0)$ has non-renormalizable combinatorics. Consider the parameter ray of angle $-2y_0$ in the complement to the Mandelbrot set. For any value c on the boundary of this ray, the combinatorics of the Yoccoz puzzle for p_c is the same as that for $L(-2y_0)$, in particular, p_c is non-renormalizable. It follows that p_c is modeled by $L(-2y_0)$, and that f is a mating of $z \mapsto z^2 - 1$ with p_c.

6.5. **Proof of Theorem B**.** Let $f = f_a$ be such that $-1 \in \partial\Omega_0$. We need to prove that f is an anti-mating of $z \mapsto z^2$ with another quadratic polynomial. We just sketch a proof skipping some details.

Consider the following family of quartic polynomials

$$q_b(z) = bz^2(z+2)^2$$

parameterized by a single complex parameter b. For this family, 0 is a superattracting fixed point, and the point -2 is a critical point that maps to 0. Finally, -1 is a "free" critical point.

Clearly, if b is small, then both -2 and -1 belong to the immediate basin of attraction of 0. Let H be the hyperbolic component in the b-plane containing small values of b. By the same methods as in [5, 20], one can show that the boundary of H is locally connected. Define the parameter ray of angle θ in H as the set of all parameter values b such that the critical value b belongs to the interior ray of angle θ emanating from 0. All parameter rays in H land. Consider the landing point b_0 of the parameter ray of angle θ_0, where θ_0 is as in Theorem B*.

It is not hard to see that the quartic polynomial q_{b_0} is modeled by the quartic invariant lamination L from Subsection 2.3. From the construction of the two-sided lamination $2L(x_0)$ it is clear that this lamination models the anti-mating of the quadratic polynomial $\sqrt{q_{b_0}}$ and $z \mapsto z^2$.

6.6. **Proof of Theorem C.** In this subsection, we conclude the proof of Theorem C, stating that all external parameter rays land. For periodic angles, this was done by Mary Rees in [17]. Periodic external parameter rays land at parabolic points. For strictly pre-periodic parameter rays, the argument is essentially the same as in Proposition 4.13. The corresponding landing points represent rational maps, for which the critical point -1 is strictly pre-periodic. Thus we can concentrate on the case of irrational angle θ_0.

Consider an external parameter ray $\mathcal{R}$ of angle θ_0. Let a and a' be two points on the boundary of $\mathcal{R}$. First note that, by Proposition 5.3 and Theorem B, the maps f_a and $f_{a'}$ are topologically conjugate (since they admit the same topological model). In particular, by Theorem B*, they are matings of $z \mapsto z^2 - 1$ with the same quadratic polynomial. From the Main Theorem of [3] it now follows that $a = a'$.

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