A PRIORI BOUNDS FOR SOME INFINITELY RENORMALIZABLE QUADRATICS: II. DECORATIONS.

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Abstract. A decoration of the Mandelbrot set M is a part of M cut off by two external rays landing at some tip of a satellite copy of M attached to the main cardioid. In this paper we consider infinitely renormalizable quadratic polynomials satisfying the decoration condition, which means that the combinatorics of the renormalization operators involved is selected from a finite family of decorations. For this class of maps we prove *a priori* bounds. They imply local connectivity of the corresponding Julia sets and the Mandelbrot set at the corresponding parameter values.

Contents

1. Introduction

A *decoration* of the Mandelbrot set M (called also a *Misiurewicz limb*) $\mathcal{L}$ is a part of M cut off by two external rays landing at some tip of a satellite copy of M attached to the main cardioid, see Figure 1.1 (see §2.1 for the precise dynamical definition). In this paper we consider infinitely renormalizable quadratic polynomials satisfying the decoration condition, which means that the combinatorics of the renormalization operators involved is selected from a finite family of decorations $\mathcal{L}_k$. (For instance, real infinitely renormalizable maps satisfy a decoration condition if and only of non of the renormalizations is of doubling type.)

An infinitely renormalizable quadratic map f is said to have *a priori bounds* if its renormalizations can be represented by quadratic-like maps $R^n f : U_n \to V_n$ with $\mathrm{mod}(V_n \smallsetminus U_n) \geq \varepsilon > 0$, $n = 1, 2 \ldots$.

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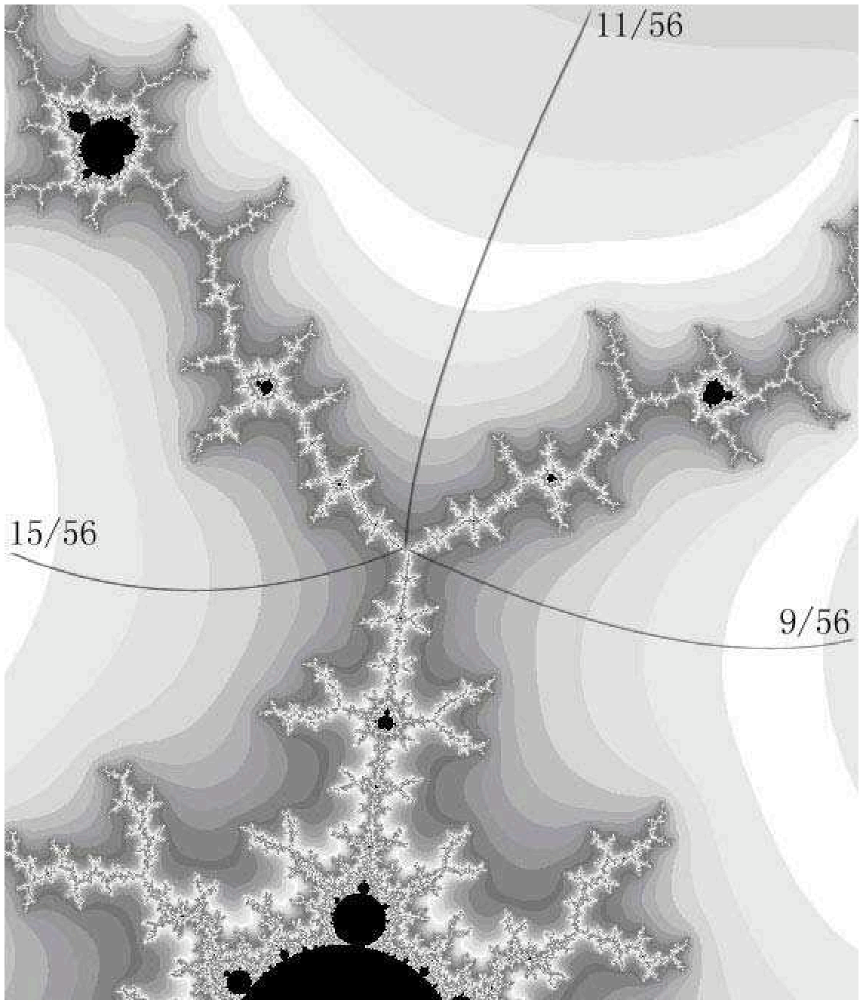

FIGURE 1.1. Two decorations of the Mandelbrot set. The little Mandelbrot sets inside specify renormalization combinatorics of type $(3, 1)$.

Our goal is to prove the following result:

Main Theorem. *Infinitely renormalizable quadratic maps satisfying the decoration condition have a priori bounds.*

By [L], this implies:

Corollary 1.1. *Let $f_c : z \mapsto z^2 + c$ be an infinitely renormalizable quadratic map satisfying the decoration condition. Then the Julia set $J(f_c)$ is locally connected, and the Mandelbrot set M is locally connected at c.*

In this paper we will deal only with the case of sufficiently high periods:

Theorem 1.2. *Given finitely many decorations $\mathcal{L}_k$, there exists a $\underline{p}$ such that any infinitely renormalizable quadratic map satisfying the decoration condition with decorations $\mathcal{L}_k$ and renormalization periods $p \geq \underline{p}$ has a priori bounds.*

The complementary case of "bounded combinatorics" is dealt in [K].

Remark 1.1. Theorem 1.2 sounds similar in spirit to the *a priori* bounds of [L]. However, the "high type" condition of [L] is stronger then the above high period condition, while the "secondary limb condition" of [L] is weaker than the decoration condition. Also, our proof of Theorem 1.2 is compatible with the proof of [K], so that they can be combined into the Main Theorem.

Let us now outline the structure of the paper.

In the next section, §2, we will describe a necessary combinatorial set-up in the framework of the Yoccoz puzzle. Besides a well-known material, it includes the construction of the *modified principal nest* from [KL2] needed for dealing with maps of "high type".

In §3 we summarize necessary information about *pseudo-quadratic-like maps* defined in [K], and introduce a *pseudo-puzzle* by applying the "pseudo-functor" to the puzzle. In this way we make domains of the return maps more canonical, which spares us from the need to control geometry of external rays.

From now on, the usual puzzle will serve only as a combinatorial frame, while all the geometric estimates will be made on the pseudo-puzzle. This is needed for this paper per se, as well as for making connection to the case of bounded combinatorics [K]. Only at the last moment (§5.7) we return back to the standard quadratic-like context.

In §4 we formulate the analytical results of [KL1], the Quasi-Additivity Law and the Covering Lemma, in the pseudo context. They will be our main analytical tools.

In §5 we prove the main results of the paper. To prove *a priori* bounds, we show that if some renormalization has a small modulus, then this modulus will improve on some deeper level. The main place where the decoration condition plays the role is on the top of the puzzle, when we compare the modulus of the first annulus of the pseudo-puzzle to the modulus of the original pseudo-quadratic-like map.

Remark 1.2. Strictly speaking, bounded combinatorics treated in [K] and high combinatorics treated by Theorem 1.2 do not cover the oscillating combinatorial types. However, these theorems follow from results on moduli improving that together cover everything.

Remark 1.3. Our proof of *a priori bounds* (Main Theorem) applies without changes in the case of unicritical maps of higher degree. However, the proof of MLC at the corresponding parameters (Corollary 1.1) given in [L] exploits some special geometric features of quadratic maps. In [C] part of [L] is combined with a new method developed in [AKLS] to prove Corollary 1.1 in the higher degree case as well.

1.1. **Terminology and Notation.** $\mathbb{N} = \{1, 2, \dots\}$ is the set of natural numbers; $\mathbb{Z}_{\geq 0} = \mathbb{N} \cup \{0\}$; $\mathbb{D} = \{z : |z| < 1\}$ is the unit disk, and $\mathbb{T}$ is the unit circle; $\mathbb{A}(r, R) = \{z : r < |z| < R\}$ is the annulus of modulus $\frac{1}{2\pi} \log(R/r)$; $\Pi(h) = \{z | 0 < \Im z < h\}$ is the horizontal strip.

A *topological disk* means a simply connected domain in some Riemann surface S. A *continuum* K is a connected closed subset in S. It is called *full* if all components of $S \smallsetminus K$ are unbounded. We say a subset K of a plane is an FJ-set (for "filled Julia set") if K is compact, connected, and full.

We let $\mathrm{orb}(z) \equiv \mathrm{orb}_g(z) = (g^n z)_{n=0}^{\infty}$ be the *orbit* of z under a map g.

Given a map $g : U \to V$ and an open topological disk $D \subset V$, components of $g^{-1}(D)$ are called *pullbacks* of D under g. If the disk D is closed, we define pullbacks of D as the closures of the pullbacks of $\mathrm{int}\, D$.[1] In either case, given a connected set $X \subset g^{-1}(\mathrm{int}\, D)$, we let $g^{-1}(D)|X$ be the pullback of D containing X.

We let $x \oplus y = (x^{-1} + y^{-1})^{-1}$ be the *harmonic sum* of x and y (it is conjugate to the ordinary sum by the inversion map $x \mapsto x^{-1}$). Similarly, $x \ominus y = (x^{-1} - y^{-1})^{-1}$ stands for the harmonic difference.

1.2. **Acknowledgement.** We thank Tao Li for making Figure 1.1. This work has been partially supported by the NSF, NSERC, the Guggenheim and Simons Foundations. Part of it was done during the authors' visit to the IMS at Stony Brook and the Fields Institute in Toronto. We are thankful to all these Institutions and Foundations.

2. Yoccoz puzzle, decorations, and the Modified Principal Nest

Let $(f_\lambda : U'_\lambda \to U_\lambda)$ be a quadratic-like family over a disk $\Lambda \subset \mathbb{C}$. Assume that this family is good enough (proper and unfolded), so that the associated Mandelbrot set $M = M(f_\lambda)$ is canonically homeomorphic to the standard Mandelbrot set (see [DH]). In fact, most of the time we will be dealing with a single map $f = f_\lambda$ from our family, so that we will usually suppress the label λ in the notation. (We need a one parameter family only to introduce different combinatorial types of the maps under consideration.)

We assume that the domains U' and U are smooth disks, f is even, and we normalize f so that 0 is its critical point.

We let $U^m = f^{-m}(U)$. The boundary of U^m is called the *equipotential of level m*.

[1] Note that the pullbacks of a closed disk D can touch one another, so they are not necessarily connected components of $g^{-1}(D)$.

2.1. **Top of the Yoccoz puzzle and decorations.** By means of straightening, we can define external rays for f. They form a foliation of $V \smallsetminus K(f)$ orthogonal to the equipotential ∂U. The map f has one *non-dividing* fixed point β (landing point of the external ray with angle 0), and one *dividing* fixed point α. There are $\mathbf{q} > 1$ external rays $\mathcal{R}_i$ landing at α which are cyclically permuted by the dynamics with rotation number $\mathbf{p}/\mathbf{q}$, see [M2] ($\mathbf{p}/\mathbf{q}$ is also called the *combinatorial rotation number* of α). These rays divide U into $\mathbf{q}$ (closed) topological disks Y_i^0 called the *Yoccoz puzzle pieces* of depth 0. Let $Y^0 \equiv Y_0^0$ stand for the critical puzzle piece, i.e., the one containing 0.

Let us consider $2\mathbf{q}$ rays of $f^{-1}(\cup \mathcal{R}_i)$. They divide U' into $2\mathbf{q}-1$ (closed) disks called Yoccoz puzzle pieces of depth 1. Let Y^1 stand for the critical puzzle piece of depth 1. There are also $\mathbf{q}-1$ puzzle pieces Y_i^1 of depth 1 contained in the corresponding off-critical pieces of depth 0. All other puzzle pieces of depth 1 will be denoted Z_i^1. They are attached to the symmetric point $\alpha' = -\alpha$.

The puzzle pieces will be labeled in such a way that $f(Y_i^1) = Y_{i+1}^0$, $i = 0, \dots, \mathbf{q}-1$, and $Z_i^1 = -Y_i^1$. We let

$$L = \bigcup_{i=1}^{\mathbf{q}-1} Y_i^1; \quad R = -L = \bigcup_{i=1}^{\mathbf{q}-1} Z_i^1.$$

Puzzle pieces Y_j^m of depth m are pullbacks of $f^{-m}(Y_i^0)$. They tile the neighborhood of $K(f)$ bounded by the equipotential ∂U^m. Each of them is bounded by finitely many arcs of this equipotential and finitely many external rays of $f^{-m}(\mathcal{R}_i)$. If $f^m(0) \neq \alpha$, then there is one puzzle piece of depth m that contains the critical point 0. It is called *critical* and is labeled as $Y^m \equiv Y_0^m$. These pieces are nested around the origin:

$$Y^0 \supset Y^1 \supset Y^2 \cdots \ni 0.$$

Let us consider a puzzle piece $Y = Y_i^m$. Different arcs of ∂Y meet at the *corners* of Y. The corners where two external rays meet will be called *vertices* of Y; they are f^m-preimages of α. Let $K_Y = K(f) \cap Y$. It is a closed connected set that meets the boundary ∂Y at its vertices. Moreover, the external rays meeting at a vertex $v \in \partial Y$ chop off from $K(f)$ a continuum S_Y^v, the component of $K(f) \smallsetminus \operatorname{int} Y$ containing v.

The critical value $f^q(0)$ belongs to the puzzle piece Y^0. If in fact it belongs to Y^1 then the map $Y^{\mathbf{q}+1} \to Y^1$ is a double branched covering. It is not a quadratic-like map, though, since the boundaries of Y^1 and $Y^{\mathbf{q}+1}$ overlap over four external rays landing at α and α'. However, by slight "thickening" of the domain of this map (see [M1]), it can be turned into a quadratic-like map g such that

$$K(g) = \{z : f^{\mathbf{q}m} z \in Y^1, m = 0, 1, 2, \dots\}.$$

The map f is called *satellite renormalizable* (or, *immediately renormalizable*) if the Julia set $K(g)$ is connected, i.e., if the critical point never escapes Y^1:

$$f^{\mathbf{q}m}(0) \in Y^1, \quad m = 0, 1, 2 \dots .$$

The set of immediately renormalizable parameter values (with a given combinatorial rotation number $\mathbf{p}/\mathbf{q}$) assemble a *satellite copy* $M_{\mathbf{p}/\mathbf{q}}$ *of* M attached to the main cardioid at the parabolic point with rotation number $\mathbf{p}/\mathbf{q}$. The parameters $t \in M_{\mathbf{p}/\mathbf{q}}$ for which the critical point eventually lands at α (i.e., $f_t^{\mathbf{q}n} = \alpha'$ for some $n \in \mathbb{N}$) are called the *tips* of $M_{\mathbf{p}/\mathbf{q}}$.

If f is not satellite renormalizable, then there exists an $n \in \mathbb{N}$ such that $f^{\mathbf{qn}}(0)$ belongs to some puzzle piece int $Z^1_{\boldsymbol{\kappa}}$. Let $\mathbf{n}$ be the smallest such n. In this case, we let

$$V^0 = f^{-\mathbf{nq}}(Z^1_{\boldsymbol{\kappa}})|0 = Y^{\mathbf{nq}+1}.$$

Each puzzle piece Z^1_j has 2^m univalent pullbacks under the 2^m-covering $f^{\mathbf{q}m} : Y^{\mathbf{q}m} \to Y^0$, $m = 1, \dots, \mathbf{n}-1$. We label these pullbacks (for all j) as $Z_i^{1+\mathbf{q}m}$. Then

$$f^{\mathbf{q}m}(0) \in Z^{\mathbf{q}(\mathbf{n}-m)+1}_{\boldsymbol{\kappa}_m}, \quad m = 1, \dots, \mathbf{n}, \tag{2.1}$$

for some sequence $\bar\kappa = (\kappa_1, \dots, \kappa_n = \boldsymbol{\kappa})$ called the *escape route* of the critical point. The escape route specifies the tip $t = t_{\kappa_1 \dots \kappa_{\mathbf{n}-1}}$ of M such that f_t satisfies (2.1) for $m < \mathbf{n}$, while $f^{\mathbf{qn}} = \alpha'$.

There are $\mathbf{q}$ parameter rays landing at each tip t of $M_{\mathbf{p}/\mathbf{q}}$. They chop off $\mathbf{q}-1$ *decorations* $\mathcal{L}_{\bar\kappa}$ (the components of $M \setminus \{t\}$ that do not intersect the main cardioid) from M. The limb $\mathcal{L}_{\bar\kappa}$ attached to t is specified by the puzzle piece $Z^1_{\boldsymbol{\kappa}}$ containing $f^{\mathbf{qn}}(0)$. Note that there are only finitely many decorations with bounded $\mathbf{q}$ and $\mathbf{n}$.

Let $P = Y^{(\mathbf{n}-1)\mathbf{q}+1}$. The piece P has $2^{\mathbf{n}}$ vertices each of which is a preimage of α of some depth $\mathbf{q}\,m$ with $m \le \mathbf{n}$ (and it takes into account *all* preimages of α in Y^1 up to depth $\mathbf{qn}$).

Remark 2.1. Any $f^{\mathbf{q}m}$-preimage $t \in Y^1$ of α, $m \le \mathbf{n}$, can be naturally labeled by a dyadic number $i/2^m \in \mathbb{Q}/\mathbb{Z}$ (with odd i). Here α is labeled by 0, α' is labeled by $1/2$, and in general, the dyadic expansion of i is $0.\,\varepsilon_1 \dots \varepsilon_m\,1$, where ε_l is equal $+1$ or -1 depending on whether the rays landing at $f^{\mathbf{q}(l-1))}(t)$ are "below" the chord connecting α and α', or above it.Then the pullbacks of Z^j attached to t can be labeled as $Z_j(i/2^m)$, $j = 1, \dots, \mathbf{q}-1$. A decoration assumes labeling $Z_j(i/2^{\mathbf{n}})$ if $f^{\mathbf{q}}(0) \in Z_j(i/2^{\mathbf{n}})$.

Note that $f^{\mathbf{q}}(P) \supset P$ and the critical value $f^{\mathbf{q}}(0)$ does not belong to P. Hence P has two univalent $f^{\mathbf{q}}$-pullbacks, Q_L and Q_R (of depth $\mathbf{qn}+1$), inside P. The puzzle piece Q_L is attached to the fixed point α while Q_R is attached to α'. Each of them shares two external rays with V^0.

Lemma 2.1. *For any vertex v of P, there exists a puzzle piece $Q^v \subset P$ of depth $(2\mathbf{n}-1)\mathbf{q}+1$ attached to the boundary rays of P landing at v, which is a univalent $f^{\mathbf{nq}}$-pullback of P. Moreover, these puzzle pieces are pairwise disjoint.*

Proof. Let $g = f^{\mathbf{q}}|\,Q_L \cup Q_R$. The domain of $g^{\mathbf{n}}$ consists of $2^{\mathbf{n}}$ components each of which is a univalent pullback of P. Each of these components contains a single $g^{\mathbf{n}}$-preimage v of α, and is attached to the pair of the boundary external rays of P landing at v. This is the desired puzzle piece Q^v. ∎

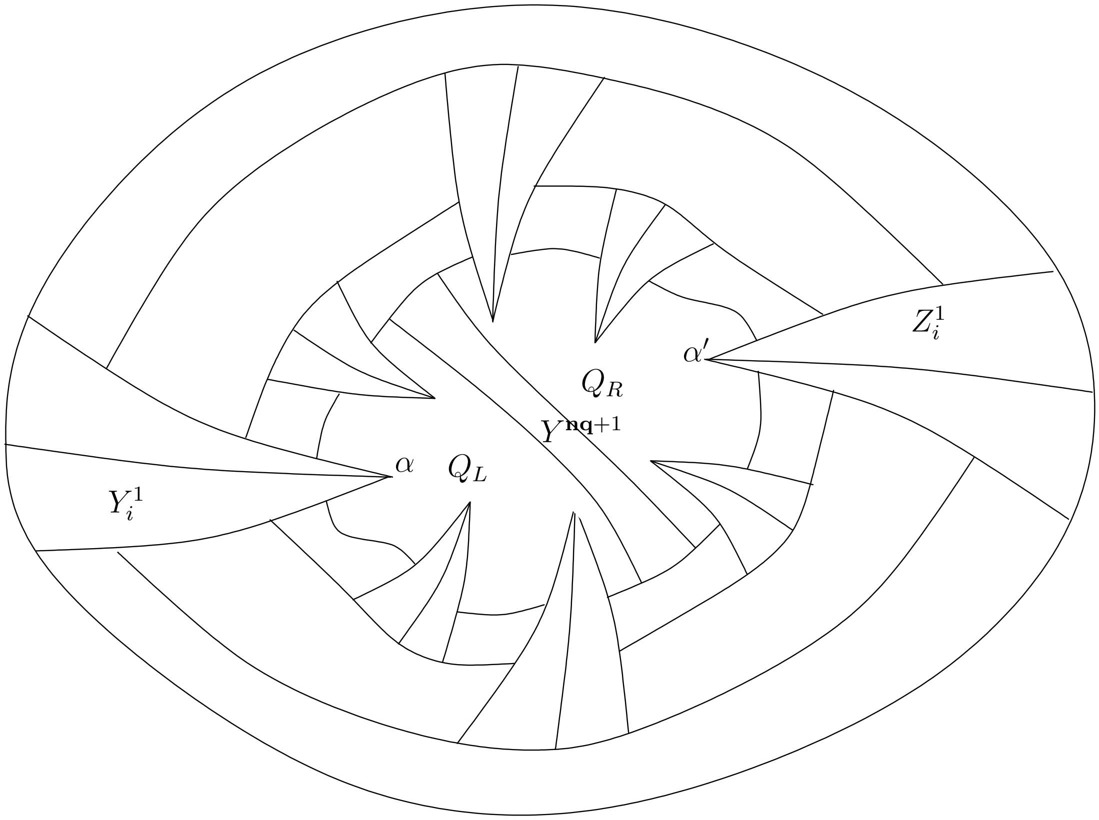

FIGURE 2.1. The top of the Yoccoz puzzle

Given two vertices, v and w, of P, we let $T_P^{vw} = K_P \setminus (Q^v \cup Q^w)$. Notice that T^{vw} *separates* v from w in the sense that v and w belong to different components of $K_P \setminus T_{vw}$.

2.2. **Modified principle nest.** Given a critical puzzle piece V, let us consider the first return $f^l 0$, $l \in \mathbb{N}$, of the critical point to V (whenever it exists). The corresponding pullback $W = f^{-l}(V)|0$ of V is called the *central domain of the first return map to* V, or briefly, the *first child* of V. Under these circumstances, $W \subset V$ and the first return map $f^l : W \to V$ is a double branched covering.

Under the above circumstances, we also consider the the first moment $k \in \mathbb{N}$ such that $f^{kl}(0) \notin W$ and then the first return $f^t(0) \in W$, $t > kl$, back to W (whenever these moments are well defined). We call it the *fine return* to W, and the corresponding pullback $A = f^{-t}(W)|0$ the *fine child* of W. The map $f^t : A \to W$ is a double branched covering. Note that if $f^l(0) \notin W$, the fine return coincides with the first return.

In [KL2] we have constructed a *(Modified) Principle Nest* of critical puzzle pieces

$$E^0 \Supset E^1 \Supset E^1 \Supset \cdots \Supset E^{\chi-1} \Supset E^{\chi}$$

and corresponding quadratic-like maps $g_n : E^n \to E^{n-1}$. Here for odd n, E^n is the first child of E^{n-1} and $g_n : E^n \to E^{n-1}$ is the corresponding first return map. For odd n, E^n is the fine child of E^{n-1} and $g_n : E^n \to E^{n-1}$ is the corresponding fine return map. We let $g \equiv g_0$.

If the map f is renormalizable then the Principle Nest terminates at some odd level χ. In this case, the last quadratic-like map $g_\chi : E^\chi \to E^{\chi-1}$ has connected Julia set and represents the *primitive renormalization* Rf of f. The renormalization level χ is also called the *height* of the nest.

Primitively renormalizable parameter values assemble a maximal *primitive copy* M' of the Mandelbrot set M. This copy specifies the *combinatorics* of the renormalization in question. In particular, it determines the parameters $\mathbf{q}$, $\mathbf{n}$, the height χ, and the renormalization period p.

In what follows we will assume that f is primitively renormalizable. We let $\mathcal{K} = K(\mathcal{R}f)$ be the *little (filled) Julia set* of f, and we let p be the renormalization period, i.e., $g_\chi = f^p$ so that $f^p(\mathcal{K}) = \mathcal{K}$. We let $\mathcal{K}_i = f^i(\mathcal{K})$, where i is taken mod p, which are also called "little Julia sets".

It is important to note that the maps g_n admit analytic *extensions* $\tilde{E}^n \to \hat{E}^{n-1}$ such that $E^n \subset \tilde{E}^n \subset \hat{E}^n \subset E^{n-1}$ and for odd n, $\hat{E}^n = E^{n-1}$ [KL2], §2.4. For $n = 0$, we let $\hat{E}^0 = \tilde{E}^0 = Y^{\mathbf{qn}}$. Then $f^{\mathbf{qn}} : \tilde{E}^0 \to Y^0$ is a branched covering of degree $2^{\mathbf{n}}$.

The following useful observation will be used many times:

Lemma 2.2 (Telescope). *Let X_k be a sequence of topological disks, $k = 0, 1, \dots, m$, and let $\phi_k : X_k \to \phi(X^k)$ be branched coverings of degree d_k such that $\phi(X_k) \supset X_{k+1}$. Let $\Phi = \phi_{n-1} \circ \cdots \circ \phi_0$ (wherever it is defined), and let $P \subset X_0$ be a component of its domain of definition. Then $\Phi : P \to V_n$ is a branched covering of degree at most $d_0 \cdots d_{n-1}$.*

If the renormalization Rf is also renormalizable then f is called twice renormalizable, and $R^2 f$ stands for its second renormalization. Proceeding this way, we can define *infinitely renormalizable* maps f, and let $R^n f$ be their n-fold renormalizations. The *combinatorics* of an infinitely renormalizable map is a sequence of little Mandelbrot copies $M^{(n)}$ that determine the combinatorics of the renormalizations $R^n f$. It determines the sequence of the parameters $\mathbf{q}_n$, $\mathbf{n}_n$, the heights χ_n, and the periods p_n of the corresponding renormalizations.

We say that an infinitely renormalizable f satisfies the *decoration condition* if all the little copies $M^{(n)}$ belong to finitely many decorations $\mathcal{L}_k$. Equivalently, *the parameters $\mathbf{q}_n$ and $\mathbf{n}_n$ are bounded.*

2.3. **Geometric puzzle pieces.** In what follows we will deal with more general puzzle pieces.

Given a puzzle piece Y_i^m, let $Y_i^{m,l}$ stand for a Jordan disk bounded by the same external rays as Y_j^m and arcs of equipotentials of level l (so $Y_i^{m,m} = Y_i^m$). Such a disk will be called a puzzle piece of *bidepth* (m, l).

A *geometric puzzle piece* of bidepth (m,l) is a closed Jordan domain which is the union of several puzzle pieces of the same bidepth. As for ordinary pieces, a pullback of a geometric puzzle piece of bidepth (m,l) under some iterate f^k is a geometric puzzle piece of bidepth $(m+k,l+k)$. Note also that if P and P' are geometric puzzle pieces with[2] $\operatorname{bidepth} P \geq \operatorname{bidepth} P'$ and $K_P \subset K_{P'}$ then $P \subset P'$.

The family of geometric puzzle pieces of bidepth (m,l) will be called $\mathcal{Y}^m(l)$. Given a geometric puzzle piece $Y \in \mathcal{Y}^m(l)$, we let $Y(k)$ be the puzzle piece bounded by the same external rays as Y truncated by the equipotential of level k. (In particular, $Y(l) = Y$.)

Any puzzle piece $Y \in \mathcal{Y}^m(l)$ admits the following combinatorial representation. Let θ_i be the cyclically ordered angles of the external rays $\mathcal{R}_i$ that bound Y. Let us consider the straight rays R_i in $\mathbb{C} \setminus \mathbb{D}$ of angles θ_i truncated by the circle $\mathbb{T}_r$ of radius $r = 1/2^l$. If two consecutive rays, $\mathcal{R}_i$ and $\mathcal{R}_{i+1}$, land at the same vertex of Y, let is connect R_i to R_{i+1} with a hyperbolic geodesic in $\mathbb{D}$. Otherwise $\mathcal{R}_i$ and $\mathcal{R}_{i+1}$ are connected with an equipotential arc. Then let us connect R_i to R_{i+1} with the appropriate arc of $\mathbb{T}_r$. We obtain a Jordan curve that bounds the *combinatorial model* M_Y of Y.

The arcs ω_i of $\mathbb{T} \cap M_Y$ correspond to the "external arcs" of the Julia piece K_Y. They have length $2\pi\lambda$, where λ is called the *combinatorial length* of the corresponding external arc of K_Y. In case Y is a dynamical puzzle piece, all the external arcs of Y have the same combinatorial length

$$\frac{2^k}{(2^{\mathbf{q}}-1)2^m}, \quad k \in \{0, 1, \dots, q-1\}, \tag{2.2}$$

where the choice of k depends on the puzzle piece $f^m(Y)$ of depth 0 (For instance, $k=0$ when $f^m(Y)$ contains the critical value $f(0)$, while $k = q-1$ when $f^m(Y)$ contains the critical point 0.)

It follows that for a geometric puzzle piece Y of depth m, the combinatorial length of its external arcs is at least $2^{-(\mathbf{q}+m)}$.

Let us now consider a geometric puzzle piece $Z^0 = -Y^0$ of bidepth (1,0).

Lemma 2.3. *Let $z \in K(f)$, $f^{\mathbf{qn}}z \in Z^0$ and let $P = f^{-\mathbf{qn}}(Z^0)|z$. Then $P \Subset \operatorname{int} Y^0$ or $P \Subset \operatorname{int} Z^0$.*

Proof. P is a geometric puzzle piece of bidepth $(\mathbf{nq}+1, \mathbf{nq})$. But $E^0 = Y^{\mathbf{nq}+1}$ is a puzzle piece of depth $\mathbf{nq}+1$ such that $f^{\mathbf{qn}}(E^0) = Z_{\boldsymbol{\kappa}}$, where $\operatorname{int} Z_{\boldsymbol{\kappa}} \cap Z^0 = \emptyset$. It follows that $P \cap \operatorname{int} E^0 = \emptyset$. But $K(f) \setminus \operatorname{int} E^0$ consists of two 0-symmetric connected components $X_L \supset L \cap K(f)$ and $X_R \supset R \cap K(f)$. We conclude that K_P is contained in one of these components, and hence it is contained in one of the sets K_{Z^0} or K_{Y^0}. As

$$\operatorname{bidepth} P \geq (1,0) = \operatorname{bidepth} Z^0 \geq (0,0) = \operatorname{bidepth} Y^0,$$

P is contained in one of the puzzle pieces Z^0 or Y^0. □

[2]the inequality between bidepths is understood componentwise

2.4. **Many happy returns.** Here we will summarize the combinatorial construction of [KL2], §1.9, that will lead to the moduli improvement in the case of high type.

Fix an arbitrary m, let N be the smallest even integer which is bigger than $\log_2 m + 5$, and take any odd level $n \geq N$. Then there exists $m/2$ returns $\Lambda_k = g^{l_k}(E^n)$ of the domain E^n to E^{n-N} with the following properties. For any domain Λ_k, the map $\Psi_k = g^{l_k} : E^n \to \Lambda_k$ admits a holomorphic extension to a branched covering

$$\Psi_k : (\Upsilon_k, \Delta_k, E^n) \to (E^{n-N-1}, \Lambda'_k, \Lambda_k) \tag{2.3}$$

such that:

(P1) $\deg(\Psi_k : \Upsilon_k \to E^{n-N-1}) \leq 2^{N+m}$;
(P2) $\deg(\Psi_k : \Delta_k \to \Lambda'_k) \leq d^5$;
(P3) $\Upsilon_k \subset E^{n-1}$;
(P4) There is a level $i \in [n-5, n-1]$ such that each pair of disks (Λ'_k, Λ_k) is mapped univalently onto $(\hat{E}^i, E^i)$ under some iterate f^t, $t = t(k)$;
(P5) The buffers $\Lambda'_k \Subset E^{n-N}$ are pairwise disjoint.

3. Pseudo-quadratic-like maps and pseudo-puzzle

3.1. **Pseudo-quadratic-like maps.** For a more general and detailed discussion of ψ-ql maps, see [K].

Suppose that $\mathbf{U}'$, $\mathbf{U}$ are disks, and $i : \mathbf{U}' \to \mathbf{U}$ is a holomorphic immersion, and $f : \mathbf{U}' \to \mathbf{U}$ is a degree d holomorphic branched cover. Suppose further that there exist full continua $K \Subset \mathbf{U}$ and $K' \Subset \mathbf{U}'$ such that $K' = i^{-1}(K) = f^{-1}(K)$. Then we say that $F = (i, f) : \mathbf{U}' \to (\mathbf{U}, \mathbf{U})$ is a *ψ-quadratic-like* (*ψ-ql*) map with filled Julia set K.

Lemma 3.1 ([K])**.** *Let $F = (i, f)\colon \mathbf{U}' \to \mathbf{U}$ be a ψ-ql map of degree d with filled Julia set K. Then i is an embedding in a neighborhood of $K' \equiv f^{-1}(K)$, and the map $g \equiv f \circ i^{-1}\colon U' \to U$ near K is quadratic-like.*

Moreover, the domains U and U' can be selected in such a way that $\operatorname{mod}(U \smallsetminus i(U')) \geq \mu(\operatorname{mod}(\mathbf{U} \smallsetminus K)) > 0$.

There is a natural ψ-ql map $\mathbf{U}^n \to \mathbf{U}^{n-1}$, the "restriction" of (i, f) to $\mathbf{U}^n$. Somewhat loosely, we will use the same notation $F = (i, f)$ for this restriction.

Let us normalize the ψ-quadratic-like maps under consideration so that $\operatorname{diam} K' = \operatorname{diam} K = 1$, both K and K' contain 0 and 1, 0 is the critical point of f, and $i(0) = 0$. Let us endow the space of ψ-quadratic-like maps (considered up to independent rescalings in the domain and the range) with the Carathéodory topology. In this topology, a sequence of normalized maps $(i_n, f_n) : \mathbf{U}'_n \to \mathbf{U}_n$ converges to $(i, f) : \mathbf{U}' \to \mathbf{U}$ if the pointed domains $(\mathbf{U}'_n, 0)$ and $(\mathbf{U}_n, 0)$ converge to $\mathbf{U}'$ and $\mathbf{U}$ respectively, and the maps i_n, f_n converge respectively to i, f, uniformly on compact subsets of $\mathbf{U}'$.

Lemma 3.2 (compare [McM]). *Let $\mu > 0$. Then the space of ψ-PL maps $F = (i, f) : \mathbf{U}' \to \mathbf{U}$ such that the Julia set K is connected and* $\operatorname{mod}(\mathbf{U} \setminus K) \geq \mu$ *is compact.*

Proof. Let $X_n = i^{-1}\{0,1\}$, $Y_n = f^{-1}\{0,1\}$. Note that both sets consist of at most 2 points and are contained in $\mathbb{D}$.

Then we can select a subsequence of domains $\mathbf{U}'_n$, $\mathbf{U}_n$ Carathéodory converging to some domains $\mathbf{U}'$, $\mathbf{U}$, while the sets X_n and Y_n converge in the Hausdorff metric to some sets $X \subset \mathbf{U}'$ and $Y \subset \mathbf{U}'$ that consist of at most two points and are contained in $\mathbb{D}$ (we will keep the same notation for the subsequence). Since the maps $i_n | \mathbf{U}'_n \setminus X_n$ and $f_n | \mathbf{U}'_n \setminus Y_n$ do not assume values 0 and 1, they form normal families on $\mathbf{U}' \setminus X$. Since these families are bounded on the sets K_n, they are uniformly bounded on compact sets of $\mathbf{U}' \setminus (X \cup Y)$. By the Maximum Principle, they are normal on the whole domain $\mathbf{U}'$.

Let i and f be some limit functions of the sequences i_n and f_n. These functions are non-constant since they assume values 0 and 1. Then i is an immersion as a non-constant limit of immersions. Also, $f : \mathbf{U}' \to \mathbf{U}$ is a branched covering of degree at most 2. Moreover, $K' \Subset \mathbf{U}$ since $\operatorname{mod}(\mathbf{U}' \setminus K') \geq \mu/2$. Hence $0 \in \mathbf{U}'$, and it is a critical point of f. It follows that $\deg f = 2$, and we are done. □

3.2. **Pseudo-puzzle.**

3.2.1. *Definitions.* Let $(i, f) : \mathbf{U}' \to \mathbf{U}$ be a ψ-ql map. By Lemma 3.1, it admits a quadratic-like restriction $U' \to U$ to a neighborhood of its (filled) Julia set $K = K_{\mathbf{U}}$. Here U' is embedded to U, so we can identify U' with $i(U')$ and $f : U' \to U$ with $f \circ i^{-1}$.

Assume that K is connected and both fixed points of f are repelling. Then we can cut U by external rays landing at the α-fixed point and consider the corresponding Yoccoz puzzle.

Given a (geometric) puzzle piece Y of depth m, recall that K_Y stands for $Y \cap K(f)$ and $S_Y = \operatorname{cl}(K(f) \setminus K_Y)$. Let $\mathcal{Y}$ stand for the space of paths $\delta : [0,1] \to \mathbf{U}^m \setminus S_Y$ such that:

- $\delta(0) \in Y$,
- if $\delta(t) \in K_Y$, then the restriction $\delta | [0,t]$ is homotopic rel endpoints to a path contained in Y.[3]

Let $\mathbf{Y}$ be the space of paths $\delta \in \mathcal{Y}$ modulo homotopy through $\mathcal{Y}$ with $\delta(1)$ fixed. Define the projection $\pi_Y : \mathbf{Y} \to \mathbf{U}^m$ by $[\delta] \mapsto \delta(1)$. One can see that $\mathbf{Y}$ is a Riemann surface, and π_Y is an immersion such that Y lifts to a disk $\hat{Y} \subset \mathbf{Y}$ which is homeomorphically projected onto Y. Thus, we can identify $\hat{Y}$ with Y; in particular, K_Y is embedded into $\mathbf{Y}$.

[3]This condition can be replaced with a more restrictive one: After the first exit from Y, the path never intersect the Julia set $K(f)$ (though it is allowed to return back to Y).

The Riemann surface $\mathbf{Y}$ will be called the *pseudo-piece* ("ψ-piece") associated with Y.

The ψ-pieces can also be defined in a different way. Let us consider the topological annulus $A = \mathbf{U}^m \setminus K(f)$ and its universal covering $\hat{A}$. Let Y_i be the components of $Y \setminus K_Y$. There are finitely many of them, and each Y_i is simply connected. Hence they can be embedded into $\hat{A}$. Select such an embedding $e_i : Y_i \to \hat{A}_i$ where $\hat{A}_i$ stands for a copy of $\hat{A}$. Then the ψ-piece is obtained by gluing the A_i to Y by means of e_i, i.e., $\mathbf{Y} = Y \sqcup_{e_i} \hat{A}_i$.

Lemma 3.3. *The above two definitions of ψ-pieces are equivalent.*

Proof. Let $\mathbf{Y}$ be a ψ-piece according to the first definition. The puzzle piece Y is embedded into $\mathbf{Y}$ by associating to a point $y \in Y$ the constant path $\delta(t) \equiv y$.

Let us realize the universal covering $\hat{A}_i \to A$ as the space of paths in A that begin in Y_i rel homotopy through such paths fixing the terminal endpoint. (This realization is legitimate since Y_i is simply connected.) This provides us with an embedding $\phi_i : \hat{A}_i \to \mathbf{Y}$

The embeddings ϕ_i have disjoint images. Indeed, all points of $\partial Y \cap K(f)$ are dividing and thus belong to S_Y. Hence, if we take two paths $\delta_1 : [0,1] \to \mathbf{U}^m \setminus K(F)$ and $\delta_2 : [0,1] \to \mathbf{U}^m \setminus K(f)$ as above representing points in $\hat{A}_i$ and $\hat{A}_j$ $(i \neq j)$ with a common endpoint, then they "surround" some piece of S_Y, and hence represent different points in $\mathbf{Y}$.

Moreover, the image $\phi_i(\hat{A}_i)$ overlaps with Y by Y_i. Hence we obtain an embedding of $Y \sqcup_{e_i} \hat{A}_i$ into $\mathbf{Y}$.

Let us show that this embedding is surjective. Take a path $\delta \in \tilde{\mathcal{Y}}$ representing some point of $\mathbf{Y}$, and let $\tau \in [0,1]$ be the last parameter for which $\delta(\tau) \in K_Y$. Since the path $\delta : [0,\tau] \to \mathbf{U}^m$ is trivial (i.e., it can be pulled to Y in $\mathbf{U}^m \setminus S_Y$ rel endpoints), the restriction $\delta : [\tau, 1] \to \mathbf{U}^m \setminus S_Y$ (appropriately reparametrized) represents the same point in $\mathbf{Y}$ as the original path. Moreover, if $\tau \neq 1$, we can replace it with an equivalent path $\delta : [\tau+\varepsilon, 1] \to \mathbf{U}^m \setminus S_Y$ which is disjoint from the Julia set $K(f)$. As the latter path represents a point in some $\hat{A}_i$, we are done. □

3.2.2. *Naturality.*

Lemma 3.4. (i) *Consider two puzzle pieces Y and Z such that the map $f : Y \to Z$ is a branched covering of degree k (where $k = 1$ or $k = 2$ depending on whether Y is off-critical or not). Then there exists an induced map $\mathbf{f} : \mathbf{Y} \to \mathbf{Z}$ which is a branched covering of the same degree k.*

(ii) *Given two puzzle pieces $Y \subset Z$, the inclusion $i : Y \to Z$ extends to an immersion $\mathbf{i} : \mathbf{Y} \to \mathbf{Z}$.*

Proof. Both properties follow easily from either definition of the ψ-pieces. Let us, for instance, use the second definition.

(i) Let $\operatorname{depth} Y = m$, $\operatorname{depth} Z = m-1$. Let us consider the degree k branched covering

$$f : (\mathbf{U}^m, Y, K_Y) \to (\mathbf{U}^{m-1}, Z, K_Z).$$

The components Y_i of $Y \smallsetminus K_Y$ are univalently mapped onto components $Z_{j(i)}$ of $Z \smallsetminus K_Z$, where the map $j = j(i)$ is k-to-1. This map extends to an isomorphism map $\hat{A}_i \to \hat{B}_j$ of the corresponding universal coverings, which glue together into a branched covering $\mathbf{Y} \to \mathbf{Z}$ of degree k.

(ii) Let $\operatorname{depth} Y = m$, $\operatorname{depth} Z = n$. Let us consider the immersion

$$i : (\mathbf{U}^m, Y, K_Y) \to (\mathbf{U}^n, Z, K_Z).$$

The components Y_i are embedded by i into some components $Z_{j(i)}$, where the map $j = j(i)$ is surjective but not necessarily injective. These embeddings extend to immersions $\hat{A}_i \to \hat{B}_j$ that glue together into an immersion $\mathbf{Y} \to \mathbf{Z}$. □

3.2.3. *Moduli.* Given two puzzle pieces $Z \Subset Y$, we let

$$\mathbf{mod}\,(Y, Z) = \operatorname{mod}(\mathbf{Y} \smallsetminus K_Z).$$

Lemma 3.4 implies:

Lemma 3.5. (i) *Consider two pairs of puzzle pieces (Y', Y) and (Z', Z) such that the map $f : (Y', Y) \to (Z', Z)$ is a branched covering of degree k (on both domains). Then*

$$\mathbf{mod}\,(Z', Z) = k\,\mathbf{mod}\,(Y', Y).$$

(ii) *Given a nest of three puzzle pieces $W \subset Z \subset Y$, we have*

$$\mathbf{mod}\,(Z, W) \leq \mathbf{mod}\,(Y, W).$$

3.2.4. *Boundary of puzzle pieces.* Let us mention in conclusion, that the ideal boundary of a pseudo-puzzle $\mathbf{Y}$ is tiled by (finitely many) arcs $\lambda_i \subset \partial \hat{A}_i$ that cover the ideal boundary of $\mathbf{U}^m$ (where $m = \operatorname{depth} Y$) and arcs $\xi_i, \eta_i \subset \partial \hat{A}_i$ mapped onto the Julia set $J(f)$. The arc λ_i meets each ξ_i, η_i at a single boundary point corresponding to a path $\delta : [0, 1) \mapsto A$ that wraps around $K(f)$ infinitely many times, while η_i meets ξ_{i+1} at a vertex $v_i \in Y \cap K(f)$. We say that the arcs λ_i form the *outer boundary* (or "O-boundary") $\partial_O \mathbf{Y}$ of the puzzle piece $\mathbf{Y}$, while the arcs ξ_i and η_i form its *J-boundary* $\partial_J \mathbf{Y}$. Given a vertex $v = v_i$ of a puzzle piece Y, let $\partial^v \mathbf{Y} = \eta_i \cup \xi_{i+1}$ stand for the part of the J-boundary of $\mathbf{Y}$ attached to v.

Note that the immersion constructed in Lemma 3.4 extends continuously to the boundary of the puzzle piece $\mathbf{Y}$. (However, $\mathbf{i}(\partial \mathbf{Y})$ is not contained in $\partial \mathbf{Z}$, unless $Z = Y$.) In what follows we will assume this extension without further comment.

A *multicurve* in some space X is a continuous map $\gamma : \cup_{k=1}^l [s_k, t_k] \to X$ parametrized by a finite union of disjoint intervals $[s_k, t_k] \subset \mathbb{R}$.[4] Note that multicurves are ordered. A multicurve in a puzzle piece $\mathbf{Y}$ is called *horizontal* if

$$\gamma(s_1) \in \partial^{v_0} \mathbf{Y}, \quad \gamma(t_k), \gamma(s_{k+1}) \in \partial^{v_k} \mathbf{Y},\ k = 1, \ldots, l-1, \quad \gamma(t_l) \in \partial^{v_l} \mathbf{Y}$$

[4]We allow that the boundary points of a multicurve in a pseudo puzzle $\mathbf{Y}$ belong to $\partial_J \mathbf{Y}$.

for some vertices v_k of $\mathbf{Y}$, $k = 0, \dots, l$. We say that such a multicurve "connects" $\partial^{v_0}\mathbf{Y}$ to $\partial^{v_l}\mathbf{Y}$. The following statement motivates introduction of multicurves:

Lemma 3.6. *Let v and w be two vertices of a geometric puzzle piece $Y \in \mathcal{Y}^m(l)$. Then any curve γ in $\mathbf{U}^l$ connecting S_Y^v to S_Y^w contains a multicurve γ' that lifts to a multicurve γ^* in $\mathbf{Y}$ connecting $\partial_v\mathbf{Y}$ to $\partial_w\mathbf{Y}$.*

Given two vertices v and w of Y, let $\mathcal{G}_Y(v, w)$ stand for the family of horizontal multicurves in $\mathbf{Y}$ connecting $\partial^v(\mathbf{Y})$ to $\partial^w(\mathbf{Y})$. Finally, let

$$\mathbf{d}_Y(v, w) = \mathcal{L}(\mathcal{G}_Y(v, w))$$

stand for the extremal distance between the corresponding parts of J-boundary of $\mathbf{Y}$.

Lemma 3.7. *If $f|Y$ is univalent, then $\mathbf{d}_Y(v, w) = \mathbf{d}_{f(Y)}(fv, fw)$.*

4. Quasi-Additivity Law and Covering Lemma

Let us now formulate two analytic results which will play a crucial role in what follows. The first one appeares in §2.10.3 of [KL1]:

Quasi-Additivity Law. *Fix some $\eta \in (0, 1)$. Let $\mathbf{V}$ be a topological disk, let $K_i \Subset \mathbf{V}$, $i = 1, \dots, m$, be pairwise disjoint full compact continua, and let $\phi_i : \mathbb{A}(1, r_i) \to \mathbf{V} \setminus \cup K_j$ be holomorphic annuli such that each ϕ_i is an embedding of some proper collar of $\mathbb{T}$ to a proper collar of ∂K_i. Then there exists a $\delta_0 > 0$ (depending on η and m) such that:*
If for some $\delta \in (0, \delta_0)$, $\mathrm{mod}(\mathbf{V}, K_i) < \delta$ while $\log r_i > 2\pi\eta\delta$ for all i, then

$$\mathrm{mod}(\mathbf{V}, \cup K_i) < \frac{2\eta^{-1}\delta}{m}.$$

The next result appears in §3.1.5 of [KL1]:

Covering Lemma. *Fix some $\eta \in (0, 1)$. Let us consider two topological disks $\mathbf{U}$ and $\mathbf{V}$, two full continua $A' \subset \mathbf{U}$ and $B' \subset \mathbf{V}$, and two compact subsets, $A \Subset A'$ and $B \Subset B'$, of topological type bounded by T.*[5]
Let $f : \mathbf{U} \to \mathbf{V}$ be a branched covering of degree D such that A' is a component of $f^{-1}(B')$, and A is the union of some components of $f^{-1}(B)$. Let $d = \deg(f : A' \to B')$.
Let B' be also embedded into another topological disk $\mathcal{B}'$. Assume $\mathcal{B}'$ is immersed into $\mathbf{V}$ by a map i in such a way that $i|\, B' = \mathrm{id}$, $i^{-1}(B') = B'$, and $i(\mathcal{B}') \setminus B'$ does not contain the critical values of f.
Under the following "Collar Assumption":

$$\mathrm{mod}(\mathcal{B}', B) > \eta\, \mathrm{mod}(\mathbf{U}, A),$$

if

$$\mathrm{mod}(\mathbf{U}, A) < \varepsilon(\eta, T, D)$$

[5]In applications, A and B will be full continua, so $T = 1$.

then

$$\operatorname{mod}(\mathbf{V}, B) < 2\eta^{-1}d^2 \operatorname{mod}(\mathbf{U}, A).$$

5. Improving the moduli

In this section C will stand for the maximum of the constants in the Quasi-Additivity Law and the Covering Lemma.

5.1. **High type.** Let us begin with a simple estimate that compares moduli on consecutive odd levels of the Principal Nest:

Lemma 5.1. *For any odd n, we have:*

$$\mathbf{mod}\,(E^{n-3}, E^{n-2}) \le 4\,\mathbf{mod}\,(E^{n-1}, E^n)$$

and

$$\mathbf{mod}\,(Y^0, R) \le 2^{\mathbf{n}+1}\,\mathbf{mod}\,(E^0, E^1).$$

Proof. By Lemma 3.5,

$$\mathbf{mod}\,(\hat{E}^{n-1}, E^{n-1}) = 2\,\mathbf{mod}\,(\tilde{E}^n, E^n) \le 2\,\mathbf{mod}\,(E^{n-1}, E^n)$$

and

$$\mathbf{mod}\,(\hat{E}^{n-1}, E^{n-1}) \ge \mathbf{mod}\,(\tilde{E}^{n-1}, E^{n-1})$$

$$= \frac{1}{2}\,\mathbf{mod}\,(\hat{E}^{n-2}, E^{n-2}) = \frac{1}{2}\,\mathbf{mod}\,(E^{n-3}, E^{n-2}),$$

and the first estimate follows.

The second estimate is similar. The puzzle piece $E^1 \equiv Y^m$ is mapped with degree 2 onto E^0, and this map admits degree 2 extension $\tilde{E}^1 \to Y^{q\mathbf{n}} \equiv \hat{E}^0$, where $\tilde{E}^1 = Y^{m-1}$. Then E^0 is mapped onto Z^1_κ by degree 2 map $f^{q\mathbf{n}}$. This map admits degree $2^{\mathbf{n}}$ extension $Y^{q\mathbf{n}} \to Y^0$. It follows that

$$\mathbf{mod}\,(Y^0, R) \le \mathbf{mod}\,(Y^0, Z^1_\kappa)$$

$$= 2^{\mathbf{n}+1}\,\mathbf{mod}\,(Y^{m-1}, E^1) \le 2^{\mathbf{n}+1}\,\mathbf{mod}\,(E^0, E^1).$$

□

The following lemma tells us that if some principal modulus is very small then it should be even smaller on some preceding level of the Principal Nest:

Lemma 5.2. *There exist absolute $N \in \mathbb{N}$ and $\varepsilon > 0$ such that: If on some odd level $n \ge N$, $\mathbf{mod}\,(E^{n-1}, E^n) < \varepsilon$, then on some previous odd level $n - s \in [n - N, n - 1]$ we have:*

$$\mathbf{mod}\,(E^{n-s-1}, E^{n-s}) < \frac{1}{2}\,\mathbf{mod}\,(E^{n-1}, E^n). \tag{5.1}$$

Proof. Let us fix some integer $m > C^3 2^{28}$. Let N be the smallest odd integer that is bigger than $\log_2 m + 5$. Take any odd level $n \geq N$. For each k, let us consider the associated 3-domain branched covering Ψ_k (2.3)

$$\Psi_k : (\Upsilon_k, \Delta_k, E^n) \to (E^{n-N}, \Lambda'_k, \Lambda_k).$$

Let us consider two cases:

Case 1. Assume that for some domain Λ_k,

$$\mathbf{mod}\,(\Lambda'_k, \Lambda_k) \leq \frac{1}{4}\,\mathbf{mod}\,(E^{n-1}, E^n).$$

By Property (P4) and Lemma 3.5, $\mathbf{mod}\,(\Lambda'_k, \Lambda_k) = \mathbf{mod}\,(\hat{E}^i, E^i)$. If i is odd then $\hat{E}^i = E^{i-1}$, and we obtain the desired estimate with $s = n - i \in [1, 5]$:

$$\mathbf{mod}\,(E^{n-1}, E^n) \geq 4\,\mathbf{mod}\,(E^{i-1}, E^i).$$

If i is even, then

$$\mathbf{mod}\,(\hat{E}^i, E^i) \geq \mathbf{mod}\,(\tilde{E}^i, E^i) = \frac{1}{2}\mathbf{mod}\,(\hat{E}^{i-1}, E^{i-1}) = \frac{1}{2}\mathbf{mod}\,(E^{i-2}, E^{i-1}),$$

and we conclude that

$$\mathbf{mod}\,(E^{n-1}, E^n) \geq 2\,\mathbf{mod}\,(E^{i-2}, E^{i-1}).$$

Case 2. Assume that for all Λ_k,

$$\mathbf{mod}\,(\Lambda'_k, \Lambda_k) \geq \frac{1}{4}\mathbf{mod}\,(E^{n-1}, E^n) \geq \frac{1}{4}\mathbf{mod}\,(\Upsilon_k, E^n) \tag{5.2}$$

(where the second estimate follows from the inclusion $\Upsilon_k \subset E^{n-1}$). By Lemma 3.4, there exists a natural covering map

$$\mathbf{\Psi}_k : (\mathbf{\Upsilon}_k, K_{\Delta_k}, K_{E^n}) \to (\mathbf{E}^{n-N-1}, K_{\Lambda'_k}, K_{\Lambda_k}),$$

and a natural immersion $i : \mathbf{\Lambda}'_k \to \mathbf{E}^{n-N}$. Note that $i(\mathbf{\Lambda}'_k) \smallsetminus K_{\Lambda'_k}$ does not contain the critical values of $\mathbf{\Psi}_k$, since the latter are contained in the Julia set $K(f)$. Moreover, equation (5.2) provides us with the Collar Assumption that allows us to apply the Covering Lemma to the map $\mathbf{\Psi}_k$. If ε is sufficiently small, it yields:

$$\mathbf{mod}\,(E^{n-N-1}, \Lambda_k) \leq C2^{12}\mathbf{mod}\,(\Upsilon_k, E^n) \leq C\,2^{12}\,\mathbf{mod}\,(E^{n-1}, E^n). \tag{5.3}$$

Estimates (5.2) and (5.3) show that the Quasi-Additivity Law is applicable to the family of islands K_{Λ_k} in $\mathbf{E}^{n-N-1}$ with $\eta^{-1} = C2^{14}$. Since there are at least $m/2$ domains $\Lambda_k \subset \Lambda'_k \subset E^{n-N}$, it implies:

$$\mathbf{mod}\,(E^{n-N-1}, E^{n-N}) \leq \frac{C^3 2^{27}\,\mathbf{mod}\,(E^{n-1}, E^n)}{m} < \frac{1}{2}\,\mathbf{mod}\,(E^{n-1}, E^n),$$

and we are done. □

Lemma 5.3. *There exist absolute constants $C > 0$, $\rho \in (0,1)$ and $\varepsilon > 0$ such that if for some odd n, $\mathbf{mod}\,(E^{n-1}, E^n) < \varepsilon$, then*

$$\mathbf{mod}\,(E^0, E^1) < C\rho^n \mathbf{mod}\,(E^{n-1}, E^n).$$

and

$$\mathbf{mod}\,(Y^0, R) \le C2^{\mathbf{n}}\rho^n \mathbf{mod}\,(E^{n-1}, E^n).$$

Proof. By Lemma 5.2, there exists an odd level $l < N$ such that

$$\mathbf{mod}\,(E^{l-1}, E^l) \le \left(\frac{1}{2}\right)^{[n/N]},$$

which together with Lemma 5.1 implies the desired estimates. □

5.2. **Frequent R-returns.** Let us consider the map

$$f^l = f^{q\mathbf{n}+1} \circ g_1 \circ \cdots \circ g_{\chi-1} \,:\, E^{\chi-1} \to Y^0 \tag{5.4}$$

and the trajectory $\mathcal{O} = \{\mathcal{K}_i\}_{i=l}^{l+p-1}$ of the little Julia set $\mathcal{K}$. Let $i_1, i_2, \dots$ be the moments in $\mathcal{O}$ for which $\mathcal{K}_i \subset R$.

Lemma 5.4. *Let $\rho > 0$, $\bar\chi \in \mathbb{N}$. Take some integer $m \ge C^3 2^{30}/\rho$, and let $\underline{p} = m^2\mathbf{qn}$. Assume that the little Julia set frequently visits R:*

$$i_{k+1} - i_k \le m\mathbf{qn}, \quad k = 1, 2, \dots, m. \tag{5.5}$$

If $\chi \le \bar\chi$ while $p \ge \underline{p}$, then

$$\mathbf{mod}\,(Y^0, R) \le \rho\, \mathrm{mod}(\mathbf{E}^{\chi-1}, \mathcal{K}),$$

provided $\mathrm{mod}(\mathbf{E}^{\chi-1}, \mathcal{K}) < \varepsilon(\mathbf{n}; \bar\chi, \rho)$.

Proof. The map (5.4) has degree $2^\chi \le 2^{\bar\chi}$. By Lemma 2.9 of [KL2], $\deg(f^l|\, E^\chi) \le 32$, and hence $l \le 5p$.

By (5.5), $i_m - i_1 < m^2\mathbf{qn} \le p$, so the moments i_k are pairwise non-congruent mod p. Hence the little Julia sets $\mathcal{K}_{i_1}, \dots, \mathcal{K}_{i_m}$ are all distinct.

Since $\mathcal{O}$ has length p, there is only one critical Julia set in $\mathcal{O}$. Hence $\deg(f^{i_k} : \mathcal{K} \to \mathcal{K}_{i_k})$ is at most 64, so that $i_k \le 6p$, $k = 1, \dots, m$.

On the other hand, $\mathcal{K}_{i_k}$ is contained in a puzzle piece in R which is mapped under $f^{i_{k+1}-i_k}$ onto Y^0 with degree at most $2^{m\mathbf{n}}$. It follows by the Telescope Lemma 2.2 that there is a puzzle piece $\Upsilon_k \subset E^{\chi-1}$ which is mapped under f^{i_k} onto Y^0 with degree at most $2^{\bar\chi + km\mathbf{n}} \le 2^{\bar\chi + m^2\mathbf{n}} \equiv D$.

We would like to apply the Covering Lemma to the corresponding map

$$\mathbf{f}^{i_k} : (\boldsymbol{\Upsilon}_k, \mathcal{K}) \to (\mathbf{Y}^0, \mathcal{K}_{i_k})$$

of degree at most D. To this end we need collars around $\mathcal{K}_{i_k}$. Let Ω be the critical pullback of E^χ under f^{6p}. Then we let $\Lambda'_k = f^{i_k}(\Omega))$. Since the moments i_k are pairwise non-congruent mod p and $i_k \le 6p$, the puzzle pieces Λ'_k are contained in different domains of the orbit $f^t(E^\chi)$, $t = 0, 1 \dots, p-1$. Hence they are pairwise disjoint. Moreover, by Lemma 3.5,

$$\text{mod}(\mathbf{\Lambda}'_k, \mathcal{K}_{i_k}) \geq \text{mod}(\mathbf{\Omega}, \mathcal{K}) = \frac{1}{4}\,\text{mod}(\mathbf{E}^{\chi-1}, \mathcal{K}) \geq \frac{1}{4}\,\text{mod}(\mathbf{\Upsilon}_k, \mathcal{K}). \tag{5.6}$$

This provides us with the desired Collar Assumption. By the Covering Lemma,

$$\text{mod}(\mathbf{Y}^0, \mathcal{K}_{i_k}) \leq C2^{14}\,\text{mod}(\mathbf{\Upsilon}_k, \mathcal{K}) \leq C2^{14}\,\text{mod}(\mathbf{E}^{\chi-1}, \mathcal{K}).$$

The last two estimates show that the Quasi-Additivity Law is applicable to the family of islands $\mathcal{K}_{i_k}$ in $\mathbf{Y}^0$ (with $\eta^{-1} = C2^{16}$):

$$\mathbf{mod}\,(Y^0, R) \leq \frac{C^3 2^{30}\,\text{mod}(\mathbf{E}^{\chi-1}, \mathcal{K})}{m} \leq \rho\,\text{mod}(\mathbf{E}^{\chi-1}, \mathcal{K}),$$

provided $\text{mod}(\mathbf{E}^{\chi-1}, \mathcal{K}) < \varepsilon(D) = \varepsilon(\mathbf{n}; \bar{\chi}, \rho)$ and we are done. □

5.3. **Many consecutive returns to L.** Here the set-up is the same as in the previous section, but we will assume that there is a gap in returns of the little Julia sets to R:

Lemma 5.5. *Let ρ, $\bar{\chi}$, m, and $\underline{p}$ be as in Lemma 5.4. Assume there is $k \leq m$ such that*

$$i_{k+1} - i_k > m\mathbf{qn}. \tag{5.7}$$

If $\chi \leq \bar{\chi}$ while $p \geq \underline{p}$, then

$$\mathbf{mod}\,(Z^0, L) \leq \rho\,\text{mod}(\mathbf{E}^{\chi-1}, \mathcal{K}),$$

provided $\text{mod}(\mathbf{E}^{\chi-1}, \mathcal{K}) < \varepsilon(\mathbf{n}; \mathbf{q}, \bar{\chi})$.

Proof. Under our assumption (5.7) the Julia set returns frequently to L:

$$\mathcal{K}_{i_k + j\mathbf{qn}} \subset L, \quad j = 1, \dots, m.$$

Let $P_j \ni f^{i_k} z$ be the pullback of Z^0 under $f^{j\mathbf{qn}}$. By Lemma 2.3 and the Telescope Lemma, $P_j \subset Y^0$.

Let Υ_j be the further pullback of P_j under f^{i_k}, and let

$$\Psi_j = f^{i_k + j\mathbf{qn}} : \Upsilon_j \to Z^0.$$

Then $\Upsilon_j \subset E^{\chi-1}$ and $\deg \Psi_j \leq 2^{\bar{\chi}+\underline{p}}$.

The rest of the argument is the same as for Lemma 5.4: the Covering Lemma implies that for $j = 1, \dots, m$,

$$\text{mod}(\mathbf{Z}_0, \mathcal{K}_{i_k + j\mathbf{qn}}) \leq C2^{14}\,\text{mod}(\mathbf{\Upsilon}_k, \mathcal{K}) \leq C2^{14}\,\text{mod}(\mathbf{E}^{\chi-1}, \mathcal{K}),$$

and by the Quasi-Additivity Law,

$$\mathbf{mod}\,(Z^0, L) \leq \frac{C^3 2^{30}\,\text{mod}(\mathbf{E}^{\chi-1}, \mathcal{K})}{m} \leq \rho\,\text{mod}(\mathbf{E}^{\chi-1}, \mathcal{K}).$$

□

Note that by symmetry, $\mathbf{mod}\,(Z^0, L) = \mathbf{mod}\,(Y^0, R)$. Putting together Lemmas 5.3, 5.4 and 5.5, we obtain:

Corollary 5.6. *For any parameters* $\mathbf{q}, \mathbf{n}$ *of a decoration and any* $\rho > 0$*, there exists* $\underline{p} \in \mathbb{N}$ *and* $\varepsilon > 0$ *such that*

$$\mathbf{mod}\,(Y^0, R) \leq \rho \,\mathrm{mod}(\mathbf{E}^{\chi-1}, \mathcal{K}),$$

provided $p \geq \underline{p}$ *and* $\mathrm{mod}(\mathbf{E}^{\chi-1}, \mathcal{K}) < \varepsilon$*.*

5.4. **Comparison of** $\mathbf{mod}\,(Y^0, R)$ **with** $\mathbf{d}_{Y^1}(\alpha, \alpha')$**.** Let

$$\mu := \min(\mathbf{mod}\,(U, K), 1/2).$$

Lemma 5.7.

$$\mathbf{d}_{Y^1}(\alpha, \alpha') \leq \mathbf{mod}\,(Y^0, R) \ominus \frac{1}{2}\mu.$$

Proof. The boundary of $\mathbf{Y}^0$ consists of two parts (see the end of §3.2): the J-boundary $\partial_J \mathbf{Y}^0 = \xi \cup \eta$ attached to α and the outer arc $\lambda = \partial_O \mathbf{Y}^0$ that covers the ideal boundary of $\mathbf{U}$. Let $\mathcal{G}^h$ stand for the family of curves in the annulus $\mathbf{Y}^0 \smallsetminus K_R$ connecting K_R to the J-boundary, while $\mathcal{G}^v$ stand for the family of curves in the same annulus connecting K_R to λ. By the Parallel Law,

$$\mathcal{L}(\mathcal{G}^h) \leq \mathbf{mod}\,(Y^0, R) \ominus \mathcal{L}(\mathcal{G}^v).$$

Let Π stand for the rectangle uniformizing $\mathbf{Y}^0 \smallsetminus K_{Y^0}$ whose horizontal sides correspond to K_{Y^0} and λ, and vertical sides correspond to ξ and η. We let ω be the horizontal side of Π corresponding to K_{Y^0}. Since any curve of the family $\mathcal{G}^v$ overflows some curve connecting K_{Y^0} to λ in $\mathbf{Y}^0 \smallsetminus K_{Y^0}$ (and thus representing a vertical curve in Π), we have:

$$\mathcal{L}(\mathcal{G}^v) \geq \mathrm{mod}\,\Pi.$$

But by definition of the pseudo-puzzle, the domain $\mathbf{Y}^0 \smallsetminus K_{Y^0}$ covers the annulus $\mathbf{U} \smallsetminus K$ extending to an embedding on K_{Y^0}. Let us uniformize $\mathbf{U} \smallsetminus K$ by a round annulus $\mathbb{A}$. It follows that the rectangle Π covers $\mathbb{A}$ in such a way that $\omega \subset \partial \Pi$ is embedded into $\partial \mathbb{A}$. By Lemma 6.6 from the Appendix,

$$\mathrm{mod}\,\Pi \geq \frac{\mu}{2}.$$

Putting the above three estimates together, we obtain:

$$\mathcal{L}(\mathcal{G}^h) \leq \mathbf{mod}\,(Y^0, R) \ominus \frac{\mu}{2}. \tag{5.8}$$

On the other hand, let us consider the family $\mathcal{H}$ of horizontal curves in the puzzle piece $\mathbf{Y}^1$ connecting $\partial_\alpha \mathbf{Y}^1$ to $\partial_{\alpha'} \mathbf{Y}^1$. Let $\phi : \mathbf{Y}^1 \to \mathbf{Y}^0$ be the natural immersion. Under ϕ, the boundary $\partial_\alpha \mathbf{Y}^1$ is mapped homeomorphically onto $\partial_\alpha \mathbf{Y}^0$. It follows that any curve γ of $\mathcal{G}^h$ contains an arc that can be lifted by ϕ to some curve of $\mathcal{H}$. Indeed, orient γ so that it begins on $\partial_\alpha \mathbf{Y}^0$. Then a maximal lift of γ that begins on $\partial_\alpha \mathbf{Y}^1$ must end on $\partial_{\alpha'} \mathbf{Y}^1$.

By Corollary 6.2, $\mathcal{L}(\mathcal{H}) \leq \mathcal{L}(\mathcal{G}^h)$. Together with (5.8), this yields the desired inequality. □

5.5. **Skipping over.** In this section we will show that not many curves can skip some piece of the Julia set.

Let $Y \in \mathcal{Y}^m(l)$ be a geometric puzzle piece of bidepth (m, l), and let A be a component of $Y \smallsetminus K_Y$. Let $\partial_J \hat{A} = \partial_J \mathbf{Y} \cap \hat{A}$. Recall that it consists of two components. Let $\mathcal{C}_A$ stand for the family of curves in $\hat{A}$ connecting different components of $\partial_J \hat{A}$, and let

$$\mathbf{d}_A = \mathcal{L}(\mathcal{C}_A).$$

Let $\dfrac{1}{2\pi} \log r = 2^{-(m+\mathbf{q})}\mu$. Then the annulus $\mathbf{U}^{m+\mathbf{q}} \smallsetminus K$ can be uniformized by the round annulus $\mathbb{A}(1, r)$, and under this uniformization, the set K_Y gets represented on the unit circle $\mathbb{T}$ as the union of arcs ω_i of length

$$|\omega_i| \geq 2\pi \cdot 2^{-(\mathbf{q}+m)}. \tag{5.9}$$

Indeed, the covering map $\mathbf{f}^{m+\mathbf{q}} : \mathbf{U}^{m+\mathbf{q}} \smallsetminus K \to \mathbf{U} \smallsetminus K$ is turned into $z \mapsto z^{2^{m+\mathbf{q}}}$ under the above uniformization of $\mathbf{U}^{m+\mathbf{q}} \smallsetminus K$ and the uniformization of $\mathbf{U} \smallsetminus K$ by $\mathbb{A}(1, e^{2\pi\mu})$ (appropriately normalized). Since under this map, every arc ω_i covers the whole circle, the length of ω_i is at least 2π times its combinatorial length (2.2).

Lemma 5.8. *Let $Y \in \mathcal{Y}^m(m+\mathbf{q})$ be a geometric puzzle piece of bidepth $(m, m+\mathbf{q})$, and let A be a component of $Y \smallsetminus K_Y$. Then*

$$\mathbf{d}_A \geq \frac{1}{\mu}.$$

Proof. We can uniformize $\hat{A}$ by the horizontal strip $\Pi = \Pi(2^{-(m+\mathbf{q})})\mu$ in such a way that the upper boundary of Π covers the O-boundary of $\mathbf{U}^{m+\mathbf{q}}$, and the group of deck transformations is generated by the translation $z \mapsto z+1$. By (5.9), the Julia set $K_Y \subset \partial A$ is represented as an interval I on $\mathbb{R}$ of length at least $2^{-(\mathbf{q}+m)}$.

Let us view Π as a quadrilateral with horizontal sides I and the top of Π. Then

$$\mathcal{L}(\mathcal{C}_A) = \frac{1}{\operatorname{mod} \Pi} \geq \frac{1}{\mu},$$

where the last estimate comes from the simple right-hand side estimate of Lemma 6.5, and we are done. □

Lemma 5.9. *Let $Y \in \mathcal{Y}^m(0)$ be a geometric puzzle piece of bidepth $(m, 0)$, and let A be a connected component of $Y \smallsetminus K_Y$. Then*

$$\mathbf{d}_A \geq \frac{\mu}{2^{m+\mathbf{q}+2}}.$$

Proof. Let $A(m+\mathbf{q})$ be the component of $Y(m+\mathbf{q}) \smallsetminus K_Y$ contained in A, and let $\phi : \hat{A}(m+\mathbf{q}) \to \hat{A}$ be the natural immersion. It extends to the identity on $K_Y \cap \partial \hat{A}(m+\mathbf{q})$.

Let us realize $\hat{A}$ as the strip $\Pi(\mu)$ that covers $\mathbf{U} \smallsetminus K$, with the group of deck transformations generated by $z \mapsto z+1$. Let us consider the interval $I \subset \mathbb{R}$ representing $K_Y \cap \partial A$, and let J be the left-adjacent interval of length 1.

Let us orient the curves $\gamma \in \mathcal{C}_A$ so that they begin on the left-hand side of I. Then any curve $\gamma \in \mathcal{C}_A$ contains the maximal initial arc γ' that can be lifted by ϕ to a curve γ^* in $\hat{A}(m+\mathbf{q})$. Accordingly, we can split the family of curves $\gamma \in \mathcal{C}_A$ into three subfamilies:

- $\mathcal{H}_1$ consists of the curves γ such that $\gamma' = \gamma$; then $\gamma^* \in \mathcal{C}_{A(m+\mathbf{q})}$;
- $\mathcal{H}_2$ consist of the curves that begin in J and whose lift γ^* terminates on the O-boundary of $\hat{A}(m+\mathbf{q})$;
- $\mathcal{H}_3$ consists of the curves that begin on the left-hand side of J.

Let us estimate the extremal length of each of these families.

Since $\mathcal{H}_1 = \phi(\mathcal{C}_{A(m+\mathbf{q})})$,

$$\mathcal{L}(\mathcal{H}_1) \geq \mathcal{L}(\mathcal{C}_{A(m+\mathbf{q})}) \geq \frac{1}{\mu},$$

where the first estimate follows from Lemma 6.1, and the second follows from Lemma 5.8.

Let $\mathcal{T}$ be the family of curves in $\hat{A}(m+\mathbf{q})$ that begin on J and end on the O-boundary of $\hat{A}(m+\mathbf{q})$. By Corollary 6.2 and Lemma 6.5,

$$\mathcal{L}(\mathcal{H}_2) \geq \mathcal{L}(\mathcal{T}) \geq \frac{\mu}{2^{m+\mathbf{q}+1}}.$$

To estimate the extremal length of $\mathcal{H}_3$, endow the rectangle $Q = J \times [0,\mu] \subset \Pi(\mu)$ with the Euclidean metric λ. Since any curve $\gamma \in \mathcal{H}_3$ horizontally overflows Q, it has λ-length at least 1. Hence

$$\mathcal{L}(\mathcal{H}_3) \geq \frac{1}{\operatorname{area} Q} = \frac{1}{\mu}.$$

Incorporating the last three estimates into the Parallel Law, we obtain the desired:

$$\mathcal{L}(\mathcal{C}_Y^{\varepsilon}) \geq \frac{1}{\mu + 2^{m+\mathbf{q}+1}\mu^{-1} + \mu} \geq \frac{\mu}{2^{m+\mathbf{q}+2}}.$$

□

Let us consider two vertices, v and w, of a geometric puzzle piece Y. Let $Z \subset Y$ be a puzzle piece of depth m that separates v from w. We say that a multicurve in $\mathbf{Y}$ connecting $\partial^v \mathbf{Y}$ to $\partial^w \mathbf{Y}$ *skips over* K_Z if one of its components does not cross K_Z.

Corollary 5.10. *Under the above circumstances, let $\mathcal{T}$ be be the family of multicurves in $\mathbf{Y}$ connecting $\partial^v \mathbf{Y}$ to $\partial^w \mathbf{Y}$ that skip over K_Z. Then*

$$\mathcal{L}(\mathcal{T}) \geq \frac{\mu}{2^{2m+\mathbf{q}+2}}.$$

Proof. The piece Z has at most 2^m components A_i of $Z \setminus K_Z$. If a multicurve γ in $\mathbf{Y}$ that skips over K_Z then it contains an arc γ' that lifts to a curve γ^* in some family $\mathcal{G}_{A_j}$. Let $\mathcal{C}_j$ be the corresponding subfamily of $\mathcal{T}$. By Lemma 5.9 (together with Corollary 6.2),

$$\mathcal{L}(\mathcal{C}_j) \geq \mathbf{d}_{A_j} \geq \frac{\mu}{2^{m+\mathbf{q}+2}}.$$

The Parallel Law concludes the proof. ☐

Let us now consider the puzzle piece $P = Y^{(\mathbf{n}-1)\mathbf{q}+1}$, together with the corresponding pseudo-piece $\mathbf{P}$, and the family of puzzle pieces $Q^v \subset P$ from Lemma 2.1. Recall that $T^{vw} = K_P \setminus (Q^v \cup Q^w)$. Given two vertices v and w of P, let $\hat{\mathcal{G}}_P^{vw}$ stand for the family of multicurves in $\mathbf{P}$ connecting $\partial_v \mathbf{P}$ to $\partial_w \mathbf{P}$ that *do not skip over* T_{vw}. By Corollary 5.10,

$$\mathcal{L}(\mathcal{G} \setminus \hat{\mathcal{G}}_P^{vw}) \geq C^{-1}\mu, \tag{5.10}$$

where here and below, C stands for a constant that depend only on $\mathbf{q}$ and $\mathbf{n}$.

5.6. **Separation between L and R.** In this section we will show that the modulus $\mathbf{d}_{Y^1}(\alpha, \alpha')$ that measures the extremal distance between L and R is comparable with μ.

Let Y be a geometric puzzle piece. For two vertices v and w of Y, we let

$$\mathbf{W}_Y(v, w) = \mathcal{W}(\mathcal{G}_Y^{vw}).$$

We define the *pseudo-conductance* of Y as

$$\mathbf{W}_Y = \sup_{v,w} \mathbf{W}_Y(v, w),$$

where the supremum is taken over all pairs of the vertices of Y.

Lemma 5.11. *For the puzzle piece $P = Y^{(\mathbf{n}-1)\mathbf{q}+1}$ we have:*

$$W_P \leq \frac{C}{\mu}.$$

Proof. Along with the above conductance of P, let us consider

$$\hat{\mathbf{W}}_P(v, w) = \mathcal{W}(\hat{\mathcal{G}}_P^{vw}); \quad \hat{\mathbf{W}}_P = \sup_{v,w} \hat{\mathbf{W}}_P(v, w).$$

By (5.10),

$$\mathbf{W}_P \leq \hat{\mathbf{W}}_P + \frac{C}{\mu}. \tag{5.11}$$

Take a pair of vertices, v and w. Let $Q^v \cap T^{vw} = \{v'\}$ and $Q^w \cap T^{vw} = \{w'\}$. Recall that depth of the puzzle pieces Q^v and Q^w is equal to $r = (2\mathbf{n}-1)\mathbf{q}+1$, and so depends only on $\mathbf{q}$ and $\mathbf{n}$. Let $\mathcal{E}^r$ be the lift of the equipotential of level r to $\mathbf{P}$.

For any horizontal multicurve $\gamma \in \hat{\mathcal{G}}_P^{vw}$, one of the following two possibilities can occur:

• γ crosses the equipotential $\mathcal{E}^r$, and hence it contains an arc γ' connecting $\mathcal{E}^r$ to T^{vw}; By Lemma 6.6 and the Parallel Law, the width of this family of curves is bounded by $2^{r+\mathbf{n}}/\mu$ (here $2^{\mathbf{n}-1}$ is a bound on the number of connected components of $P \smallsetminus K_P$);

• contains two disjoint multicurves, δ^v and δ^w, that do not cross $\mathcal{E}^r$ and such that δ^v connects $\partial^v\mathbf{P}$ to T^{vw}, while δ^w connects T^{vw} to $\partial^w\mathbf{P}$. Then δ^v contains a multicurve that can be lifted to a horizontal multicurve in $\mathbf{Q}^v$ connecting $\partial^v\mathbf{Q}^v$ to $\partial^{v'}\mathbf{Q}^v$, an similarly for δ^w.

By the Series and Parallel Laws,

$$\hat{\mathbf{W}}_P(v,w) \leq \mathbf{W}_{Q^v}(v,v') \oplus \mathbf{W}_{Q^w}(w,w') + \frac{2^{r+\mathbf{n}}}{\mu} \leq \mathbf{W}_{Q^v} \oplus \mathbf{W}_{Q^w} + \frac{2^{r+\mathbf{n}}}{\mu}.$$

But $\mathbf{W}(Q^v) = \mathbf{W}(Q^w) = \mathbf{W}(P)$ since Q^v and Q^w are univalent pullbacks of P. Hence

$$\mathbf{W}_{Q^v} \oplus \mathbf{W}_{Q^w} \leq \frac{1}{2}\mathbf{W}_P.$$

Putting the last two estimates together and taking the supremum over all pairs of vertices (v,w) of P, we conclude that

$$\hat{\mathbf{W}}_P \leq \frac{1}{2}\mathbf{W}_P + \frac{2^{r+\mathbf{n}}}{\mu}.$$

Together with (5.11) it yields:

$$\mathbf{W}_P \leq \frac{1}{2}\mathbf{W}_P + \frac{C}{\mu},$$

and the conclusion follows. □

Proposition 5.12. $\mathbf{d}_{Y^1}(\alpha,\alpha') \geq C^{-1}\mu$.

Proof. Since the map $f^{(\mathbf{n}-1)\mathbf{q}} : \mathbf{P} \to \mathbf{Y}^1$ is a branched covering that maps $\partial_0\mathbf{P}$ to $\partial_0\mathbf{Y}^1$, any curve $\gamma \in \mathcal{G}^{\alpha\alpha'}_{Y^1}$ can be lifted to a curve $\gamma^* \in \cup\mathcal{G}^{vw}_P$, where the union is taken over all pairs of vertices of P. Hence

$$\mathcal{L}(\mathcal{G}^{\alpha\alpha'}_{Y^1}) \geq \bigoplus_{v,w} \mathcal{L}(\mathcal{G}^{vw}_P) \geq \frac{1}{N\mathbf{W}_P},$$

where N is the number of pairs (v,w). The conclusion follows. □

Lemma 5.7 and Proposition 5.12 imply:

Corollary 5.13. $\dfrac{\mu}{2} \leq C\,\mathbf{mod}\,(Y^0,R)$.

Corollary 5.14. *Let $f : (\mathbf{U},K) \to (\mathbf{U},K)$ be a renormalizable ψ-quadratic-like map with decoration parameters $(\mathbf{q},\mathbf{n})$, and let $f' = f^p : (\mathbf{U}',K') \to (\mathbf{U}',K')$ be its first renormalization. Then*

$$\min\{\mathrm{mod}(\mathbf{U},K), 1/2\} \leq C\ \mathrm{mod}(\mathbf{U}',K'),$$

where $C = C(\mathbf{q},\mathbf{n})$.

Proof. This follows from Lemma 5.1 and Corollary 5.13 by noticing that $(\mathbf{E}^{\chi-1}, \mathcal{K}) = (\mathbf{U}', K')$. □

5.7. **Conclusion.** Everything is now prepared for the main results. Corollary 5.6 and Corollary 5.13 imply:

Theorem 5.15 (Improving of the moduli: bounded decoration parameters)**.** *For any parameters* $\bar{\mathbf{q}}, \bar{\mathbf{n}}$ *and any* $\rho > 0$*, there exist* $\underline{p} \in \mathbb{N}$ *and* $\varepsilon > 0$ *with the following property. Let* $f : (\mathbf{U}, K) \to (\mathbf{U}, K)$ *be a renormalizable* ψ*-quadratic-like map with decoration parameters* $(\mathbf{q}, \mathbf{n}) \le (\bar{\mathbf{q}}, \bar{\mathbf{n}})$*, and let* $f' = f^p : (\mathbf{U}', K') \to (\mathbf{U}', K')$ *be its first renormalization. Then*

$$\{p \ge \underline{p} \ \text{ and } \ \mathrm{mod}(\mathbf{U}', K') < \varepsilon\} \Rightarrow \mathrm{mod}(\mathbf{U}, K) < \rho\, \mathrm{mod}(\mathbf{U}', K').$$

Remark 5.1. The logic of this theorem can be adjusted so that it would sound more like an "improvement in the future" rather than "worsening in the past":

For any parameters $\bar{\mathbf{q}}, \bar{\mathbf{n}}$ *of a Misuirewicz limb, there exists* $\underline{p} \in \mathbb{N}$ *and* $\varepsilon > 0$ *such that*

$$\mathrm{mod}(\mathbf{U}', K') \ge 2\ \mathrm{mod}(\mathbf{U}, K)$$

provided $p \ge \underline{p}$ *and* $\mathrm{mod}(\mathbf{U}, K) < \varepsilon/2$.

Theorem 5.15, together with Lemma 3.1, implies Theorem 1.2 from the Introduction.

To derive the Main Theorem, we will combine Theorem 5.15 with the following result (Theorem 9.1 from [K]):

Theorem 5.16 (Improving of the moduli: bounded period)**.** *For any* $\rho \in (0, 1)$*, there exists* $\underline{p} = \underline{p}(\rho)$ *such that for any* $\bar{p} \ge \underline{p}$*, there exists* $\varepsilon = \varepsilon(\bar{p}) > 0$ *with the following property. Let* $f : (\mathbf{U}, K) \to (\mathbf{U}, K)$ *be primitively renormalizable* ψ*-quadratic-like map, and let* $f' = f^p = (\mathbf{U}', K') \to (\mathbf{U}', K')$ *be the corresponding renormalization. Then*

$$\{\underline{p} \le p \le \bar{p} \ \text{ and } \ \mathrm{mod}(\mathbf{U}' \smallsetminus K') < \varepsilon\} \Rightarrow \mathrm{mod}(\mathbf{U} \smallsetminus K) < \rho\, \mathrm{mod}(\mathbf{U}' \smallsetminus K').$$

Remark 5.2. Unlike Theorem 5.15, in Theorem 5.16 the map f' *is not necessarily the first renormalization of* f. On the other hand, in Theorem 5.16, the scale ε *depends on the upper bound* $\bar{p}$, while in Theorem 5.15 it does not.

We say that an infinitely renormalizable ψ-ql map f belongs to the *decoration class* $(\bar{\mathbf{q}}, \bar{\mathbf{n}})$ if the decoration parameters $(\bar{\mathbf{q}}_n, \bar{\mathbf{n}}_n)$ of the renormalizations $R^n f$ are all bounded by $(\bar{\mathbf{q}}, \bar{\mathbf{n}})$.

Let us now put the above two theorems together:

Corollary 5.17. *For any* $(\bar{\mathbf{q}}, \bar{\mathbf{n}})$*, there exist an* $\varepsilon > 0$ *and* $l \in \mathbb{N}$ *with the following property. For any infinitely renormalizable* ψ*-ql map* f *of decoration class* $(\bar{\mathbf{q}}, \bar{\mathbf{n}})$ *with renormalizations* $R^n f : (\mathbf{U}_n, K_n) \to (\mathbf{U}_n, K_n)$*, if* $\mathrm{mod}(\mathbf{U}_n \smallsetminus K_n) < \varepsilon$*,* $n \ge l$*, then* $\mathrm{mod}(\mathbf{U}_{n-l} \smallsetminus K_{n-l}) < \mathrm{mod}(\mathbf{U}_n \smallsetminus K_n)/2$.

Proof. Given an infinitely renormalizable ψ-ql map f with renormalizations $R^n f : (\mathbf{U}_n, K_n) \to (\mathbf{U}_n \setminus K_n)$, we let $\mu_n(f) = \operatorname{mod}(\mathbf{U}_n, K_n)$. Arguing by contradiction, we find a sequence f_i of infinitely renormalizable ψ-ql maps of decoration class $(\bar{\mathbf{q}}, \bar{\mathbf{n}})$ and sequences $\varepsilon_i \to 0$ and $n(i) \to \infty$ such that:

P1: $\mu_{n(i)}(f_i) < \varepsilon_i$;

P2: $\mu_{n(i)}(f_i) < 2\mu_k(f_i)$, $k = 0, 1, \ldots, n(i) - 1$.

Let $R^n f_i$ be the renormalization of $R^{n-1} f_i$ with period $p_n(f_i)$. Applying then the diagonal process, we can also assume the following property:

P3: $p_{n(i)-s}(f_i) \to \pi_s \in \mathbb{N} \cup \{\infty\}$ for $s = 0, 1, \ldots$.

We let $\bar{s} \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ be the first moment for which $\pi_s = \infty$ (with understanding that $\bar{s} = \infty$ if such a moment does not exist).

Let us consider two cases:

Case 1: $\bar{s} < \infty$. Applying consecutively Corollary 5.14, we conclude that for sufficiently big i,

$$\mu_{n(i)-s}(f_i) \leq C^s \mu_{n(i)}(f_i), \quad s \leq \bar{s}.$$

Let $\rho \in (0, 1/2C^{\bar{s}})$. By Theorem 5.15, for all sufficiently big i,

$$\mu_{n(i)-\bar{s}-1}(f_i) \leq \rho\, \mu_{n(i)-\bar{s}}(f_i).$$

Putting the last two estimates together, we conclude that for all sufficiently big i,

$$\mu_{n(i)-\bar{s}-1}(f_i) < \frac{1}{2}\mu_{n(i)}(f_i),$$

contradicting assumption (P2).

Case 2: $\bar{s} = \infty$. Take an s such that

$$\bar{p} \equiv \pi_0 \pi_1 \ldots \pi_s > \underline{p},$$

where $\underline{p} = \underline{p}(1/2)$ comes from Theorem 5.16. By this theorem, for sufficiently big i,

$$\mu_{n(i)-s-1}(i) < \frac{1}{2}\mu_{n(i)}(i),$$

contradicting again assumption (P2). □

We are ready to prove the Main Theorem, in an important refined version. We say that a family $\mathcal{M}$ of little Mandelbrot copies (and the corresponding renormalization combinatorics) has *beau*[6] *a priori* bounds if there exists an $\varepsilon = \varepsilon(\mathcal{M}) > 0$ and a function $N : \mathbb{R}_+ \to \mathbb{N}$ with the following property. Let $f : U \to V$ be a quadratic-like map with $\operatorname{mod}(V \setminus U) \geq \delta > 0$ that is at least $N = N(\delta)$ times renormalizable. Then for any $n \geq N$, the n-fold renormalization of f can be represented by a quadratic-like map $R^n f : U_n \to V_n$ with $\operatorname{mod}(V_n \setminus U_n) \geq \varepsilon$.

Beau Bounds (Refined Main Theorem). *For any parameters* $(\bar{\mathbf{q}}, \bar{\mathbf{n}})$*, the family of renormalization combinatorics of decoration class* $(\bar{\mathbf{q}}, \bar{\mathbf{n}})$ *has beau a priori bounds.*

[6] According to Dennis Sullivan, "beau" stands for "bounded and eventually universal".

Proof. Let $\varepsilon > 0$ and l come from Corollary 5.17, and $C > 0$ comes from Corollary 5.14. We will use notation $\mu_n(f)$ from the proof of Corollary 5.17. Assume that for some $\delta > 0$, there is a sequence of ψ-ql maps f_i in question with $\mu_0(f_i) \geq \delta$, while $\mu_{n(i)}(f_i) < \varepsilon$, where $n(i) \to \infty$. Let $n(i) = k_i l + r_i$ where $0 \leq r_i < l$. Then by Corollaries 5.17 and 5.14,

$$\mu_0(f_i) \leq C^l \varepsilon / 2^{r_i} \to 0 \quad \text{as } i \to \infty.$$

This contradiction proves the beau bounds for the moduli $\mu_n(f)$ of ψ-ql maps. The beau bounds for ordinary quadratic-like maps follow by Lemma 3.1. □

6. Appendix: Extremal length and width

Given a family curves $\mathcal{G}$ on a Riemann surface S and a conformal metric μ on S, we let $\mu(\gamma)$ be the μ-length of a curve $\gamma \in \Gamma$, $\mu(\Gamma)$ be the infimum of these lengths, area_μ be the corresponding measure, and $\mathcal{L}(\mathcal{G})$ and $\mathcal{W}(\mathcal{G}) = \mathcal{L}(\mathcal{G})^{-1}$ be respectively the *extremal length* and *width* of $\mathcal{G}$: see [A] or the Appendices [KL1, K] for the precise definitions. The most basic properties of these conformal invariants, the *Parallel* and *Series Laws* can also be found in these sources.

6.1. **Transformation rules.** Both extremal length and extremal width are conformal invariants. More generally, we have:

Lemma 6.1. *Let $f : U \to V$ be a holomorphic map between two Riemann surfaces, and let $\mathcal{G}$ be a family of curves on U. Then*

$$\mathcal{L}(f(\mathcal{G})) \geq \mathcal{L}(\mathcal{G}).$$

See Lemma 4.1 of [KL1] for a proof.

Corollary 6.2. *Under the circumstances of the previous lemma, let $\mathcal{H}$ be a family of curves in V satisfying the following lifting property: any curve $\gamma \in \mathcal{H}$ contains an arc that lifts to some curve in $\mathcal{G}$. Then $\mathcal{L}(\mathcal{H}) \geq \mathcal{L}(\mathcal{G})$.*

See Corollary 10.3 of [K] for a proof.

Given a compact subset $K \subset \operatorname{int} U$, the *extremal distance*

$$\mathcal{L}(U, K) \equiv \operatorname{mod}(U, K)$$

(between ∂U and K) is defined as $\mathcal{L}(\mathcal{G})$, where $\mathcal{G}$ is the family of curves connecting ∂U and K. In case when U is a topological disk and K is connected, we obtain the usual modulus $\operatorname{mod}(U \smallsetminus K)$ of the annulus $U \smallsetminus K$. We let $\mathcal{W}(U, K) = \mathcal{L}^{-1}(U, K)$.

Lemma 6.3. *Let $f : U \to V$ be a branched covering between two compact Riemann surfaces with boundary. Let A be an archipelago in U, $B = f(A)$, and assume that $f : A \to B$ is a branched covering of degree d. Then*

$$\operatorname{mod}(V, B) \geq d \operatorname{mod}(U, A).$$

See Lemma 4.3 of [KL1] for a proof.

Lemma 6.4. *Let* (U, A) *and* (V, B) *be as above, and let* $f : U \smallsetminus A \to V \smallsetminus B$ *be a branched covering of degree* N*. Then*

$$\mathrm{mod}(V, B) = N\ \mathrm{mod}(U, A).$$

See [A] for a proof.

6.2. **Strips and quadrilaterals.**

Lemma 6.5. *Let us consider a horizontal strip* $\Pi(h)$ *and an interval* $I = (x, x+a) \subset \mathbb{R}$*. We view* Π *as a quadrilateral with horizontal sides* I *and* $\mathbb{R} + ih$*. Then*

$$\frac{h}{2a} \leq \mathrm{mod}\,\Pi \leq \frac{h}{a},$$

provided $h/a \leq 1/2$ *or* $\mathrm{mod}\,\Pi \leq 1/4$ *(for the left-hand side inequality).*

Proof. By definition, $\mathrm{mod}\,\Pi$ is the extremal length of the family of curves connecting I to $\mathbb{R} + ih$. This family contains the family $\mathcal{G}'$ of vertical curves in the Euclidean rectangle with horizontal sides I and $I + ih$. Hence $\mathcal{L}(\mathcal{G}) \leq \mathcal{L}(\mathcal{G}') = h/a$.

To prove the left-hand side inequality, let us consider a Euclidean rectangle Q with vertices $x - h, x + a + h, x + a + h + ih, x - h + ih$ endowed with the Euclidean metric μ. Any curve of $\mathcal{G}$ has μ-length at least h. Hence

$$\mathcal{L}(\mathcal{G}) \geq \frac{h^2}{\mathrm{area}_\mu(Q)} = \frac{t}{1+2t}, \quad \textit{where } t = h/a.$$

We see that $\mathcal{L}(\mathcal{G}) \geq t/2$ for $t \leq 1/2$, while $\mathcal{L}(\mathcal{G}) > 1/4$ otherwise. The conclusion follows □

Lemma 6.6. *Let* Π *and* I *be as in the previous lemma. Let* $C = \Pi/\,l\,\mathbb{Z}$ *be a cylinder covered by* Π *so that* I *is embedded into the bottom of* C*. Then*

$$\mathrm{mod}\,\Pi \geq \frac{1}{2} \min(\mathrm{mod}\,C,\ 0.5).$$

Proof. Since the covering $\Pi \to C$ is an embedding on I, we have: $a \leq l$. Then by the previous lemma we obtain:

$$\mathrm{mod}\,\Pi \geq \frac{h}{2a} \geq \frac{h}{2l} = \frac{1}{2}\,\mathrm{mod}\,C.$$

□

6.3. **Holomorphic and embedded annuli.** Let S be a hyperbolic Riemann surface with boundary with a preferred component σ of ∂S. We assume that S has finite topological type and is not the punctured disk. A *holomorphic annulus* in $\mathcal{A}$ is a holomorphic map $A : \mathbb{A}(1, r) \to S$ that extends to a homeomorphism $\phi : \mathbb{T} \to \sigma$. We let $\mathrm{mod}\,A = \mathrm{mod}\,\mathbb{A}(1, r)$.

The family of holomorphic annuli contains a subfamily of *embedded annuli.* Among embedded annuli, there is an annulus A_* of maximal modulus, which has nice special properties. Namely, let us uniformize A_* by a flat cylinder $C = \Pi(h)/\mathbb{Z}$. Then the quadratic differential dz^2 on C is the pull-back of some quadratic differential q on S. Moreover, the uniformization $C \to A_*$ extends continuously to the upper boundary $C^+ = \mathbb{R} + ih/\mathbb{Z}$ of C (minus finitely many points corresponding to the punctures of S), and induces there an equivalence relation $\tau_k : \alpha_k \to \alpha'_k$, where (τ_k) is a finite family of isometries between pairs of disjoint arcs in C^+. The images of these arcs, $\lambda_k = i(\alpha_k) = i(\alpha'_k)$, are *horizontal separatricies* of q. (It is a version of Strebel's Theorem, see e.g., [GL, §11]).

Lemma 6.7. *For any holomorphic annulus $A : \mathbb{A}(1,r) \to S$, we have:*

$$\operatorname{mod} A \le 16 \operatorname{mod} A_*.$$

Proof. Let us consider a family $\mathcal{G}$ of non-trivial proper curves γ in S that begin in σ.[7] Then any curve $\gamma \in \mathcal{G}$ contains an initial segment γ' that lifts to a vertical curve in $\mathbb{A}(1,r)$. By Corollary 6.2,

$$\operatorname{mod} A \le \mathcal{L}(\mathcal{G}). \tag{6.1}$$

Let us now take any conformal metric μ on S, and let $l = \mu(\mathcal{G})$. For any vertical curve δ in A_*, two possibilities can occur:

- δ ends on ∂S. Then $\delta \in \mathcal{G}$ and hence $\mu(\delta) \ge l$.
- δ ends on some separatrix $i(\alpha_k)$. Then there is another vertical curve λ in A_* that ends at the same point as δ. The concatenation of δ and λ is a curve of family $\mathcal{G}$. Hence one of the curves, δ or λ, is "long", i.e., it has μ-length at least $l/2$.

It follows that at least one half of the vertical curves in A_* are long. Let $I \subset C^+$ be the set of endpoints of i^{-1}(long curves). We can now proceed as in the classical Grötztsch estimate. By the Cauchy-Schwarz Inequality,

$$h \operatorname{area}_\mu(C) = \operatorname{area}(C) \int_C \operatorname{area}_\mu \, dx\, dy \ge \left(\int_I dx \int_0^h \mu(x,y) dy \right)^2 \ge \left(\frac{l}{4} \right)^2,$$

which implies

$$\mathcal{L}_\mu(\mathcal{G}) = l^2 / \operatorname{area}_\mu(C) \le 16h = 16 \operatorname{mod} A_*.$$

Since this is valid for any conformal metric μ, we conclude that $\mathcal{L}(\mathcal{G}) \le 16 \operatorname{mod} A_*$. Together with (6.1), this gives us the desired estimate. ☐

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