EXTREMAL MAPS OF THE UNIVERSAL HYPERBOLIC SOLENOID

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Abstract. We show that the set of points in the Teichmüller space of the universal hyperbolic solenoid which do not have a Teichmüller extremal representative is generic (that is, its complement is the set of the first kind in the sense of Baire). This is in sharp contrast with the Teichmüller space of a Riemann surface where at least an open, dense subset has Teichmüller extremal representatives. In addition, we provide a sufficient criteria for the existence of Teichmüller extremal representatives in the given homotopy class. These results indicate that there is an interesting theory of extremal (and uniquely extremal) quasiconformal mappings on hyperbolic solenoids.

1. Introduction

The Teichmüller space $T(S)$ of a Riemann surface S consists of all marked complex structures on S. A *marked complex structure* on S is a homotopy class of quasiconformal maps from S to an arbitrary Riemann surface up to post composition by a conformal map. The homotopy class $[id]$ of the identity map $id : S \to S$ is the *basepoint* of $T(S)$. The distance between the basepoint and the homotopy class $[f]$ of a quasiconformal map $f : S \to S_1$ is the infimum of the logarithms of quasiconformal constants over all quasiconformal maps in the marked complex structure(≡homotopy class $[f]$) of f. A quasiconformal map $f_1 : S \to S_1$ in the homotopy class $[f]$ is called *extremal* if its quasiconformal constant $K(f_1)$ is equal to the infimum of the quasiconformal constants over all maps in $[f]$. In this case, the (Teichmüller) distance between $[id]$ and $[f]$ is simply:

$$dist([id],[f]) = \log K(f_1).$$

If μ is a Beltrami coefficient of an extremal map $f : S \to S_1$, then each quasiconformal map $f^{t\mu}$ with the Beltrami coefficient $t\mu$, $\frac{-1}{\|\mu\|_\infty} < t < \frac{1}{\|\mu\|_\infty}$, is extremal as well. The path $t \mapsto [f^{t\frac{|\varphi|}{\varphi}}] \in T(S)$ is a geodesic for the above distance.

Teichmüller's fundamental result [18] states that each marked complex structure of an analytically finite (closed with at most finitely many points deleted) Riemann surface S contains a unique extremal map with Beltrami coefficient $k\frac{|\varphi|}{\varphi}$, where φ is a holomorphic quadratic differential on S and $0 < k < 1$. The natural parameter for φ partitions S into Euclidean rectangles and the extremal map is an affine stretching on each rectangle. Teichmüller's theorem is a highly non-trivial generalization of a result of Grötzsch concerning a single rectangle. We say that such extremal maps, their Beltrami coefficients and their corresponding geodesics $t \mapsto [f^{t\frac{|\varphi|}{\varphi}}]$ are of *Teichmüller type*.

The Teichmüller theorem completely answers all questions about extremal and uniquely extremal quasiconformal mappings for analytically finite Riemann surfaces. However, for Riemann surfaces that are not analytically finite there exists a rich theory about extremal mappings. A modern approach to extremal maps for arbitrary Riemann surfaces started with Reich and Strebel. They showed that Teichmüller maps are extremal on arbitrary Riemann surfaces by generalizing the original approach of Grötzsch to Riemann surfaces. Using, what is today called Reich-Strebel inequality and results of Hamilton [5] and Krushkal [6], they characterized extremal quasiconformal maps [12]. It is worth noting that every marked complex structure of a Riemann surface contains an extremal map by the pre-compactness of a family of normalized K-quasiconformal maps. Strebel [15] showed that not every extremal map is of Teichmüller type and that

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there could be more than one extremal map in a given homotopy class on analytically infinite Riemann surfaces. The corresponding characterization for uniquely extremal maps has been obtained by Bozin, Lakic, Markovic and Mateljevic [1] (see [8], [13], [2], [20] for some applications of these results).

Sullivan [17] introduced the universal hyperbolic solenoid $\mathcal{H}$ as the inverse limit of the system of unbranched finite degree covers of a compact surface. The universal hyperbolic solenoid $\mathcal{H}$ is a compact space which is locally homeomorphic to a 2-disk times a Cantor set. Sullivan [17] introduced a complex structure on $\mathcal{H}$ and showed that the Teichmüller space $T(\mathcal{H})$ of the solenoid is a separable complex Banach manifold. Nag and Sullivan [9] observed that $T(\mathcal{H})$ embeds in the universal Teichmüller space as a closure of the union of Teichmüller spaces of all compact Riemann surfaces. In the remark at the end of Section 2, we use the embedding to give an alternative definition of $T(\mathcal{H})$ as a subset of the universal Teichmüller space and the metric on it is given in terms of this subset (this description uses terms more familiar with the standard Teichmüller theory). The questions that we consider can be directly restated in these terms.) The Teichmüller distance on $T(\mathcal{H})$ is defined similar to above and we consider the questions about the existence and the structure of extremal quasiconformal maps. Unlike for Riemann surfaces, the existence of extremal maps on the solenoid is not guaranteed, and in fact it is an interesting open problem whether they always exist. In [14], it is showed that Teichmüller type maps are uniquely extremal. The question was raised whether each marked complex structure contains a Teichmüller map. If the answer were yes, this would show both the existence and the uniqueness of the extremal maps.

There are reasons why one would expect to have a positive answer. The solenoid $\mathcal{H}$ is a compact space and $T(\mathcal{H})$ is the closure of the union of Teichmüller spaces of all compact Riemann surfaces. (Recall that by Teichmüller's theorem each marked complex structure on a compact surface contains a Teichmüller map.) On the other hand, Lakic [7] showed that the set of points in the Teichmüller space of an analytically infinite Riemann surface which contain a Teichmüller extremal map is open and dense. The set of points which do not have a Teichmüller representative is nowhere dense and closed, in particular of first kind in the sense of Baire. By analogy, one would expect the subset of $T(\mathcal{H})$ without Teichmüller extremal maps is at most of first kind, if not empty. We obtain a somewhat unexpected result:

Theorem 1. *The set of points in the Teichmüller space $T(\mathcal{H})$ of the universal hyperbolic solenoid $\mathcal{H}$ which do not have a Teichmüller extremal representative is generic in $T(\mathcal{H})$. That is, the set of points that do have a Teichmüller representative is of the first kind in the sense of Baire with respect to the Teichmüller metric.*

By the above theorem, a large set of points in $T(\mathcal{H})$ do not have a Teichmüller extremal representative. It is interesting to determine when a given marked complex structure has a Teichmüller extremal representative. A sufficient condition for the case of infinite Riemann surfaces is given by Strebel's Frame Mapping Condition [16]. This condition depends on the existence of ends of the Riemann surface. Since the universal hyperbolic solenoid $\mathcal{H}$ is a compact space, it has no ends. However, we obtain a sufficient condition, where a given complex structure has a Teichmüller representative if it is well approximated with locally constant (rational) complex structures (see below for the definition of locally constant complex structures).

A complex structure on an arbitrary compact Riemann surface lifts to a special complex structure on the solenoid $\mathcal{H}$, namely a transversely locally constant complex structure. Each transversely locally constant marked complex structure on $\mathcal{H}$ has a Teichmüller extremal representative which comes by lifting the extremal Teichmüller representative from the surface. Therefore, it is interesting to consider only non transversely locally constant complex structures. In Section 3 we define a notion of being *well-approximated by transversely locally constant complex structures*. (Recall that the transversely locally constant complex structures are dense in $T(\mathcal{H})$ by [17]. Therefore each non transversely locally constant complex structure is approximated by transversely locally constant complex structures.) We show:

Theorem 2. *If a non locally transversely constant marked complex structure is well-approximated by transversely locally constant complex structures then it contains a Teichmüller extremal representative.*

The immediate consequence of Theorem 1 and Theorem 2 is

Corollary 1. *The set of points in $T(\mathcal{H})$ which are not well-approximated by transversely locally constant marked complex structures is generic in $T(\mathcal{H})$.*

This is analogous with the fact that the set of real numbers which are not well approximated by rational numbers is of full measure. We remark that the existence of extremal maps in arbitrary marked complex structure is open. Further, the existence of a geodesic connecting the basepoint with any marked complex structure is implied by the existence of an extremal map but it is not necessarily equivalent to it. This is illustrated by an example of L. Zhong [19] for Riemann surfaces.

2. Preliminaries

We define the *universal hyperbolic solenoid* $\mathcal{H}$ introduced by Sullivan [17], see also [10] and [14]. Let (S,x) be a fixed compact surface of genus at least 2 with a basepoint. Consider all finite sheeted unbranched coverings $\pi_i:(S_i,x_i)\to(S,x)$ such that $\pi_i(x_i)=x$. There is a natural *partial ordering* $\geq$ given by $(S_j,x_j)\geq(S_i,x_i)$ whenever there exists a finite unbranched covering $\pi_{i,j}:(S_j,x_j)\to(S_i,x_i)$ such that $\pi_{i,j}(x_j)=x_i$ and $\pi_j=\pi_i\circ\pi_{i,j}$. Given any two finite unbranched covers (S_i,x_i) and (S_j,x_j) of (S,x), there exists a third finite unbranched cover (S_k,x_k) which covers both such that $\pi_i\circ\pi_{i,k}=\pi_j\circ\pi_{j,k}=\pi_k$, $\pi_{i,k}(x_k)=x_i$ and $\pi_{j,k}(x_k)=x_j$. In other words, $(S_k,x_k)\geq(S_i,x_i)$ and $(S_k,x_k)\geq(S_j,x_j)$. Thus the system of covers is inverse directed and the inverse limit is well defined. The universal hyperbolic solenoid is by definition

$$\mathcal{H}=\lim_{\leftarrow}(S_i,x_i).$$

We give an alternative definition for the universal hyperbolic solenoid $\mathcal{H}$. Denote by Δ the unit disk. Let G be a Fuchsian group such that Δ/G is a compact Riemann surface of genus at least two. Let G_n be the intersection of all subgroups of G of index at most n. Then G_n is a finite index characteristic subgroup of G. The *profinite metric* on G is defined by

$$d(\alpha,\beta)=\max_{\alpha\beta^{-1}\in G_n}\frac{1}{n}$$

for all $\alpha,\beta\in G$. The profinite completion $\hat{G}$ of G is a compact topological group homeomorphic to a Cantor set. The action of G on the product $\Delta\times\hat{G}$ is defined by $\gamma(z,t):=(\gamma(z),t\gamma^{-1})$ for all $\gamma\in G$ and $(z,t)\in\Delta\times\hat{G}$. The universal hyperbolic solenoid is $\mathcal{H}:=(\Delta\times\hat{G})/G$. For more details on this definition see [10].

The solenoid $\mathcal{H}$ is a compact space locally homeomorphic to a 2-disk times a Cantor set. Path components of $\mathcal{H}$ are called *leaves*. Each leaf is dense in $\mathcal{H}$ and homeomorphic to the unit disk. The profinite group completion $\hat{G}$ supports a unique left and right translation invariant measure m of full support (the Haar measure). The Haar measure m induces a holonomy invariant measure on the solenoid $\mathcal{H}=(\Delta\times\hat{G})/G$. This measure allows for the integration of quantities which induce local measures on leaves of $\mathcal{H}$, e.g. the absolute value of a quadratic differential.

The complex structure on $\mathcal{H}$ is given by an assignment of holomorphic charts on leaves(making the leaves holomorphic to the unit disk) which vary continuously for the transverse variation in local charts [17]. From the results of Candel [3], it follows that $\mathcal{H}$ supports hyperbolic metric for each conformal class (induced by a complex structure). Note that by fixing a Fuchsian group G the solenoid $\mathcal{H}=(\Delta\times\hat{G})/G$ has already induced complex structure and hyperbolic metric from the unit disk Δ (see [10]). The induced complex structure is locally constant in the transverse direction and all locally transversely constant complex structures on $\mathcal{H}$ arise in this way [17], [9].

The Teichmüller space $T(\mathcal{H})$ consists of all differentiable quasiconformal maps (which are continuous for the transverse variation in local charts) from fixed complex solenoid $\mathcal{H}=(\Delta\times\hat{G})/G$ onto an arbitrary complex solenoid modulo homotopy and post-composition by conformal maps. The requirement of the continuity for the transverse variation in local charts can be achieved by requiring the differentiable maps to vary continuously in the C^∞-topology on C^∞ maps (for more details see [14]). Equivalently, the Teichmüller space $T(\mathcal{H})$ is the space of all smooth Beltrami coefficients which are continuous for the transverse variations in the C^∞-topology modulo the above condition. Thus a point in $T(\mathcal{H})$ is an equivalence class $[\mu]$ of a smooth Beltrami coefficient μ on $\mathcal{H}$. The main point is that the leafwise Beltrami equations give a transversely continuous solution (see [14]). It is also possible to weaken the condition on differentiability of the Beltrami coefficients as long as they are leafwise equivalent(as elements of the universal Teichmüller space) to the restriction on leaves of a smooth Beltrami coefficient on $\mathcal{H}$ (for more details see [14]). A Beltrami coefficient μ on $\mathcal{H}$ is *extremal* if $\|\mu\|_\infty = \inf_{\nu\in[\mu]} \|\nu\|_\infty$. A Beltrami coefficient μ is of *Teichmüller* type if $\mu = k\frac{|\varphi|}{\varphi}$ for a holomorphic quadratic differential $0 \neq \varphi$ and $0 < k < 1$. They are corresponding to the similar notions for quasiconformal maps given in Introduction.

The space $A(\mathcal{H})$ consists of all holomorphic quadratic differentials on $\mathcal{H}$ which are continuous for the transverse variation in local charts. The Bers norm of $\varphi \in A(\mathcal{H})$ is given by $\|\varphi\|_{Bers} := \|\varphi\rho^{-2}\|_\infty$, where ρ is the hyperbolic length element on the leaves of $\mathcal{H}$. The space $A(\mathcal{H})$ is a complex Banach space for the Bers norm and the closure of $A(\mathcal{H})$ for the L^1-norm given by $\|\varphi\|_{L^1} = \int_{\mathcal{H}} |\varphi| dm$ is the space of all integrable quadratic differentials which are holomorphic(and defined) on almost all leaves of $\mathcal{H}$ without transverse continuity requirement(see [14]).

The space $N(\mathcal{H})$ of *infinitesimally trivial Beltrami differentials* consists of all smooth Beltrami differentials μ such that $\int_{\mathcal{H}} \mu\varphi dm = 0$, for all $\varphi \in A(\mathcal{H})$. In fact, a smooth Beltrami differential μ is infinitesimally trivial if and only if there exists a path of smooth Beltrami coefficients $t \mapsto \nu_t$ such that $\nu_t = t\mu + o(t)$ and ν_t is a trivial deformation of $\mathcal{H}$, i.e. the quasiconformal map f^{ν_t} is homotopic to the identity (see [14]).

The tangent space to the Teichmüller space $T(\mathcal{H})$ at the base point is given by the space $L^\infty(\mathcal{H})$ of smooth Beltrami differentials on $\mathcal{H} = (\Delta \times \hat{G})/G$ modulo the space $N(\mathcal{H})$ of infinitesimally trivial Beltrami differentials(for details see [14]). (Our terminology assumes that any Beltrami coefficient μ satisfies $\|\mu\|_\infty < 1$, while any Beltrami differential μ satisfies $\|\mu\|_\infty < \infty$.) A Beltrami differential μ on $\mathcal{H}$ is *infinitesimally extremal* if $\|\mu\|_\infty = \inf_\nu \|\nu\|_\infty$, where the infimum is over all ν such that $\mu - \nu \in N(\mathcal{H})$. The tangent space $L^\infty(\mathcal{H})/N(\mathcal{H})$ is Banach in the quotient topology even though $L^\infty(\mathcal{H})$ and $N(\mathcal{H})$ are not complete (see [14]). There is a natural pairing between $L^\infty(\mathcal{H})$ and $A(\mathcal{H})$ given by

$$(\mu, \varphi) \mapsto \int_{\mathcal{H}} \mu\varphi dm.$$

The pairing descends to the pairing of $L^\infty(\mathcal{H})/N(\mathcal{H})$ and $A(\mathcal{H})$. The tangent space $L^\infty(\mathcal{H})/N(\mathcal{H})$ embeds in the dual $A(\mathcal{H})^*$ but it is strictly smaller [14].

Remark. We give an alternative description of the Teichmüller space $T(\mathcal{H})$. Fix a Fuchsian group G such that Δ/G is a compact Riemann surface of genus at least two. A quasiconformal map $f : \Delta \to \Delta$ is said to be *almost invariant* with respect to G if $\|Belt(f \circ \gamma \circ f^{-1})\|_\infty \to 0$ as $d(\gamma, id) \to 0$. In other words, the Beltrami coefficient of f is very close to be invariant under the push forward by elements of a finite index subgroup G_n of G of some large index (i.e. n is large). In particular, a lift of a quasiconformal map from the Riemann surface Δ/G_n for any finite index subgroup G_n of G is almost invariant for G. The Teichmüller space $T(\mathcal{H})$ is isomorphic to a subset of the universal Teichmüller space $T(\Delta)$ consisting of all classes with almost invariant (for G) representatives. The distance $dist([id], [f])$ is given by the infimum of the logarithm of the quasiconformal constants of all almost invariant maps homotopic to f. The question about extremal representatives can be considered in this setting as well. However, it appears that working directly on the solenoid $\mathcal{H}$ is somewhat better suited for our purposes due to the strong technical tools developed using the Reich-Strebel inequality for the solenoid [14]. In fact, one would presumably be able to replace integration in the transverse direction by the limit of the average of integrals over fundamental regions for G_n as $n \to \infty$.

3. A sufficient condition for Teichmüller maps

A Teichmüller Beltrami coefficient $\mu = k\frac{|\varphi|}{\varphi}$, for some $0 < k < 1$ and for some holomorphic quadratic differential $\varphi \neq 0$ on the solenoid $\mathcal{H}$, is uniquely extremal in its Teichmüller class and it determines a geodesic $t \mapsto [t\frac{|\varphi|}{\varphi}]$, $t \in (-1/k, 1/k)$. Moreover, this is unique geodesic connecting the basepoint $[0]$ with $[\mu = k\frac{|\varphi|}{\varphi}]$ (see [14]).

If $\mu = k\frac{|\varphi|}{\varphi}$, $k \in \mathbb{R}^+$, is a Beltrami differential, then the linear functional $\Lambda_\mu : \psi \mapsto \int_{\mathcal{H}} \mu\psi dm$, for $\psi \in A(\mathcal{H})$, achieves its norm on the vector $\frac{\varphi}{\|\varphi\|_{L^1}}$. In that case, $\|\mu\|_\infty = k$ is equal to the norm of the functional Λ_μ and any other ν in the infinitesimal class of μ(i.e. any ν such that $\mu - \nu \in N(\mathcal{H})$) satisfies $\|\nu\|_\infty > \|\mu\|_\infty$. In other words, μ is uniquely infinitesimally extremal.

It is not, a priori, clear whether each Teichmüller (or infinitesimal) class contains a Teichmüller type Beltrami coefficient. If this is the case, this would certainly be a nice situation similar to Teichmüller spaces of compact surfaces. On the other hand, on infinite Riemann surfaces there exist Teichmüller (and infinitesimal) classes of Beltrami coefficients (and differentials) which do not contain a Teichmüller type Beltrami coefficient (and differential). Strebel [15] gave a very useful sufficient condition (called the Frame Mapping Condition) to determine when a given class contains a Teichmüller type representative.

We find a sufficient condition for a given Beltrami coefficient μ on the solenoid $\mathcal{H}$ to be equivalent to a Teichmüller type Beltrami coefficient in both infinitesimal and Teichmüller classes. We point out that the Strebel's Frame Mapping Condition depends on the non-compactness of the given Riemann surface, whereas the solenoid is a compact space. Therefore we need a different approach. If μ is a transversely locally constant Beltrami coefficient then it is a lift of a Beltrami coefficient on a Riemann surface S_i covering S for the base complex structure on $\mathcal{H}(\equiv \Delta \times \hat{G}_i/G_i$, where $S \equiv \Delta/G_i$, G_i a Fuchsian group). Since on S_i any Beltrami coefficient is equivalent to a Teichmüller type Beltrami coefficient, by lifting the corresponding holomorphic quadratic differential on S_i to $\mathcal{H}$, we obtain a Teichmüller coefficient equivalent to μ (either infinitesimally or in Teichmüller sense). Therefore we restrict our attention to non transversely locally constant Beltrami coefficients on $\mathcal{H}$ and look for a sufficient condition.

Let $S_n \equiv \Delta/G_n$ ($S_1 = S$) be a sequence of finite sheeted coverings of S such that $\cap_{n=1}^{\infty} G_n = \{id\}$ (we assume that $G_{n+1} < G_n$). One can think about S_n as an approximating sequence for $\mathcal{H}$.

Let μ be a non-trivial (in the Teichmüller sense) Beltrami coefficient on Δ/G_n such that $[\mu] = [k\frac{|\varphi|}{\varphi}]$, for $0 < k < 1$ and φ normalized such that $\|\varphi\|_{Bers} = 1$. Let $B_n : T(\Delta/G_n) \to A(\Delta/G_n)$ be the map given by $B_n([\mu]) = k\varphi$, where μ, k and φ are as above. The map B_n is continuous for the Teichmüller metric on $T(\Delta/G_n)$ and the Bers norm on the unit ball in $A(\Delta/G_n)$ (see [4]).

Definition 3.1. Let μ be a Beltrami coefficient on $\mathcal{H}$ not equivalent to a transversely locally constant Beltrami coefficient. Let S_n be an increasing sequence of finite coverings of S, φ_n a sequence of holomorphic quadratic differentials on S_n and $\tilde{\varphi}_n$ their lifts to $\mathcal{H}$. Given a sequence $0 < k_n < 1$, define a sequence of Beltrami coefficients $\mu_n = k_n\frac{|\varphi_n|}{\varphi_n}$ on S_n and their lifts $\tilde{\mu}_n = k_n\frac{|\tilde{\varphi}_n|}{\tilde{\varphi}_n}$ on $\mathcal{H}$. Assume that $[\tilde{\mu}_n] \to [\tilde{\mu}]$ for some $0 < k_n < 1$. The Teichmüller class $[\mu]$ of the Beltrami coefficient μ is *well-approximated by transversely locally constant Beltrami coefficients* if there exists a sequence μ_n as above such that

$$\sum_{n=1}^{\infty} \|B_n([\mu_n]) - B_{n+1}([\mu_{n+1}])\|_{Bers} < \infty.$$

Proof of Theorem 2. Consider the lifts $\tilde{\varphi}_n$ on $\mathcal{H}$ of holomorphic quadratic differentials φ_n on S_n. Then $\sum_{n=1}^{\infty} \|\tilde{\varphi}_n - \tilde{\varphi}_{n+1}\|_{Bers} < \infty$ by the assumption, which implies that $\tilde{\varphi}_n$ converges uniformly to a holomorphic quadratic differential ψ on $\mathcal{H}$. Note that ψ is not a lift of a holomorphic quadratic differential on S_n, for any n. Since $\psi = \tilde{\varphi}_k + \sum_{n=k}^{\infty}(\tilde{\varphi}_{k+1} - \tilde{\varphi}_k)$ and $\|\tilde{\varphi}_n\|_{Bers} = 1$, we conclude that $\psi \neq 0$. By the uniform convergence $\tilde{\varphi}_n \to \psi$, we get that μ is Teichmüller equivalent to $k\frac{|\psi|}{\psi}$, for k depending on the distance from $[0]$ to $[\mu]$. □

A similar statement can be made for the infinitesimal case. Let μ be a Beltrami differential on $\mathcal{H}$ representing a tangent vector $[\mu]$ which does not come from lifting a tangent vector of the Teichmüller space of a compact surface, i.e. the coset $\mu + N(\mathcal{H})$ does not contain a transversely locally constant Beltrami differential. Similar to Teichmüller classes, it is also true that transversely locally constant Beltrami differentials approximate each Beltrami differential on $\mathcal{H}$. (Recall that each tangent vector is a continuous linear functional on the space of holomorphic quadratic differentials $A(\mathcal{H})$ and the approximation is with respect to the dual norm.)

Let S_n be a sequence of compact Riemann surfaces "approximating" $\mathcal{H}$ as above. Define $B'_n : T(S_n) \to A(S_n)$ by $B'_n([\mu_n]) = k\varphi_n$, where $\mu_n - k\frac{|\varphi_n|}{\varphi_n} \in N(S_n)$ and $\|\varphi_n\|_{Bers} = 1$. We say that the infinitesimal class of a non transversely locally constant Beltrami differential μ on $\mathcal{H}$ is *well-approximated with transversely locally constant Beltrami differentials* if there exists a sequence of Beltrami differentials μ_n on S_n whose lifts $\tilde{\mu}_n$ on $\mathcal{H}$ approximate μ in the sense of the linear functionals on $A(\mathcal{H})$ such that $\sum_{n=1}^{\infty} \|B'_n([\mu_n]) - B'_{n+1}([\mu_{n+1}])\|_{Bers} < \infty$. We obtain an analogous statement to Theorem 2(and similar proof) for the infinitesimal class.

Theorem 2'. *If a non transversely locally constant infinitesimal class of a Beltrami differential on the universal hyperbolic solenoid is well-approximated by transversely locally constant infinitesimal classes of Beltrami differentials then it is infinitesimally equivalent to a Teichmüller Beltrami differential.*

4. Teichmüller classes without Teichmüller representatives

In this section we consider the question of existence of Teichmüller representatives for arbitrary Teichmüller classes in $T(\mathcal{H})$. We show that Teichmüller representative does not always exist. It is true that there exists a dense subsets of points in $T(\mathcal{H})$ which have Teichmüller representative by the density of transversely locally constant structures on $\mathcal{H}$ (which come from complex structures on finite sheeted covers of S).

Lakic [7] showed that even though not all Teichmüller classes of Beltrami coefficients on infinite Riemann surfaces have Teichmüller representative, the one that do form an open, dense subset of the corresponding Teichmüller space. Therefore, for infinite Riemann surfaces this set is quite large and for finite Riemann surfaces it equals the whole Teichmüller space.

We show that, quite unexpectedly, for $T(\mathcal{H})$ the set of points which do not have Teichmüller Beltrami coefficient representative is generic. This means that the set of elements in $T(\mathcal{H})$ that do have a Teichmüller Beltrami representative is contained in a countable union of closed nowhere dense subsets of $T(\mathcal{H})$.

Let $Arg : \mathbb{C} - \{0\} \to (-\pi, \pi]$ be the standard argument function defined on non-zero complex numbers. Let $\Delta_r = \{z \in \mathbb{C};\ |z| < r\}$. Given a holomorphic function $f : \Delta_1 \to \mathbb{C}$ and $0 < r < 1$, denote by $\|f|_{\Delta_r}\|_{Bers}$ the supremum of $|f(z)|\rho^{-2}(z)$ over Δ_r, where ρ is the hyperbolic length density on the unit disk $\Delta = \Delta_1$. Let $\|f\|_{Bers}$ denote the Bers norm, namely the supremum of $|f(z)|\rho^{-2}(z)$ over the unit disk Δ_1 if it exists. Given a measurable set $S \subset \Delta_1$, denote by $|S|$ its Euclidean area. In what follows, we use the following lemma.

Lemma 4.1. *Let $\epsilon > 0$, $0 < r < 1$, $N > M > 0$ and let φ, ψ be two holomorphic functions on the unit disk Δ_1 such that $\|\varphi\|_{Bers}, \|\psi\|_{Bers} \leq N$, $\|\varphi|_{\Delta_r}\|_{Bers} \geq M$. Assume that there exists a measurable set $S \subset \Delta_1$ with $|S| = p > 0$ such that $\|Arg(\frac{\psi}{\varphi})|_S\|_\infty \leq \epsilon$. Then there exists $k > 0$ and $\delta(\epsilon, p, r, M, N) > 0$ such that*

$$\|(\psi - k\varphi)|_{\Delta_r}\|_{Bers} < \delta(\epsilon, p, r, M, N),$$

where $\delta(\epsilon, p, r, M, N) \to 0$ as $\epsilon \to 0$ for fixed p, r, M, N.

Proof. Assume that the lemma is not true for some $0 < r < 1$, $p > 0$ and $N > M > 0$. Then there exists $\delta > 0$, there exist two sequences φ_n, ψ_n of holomorphic functions which satisfy $\|\varphi_n\|_{Bers}, \|\psi_n\|_{Bers} \leq N$, $\|\varphi_n|_{\Delta_r}\|_{Bers} \geq M$ and, there exists a sequence of measurable set $S_n \subset \Delta_1$, $|S_n| = p$, such that $\|Arg(\frac{\psi_n}{\varphi_n})|_{S_n}\|_\infty \leq \frac{1}{n}$ and for each $k > 0$ there exists $z = z(k) \in \Delta_r$ with

$$|\psi_n(z) - k\varphi_n(z)| \geq \delta > 0. \tag{1}$$

We find a contradiction with the above statement.

Since $\|\varphi_n\|_{Bers}, \|\psi_n\|_{Bers} \leq N$, there exist convergent subsequences $\varphi_{n_k} \to \varphi$, $\psi_{n_k} \to \psi$ with $\|\varphi\|_{Bers}, \|\psi\|_{Bers} < \infty$. The convergence is uniform on compact subsets of Δ_1. For simplicity of notation write φ_n,ψ_n in place of $\varphi_{n_k}, \psi_{n_k}$. There exists r_1, $r < r_1 < 1$, such that $|\Delta_{r_1} \cap S_n| \geq \frac{p}{2}$ for all n. Also $\varphi_n \to \varphi$ and $\psi_n \to \psi$ uniformly on Δ_{r_1}, namely $\|(\varphi_n - \varphi)|_{\Delta_{r_1}}\|_{Bers}, \|(\psi_n - \psi)|_{\Delta_{r_1}}\|_{Bers} \to 0$ as $n \to \infty$.

Since $\|\varphi_n|_{\Delta_r}\|_{Bers} \geq M$ for each n, we have $\|\varphi|_{\Delta_r}\|_{Bers} \geq M$ and in particular φ is not identically equal to zero. If $\psi \equiv 0$, then the above inequality (1) fails by taking $k > 0$ small enough and n large enough.

Assume that ψ is not a zero function. The number of zeros of φ and ψ in Δ_{r_1} is finite. Let R be the union of disk neighborhoods of the zeros small enough such that $|S_n \cap (\Delta_{r_1} - R)| \geq \frac{p}{3}$. Given $q > 0$, define

$$D_q = \{z \in \Delta_{r_1} - R;\ |Arg(\frac{\psi}{\varphi})(z)| \leq q\}$$

and

$$D_q^n = \{z \in \Delta_{r_1} - R;\ |Arg(\frac{\psi_n}{\varphi_n})(z)| \leq q\},$$

and let $D_0 = \cap_{k=1}^{\infty} D_{\frac{1}{k}}$ and $D_0^n = \cap_{k=1}^{\infty} D_{\frac{1}{k}}^n$.

There exists n_0 such that $\|(\varphi_n - \varphi)|_{\Delta_{r_1}}\|_{Bers} \leq q$ and $\|(\psi_n - \psi)|_{\Delta_{r_1}}\|_{Bers} \leq q$ for all $n > n_0$. Then there exists a universal constant $c > 0$ such that $D_{cq} \supset D_q^n$ for $n > n_0$. In addition, $D_q^n \supset S_n \cap (\Delta_{r_1} - R)$ for $n > n_0$ whenever $\frac{1}{n_0} < q$. Therefore $|D_{cq}| \geq \frac{p}{3}$, for each $q > 0$. By the monotonicity of a positive measure, we obtain $|D_0| = \lim_{n\to\infty} |D_{\frac{1}{n}}| \geq \frac{p}{3}$.

We have $Arg(\frac{\psi}{\varphi})(z) = 0$ for all $z \in D_0$. Since $|D_0| > 0$, we obtain that $\varphi(z) = k\psi(z)$ for a fixed $k > 0$ and for all $z \in \Delta_1$. But then $\|(\varphi_n - k\psi_n)|_{\Delta_{r_1}}\|_{Bers} \to 0$ as $n \to \infty$, which again gives a contradiction with (1). □

Let A_1 be the unit sphere in L^1-norm of the space $A(\mathcal{H})$ of transversely continuous holomorphic quadratic differentials on the solenoid $\mathcal{H}$. For a given natural number N, define

$$A_1(N) = \{\varphi \in A_1;\ \|\varphi\|_{Bers} \leq N\}.$$

Let $X \subset T(\mathcal{H})$ be the set of points that do have a Teichmüller representative. For $[\mu] \in X$ let $k\frac{|\varphi|}{\varphi}$, $0 < k < 1$, $\varphi \in A_1$, be that representative. Define the map $\pi : X \to A_1$, by

$$\pi([\mu]) = \varphi,$$

for $[\mu] \neq [0]$ and

$$\pi([0]) = 0.$$

We need the following proposition.

Proposition 4.2. *Let $[\mu] \in X$, where μ is a Beltrami coefficient on $\mathcal{H}$. If $[\mu]$ is an element to the closure of the set $\pi^{-1}(A_1(N)) \cup \{[0]\}$ then $[\mu] \in \pi^{-1}(A_1(N)) \cup \{[0]\}$.*

Proof. Let $[\mu_n] \in \pi^{-1}(A_1(N))$ such that $[\mu_n] \to [\mu]$ in the Teichmüller metric. We need to show that $[\mu] \in \pi^{-1}(A_1(N)) \cup \{[0]\}$. If $[\mu] = [0]$ then we are done. Therefore, we assume that $[\mu] \neq [0]$. Without loss of generality, we assume that $\mu_n = k_n \frac{|\varphi_n|}{\varphi_n}$ and $\mu = k\frac{|\varphi|}{\varphi}$ for $\varphi_n, \varphi \in A_1$. Then $k_n \to k$ as $n \to \infty$ by our assumption. We have that $\varphi_n \in A_1(N)$ and we need to show that $\varphi \in A_1(N)$.

There exist $\nu_n \in [\mu_n]$ such that $\|Belt((f^{\nu_n})^{-1} \circ f^{\mu})\|_\infty \to 0$ as $n \to \infty$ because $[\mu_n] \to [\mu]$. Then $\nu_n = \mu + o(1)$, where $\|o(1)\|_\infty \to 0$ as $n \to \infty$. Also $\|\nu_n\|_\infty > k_n$ by the unique extremality of μ_n [14], and we apply the δ-inequality of [1] to ν_n and μ_n:

$$\int_{\mathcal{H}} \Big| \frac{\tilde{\nu}_n(f_n) - \tilde{\mu}_n(f_n)}{1 - \overline{\tilde{\nu}_n(f_n)}\tilde{\mu}_n(f_n)} \Big|^2 |\psi| dm \leq C\Big(\|\nu_n\|_\infty - Re \int_{\mathcal{H}} \nu_n \psi dm\Big),$$

for all $\psi \in A_1$, where $f_n = f^{\nu_n}$ and where $\tilde{\nu}_n, \tilde{\mu}_n$ are the Beltrami coefficients of the inverse maps of f^{ν_n}, f^{μ_n}, respectively. The proof of the δ-inequality for the solenoid follows the same lines as the proof for

Riemann surfaces using the Reich-Strebel inequality for the solenoid [14]. Note that $\tilde{\nu}_n(f_n) = -\nu_n\lambda_{\nu_n}$ and $\tilde{\mu}_n(f_n) = -\mu_n\lambda_{\nu_n} + o(1)$, where $\lambda_{\nu_n} = \frac{\overline{(f_n)_z}}{(f_n)_z}$.

Then we obtain

$$\int_{\mathcal{H}} |\mu - \mu_n|^2 |\psi| dm \leq C_1(k - Re \int_{\mathcal{H}} \mu\psi dm) + o(1).$$

We let $\psi = \varphi$ in the above and obtain

$$\int_{\mathcal{H}} \Big| \frac{|\varphi|}{\varphi} - \frac{|\varphi_n|}{\varphi_n} \Big|^2 |\varphi| dm \to 0$$

as $n \to \infty$. This implies that

$$Re \int_{\mathcal{H}} \frac{|\varphi_n|}{\varphi_n} \varphi dm \to 1$$

as $n \to \infty$.

By the definition $\varphi = \pi([\mu])$. By our assumption, $\|\varphi_n\|_{Bers} \leq N$ and it is enough to show that $\|\varphi\|_{Bers} \leq N$. Denote by α the product of leafwise hyperbolic area measure and the transverse measure on $\mathcal{H}$. We scale the transverse measure in such fashion that $\alpha(\mathcal{H}) = 1$. Define $\mathcal{H}_{n,\epsilon} = \{x \in \mathcal{H} : |Arg(\frac{\varphi_n(x)}{\varphi(x)})| < \epsilon\}$. We show that $\lim_{n\to\infty} \alpha(\mathcal{H}_{n,\epsilon}) = 1$, for all $\epsilon > 0$. Assume on the contrary that $\liminf_{n\to\infty} \alpha(\mathcal{H}_{n,\epsilon}) \leq 1 - \delta$, for $\delta > 0$. Then we have

$$Re \int_{\mathcal{H}} \frac{|\varphi_n|}{\varphi_n} \varphi dm \leq \int_{\mathcal{H}_{n,\epsilon}} |\varphi| dm + \cos\epsilon \int_{\mathcal{H}-\mathcal{H}_{n,\epsilon}} |\varphi| dm.$$

The zeros of φ make a closed subset of $\mathcal{H}$ which is leafwise discrete and whose α-measure is 0. Moreover, there exists an open neighborhood U of the zeros of arbitrary small α-measure whose intersection with any leaf consists of hyperbolic disks. For any such neighborhood U, we have $C(U) = \inf_{(z,t)\in\mathcal{H}-U} |\varphi(z,t)\rho^{-2}(z,t)| > 0$. From the above we get, for U small enough, that

$$\liminf_{n\to\infty} Re \int_{\mathcal{H}} \frac{|\varphi_n|}{\varphi_n} \varphi dm \leq 1 - (1-\cos\epsilon)(\delta - \alpha(U))C(U) < 1$$

which contradicts $\int_{\mathcal{H}} \frac{|\varphi_n|}{\varphi_n} \varphi dm \to 1$ as $n \to \infty$. Therefore $\lim_{n\to\infty} \alpha(\mathcal{H}_{n,\epsilon}) = 1$ as $n \to \infty$.

We fix $\delta > 0$. By the above, there exists n_1 such that $\alpha(\mathcal{H}_{n_1,\delta/2}) > 1 - \delta/2$. Further, there exists $n_2 \geq n_1$ such that $\alpha(\mathcal{H}_{n_2,\delta/4}) > 1 - \delta/4$, and so on. In general, we find n_j such that $n_j \geq n_{j-1}$ and $\alpha(\mathcal{H}_{n_j,\delta/2^j}) > 1 - \delta/2^j$. Define $\mathcal{H}_0 = \cap_{j=1}^{\infty} \mathcal{H}_{n_j,\delta/2^j}$. Then $\alpha(\mathcal{H}_0) \geq 1 - \delta$ and the sequence $Arg(\frac{\varphi_{n_j}}{\varphi})$ converges to zero uniformly on $\mathcal{H}_0$. For simplicity of notation, rename the sequence φ_{n_j} to φ_n.

Consider lifts $\tilde{\varphi}, \tilde{\varphi}_n$ of φ, φ_n to the universal covering $\Delta_1 \times \hat{G}$ of $\mathcal{H}$. Recall that for a finite index subgroup G_k of G we have $\Delta_1 \times \hat{G}_k / G_k \equiv \mathcal{H}$. Given $\epsilon > 0$ there exists k such that $\sup_{z\in\Delta_1} |\tilde{\varphi}(z,t_1) - \tilde{\varphi}(z,t_2)|\rho^{-2}(z) < \epsilon$, for all $t_1, t_2 \in \hat{G}_k$. Let ω_k be a fundamental polygon for G_k. Then $\omega_k \times \hat{G}_k$ is a fundamental set for the action of G_k on $\Delta_1 \times \hat{G}_k$.

Let $c_k = \text{h-area}(\omega_k)$. Then $m(\hat{G}_k) = 1/c_k$ because of the normalization $\alpha(\mathcal{H}) = 1$. Given $t \in \hat{G}_k$, denote by $\omega_{k,t}^{\delta,n}$ the set of all $(z,t) \in \omega_k \times \hat{G}_k$ such that $|Arg(\frac{\tilde{\varphi}_n}{\tilde{\varphi}})(z,t)| < \delta$. Let $\hat{G}_k^{\delta_1,n} = \{t \in \hat{G}_k;\ \text{h-area}(\omega_{k,t}^{\delta,n}) < \delta_1\}$. Then we obtain

$$1 - \delta \leq \alpha(\mathcal{H}_0) \leq \delta_1 m(\hat{G}_k^{\delta_1,n}) + c_k(1/c_k - m(\hat{G}_n^{\delta_1,n}))$$

which implies

$$m(\hat{G}_k^{\delta_1,n}) \leq \frac{\delta}{c_k - \delta_1}.$$

The above implies that $\omega_{k,t}^{\delta,n}$ has Lebesgue measure bounded from below for each $t \in \hat{G}_k - \hat{G}_k^{\delta_1,n}$ and Lemma 4.1 applies to such t. Thus, by Lemma 4.1, there exists a sequence of functions $k_n : \hat{G}_k - \hat{G}_k^{\delta_1,n} \to \mathbb{R}^+$ such that

$$(2) \qquad \|(\tilde{\varphi}_n - k_n\tilde{\varphi})|_{\omega_k\times(\hat{G}_k-\hat{G}_k^{\delta_1,n})}\|_{Bers} \leq d_n \to 0$$

as $n \to \infty$.

We claim that there exists a sequence $t_n \in \hat{G}_k - \hat{G}_k^{\delta_1,n}$ such that $\limsup_{n\to\infty} k_n(t_n) \geq 1$. Suppose on the contrary that there exists $c > 0$ such that $\|k_n\|_\infty \leq 1 - c$ for all large n. From (2) and by the above assumption, we obtain

$$\int_{\omega_k\times(\hat{G}_k-\hat{G}_k^{\delta_1,n})} |\tilde{\varphi}_n| dm - d_n \leq \int_{\omega_k\times(\hat{G}_k-\hat{G}_k^{\delta_1,n})} k_n|\tilde{\varphi}|dm \leq (1-c)\int_{\omega_k\times(\hat{G}_k-\hat{G}_k^{\delta_1,n})} |\tilde{\varphi}|dm.$$

This implies

$$(3) \qquad \int_{\omega_k\times(\hat{G}_k-\hat{G}_k^{\delta_1,n})} |\tilde{\varphi}_n| dm/(1-c) - d_n/(1-c) \leq \int_{\omega_k\times(\hat{G}_k-\hat{G}_k^{\delta_1,n})} |\tilde{\varphi}|dm.$$

Note that

$$(4) \qquad \int_{\omega_k\times\hat{G}_k^{\delta_1,n}} |\tilde{\varphi}_n| dm \leq \|\varphi_n\|_{Bers}\ \alpha(\omega_k \times \hat{G}_k^{\delta_1,n}) \leq Nc_k\frac{\delta}{c_k-\delta_1} \to 0$$

as $\delta \to 0$, for fixed k and δ_1, and uniformly in n. If we take n large enough and δ small enough in (3) and (4), we get that $\int_{\mathcal{H}} |\varphi|dm > 1$ which is a contradiction with our choice of φ.

Therefore, there exists t_n such that $\limsup_{n\to\infty} k_n(t_n) \geq 1$. From (2) we get that $\|\tilde{\varphi}|_{\omega_k\times t_n}\|_{Bers} \leq \|\tilde{\varphi}_n|_{\omega_k\times t_n}\|_{Bers}/k_n(t_n) + d_n/k_n(t_n)$. Consequently, we have that $\|\varphi\|_{Bers} \leq N + \epsilon$ by letting $n \to \infty$. Since ϵ was arbitrary, we get that $\|\varphi\|_{Bers} \leq N$. □

Before proving the Theorem 1. we need to prove the next lemma.

Let S_0 be a compact Riemann surface of genus two at least two. Let γ be a non-diving simple closed geodesic (in the corresponding hyperbolic metric). We cut S_0 along γ to obtain a bordered hyperbolic surface S_0^b. Denote by S_n a compact Riemann surface obtained by gluing n copies of S_0^b along their boundaries such that S_n is $\mathbb{Z}_n$-cover of S_0. Given r, $0 < r < 1$, we denote by $R_{n,r}$ a subsurface of S_n which consists of $[rn]$ consecutive copies of S_0^b in S_n, where $[rn]$ is the smallest integer less that or equal to rn. The hyperbolic metric on S_0 lifts to a unique hyperbolic metric on S_n and both hyperbolic metrics lift to a transversely locally constant hyperbolic metric on $\mathcal{H}$. Denote by $\tilde{R}_{n,r}$ the lift of $R_{n,r}$ to $\mathcal{H}$. It is then clear that $\alpha(\tilde{R}_{n,r}) = [nr]/n$.

If f is a leafwise quadratic differential on $\mathcal{H}$, we define $\|f\|_{L^1} := \int_{\mathcal{H}} |f|dm$. Define a non-holomorphic quadratic differential on $\mathcal{H}$ by

$$\tilde{\varphi}_n = \begin{cases} 0, & \text{on} \quad \mathcal{H} - \tilde{R}_{n,r} \\ \tilde{\varphi}_0, & \text{on} \quad \tilde{R}_{n,r} \end{cases}$$

Note that $\tilde{\varphi}_n$ is the lift of

$$\varphi_n = \begin{cases} 0, & \text{on} \quad S_n - R_{n,r} \\ \varphi_0, & \text{on} \quad R_{n,r} \end{cases}$$

Lemma 4.3. *There exists a holomorphic quadratic differential $\tilde{\psi}_n \in A(\mathcal{H})$ such that*

$$\Big\| \frac{\tilde{\psi}_n}{\|\tilde{\psi}_n\|_{L^1}} - \frac{\tilde{\varphi}_n}{\|\tilde{\varphi}_n\|_{L^1}} \Big\|_{L^1} \to 0,$$

as $n \to \infty$, for fixed r. Moreover, we can choose $\tilde{\psi}_n$ to be the lift of a holomorphic quadratic differential ψ_n on S_n.

Proof. Let $A(S_n)$ denote the space of holomorphic quadratic differentials on S_n. We define a linear functional $\sigma : A(S_n) \to \mathbb{C}$ by

$$\sigma(f) = \int_{S_n} \bar{\varphi}_n \rho^{-2} f,$$

for $f \in A(S_n)$. It is a standard fact for Riemann surfaces that there exists a unique $\psi_n \in A(S_n)$ such that $\sigma(f) = \int_{S_n} \bar{\psi}_n \rho^{-2} f$, for all $f \in A(S_n)$. Denote by $\tilde{\psi}_n$ the lift of ψ_n to $\mathcal{H}$.

We consider c-neighborhood $U_n(c)$ of the two boundary curves of $R_{n,r}$ in the hyperbolic metric on S_n, for $c > 0$. We claim that for any $\epsilon > 0$ there exist n_0 and $c > 0$ such that

$$\rho^{-2}|\varphi_n - \psi_n| < \epsilon \tag{5}$$

on $S_n - U_n(c)$ for all $n > n_0$.

To show the claim, we assume that it is not true (and arrive at a contradiction). Then there exist $\epsilon > 0$, a sequence $c_n > 0$ and a sequence $z_n \in S_n - U_n(c_n)$ such that $c_n \to \infty$ as $n \to \infty$ and

$$\rho^{-2}(z_n)|\varphi_n(z_n) - \psi_n(z_n)| \geq \epsilon \tag{6}$$

for all n. We arrange that either $z_n \in S_n - R_{n,r}$ or $z_n \in R_{n,r}$ for all n after possibly taking a subsequence. Consider a $\mathbb{Z}$-cover $\tilde{S}$ of S_0 which is made by gluing together infinitely many S_0^b. Note that $\tilde{S}$ is also $\mathbb{Z}$-cover of each S_n. We arrange that the covering maps $\tilde{S} \to S_n$ have a lift of z_n in a fixed copy of S_0^b in $\tilde{S}$. We denote by $\tilde{\varphi}_n$, $\tilde{\psi}_n$ the lifts of φ_n, ψ_n to $\tilde{S}$ as well as to $\mathcal{H}$ and the meaning should be read from the context. Define a linear functional on $A(\tilde{S})$ by

$$\tilde{\sigma}_n(f) := \int_{\tilde{S}} \bar{\tilde{\varphi}}_n \, \rho^{-2} f,$$

for all $f \in A(\tilde{S})$. Then $\tilde{\sigma}_n$ satisfies

$$\tilde{\sigma}_n(f) = \int_{\tilde{S}} \bar{\tilde{\psi}}_n \, \rho^{-2} f$$

for all $f \in A(\tilde{S})$ by the Bers' reproducing formula. Note that $A(S_n)$ does not lift to a subset of $A(\tilde{S})$.

By the choice of the covering maps $\tilde{S} \to S_n$, we have that either $\tilde{\varphi}_n \to \tilde{\varphi}_0$ on $\tilde{S}$ uniformly on compact subsets if $z_n \in R_{n,r}$, or $\tilde{\varphi}_n \to 0$ uniformly on compact subsets otherwise. Then either

$$\lim_{n\to\infty} \int_{\tilde{S}} \bar{\tilde{\varphi}}_n \, \rho^{-2} f = \int_{\tilde{S}} \bar{\tilde{\varphi}}_0 \, \rho^{-2} f$$

for all $f \in A(\tilde{S})$ in the first case or

$$\lim_{n\to\infty} \int_{\tilde{S}} \bar{\tilde{\varphi}}_n \, \rho^{-2} f = 0$$

in the second case.

It is clear that $\|\tilde{\psi}_n\|_{Bers} \leq 3\|\varphi_0\|_{Bers} < \infty$ for all n because $\|\tilde{\sigma}_n\| \leq \|\varphi_0\|_{Bers}$. Therefore, $\tilde{\psi}_n$ has a subsequence which converges uniformly on compact subsets of $\tilde{S}$ to a holomorphic quadratic differential $\tilde{\psi} \in A(\tilde{S})$. Then

$$\lim_{n\to\infty} \int_{\tilde{S}} \bar{\tilde{\psi}}_n \, \rho^{-2} f = \int_{\tilde{S}} \bar{\tilde{\psi}} \, \rho^{-2} f. \tag{7}$$

On the other hand, $\tilde{\psi} \neq \tilde{\varphi}_0$ in the first case and $\tilde{\psi} \neq 0$ in the second case by (6) and the fact that the inequality prevails in the lifts to a compact subset of $\tilde{S}$. Thus we obtain two different presentations for the limiting linear functional. This is a contradiction to the uniqueness of the presentation of linear functionals in the above form. Therefore, we showed that given $\epsilon > 0$ there exists $c > 0$ such that $\rho^{-2}|\varphi_n - \psi_n| < \epsilon$ in $S_n - U_n(c)$ for all n large enough.

Note that the hyperbolic area of $U_n(c)$ is constant in n for a fixed $c > 0$ by our choice of coverings. Since the genus of S_n goes to infinity as $n \to \infty$, we conclude that $\alpha(\tilde{U}_n(c)) \to 0$ as $n \to \infty$, where $\tilde{U}_n(c)$ is the lift of $U_n(c)$ to $\mathcal{H}$. Then by (5) and by the above, we get

$$\begin{aligned} &\textstyle\int_{\mathcal{H}} |\tilde{\varphi}_n - \tilde{\psi}_n| dm \leq \int_{\mathcal{H}-(\tilde{U}_n(c))} |\tilde{\varphi}_n - \tilde{\psi}_n| \rho^{-2}\rho^2 dm + \int_{\tilde{U}_n(c)} |\tilde{\varphi}_n - \tilde{\psi}_n| \rho^{-2}\rho^2 dm \leq \\ &\quad \leq \epsilon\alpha(\mathcal{H} - U_n(c)) + M\alpha(U_n(c)) \to 0 \end{aligned}$$

as $n \to \infty$. In other words, we showed that $\lim_{n\to\infty} \|\tilde{\varphi}_n - \tilde{\psi}_n\|_{L^1} = 0$.

It is clear that $\|\tilde{\varphi}_n\|_{L^1} = \frac{[rn]}{n}\|\varphi_0\|_{L^1(S_0)}$. By the above, we also get that $\|\tilde{\psi}_n\|_{L^1} - \frac{[rn]}{n}\|\varphi_0\|_{L^1(S_0)} \to 0$ as $n \to \infty$. This implies that both $\|\tilde{\varphi}_n\|_{L^1}$ and $\|\tilde{\psi}_n\|_{L^1}$ are bounded from below independently of n. Moreover, their difference converges to 0 as $n \to \infty$. Thus we obtain

$$\Big\|\frac{\tilde{\psi}_n}{\|\tilde{\psi}_n\|_{L^1}} - \frac{\tilde{\varphi}_n}{\|\tilde{\varphi}_n\|_{L^1}}\Big\|_{L^1} \leq \frac{k[nr]}{n\|\varphi_0\|_{L^1(S_0)}}\|\tilde{\varphi}_n - \tilde{\psi}_n\|_{L^1} \to 0,$$

for some constant $k > 0$ as $n \to \infty$. □

Proof of Theorem 1. To prove that the set of points which do not have a Teichmüller representative is generic we need to prove that the set X is of first category. Recall that

$$X = \{[0]\} \cup (\cup \pi^{-1}(A_1(N))),$$

where the second union is over all $N \in \mathbb{N}$. It is enough to show that each of the sets $\pi^{-1}(A_1(N))$ is of the first category. We prove this by contradiction.

Assume that for some N the set $\pi^{-1}(A_1(N))$ is of the second category. Then the closure $\pi^{-1}(A_1(N))^c$ has non-empty interior. Moreover, by Proposition 4.2 every element in $\pi^{-1}(A_1(N))^c$ that has a Teichmüller representative must be in $\pi^{-1}(A_1(N))$. Let $[\mu = k\frac{|\varphi|}{\varphi}]$ be a transversely locally constant Beltrami coefficient on $\mathcal{H}$ such that the point $[\mu]$ is an element of the interior of the set $\pi^{-1}(A_1(N))^c$. We assume that μ is lifted from a surface S_0, that is $\varphi \in A_1$ is a lift of a holomorphic quadratic differential φ_0 on a closed Riemann surface S_0 of genus at least two. We will show that there exists a transversely locally constant sequence $[\mu_n] \to [\mu]$ such that $\xi_n = \pi([\mu_n])$ are unbounded in the Bers norm. Since for n large enough we have that $[\mu_n]$ is an element of the interior of the set $\pi^{-1}(A_1(N))^c$ we will obtain a contradiction.

We keep the notation S_n for $\mathbb{Z}_n$ cover of S_0 and $R_{n,r}$ for $[rn]/n$ proportion of S_n as above. Consider a Beltrami coefficient

$$\mu_n = \begin{cases} k\frac{|\varphi|}{\varphi}, & \mathcal{H} - \tilde{R}_{n,r} \\ (1+r)k\frac{|\varphi|}{\varphi}, & \tilde{R}_{n,r} \end{cases} \tag{8}$$

for $r > 0$ small enough such that $\|\mu_n\|_\infty < 1$. The Beltrami coefficient μ_n is not smooth at the lift of two boundary curves of $R_{n,r}$ to the solenoid $\mathcal{H}$ and it can be smoothly approximated in arbitrary small area neighborhoods of the lift. Therefore, we can work with μ_n as well and we refer the reader to [14] for more details.

Let $\lambda^n(\psi) = \int_{\mathcal{H}} \mu_n \psi dm$, for $\psi \in A(\mathcal{H})$, be the corresponding linear functional. Denote by $\tilde{\varphi}_n$ the non-holomorphic quadratic differential which is the lift of

$$\varphi_n = \begin{cases} \varphi_0, & R_{n,r} \\ 0, & S_n - R_{n,r} \end{cases}. \tag{9}$$

By Lemma 4.3, there exist holomorphic quadratic differentials $\tilde{\psi}_n$ on $\mathcal{H}$ such that

$$\Big\|\frac{\tilde{\psi}_n}{\|\tilde{\psi}_n\|_{L^1}} - \frac{\tilde{\varphi}_n}{\|\tilde{\varphi}_n\|_{L^1}}\Big\|_{L^1} \to 0 \tag{10}$$

as $n \to \infty$.

Denote by $\|\lambda_n\| = \sup_{\psi \in A_1} |\lambda_n(\psi)|$ the operator norm of λ_n. Then $\|\lambda_n\| \leq (1+r)k$ because $\|\mu_n\|_\infty = (1+r)k$. Since $\lambda_n(\tilde{\varphi}_n/\|\tilde{\varphi}_n\|_{L^1}) = (1+r)k$ and by (10), we have that $\|\lambda_n\| \to (1+r)k$ as $n \to \infty$. There exists $\xi_n \in A_1$ (which is the lift of a holomorphic quadratic differential on S_n) and there exists $l_n > 0$ such that $l_n\frac{|\xi_n|}{\xi_n} \in [\mu_n]$.

Let $k(\mu_n)=\inf_{\nu\in[\mu_n]}\|\nu\|_\infty$. The Teichmüller contraction inequality [14] applied to μ_n gives

$$\|\mu_n\|_\infty - k(\mu_n)\leq C(\|\mu_n\|_\infty - \sup_{\psi\in A_1} Re\int_{\mathcal{H}}\mu_n\psi dm)$$

where $C>0$ is a fixed constant. The right hand side of the above inequality converges to zero as $n\to\infty$. Therefore the left hand side converges to zero as well. Since $\|\mu_n\|_\infty=(1+r)k$, then we have $k(\mu_n)=l_n\to (1+r)k$ as $n\to\infty$.

We use another standard formula which is an easy consequence of the Reich-Strebel inequality developed for the solenoid in [14] in the course of proof of the Teichmüller contraction. Namely, we get

$$\frac{1+l_n}{1-l_n}\leq\int_{\mathcal{H}}\frac{|1-\mu_n\frac{\xi_n}{|\xi_n|}|^2}{1-|\mu_n|^2}|\xi_n|dm$$

where $l_n\frac{|\xi_n|}{\xi_n}\in[\mu_n]$ is an important condition (the formula is not true for arbitrary holomorphic quadratic differential). Further,

$$\int_{\mathcal{H}}\frac{|1-\mu_n\frac{\xi_n}{|\xi_n|}|^2}{1-|\mu_n|^2}|\xi_n|dm\leq\int_{\mathcal{H}-\tilde{R}_{n,r}}\frac{(1+k)^2}{1-k^2}|\xi_n|dm+\int_{\tilde{R}_{n,r}}\frac{[1+(1+r)k]^2}{1-[(1+r)k]^2}|\xi_n|dm$$

which implies that

$$\frac{1+l_n}{1-l_n}\leq\frac{1+k}{1-k}\int_{\mathcal{H}-\tilde{R}_{n,r}}|\xi_n|dm+\frac{1+(1+r)k}{1-(1+r)k}\int_{\tilde{R}_{n,r}}|\xi_n|dm.$$

From the above inequality and by $l_n\to(1+r)k$, we get that $\lim_{n\to\infty}\int_{\mathcal{H}-\tilde{R}_{n,r}}|\xi_n|dm=0$, for each $0<r<1$. To see this assume on the contrary that $\limsup_{n\to\infty}\int_{\mathcal{H}-\tilde{R}_{n,r}}|\xi_n|dm=\delta>0$. Then by taking $\limsup_{n\to\infty}$ in the above inequality, we obtain $\frac{1+l_n}{1-l_n}\leq\frac{1+k}{1-k}\delta+\frac{1+(1+r)k}{1-(1+r)k}(1-\delta)$ which is impossible. By Cantor diagonal argument, there exists a sequence $r_n\to 0$, $0<r_n<1$, such that $nr_n\to\infty$ and $\int_{\mathcal{H}-\tilde{R}_{n,r_n}}|\xi_n|dm\to 0$ as $n\to\infty$. Clearly $\lim_{n\to\infty}\alpha(\tilde{R}_{n,r_n})=0$. For simplicity, we write $\tilde{R}_n=\tilde{R}_{n,r_n}$.

Finally, we assume that $\|\xi_n\|_{Bers}\leq N$. Then we have that

$$1=\|\xi_n\|_{L^1}\leq\int_{\mathcal{H}-\tilde{R}_n}|\xi_n|dm+\|\xi_n\|_{Bers}\ \alpha(\tilde{R}_n)=\int_{\mathcal{H}-\tilde{R}_n}|\xi_n|dm+N\alpha(\tilde{R}_n)\to 0$$

as $n\to\infty$. This is a contradiction. Therefore ξ_n is not bounded in the Bers norm.
□

5. Infinitesimal Teichmüller classes without Teichmüller representatives

A tangent vector to $T(\mathcal{H})$ at the basepoint $[0]$ is represented by a Beltrami differential μ. It defines a continuous linear functional on $A(\mathcal{H})$ via the natural pairing. If μ is infinitesimally equivalent to $k\frac{|\varphi|}{\varphi}$ (i.e. $\mu-k\frac{|\varphi|}{\varphi}\in N(\mathcal{H})$) then the linear functional achieves norm on the unique vector $\frac{\varphi}{\|\varphi\|_{L^1}}\in A(\mathcal{H})$. A question whether each infinitesimal Teichmüller class contains a Beltrami coefficient of the Teichmüller type $k\frac{|\varphi|}{\varphi}$, for $k\in\mathbb{R}^+$ and $0\neq\varphi\in A(\mathcal{H})$, is analogous to the question of the existence of Teichmüller representative for marked complex structures. An equivalent question is whether the induced linear functional achieves its norm on $A(\mathcal{H})$. We obtain an analogous result to Theorem 1:

Theorem 3. *The set of points in the tangent space of $T(\mathcal{H})$ at the basepoint $[0]$ which do not achieve its norm on $A(\mathcal{H})$ is generic.*

Proof. Denote by $B(\mathcal{H}) = L^\infty(\mathcal{H})/N(\mathcal{H})$ the tangent space to $T(\mathcal{H})$ at the basepoint. Let $X \subset B(\mathcal{H})$ be the set of points that do achieve its norm. That is, each $\lambda \in X$ achieves its norm on some $\varphi \in A_1$. We define $\pi : X \to A_1$ by $\pi(\lambda) = \varphi$ if λ achieves its norm on $\varphi \in A_1$. Then we have $X = \cup_{N=1}^{\infty} \pi^{-1}(A_1(N)) \cup \{[0]\})$.

We show that if $\lambda \in \pi^{-1}(A_1(N))^c$ achieves its norm then $\lambda \in \pi^{-1}(A_1(N))$ Let $\lambda_n \in \pi^{-1}(A_1(N))$ and assume that $\lambda_n \to \lambda$. Let $\mu_n = k_n \frac{|\varphi_n|}{\varphi_n}$ be the Teichmüller Beltrami differential representing λ_n, where $\varphi_n \in A_1$ and let $\mu = k\frac{|\varphi|}{\varphi}$ be the Teichmüller Beltrami differential representing μ, where $\varphi \in A_1$. Then $k_n \to k$ and

$$\int_{\mathcal{H}} k_n \frac{|\varphi_n|}{\varphi_n} \psi dm \to \int_{\mathcal{H}} k \frac{|\varphi|}{\varphi} \psi dm$$

as $n \to \infty$ for all $\psi \in A(\mathcal{H})$. By letting $\psi = \varphi$ in the above, we get

$$\int_{\mathcal{H}} \frac{|\varphi_n|}{\varphi_n} \varphi dm \to 1$$

as $n \to \infty$. In the proof of Proposition 4.2, we showed that the above convergence implies that $\varphi \in A_1(N)$, which proves the claim.

Same as in the proof of Theorem 1 , we prove that X is of the first category. It is enough to prove that each $\pi^{-1}(A_1(N))$ is of the first kind. We do this by contradiction.

Assume that for some N the set $\pi^{-1}(A_1(N))$ is of the second kind. Therefore, the closure $\pi^{-1}(A_1(N))^c$ has non-empty interior. Then there exists $\lambda_0 \in (\pi^{-1}(A_1(N)))^\circ$ which is transversely locally constant. This implies that any sequence of transversely locally constant λ_n which converge to λ_0 must have bounded Bers norm. We find a contradiction with this statement by constructing a convergent sequence with unbounded Bers norm below.

Assume that $\lambda_0 \in B(\mathcal{H})$ achieves its norm on $\tilde{\varphi}_0 \in A_1$, where $\tilde{\varphi}_0$ is the lift of a holomorphic quadratic differential φ_0 on a Riemann surface S_0. As in Section 4, define a non-holomorphic quadratic differential on $\mathcal{H}$ by

$$\tilde{\varphi}_n = \begin{cases} 0, & \text{on} \quad \mathcal{H} - \tilde{R}_{n,r} \\ \tilde{\varphi}_0, & \text{on} \quad \tilde{R}_{n,r} \end{cases}$$

Note that $\tilde{\varphi}_n$ is the lift of

$$\varphi_n = \begin{cases} 0, & \text{on} \quad S_n - R_{n,r} \\ \varphi_0, & \text{on} \quad R_{n,r} \end{cases}$$

By Lemma 4.3, there exist a sequence $\tilde{\psi}_n \in A(\mathcal{H})$ such that $\left\| \frac{\tilde{\psi}_n}{\|\tilde{\psi}_n\|_{L^1}} - \frac{\tilde{\varphi}_n}{\|\tilde{\varphi}_n\|_{L^1}} \right\|_{L^1} \to 0$ as $n \to \infty$. We define $\psi_n := \frac{\tilde{\psi}_n}{\|\tilde{\psi}_n\|_{L^1}}$, i.e. ψ_n is a positive multiple of $\tilde{\psi}_n$ which belongs to A_1.

Consider linear functionals

$$\lambda_n(f) = \int_{\mathcal{H} - \tilde{R}_{n,r}} \frac{|\varphi_0|}{\varphi_0} f dm + (1+l) \int_{\tilde{R}_{n,r}} \frac{|\varphi_0|}{\varphi_0} f dm.$$

for $f \in A(\mathcal{H})$ and $l > 0$. Then we obtain

$$|\lambda_n(\psi_n)| \geq - \int_{\mathcal{H} - \tilde{R}_{n,r}} |\psi_n| dm + (1+l) \int_{\tilde{R}_{n,r}} |\psi_n| dm \to 1 + l,$$

as $n \to \infty$. Therefore, $\|\lambda_n\| \to 1 + l$ as $n \to \infty$.

Note that λ_n descends to a functional on $A(S_n)$. Thus there exists a unique $\xi_n \in A(S_n)$ on which λ_n achieves its norm. Lift a positive multiple of ξ_n to a transversely locally constant holomorphic quadratic differential $\tilde{\xi}_n$ on $\mathcal{H}$ such that $\|\tilde{\xi}_n\|_{L^1} = 1$. Then we have $\lambda_n(\tilde{\xi}_n) = \|\lambda_n\| \geq 1 + l - \epsilon$ for all n large enough depending of ϵ.

We claim that $\int_{\mathcal{H}-\tilde{R}_{n,r}}|\tilde{\xi_n}|dm\to 0$ as $n\to\infty$. Assume on the contrary that there exists $\delta>0$ such that $\limsup_{k\to\infty}\int_{\mathcal{H}-\tilde{R}_k}|\tilde{\xi_k}|dm=\delta$. Then we get

$$\limsup_{n\to\infty}\lambda_n(\tilde{\xi_n})\leq\delta+(1+l)(1-\delta)=1+l-l\delta<1+l.$$

But this is in contradiction with $\|\lambda_n\|\to 1+l$ as $n\to\infty$. Therefore $\int_{\mathcal{H}-\tilde{R}_{n,r}}|\tilde{\xi_n}|dm\to 0$ as $n\to\infty$.

By the Cantor diagonal argument, there exists a sequence $r_n\to 0$ such that $\int_{\mathcal{H}-\tilde{R}_{n,r_n}}|\tilde{\xi_n}|dm\to 0$ and $nr_n\to\infty$ as $n\to\infty$. The condition $r_n\to 0$ implies that $\alpha(\tilde{R}_{n,r_n})\to 0$ as $n\to\infty$. For simplicity of notation, we write $\tilde{R}_n=\tilde{R}_{n,r_n}$.

This implies

$$1=\|\tilde{\xi_n}\|_{L^1}=\int_{\mathcal{H}-\tilde{R}_n}|\tilde{\xi_n}|dm+\int_{\tilde{R}_n}|\tilde{\xi_n}|dm\leq\int_{\mathcal{H}-\tilde{R}_n}|\tilde{\xi_n}|dm+\|\tilde{\xi_n}\|_{Bers}\alpha(\tilde{R}_n)\to 0$$

as $n\to\infty$. This is a contradiction.
□

Remark. The results of this paper immediately generalize to the punctured solenoid introduced in [11].

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