TEICHMÜLLER THEORY OF THE PUNCTURED SOLENOID

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Abstract. The punctured solenoid $\mathcal{H}$ is an initial object for the category of punctured surfaces with morphisms given by finite covers branched only over the punctures. The (decorated) Teichmüller space of $\mathcal{H}$ is introduced, studied, and found to be parametrized by certain coordinates on a fixed triangulation of $\mathcal{H}$. Furthermore, a point in the decorated Teichmüller space induces a polygonal decomposition of $\mathcal{H}$ giving a combinatorial description of its decorated Teichmüller space itself. This is used to obtain a non-trivial set of generators of the modular group of $\mathcal{H}$, which is presumably the main result of this paper. Moreover, each word in these generators admits a normal form, and the natural equivalence relation on normal forms is described. There is furthermore a non-degenerate modular group invariant two form on the Teichmüller space of $\mathcal{H}$. All of this structure is in perfect analogy with that of the decorated Teichmüller space of a punctured surface of finite type.

1. Introduction

Sullivan [23] introduced the *universal hyperbolic solenoid* as the inverse limit of all finite unbranched covers of a compact surface with negative Euler characteristic. (Different choices of compact surface give homeomorphic solenoids since any two such surfaces admit a common cover.) The space of all complex structures on the solenoid is a version of a "universal" Teichmüller space insofar as the union of Teichmüller spaces of all compact surfaces lies naturally as a dense subset [23], [15].

Locally as a topological space, the solenoid is modeled on a product of the form (2-dimensional disk) $\times$ (Cantor set), and these charts glue together to provide a 2-dimensional foliation of the solenoid itself. Because surface groups are residually finite, each leaf of this foliation is a 2-dimensional disk, which is in fact dense in the solenoid. Deformations of geometric structures on the solenoid are typically required to be smooth/conformal/quasiconformal in the 2-disk direction and continuous in the Cantor set direction. One may follow the pattern of Ahlfors-Bers theory [1], [2] for compact surfaces in order to precisely define the Teichmüller space and modular group of the solenoid [23].

The Ehrenpreis Conjecture, which is well-known in certain circles, is that any two compact Riemann surfaces have almost conformal finite unbranched covers of the same genus. Sullivan [23], [3] noted that the Ehrenpreis Conjecture is equivalent to the statement that the (baseleaf preserving) modular group of the solenoid has dense orbits in the Teichmüller space of the solenoid. The algebraic structure of the modular group of Sullivan's solenoid is not yet well understood. (An interesting phenomenon is that any finite subgroup of the modular group of the solenoid is cyclic [14], unlike for compact surfaces.)

We modify the universal object by allowing controlled finite branching, namely, the *punctured solenoid* $\mathcal{H}$ (our universal object) is the inverse limit of all finite unbranched covers of any fixed punctured surface with negative Euler characteristic, e.g., covers of the once-punctured torus, where, in effect, branching is permitted only over the missing puncture. Equivalently again using properties of finite covers, $\mathcal{H}$ is the inverse limit over all finite-index subgroups K in the modular group $PSL_2(\mathbb{Z})$ of the tower $\mathbb{D}/K$ of covers, where $\mathbb{D}$ is the unit disk with frontier circle S^1.

Unlike Sullivan's universal hyperbolic solenoid, $\mathcal{H}$ is not a compact space; the ends are quotients of the product (horoball)$\times$(Cantor set) by the continuous action of a countable group, and the orbit of each horoball is dense in the end. The centers of the horoballs are called punctures of $\mathcal{H}$.

In analogy to the case of punctured surfaces [19], we introduce decorations at the punctures, namely, a choice of horocycle at each puncture, and we find global coordinates and a combinatorial decomposition of

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the decorated Teichmüller space of $\mathcal{H}$. Furthermore, the combinatorial action of an appropriate "baseleaf preserving" modular group of the punctured solenoid is used to give an explicit set of generators. The elements of this modular group are written as compositions of generators in "normal form". We find the relations which identify equivalent normal forms. We furthermore disprove the Ehrenpreis Conjecture in a strong sense for the decorated Teichmüller space, namely, there is an open dense subspace of the quotient by the modular group which is Hausdorff. Finally, we give a closed two form akin to the Weil-Petersson Kähler form using our coordinates, and we show it is invariant under the modular group using our generators.

With this statement of the main results of this paper complete, we turn to a somewhat more detailed description and discussion.

The Teichmüller space $T(\mathcal{H})$ of the punctured solenoid $\mathcal{H}$ is a separable complex Banach space [23] with a complete Teichmüller metric which is equal to the induced (by the complex structure) Kobayashi metric [22]. Starting from the quasiconformal definition of Teichmüller space, we use the universal covering space and covering group for $\mathcal{H}$ from [22] to give a representation-theoretic definition of $T(\mathcal{H})$ in the spirit of hyperbolic geometry (see Theorem 4.1).

The decorated Teichmüller space $\tilde{T}(\mathcal{H})$ is the space of decorated hyperbolic structures on $\mathcal{H}$, i.e., all hyperbolic metrics on $\mathcal{H}$ together with an assignment of one horocycle centered at each puncture. The analogous decorated Teichmüller spaces of punctured surfaces as well as a universal space modeled on orientation-preserving homeomorphisms of the circle modulo the Möbius group were studied [19], [18] using certain coordinates adapted to the decorated setting called *lambda lengths*; the lambda length is essentially the hyperbolic distance between horocycles (precisely the square root of twice the exponential of this signed distance, taken with a positive sign when the horocycles are disjoint).

For a punctured surface, the decorated Teichmüller space is parametrized by all positive assignments of lambda lengths, one such coordinate for each edge of any fixed "ideal triangulation" of the surface, i.e., a decomposition into triangles with vertices at the punctures (cf. Theorem B.1).

In the universal setting, the corresponding coordinates are assigned to the edges of the "Farey tesselation" of $\mathbb{D}$, which we next recall and for which we establish our notation. Fix the ideal hyperbolic triangle T with vertices 1, -1 and $-i$ on the boundary S^1 of $\mathbb{D}$ sitting in the complex plane, and let e_0 denote the real segment connecting ± 1. The group generated by hyperbolic reflections in the sides of T contains $PSL_2(\mathbb{Z})$ as the subgroup of orientation-preserving elements. The orbit of T under this group gives the edges of the Farey tesselation of $\mathbb{D}$ as illustrated in Figure 1. τ_* is regarded as a set of edges. The ideal vertices of τ_*, i.e., the asymptotic points in S^1, are naturally enumerated by $\bar{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\} \subseteq S^1$ as is also illustrated. In fact, $PSL_2(\mathbb{Z})$ acts simply transitively on the set of orientations on edges of τ_*. The edges of τ_* are also naturally enumerated by $\mathbb{Q} - \{1, -1\}$: any edge $e \in \tau_* - \{e_0\}$ separates two triangles, whose vertices are given by the endpoints of e and two other points in $\bar{\mathbb{Q}}$; among the two latter, take as the label for e the one on the opposite side of e from the edge e_0 of τ_*. We shall also typically regard e_0 as a distinguished oriented edge, or "*DOE*", from -1 to $+1$ as is also illustrated in Figure 1.

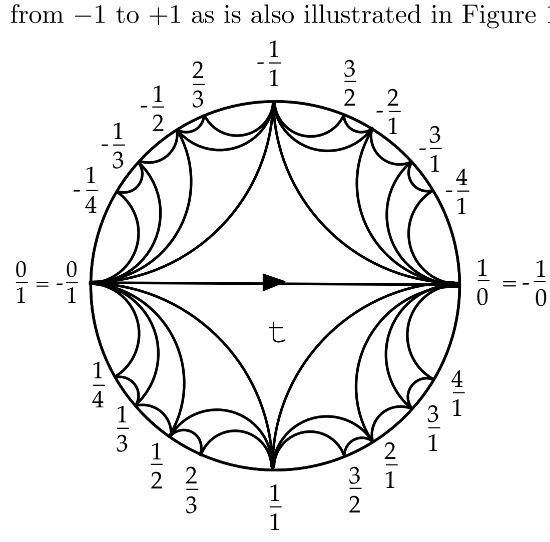

Figure 1 The Farey tesselation τ_*.

In the universal setting of [18], if f is a homeomorphism of the circle, then we derive another "tesselation with DOE" τ_f of $\mathbb{D}$ by demanding that three points in S^1 span an ideal triangle complementary to τ_f if and only if their pre-images under f span a triangle complementary to τ_*. Using rigidity of tesselations with DOE and properties of order-preserving maps of the circle, it is not hard to see that every tesselation (i.e., locally finite decomposition by geodesics into ideal triangles) arises in this way. This essentially proves that the space of orientation-preserving homeomorphisms of the circle is identified (after the choice of τ_* as a kind of basepoint) with the collection of all tesselations with DOE of $\mathbb{D}$. We may then decorate the ideal points of τ_f (i.e., choose one horocycle centered at each point of τ_f) and introduce one lambda length coordinate to each edge of τ_*.

In contrast to the case of punctured surfaces, an assignment of putative positive lambda lengths to each edge of τ_* does not necessarily correspond to a point in the universal decorated Teichmüller space, and it is an open question which weights do [18, Section 3]; on the other hand, a sufficient condition from [18, Theorems 6.3 and 6.4] is that if the lambda lengths are "pinched" in the sense that there is a constant $M > 1$ with all lambda lengths between M^{-1} and M, then there is a corresponding homeomorphism of the circle, and in fact, this homeomorphism is quasi-symmetric (the latter being joint with Sullivan in [18]). This is related to an open question by Thurston [24], namely, which measured laminations of the unit disk produce an earthquake whose boundary values give a homeomorphism of the unit circle.

Letting G denote the fundamental group of the punctured surface used to define the punctured solenoid and letting $\hat{G}$ denote its profinite completion (see Section 2), the set $Cont^G(\hat{G}, \mathbb{R}^{\tau_*}_{>0})$ of all continuous G-equivariant weights $\hat{G} \to \mathbb{R}^{\tau_*}_{>0}$ (see Section 5 for the precise definition) is in bijection with the decorated Teichmüller space $\tilde{T}(\mathcal{H})$, and the natural norm on $Cont^G(\hat{G}, \mathbb{R}^{\tau_*}_{>0})$ renders this map a homeomorphism. These are important properties of $\tilde{T}(\mathcal{H})$ in view of the problematic behavior of other universal Teichmüller spaces in this regard.

Theorem 5.3 *The assignment of lambda lengths is a homeomorphism onto*

$$\lambda : \tilde{T}(\mathcal{H}_G) \to Cont^G(\hat{G}, \mathbb{R}^{\tau_*}_{>0});$$

that is, we obtain a parametrization of $\tilde{T}(\mathcal{H}_G)$.

Whereas in the universal setting of [18], there are lambda length numbers assigned to each edge of τ_*, for the solenoid, there are continuous lambda length functions $\hat{G} \to \mathbb{R}_{>0}$ for each edge of τ_*. Furthermore, whereas the weak topology on the former corresponds to the compact-open topology on the space of homeomorphisms of the circle, it is the strong topology on lambda length functions which corresponds to the decorated Teichmüller theory of the solenoid.

The "convex hull construction" [19] is the basic construction in the decorated setting which provides combinatorial from geometric data. In the case of punctured surfaces, it gives a modular group invariant cell decomposition of decorated Teichmüller space, where cells in the decomposition are in one-to-one correspondence with decompositions of the surface into ideal polygons, or "pavings" (cf. Appendix B). In the universal setting of [18], the pinched condition on lambda lengths was shown to be sufficient to guarantee that the corresponding decomposition of $\mathbb{D}$ is again a locally finite one by ideal polygons with a possibly infinite number of sides (cf. Appendix A), and we shall again call such a decomposition a "paving" of $\mathbb{D}$. It turns out that continuity of the lambda length functions easily implies that the lambda lengths are pinched for each fixed element of $\hat{G}$ (see Lemma 6.1), so these aspects of [18] directly apply to the punctured solenoid.

We may choose a leaf of the foliation of $\mathcal{H}$ designated the "baseleaf". If τ is a paving of $\mathbb{D}$, let $\overset{\circ}{\mathcal{C}}(\tau)$ be the set of decorated hyperbolic structures on $\mathcal{H}$ so that the convex hull construction associates the decomposition τ of $\mathbb{D}$ on the baseleaf. As τ varies, this gives a decomposition of $\tilde{T}(\mathcal{H})$ which is invariant under the modular group. In contrast to the case of punctured surfaces (cf. Theorem B.5), it is not known whether each decomposition element for the solenoid is contractible, and indeed, it is not even known precisely which are non-empty.

A tractable class of pavings of $\mathbb{D}$ is provided by the "transversely locally constant", or TLC, ones. Namely, choose a subgroup K of $PSL_2(\mathbb{Z})$ of finite index without elliptics, choose a paving of the surface $\mathbb{D}/K$, and lift to the universal cover to get a K-invariant paving τ of $\mathbb{D}$ which is said to be TLC. These are obviously a very special class of pavings of $\mathbb{D}$, and yet they are generic in the following sense:

Theorem 6.2. *The subspace* $\overset{\circ}{\mathcal{C}}(\tau)$ *of* $\tilde{T}(\mathcal{H})$ *is open for each TLC triangulation* τ*, and* $\cup_\tau \overset{\circ}{\mathcal{C}}(\tau)$ *is a dense open subset of* $\tilde{T}(\mathcal{H})$*, where the union is over all TLC triangulations* τ *of* $\mathcal{H}$*.*

The *baseleaf preserving modular group* $Mod_{BLP}(\mathcal{H})$ consists of all quasiconformal baseleaf preserving self-maps of $\mathcal{H}$ up to isotopies. One achievement of this paper is to give an explicit generating set for $Mod_{BLP}(\mathcal{H})$ together with an explicit "normal form" for elements of $Mod_{BLP}(\mathcal{H})$. To this end, we first show that $Mod_{BLP}(\mathcal{H})$ acts transitively on $\{\overset{\circ}{\mathcal{C}}(\tau),\ \tau \text{ is a TLC triangulation of } \mathcal{H}\}$ and find the isotropy groups.

Theorem 7.6. $Mod_{BLP}(\mathcal{H}_G)$ *acts transitively on* $\{\overset{\circ}{\mathcal{C}}(\tau) : \tau \text{ is a TLC tesselation}\}$*. Furthermore, the isotropy subgroup in* $Mod_{BLP}(\mathcal{H}_G)$ *of* $\overset{\circ}{\mathcal{C}}(\tau)$ *is quasi-conformally conjugate to* $PSL_2(\mathbb{Z})$*.*

Fix an ideal triangulation τ of a punctured surface S. The Whitehead move on an edge $e \in \tau$ replaces e by the other diagonal of the quadrilateral in $(S-\tau)\cup e$ and keeps the remaining edges of τ fixed. In [19] (cf. Theorem B.4), a finite presentation was given for $Mod(S)$ generated by Whitehead moves and symmetries of top-dimensional cells, where certain sequences of Whitehead moves correspond to elements of $Mod(S)$; however, a single Whitehead move typically does not correspond to an element of $Mod(S)$.

We appropriately define Whitehead moves on any TLC triangulation τ of $\mathcal{H}$ in effect by performing the move K-equivariantly for some finite-index subgroup K as before. In this context, a single Whitehead move is an element of the modular group. A sequence of these elements is said to be "geometric" if there is actually a sequence of Whitehead moves along consecutive ideal triangulations underlying it. (See Section 8.) Moreover, we show that geometric sequences of Whitehead moves and $PSL_2(\mathbb{Z})$ generate the modular group of $\mathcal{H}$. A non-trivial generating set of the modular group for Sullivan's universal hyperbolic solenoid is not known.

Theorem 8.5 *Any element of the modular group* $Mod_{BLP}(\mathcal{H}_G)$ *can be written as a composition* $w\circ\gamma$*, where* $\gamma \in PSL_2(\mathbb{Z})$ *and* w *is a geometric composition of* K*-equivariant Whitehead homeomorphisms for some fixed* K*.*

We distinguish four relations on the generators above. For the detailed description, see Section 8. Three of the relations are already familiar from [18],[19]: the pentagon, commutativity and involutivity relations. (The fourth relation is related to cosets and is specific to the punctured solenoid.) In fact, the composition $\omega\circ\gamma$ in the above theorem is called a *normal form* of the element of $Mod_{BLP}(\mathcal{H})$, and we next give a necessary condition for equivalent normal forms.

Theorem 8.7 *Suppose* $\gamma_1,\gamma_2 \in PSL_2(\mathbb{Z})$ *and* ω_1,ω_2 *are two geometric Whitehead compositions with* $\omega_1\circ\gamma_1 = \omega_2\circ\gamma_2$*. Then there is some finite index subgroup* K *of* $PSL_2(\mathbb{Z})$ *with* $S=\mathbb{D}/K$ *so that up to Relations 1-4) for* K*,* $\omega_2^{-1}\circ\omega_1 = \gamma_2^{-1}\circ\gamma_1$ *is a finite composition* $\phi_1\circ\phi_2\circ\cdots\circ\phi_k$ *of automorphisms* $\phi_i \in Aut(\tau_i) < PSL_2(\mathbb{Z})$ *of tesselations* τ_i *of* S *without DOE, for* $i=1,\ldots,k$*.*

We show that an orbit of a point under $Mod_{BLP}(\mathcal{H})$ is not dense in $\tilde{T}(\mathcal{H})$, namely the Ehrenpreis Conjecture is false for $\tilde{T}(\mathcal{H})$. More precisely, we show that the space $\cup_\tau \overset{\circ}{\mathcal{C}}(\tau)/Mod_{BLP}(\mathcal{H}_G)$ is Hausdorff, where the union is over all TLC triangulations τ.

Theorem 7.7. *The quotient* $\cup_\tau \overset{\circ}{\mathcal{C}}(\tau)/Mod_P(\mathcal{H})$ *is Hausdorff, where the union is over all TLC triangulations* τ*. Moreover, no orbit of a point in* $\tilde{T}(\mathcal{H})$ *is dense.*

We finally introduce a generalization of the Weil-Petersson Kähler two form on $\tilde{T}(\mathcal{H})$ in two equivalent ways (see Proposition 9.1). This two form projects to the Teichmüller space $T(\mathcal{H})$:

Theorem 9.2. *The Weil-Petersson two form on* $\tilde{T}(\mathcal{H}_G)$ *projects to a non-degenerate two form on the Teichmüller space* $T(\mathcal{H}_G)$*. Moreover, the two form is invariant under* $Mod_{BLP}(\mathcal{H})$ *(see Theorem 9.4).*

This paper is organized as follows. Section 2 describes preliminary material including profinite completion and the universal cover of a solenoid. Section 3 then develops the Teichmüller theory of the punctured solenoid using quasiconformal mappings, while Section 4 introduces the representation-theoretic viewpoint of hyperbolic geometry. Section 5 then develops the decorated theory of the solenoid and gives the basic parametrization of decorated Teichmüller space by lambda length coordinate functions. Section 6 develops the convex hull construction in Minkowski space and leads to a canonical decomposition of decorated Teichmüller space; some of the background material as well as other basic definitions and results are surveyed in Appendix A, which probably should be read before Section 6. Section 7 then introduces the mapping class group of the punctured solenoid, presents basic results about it, and furthermore analyzes its action on the decomposition elements in decorated Teichmüller space. In Section 8, we introduce a collection of elements of the mapping class group derived from Whitehead moves, and by comparing their action on the decomposition elements with that of the modular group, we conclude that, together with $PSL_2(\mathbb{Z})$, they generate this group. Finally in Section 9, a two form parallel to the Weil-Petersson Kähler form is introduced, and its basic properties are described. The short section 10 contains closing remarks and open questions. Appendix A surveys background material on homeomorphisms of the circle from [18], and Appendix B surveys background material on punctured surfaces of finite type from [19] and explicates the presentation of their modular groups.

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2. Preliminaries

Let (S,x) be a punctured torus with a base point $x \in S - \{\text{puncture}\}$. Consider the set of all pointed unbranched finite coverings $\pi_i : (S_i,x_i) \to (S,x)$ by punctured surfaces (S_i,x_i) such that $\pi_i(x_i) = x$ for all i. There is a natural partial ordering $\leq$ on the set of coverings as follows. $(S_i,x_i) \leq (S_j,x_j)$ if $\pi_j : (S_j,x_j) \to (S,x)$ factors through $\pi_i : (S_i,x_i) \to (S,x)$, namely, if there is a pointed unbranched finite covering $\pi_{j,i} : (S_j,x_j) \to (S_i,x_i)$ such that $\pi_j = \pi_{j,i} \circ \pi_i$. The system of coverings (S_i,x_i) is an inverse system because given any two coverings there exists a third covering which is greater than or equal to both.

Definition 2.1. The punctured solenoid $\mathcal{H}$ is the inverse limit of the system of coverings of (S,x).

If we start the above construction from any other punctured surface of negative Euler characteristic, we obtain a homemorphic inverse limit. To see this, observe that any two pointed punctured surfaces of negative Euler characteristic have a common pointed cover. This implies that they have a common co-final subsystem of coverings which implies that the inverse limits are homeomorphic. The punctured solenoid $\mathcal{H}$ is an initial object for the category of punctured surfaces with negative Euler characteristic with morphisms finite unbrached covers.

$\mathcal{H}$ is locally homeomorphic to (2-disk)$\times$(Cantor set). The path components are called leaves. Each leaf is homeomorphic to the unit disk and is dense in $\mathcal{H}$. These observations are made in similar fashion to the case of the inverse limit of coverings of a compact surface of genus greater than one (Sullivan's universal hyperbolic solenoid). However, the punctured solenoid is not a compact space unlike that of Sullivan [23]. To see this, note that $\mathcal{H}$ is a closed subset of the product of all covering surfaces of the punctured torus consisting of all backward trajectories with initial points on the punctured torus. A sequence of backward trajectories with the first coordinates converging to the puncture on the base punctured torus has no convergent subsequence.

Let G be any finite index subgroup of $PSL_2(\mathbb{Z})$, so $\mathbb{D}/G$ is an orbifold. G has characteristic subgroups

$$G_N = \cap\{\Gamma < G : [\Gamma : G] \leq N\},$$

for each $N \geq 1$, and these are nested $G_N \leq G_{N+1}$. Define a metric on G by taking the distance between $\gamma, \delta \in G$ to be

$$\min\{N^{-1} : \gamma\delta^{-1} \in G_N\},$$

and define the *profinite completion* $\hat{G}$ of G as the metric completion of G, i.e., equivalence classes of Cauchy sequences in G for the above metric. Termwise multiplication of Cauchy sequences gives $\hat{G}$ the structure of a topological group. Moreover, $\hat{G}$ is a compact space homeomorphic to a Cantor set (see for example [16]). G has a natural embedding into $\hat{G}$ by mapping each $g \in G$ to the constant Cauchy sequence $[g,g,\ldots] \in \hat{G}$, and the image of G is dense in $\hat{G}$ by definition. Since G is naturally a subgroup of $\hat{G}$, we get a continuous right action of G on $\hat{G}$, i.e., $g \times [a_1,a_2,\ldots] \mapsto [a_1g^{-1},a_2g^{-1},\ldots] = [a_1,a_2,\ldots]g^{-1}$ for $g \in G$ and $[a_1,a_2,\ldots] \in \hat{G}$.

To alternatively define the punctured solenoid $\mathcal{H}$ as a topological space, for definiteness fix the once-punctured torus group $G < PSL_2(\mathbb{Z})$, the unique charactersitic subgroup of index six, let $\hat{G}$ denote its profinite completion, and define the "G-tagged solenoid"

$$\mathcal{H}_G = (\mathbb{D} \times \hat{G})/G,$$

where $\gamma \in G$ acts on $(z,t) \in \mathbb{D} \times \hat{G}$ by $\gamma(z,t) = (\gamma z, t\gamma^{-1})$. The space $\mathcal{H}_G$ is homeomorphic to $\mathcal{H}$ following the proof of [16] in the compact case.

As in definition 2.1, we can start from any finite index subgroup of G and repeat the above construction to obtain space homeomorphic to $\mathcal{H}_G$. The construction of $\mathcal{H}_G$ using a particular group G yields more structure on the leaves, namely, the hyperbolic metric on leaves. Different choices of Fuchsian groups give different hyperbolic metrics on the solenoid if the groups have no common finite index subgroups (see [15]).

The topological picture of $\mathcal{H}$ is easy to understand in the model $\mathcal{H}_G$. Namely (in the notation of the introduction), let $\gamma, \delta \in G$ be the hyperbolic generators of G which map $e_{1/2}$ and $e_{2/1}$ onto $e_{-2/1}$ and $e_{-1/2}$, respectively. Let Q be the quadrilateral in $\mathbb{D}$ with boundary sides $e_{1/2}, e_{2/1}, e_{-2/1}$ and $e_{-1/2}$. Thus, Q is a fundamental polygon for G, and g and h are side identifications of Q. The set $Q \times \hat{G}$ is a fundamental set for the action of G on $\mathbb{D} \times \hat{G}$. The side $e_{1/2} \times t$ is identified with $e_{-2/1} \times t\gamma^{-1}$ and the side $e_{2/1} \times t$ is identified with $e_{-1/2} \times t\delta^{-1}$, for all $t \in \hat{G}$. These are the only identifications by G on the fundamental set $Q \times \hat{G}$. Therefore, the punctured solenoid is obtained by "sewing" the Cantor set of polygons along their boundary sides according to the action of G on $\hat{G}$. This immediately shows that leaves are unit disks and that each leaf is dense (by the density of G in $\hat{G}$).

3. Quasiconformal definition of the Teichmüller space $T(\mathcal{H}_G)$

In this section, we define the Teichmüller space of the punctured solenoid $\mathcal{H}_G \equiv \mathbb{D} \times_G \hat{G}$ in the spirit of Ahlfors-Bers theory [2] and in such fashion that the union of the natural lifts of Teichmüller spaces of all finite sheeted covers of the modular curve is dense. The condition of continuity for the transverse variation, which is familiar from the compact case [23] and [22], requires an extra stipulation in its definition in the punctured case due to the non-compactness.

Fix a metric (which is complete with constant curvature -1) on $\mathcal{H}_G$ coming from the lift of the hyperbolic metric on the modular curve (or the covering of the modular curve by the punctured torus with the covering group $G < PSL_2\mathbb{Z}$). This is a transversely locally constant metric, namely there exists a subatlas of $\mathcal{H}_G$ with the metric being constant for the transverse variation in the charts of the subatlas. Let $\mathcal{H}_d$ be the differentiable structure on the solenoid compatible with the hyperbolic metric on $\mathcal{H}_G$.

Definition 3.1. A *hyperbolic metric* on $\mathcal{H}_d$ is an assignment of a hyperbolic metric (a complete metric of curvature -1) on each leaf that varies continuously (in the C^∞-topology) for the transversal variations in the local charts.

A hyperbolic metric on $\mathcal{H}_d$ gives a complex structure on $\mathcal{H}_d$. The complex structure varies continuously for the transverse variations on the local charts. The converse also holds [5].

Let $\mathcal{H}$ be an arbitrary complex solenoid with the underlying differentiable solenoid $\mathcal{H}_d$. As a preliminary for the definition of the Teichmüller space, we define a (differentiable) quasiconformal map from the fixed solenoid $\mathcal{H}_G$ onto $\mathcal{H}$. Recall that a leaf of the solenoid intersects any local chart infinitely many times. Each component of the intersection is called a *local leaf* of the solenoid.

A chart for $\mathcal{H}_G$ is modeled on an open set in (hyperbolic disk)$\times$(transverse Cantor set). Since the natural projection $\pi : \mathbb{D} \times \hat{G} \to (\mathbb{D} \times \hat{G})/G$ is a local homeomorphism, there exists $U \subset \mathbb{D}$ such that the set $U \times \hat{G}$ together with the map $\pi : U \times \hat{G} \to \mathcal{H}_G$ is a local chart for $\mathcal{H}_G$. In particular, we choose $U \subset \mathbb{D}$ to be a hyperbolic disk with center $0 \in \mathbb{D}$ and radius smaller than the injectivity radius of $\mathbb{D}/G$. The chart map $\pi : U \times \mathbb{D} \to \mathcal{H}_G$ is an isometry on the leaves, and the natural identification $U \times t \equiv U \times t_1$ is also an isometry, for all $t, t_1 \in \hat{G}$. For two fixed global leaves of $\mathcal{H}_G$ and a choice of their corresponding local leaves in the chart $U \times \hat{G}$, there is thus a preferred isometric identification of the two local leaves. This gives an isometric identification between two open sets of the global leaves, and this identification extends uniquely to a preferred isometric identification of the two global leaves themselves.

Definition 3.2. A homeomorphism $f : \mathcal{H}_G \to \mathcal{H}$ is said to be a *(differentiable) K-quasiconformal map* if its restriction to each leaf is a K-quasiconformal C^∞-map, if the restriction of f to any local chart varies continuously for the transversal variation in the C^∞-topology, and if f varies continuously (in the quasiconformal topology) for the transversal variation on global leaves (with two leaves identified using their local leaves in the fixed chart as above).

Remark 3.3. The quasiconformal topology is induced by the semi-norm on the space of quasiconformal maps: the distance between two quasiconformal maps f and g is induced by the sup norm of the Beltrami differential of the map $f \circ g^{-1}$. (This is only a semi-norm because the Beltrami coefficient of a conformal map vanishes identically.) The quasiconformal topology induces a topology on the quotient of the space of quasiconformal maps by post-composition with Möbius transformations, and the resulting space is Hausdorff.

Remark 3.4. The natural requirement of continuity in the C^∞-topology for the transversal variations in the local charts is given by Sullivan [23] (see also [22]) for compact Riemann surface laminations although other transverse structures are also possibly interesting. In the case of the universal hyperbolic compact solenoid (see [23], [15] and [22]), this is enough to guarantee that the union of Teichmüller spaces of compact surfaces is dense in the Teichmüller space of the compact solenoid. However, for the punctured solenoid we require the additional quasiconformal continuity because of the non-compactness.

To make sense of the identifications of the different global leaves, we are forced to start from a locally constant hyperbolic structure to make a proper choice of the identifications of the global leaves.

Definition 3.5. The *Teichmüller space* $T(\mathcal{H}_G)$ of the punctured solenoid $\mathcal{H}_G$ is the space of all differentiable quasiconformal maps $f : \mathcal{H}_G \to \mathcal{H}$, where $\mathcal{H}$ is an arbitrary hyperbolic solenoid with the underlining differentiable solenoid $\mathcal{H}_d$ up to the following equivalence relation: two maps $f_1 : \mathcal{H}_G \to \mathcal{H}_1$ and $f_2 : \mathcal{H}_G \to \mathcal{H}_2$ are *Teichmüller equivalent* if there is a hyperbolic isometry $c : \mathcal{H}_1 \to \mathcal{H}_2$ such that $f_2^{-1} \circ c \circ f_1 : \mathcal{H}_G \to \mathcal{H}_G$ is homotopic to the identity by a bounded homotopy with respect to the hyperbolic metric.

We already introduced the universal covering of $\mathcal{H}_G$ by the natural projection map $\pi : \mathbb{D} \times \hat{G} \to (\mathbb{D} \times \hat{G})/G$. Note that the restriction of π to each leaf of the covering is an isometry for the hyperbolic metrics of the leaf in the covering and the leaf of $\mathcal{H}_G$. However, countably many leaves of the covering are mapped onto a single leaf of $\mathcal{H}_G$. Given an arbitrary hyperbolic punctured solenoid $\mathcal{H}$, there exists a quasiconformal map $f : \mathcal{H}_G \to \mathcal{H}$. We next introduce a universal covering of $\mathcal{H}$ using the existence of f, similarly to [22].

Let $U \times \hat{G}$, $0 \in U$, be a local chart of $\mathcal{H}_G$ such that $f|_{(\pi(U \times \hat{G}))}$ is a homeomorphism. Choose a local chart $V \times T \to \mathcal{H}$ for $\mathcal{H}$ which contains $f(\pi(0 \times \hat{G}))$ and let $\psi : V \times T \to \mathcal{H}$ be isometric local chart map. Identify T with $\hat{G}$ via the correspondence $\hat{G} \to 0 \times \hat{G} \to f(0 \times \hat{G}) \to T$. Finally, define the universal covering $\pi_{\mathcal{H}} : \mathbb{D} \times \hat{G} \to \mathcal{H}$ as follows. Choose a continuous isometric embedding $i : V \times \hat{G} \to \mathbb{D} \times \hat{G}$ such that $i \circ \psi^{-1}(f(0 \times \hat{G})) = 0 \times \hat{G}$ and define $\pi_{\mathcal{H}}$ by extending the local isometry $i \circ \psi^{-1}$ from $V \times \hat{G} \subset \mathbb{D} \times \hat{G}$ to the global leafwise isometry from $\mathbb{D} \times \hat{G}$ to $\mathcal{H}_G$. Note that $f : \mathcal{H}_G \to \mathcal{H}$ lifts to a quasiconformal map $\tilde{f} : \mathbb{D} \times \hat{G} \to \mathbb{D} \times \hat{G}$ by the formula $\tilde{f}(z,t) = \pi_{\mathcal{H}}^{-1}(f \circ \pi(z,t), t)$, for $z \in \mathbb{D}$ and $t \in \hat{G}$. Further, the action of any $\gamma \in G$ on $\mathbb{D} \times \hat{G}$ is conjugated by $\tilde{f}$ to the action on the universal cover of $\mathcal{H}$. This conjugated action is also an isometry from one leaf onto another, but it depends on the leaf, i.e., it is not constant in $t \in \hat{G}$ (this follows from the formula for $\tilde{f}$, or see [22] for a similar argument). In the proposition below, we construct the hyperbolic punctured solenoid by constructing first the universal covering and the covering group from a Beltrami coefficient, thus reversing the above construction.

The leafwise Beltrami coefficient μ on $\mathcal{H}_G$ of a differentiable quasiconformal map $f : \mathcal{H}_G \to \mathcal{H}$ is smooth, $\|\mu\|_\infty \leq k < 1$, it varies continuously for the transversal variations in the smooth topology and $\|\mu_{t_1} - \mu_t\|_\infty \to 0$ as $t_1 \to t$, where μ_{t_1}, μ_t are the restrictions of μ to the global leaves t_1, t transversely identified as above with the unit disk $\mathbb{D}$. Conversely, we show

Proposition 3.6. *Let μ be a Beltrami coefficient on $\mathcal{H}_G$ so that μ is smooth, has L^∞-norm bounded above by $k<1$, varies continuously for transverse variations in the smooth topology, and so that $\|\mu_{t_1}-\mu_t\|_\infty \to 0$ as $t_1 \to t$ in the transverse direction. Then there exists a hyperbolic punctured solenoid $\mathcal{H}$ and a (differentiable) quasiconformal map $f:\mathcal{H}_G \to \mathcal{H}$ such that the Beltrami coefficient of f is μ. Moreover, f and $\mathcal{H}$ are unique up to post-composition with a hyperbolic isometry.*

Proof. The natural quotient map $\pi:\mathbb{D}\times\hat{G}\to\mathbb{D}\times_G\hat{G}=\mathcal{H}_G$ is the universal covering map of $\mathcal{H}_G$. Denote by $\tilde{\mu}$ the lift of μ to the universal covering; hence $\tilde{\mu}$ satisfies

$$\tilde{\mu}(\gamma z, t\gamma^{-1})\ \overline{\gamma'(z)} = \tilde{\mu}(z,t)\ \gamma'(z) \tag{1}$$

for all $\gamma\in G$.

Thus, $\tilde{\mu}(\cdot,t)$, for $t\in\hat{G}$, varies continuously with t in the norm $\|\cdot\|_\infty$ on $\mathbb{D}\equiv\mathbb{D}\times\{t\}$ by definition of μ. We solve the Beltrami equation for $\tilde{\mu}(\cdot,t)$ on each $\mathbb{D}\times\{t\}$ such that the solution $\tilde{f}_t:\mathbb{D}\times\{t\}\to\mathbb{D}\times\{t\}$ fixes 1, -1 and $-i$.

The resulting $\tilde{f}_t$ is smooth and varies continuously in the smooth and quasiconformal topology by the corresponding properties of $\tilde{\mu}$ (see [2]). By (1), the maps $\tilde{f}_t$ conjugate the action of $\gamma\in G$ on $\mathbb{D}\times\hat{G}$ into the action of $\gamma_\mu\in G_\mu$ on $\mathbb{D}\times\hat{G}$, where each γ_μ acts by isometry leafwise and varies continuously for the transverse variation. Thus, $\tilde{f}$ projects to a homeomorphism $f:\mathbb{D}\times\hat{G}/G\to\mathbb{D}\times\hat{G}/G_\mu$ which is a differentiable quasiconformal map from $\mathcal{H}_G$ to the hyperbolic punctured solenoid $\mathcal{H}=\mathbb{D}\times\hat{G}/G_\mu$. (For a similar construction, see [22, Section 2].) □

We thus obtain an equivalent definition of $T(\mathcal{H}_G)$:

Definition 3.7. The *Teichmüller space* $T(\mathcal{H}_G)$ is the space of all Beltrami coefficients on $\mathcal{H}_G$ which satisfy the conditions of Proposition 3.1 up to an equivalence. Two Beltrami coefficients are *equivalent* if their corresponding quasiconformal maps are equivalent.

Sullivan [23] showed that union of the Teichmüller spaces of all compact surfaces is dense in the Teichmüller space of the compact solenoid, and we next prove the corresponding result for the punctured solenoid. Any punctured surface of finite area is covered by the punctured solenoid, and in fact, this covering is a principal fiber bundle. The Teichmüller space of a punctured surface lifts to $T(\mathcal{H}_G)$ by lifting hyperbolic metrics on the surface to locally constant hyperbolic metrics on $\mathcal{H}_d$. For example, the hyperbolic metric on $\mathcal{H}_G$ is the lift of the hyperbolic metric on the modular curve $\mathbb{D}/PSL_2(\mathbb{Z})$, which in turn lifts to a hyperbolic metric on the punctured torus or any other surface covering the modular curve.

Theorem 3.8. *The union of natural lifts of Teichmüller spaces of all punctured hyperbolic surfaces is dense in the Teichmüller space $T(\mathcal{H}_G)$.*

Proof. It is enough to show that for any Beltrami coefficient μ on $\mathcal{H}_G$ there is a sequence of locally constant Beltrami coefficients μ_n which approximate μ in the Teichmüller topology.

Let $\tilde{\mu}$ be the lift of μ to $\mathbb{D}\times\hat{G}$, so $\tilde{\mu}$ satisfies (1). Let G_n be the intersection of all index at most n subgroups of G (see Section 2). Let P be a fundamental polygon for the action of G_n on $\mathbb{D}$. We define $\tilde{\mu}_n(z,t)=\tilde{\mu}(z,id)$ for $z\in P$ and $t\in\hat{G}_n$, and extend $\tilde{\mu}_n$ to $\mathbb{D}\times\hat{G}_n$ by the action of G_n. Since $\mathbb{D}\times\hat{G}/G\equiv\mathbb{D}\times\hat{G}_n/G_n$ and $\tilde{\mu}_n$ is close to $\tilde{\mu}$ in the Teichmüller topology on each leaf, we obtain the required locally constant sequence approximating $\tilde{\mu}$. We may replace $\tilde{\mu}_n$, for instance with the barycentric extension of its boundary values (see Douady-Earle [7]), to produce a Teichmüller equivalent Beltrami coefficient which is smooth and transversely continuous for both the smooth and quasiconformal topologies. □

4. The representation-theoretic definition of $T(\mathcal{H}_G)$

We give an alternative definition of the Teichmüller space using the universal covering and the representation of the covering group of the solenoid. This definition is motivated by the finite-dimensional Teichmüller theory of punctured surfaces. We use the universal covering construction above which was adopted from [22] to the case of the punctured solenoid.

Let us consider the collection $Hom(G \times \hat{G}, PSL_2\mathbb{R})$ of all functions $\rho : G \times \hat{G} \to PSL_2\mathbb{R}$ satisfying the following three properties:

Property 1: ρ is continuous;

Property 2 [G-equivariance]: for each $\gamma_1, \gamma_2 \in G$ and $t \in \hat{G}$, we have

$$\rho(\gamma_1 \circ \gamma_2, t) = \rho(\gamma_1, t\gamma_2^{-1}) \circ \rho(\gamma_2, t);$$

Property 3: for every $t \in \hat{G}$, there is a quasiconformal mapping $\phi_t : \mathbb{D} \to \mathbb{D}$ depending continuously on $t \in \hat{G}$ so that for every $\gamma \in G$, the following diagram commutes:

$$\begin{array}{lcl}
(z,t) & \mapsto & (\gamma z, t\gamma^{-1}) \\
\mathbb{D} \times \hat{G} & \to & \mathbb{D} \times \hat{G} \\
\phi_t \times \mathrm{id} \downarrow & & \downarrow \phi_{t\gamma^{-1}} \times \mathrm{id} \\
\mathbb{D} \times \hat{G} & \to & \mathbb{D} \times \hat{G} \\
(\phi_t(z), t) & \mapsto & (\rho(\gamma,t) \circ \phi_t(z) = \phi_{t\gamma^{-1}} \circ \gamma(z), t\gamma^{-1})
\end{array}$$

As to property 1, notice that since G is discrete, ρ is continuous if and only if it is so in its second variable only. Property 2 is a kind of homomorphism property of ρ mixing the leaves; notice in particular that taking $\gamma_2 = I$ gives $\rho(I,t) = I$ for all $t \in \hat{G}$. Property 3 mandates that for each $t \in \hat{G}$, ϕ_t conjugates the standard action of $\gamma \in G$ on $\mathbb{D} \times \hat{G}$ at the top of the diagram to the action

$$\gamma_\rho : (z,t) \mapsto (\rho(\gamma,t)z, t\gamma^{-1})$$

at the bottom, and we let $G_\rho = \{\gamma_\rho : \gamma \in G\} \approx G$. Notice that the action of G_ρ on $\mathbb{D} \times \hat{G}$ extends continuously to an action on $S^1 \times \hat{G}$. We finally define the solenoid (with marked hyperbolic structure)

$$\mathcal{H}_\rho = (\mathbb{D} \times_\rho \hat{G}) = (\mathbb{D} \times \hat{G})/G_\rho.$$

Define the group $Cont(\hat{G}, PSL_2\mathbb{R})$ to be the collection of all continuous maps $\alpha : \hat{G} \to PSL_2\mathbb{R}$, where the product of two $\alpha, \beta \in Cont(\hat{G}, PSL_2\mathbb{R})$ is taken pointwise $(\alpha\beta)(t) = \alpha(t) \circ \beta(t)$ in $PSL_2\mathbb{R}$. $\alpha \in Cont(\hat{G}, PSL_2\mathbb{R})$ acts continuously on $\rho \in Hom(G \times \hat{G}, PSL_2\mathbb{R})$ according to

$$(\alpha\rho)(\gamma, t) = \alpha(t\gamma^{-1}) \circ \rho(\gamma, t) \circ \alpha^{-1}(t).$$

We introduce the topology on $Hom(G \times \hat{G}, PSL_2\mathbb{R})$. Consider the natural metric d on $PSL_2\mathbb{R}$ induced by identifying it with the unit tangent bundle of the unit disk $\mathbb{D}$. Let $\rho_1, \rho_2 \in Hom(G \times \hat{G}, PSL_2\mathbb{R})$ and let $\gamma_1, \ldots, \gamma_j \in G$ be a generating set of G. The distance between ρ_1 and ρ_2 is given by

$$\max_{1 \le i \le j,\ t \in \hat{G}} d(\rho_1(\gamma_i, t), \rho_2(\gamma_i, t)). \tag{2}$$

This metric is not canonical, but any such two metrics induce the same topology.

The topology on $Hom'(G \times \hat{G}, PSL_2\mathbb{R}) = Hom(G \times \hat{G}, PSL_2\mathbb{R})/Cont(\hat{G}, PSL_2\mathbb{R})$ is the quotient topology of the above topology on $Hom(G \times \hat{G}, PSL_2\mathbb{R})$.

Theorem 4.1. *There is a natural homeomorphism of the Teichmüller space of the solenoid $\mathcal{H}_G$ with*

$$Hom'(G \times \hat{G}, PSL_2\mathbb{R}) = Hom(G \times \hat{G}, PSL_2\mathbb{R})/Cont(\hat{G}, PSL_2\mathbb{R}).$$

Proof. Let μ be a Beltrami coefficient on $\mathcal{H}_G$ which represents a point in $T(\mathcal{H}_G)$. Let $\tilde{\mu}$ be the lift of μ to $\mathbb{D}\times\hat{G}$ and let $\tilde{f}:\mathbb{D}\times\hat{G}\to\mathbb{D}\times\hat{G}$ be the leafwise solution of the corresponding Beltrami equation which fixes 1, -1 and $-i$ on the boundary of each leaf, as in Proposition 3.6. Since $\tilde{\mu}$ satisfies (1), then for each $\gamma\in G$ there exists $\gamma_\mu:\mathbb{D}\times\hat{G}\to\mathbb{D}\times\hat{G}$ which is leafwise a Möbius transformation such that

$$(\tilde{f}\circ\gamma)(z,t)=(\gamma_\mu\circ\tilde{f})(z,t) \tag{3}$$

for all $(z,t)\in\mathbb{D}\times\hat{G}$ and $\gamma\in G$. We note that $\tilde{f}$ fixes each leaf of $\mathbb{D}\times\hat{G}$, and that γ and γ_μ exchange leaves in the same fashion $\mathbb{D}\times\{t\}\mapsto\mathbb{D}\times\{t\gamma^{-1}\}$.

We define the representation $\rho_\mu:G\times\hat{G}\to PSL_2\mathbb{R}$ by

$$\rho_\mu(\gamma,t)=\gamma_\mu(\cdot,t) \tag{4}$$

for $\gamma\in G$ and $t\in\hat{G}$.

We check that properties 1, 2 and 3 hold. To see that ρ_μ is continuous, note that an element of $PSL_2\mathbb{R}$ is uniquely determined by specifying the image of three arbitrary points on S^1, and that it depends continuously on this image. Therefore, Property 1 holds because $\tilde{f}$ varies continuously on $S^1\times\hat{G}$ in the C^0-topology.

Property 2 follows from the computation:

$$\rho_\mu(\gamma_1\circ\gamma_2,t)=\tilde{f}\circ\gamma_1\circ\gamma_2\circ\tilde{f}^{-1}(\cdot,t)=\tilde{f}\circ\gamma_1\circ\tilde{f}^{-1}\circ\tilde{f}\circ\gamma_2\circ\tilde{f}^{-1}(\cdot,t)=$$
$$[(\tilde{f}\circ\gamma_1\circ\tilde{f}^{-1})(\cdot,t\gamma_2^{-1})]\circ[(\tilde{f}\circ\gamma_2\circ\tilde{f}^{-1})(\cdot,t)]=\rho_\mu(\gamma_1,t\gamma_2^{-1})\circ\rho_\mu(\gamma_2,t).$$

Thus, $\phi_t:=\tilde{f}(\cdot,t)$ is a quasiconformal map which satisfies Property 3.

Upon taking quotients, the assignment (4) induces a well-defined map from $T(\mathcal{H}_G)$ to $Hom'(G\times\hat{G},PSL_2\mathbb{R})=Hom(G\times\hat{G},PSL_2\mathbb{R})/Cont(\hat{G},PSL_2\mathbb{R})$. To see this, simply note that $f_t|_{S^1}$ up to post-composition with an element of $PSL_2\mathbb{R}$ depends only on the Teichmüller class of the Beltrami coefficient μ. Thus, γ_μ is determined up to the same ambiguity, and the map is well-defined on the quotients.

We prove the continuity of this map at an arbitrary $[\mu_1]\in T(\mathcal{H}_G)$, where $[\mu_1]$ denotes the Teichmüller class of the Beltrami coefficient μ_1. Let μ_2 be a Beltrami coefficients representing a points in $T(\mathcal{H}_G)$ in a small neighborhood of $[\mu_1]$ such that $\|\mu_1-\mu_2\|_\infty\to 0$ as $[\mu_2]\to[\mu_1]$. Their corresponding maps $\tilde{f}_1$ and $\tilde{f}_2$ are thus close as quasiconformal maps for each $t\in\hat{G}$, and they both fix 1, -1 and i on each leaf.

Recall that the group $G<PSL_2\mathbb{Z}$ for the punctured torus is generated by two hyperbolic elements γ_1 and γ_2 which carry $e_{1/2}$ onto $e_{-2/1}$ and $e_{2/1}$ onto $e_{-1/2}$, respectively.

It follows that G_{μ_k} are generated by their conjugates $\gamma_1^{\mu_k},\gamma_2^{\mu_k}$, for $k=1,2$. To estimate $d(\gamma_1^{\mu_1}(\cdot,t),\gamma_1^{\mu_2}(\cdot,t))$ and $d(\gamma_2^{\mu_1}(\cdot,t),\gamma_2^{\mu_2}(\cdot,t))$, it is enough to compare the corresponding images of -1, $-i$ and 1, for all $t\in\hat{G}$. To this end, we use the formula

$$\gamma_1^{\mu_k}(\cdot,t)=\tilde{f}_k(\cdot,t\gamma_1^{-1})\circ\gamma_1(\cdot,t)\circ\tilde{f}_k(\cdot,t)^{-1},$$

for all $t\in\hat{G}$, and similarly for $\gamma_2^{\mu_k}$. Thus, $\gamma_1^{\mu_k}(-1,t)=\tilde{f}_k(i,t\gamma_1^{-1})$, $\gamma_1^{\mu_k}(-i,t)=\tilde{f}_k(1,t\gamma_1^{-1})$ and $\gamma_1^{\mu_k}(1,t)=\tilde{f}_k(\gamma_1(1),t\gamma_1^{-1})$. Therefore, it is enough to show that pairs $\tilde{f}_1(i,t\gamma_1^{-1})$ and $\tilde{f}_2(i,t\gamma_1^{-1})$, $\tilde{f}_1(1,t\gamma_1^{-1})$ and $\tilde{f}_2(1,t\gamma_1^{-1})$, and $\tilde{f}_1(\gamma_1(1),t\gamma_1^{-1})$ and $\tilde{f}_2(\gamma_1(1),t\gamma_1^{-1})$ are close in the angle metric on S^1. This follows since $\tilde{f}_1$ and $\tilde{f}_2$ are properly normalized and their Beltrami coefficients are close in the supremum norm uniformly in $t\in\hat{G}$. The similar statement holds for $\gamma_2^{\mu_k}$. The continuity is proved.

Assume that $\rho_{\mu_1}=\rho_{\mu_2}=\rho$ for two Beltrami coefficients μ_1,μ_2 representing elements of $T(\mathcal{H}_G)$. We need to show that $\tilde{f}_1(\cdot,t)|_{S^1}=\tilde{f}_2(\cdot,t)|_{S^1}$ for all $t\in\hat{G}$. This implies that there is a bounded homotopy through quasiconformal maps between $\tilde{f}_1$ and $\tilde{f}_2$ which is invariant under G_ρ (the proof for the compact solenoid [14] extends directly to the punctured solenoid), which says that μ_1 and μ_2 are Teichmüller equivalent, so the map is one to one.

It is enough to show that $\tilde{f}_1(x,t)=\tilde{f}_2(x,t)$ for all $x\in\bar{\mathbb{Q}}$ and $t\in\hat{G}$. Note that the equality holds at $1/0,0/1,1/1\in\bar{\mathbb{Q}}$ by our normalization. Furthermore, at least one edge of the triangle with vertices $1/0,0/1,1/1$ is mapped onto any other edge of the Farey tesselation by an element of the once-punctured torus group G, where G is generated by two hyperbolic elements γ_1 and γ_2. We proceed inductively. Namely, assume that $x\in\bar{\mathbb{Q}}$ is one vertex of edge $e\in\tau_*$ and that at vertices of $e_1\in\tau_*$ the equality of the two maps hold for all $t\in\hat{G}$ and that $\gamma_i(e_1)=e$ for either $i=1$ or $i=2$. Let x_1 be the vertex of e_1 such that $\gamma_i(x_1)=x$. By (3), it follows that $\tilde{f}_j(x,t)=(\gamma_i^{\mu_j}\circ\tilde{f}_j)(x_1,t\gamma_i)$ for $j=1,2$. By our assumption, $\gamma_i^{\mu_1}\equiv\gamma_i^{\mu_2}$

and $\tilde{f}_1(x_1, t\gamma_i) = \tilde{f}_2(x_1, t\gamma_i)$ for all $t \in \hat{G}$. Thus, $\tilde{f}_1(x, t) = \tilde{f}_2(x, t)$ for all $t \in \hat{G}$. The inductive step is complete, and the injectivity follows.

The map is surjective because $\mathbb{D} \times \hat{G}/G_\rho$ is a quasiconformal image of $\mathcal{H}_G$. The quasiconformal map is the projection of the family $\phi_t : \mathbb{D} \to \mathbb{D}$, for $t \in \hat{G}$, to the quotients $\mathcal{H}_G$ and $\mathcal{H}_\rho$ obtained by the actions of G and G_ρ on $\mathbb{D} \times \hat{G}$.

We must finally show that the inverse map is continuous. Suppose that ρ_1 and ρ_2 are two representations of the group G which are close in the metric (2). Let $\phi_1, \phi_2 : \mathbb{D} \times \hat{G} \to \mathbb{D} \times \hat{G}$ be conjugating maps for ρ_1, ρ_2 respectively, from Property 3. It is enough to find another pair of conjugating maps $\tilde{f}_1, \tilde{f}_2$ such that the quasiconformal constant of $\tilde{f}_2 \circ \tilde{f}_1^{-1}$ converges to 1 as ρ_1 converges to ρ_2.

Let Q be the ideal hyperbolic rectangle inside $\mathbb{D}$ with vertices 1, i, -1 and $-i$, so $Q \times \hat{G}$ is a fundamental domain for the action of the punctured torus group G on $\mathbb{D} \times \hat{G}$. Let $\gamma_1, \gamma_2 \in G$ be fixed hyperbolic generators of the punctured torus group G as above. The sides of $Q \times t$ are identified to the sides of $Q \times t\gamma_1^{-1}$ and $Q \times t\gamma_2^{-1}$, for $t \in \hat{G}$. Note that $\mathbb{D} \times \hat{G}/G \equiv Q \times \hat{G}/G \equiv \mathcal{H}_G$.

Let $G_k = \rho_k(G)$ and let Q_k^t be the ideal rectangle with vertices -1, $-i$, 1 and $i_k := \phi_k(i, t)$, for $k = 1, 2$ and $t \in \hat{G}$. Thus, $F_k = \cup_{t \in \hat{G}} Q_k^t \times \{t\}$ is a fundamental set for G_k.

It is enough to find a quasiconformal map $\tilde{f}$ between F_1 and F_2 with small quasiconformal constant such that $\tilde{f} = I$ on the geodesics $e_{1/2} \times t$ and $e_{2/1} \times t$, $\tilde{f} = \rho_2(\gamma_1, t) \circ \rho_1(\gamma_1, t)^{-1}$ on $\phi_1(e_{-2/1}, t\gamma_1^{-1})$ and $\tilde{f} = \rho_2(\gamma_2, t) \circ \rho_1(\gamma_2, t)^{-1}$ on $\phi_1(e_{-1/2}, t\gamma_2^{-1})$. By assumption, the covering maps $\rho_1(\gamma_j, t)$ and $\rho_2(\gamma_j, t)$, $j = 1, 2$ are close and the rectangles Q_1^t and Q_2^t are almost equal, so there exists a quasiconformal map $\tilde{f}$ with small dilatation which satisfies above boundary conditions. Furthermore, $\tilde{f} : F_1 \to F_2$ lifts and extends to a quasiconformal self-map of $\mathbb{D} \times \hat{G}$ using the actions of the covering groups G_k, for $k = 1, 2$. Finally, taking $\tilde{f}_1 = \phi_1$ and $\tilde{f}_2 = \tilde{f} \circ \phi_1$, the continuity of the inverse map follows. □

5. The decorated Teichmüller space $\tilde{T}(\mathcal{H}_G)$

Let $\mathcal{H}_\rho$ be a hyperbolic solenoid obtained from the representation ρ with corresponding quasiconformal map $\phi : \mathcal{H}_G \to \mathcal{H}_\rho$. Fix a leaf of $\mathcal{H}_\rho$ and fix a point p on that leaf. Consider a geodesic ray in the hyperbolic metric of the leaf starting at p. If a ray leaves every compact subset of $\mathcal{H}_\rho$ then it determines a "puncture" of the solenoid $\mathcal{H}_\rho$. More precisely, a *puncture* of a hyperbolic solenoid $\mathcal{H}_\rho$ is an equivalence class of wandering rays from points of the leaf, where two rays are equivalent if they are asymptotic.

We may describe the punctures of $\mathcal{H}_\rho$ using the representation ρ. The quasiconformal map $\phi : \mathbb{D} \times \hat{G} \to \mathbb{D} \times \hat{G}$ extends continuously to a leaf-wise quasi-symmetric map $\phi : S^1 \times \hat{G} \to S^1 \times \hat{G}$. Recall that $\bar{\mathbb{Q}} \subset S^1$ parametrizes the endpoints of the standard triangulation of $\mathbb{D}$ invariant under $PSL_2\mathbb{R}$. We say that a point $(p, t) \in S^1 \times \hat{G}$ is a *ρ-puncture* if $\phi^{-1}(p, t) \in \bar{\mathbb{Q}}$, and a *puncture* of $\mathcal{H}_\rho$ itself is a G_ρ-orbit of ρ-punctures. A *ρ-horocycle* at a ρ-puncture (p, t) is the horocycle in $\mathbb{D} \times \{t\}$ centered at (p, t) and a *horocycle* on $\mathcal{H}_\rho$ is a G_ρ-orbit of ρ-horocycles.

Definition 5.1 A *decoration* on $\mathcal{H}_\rho$, or a "decorated hyperbolic structure" on $\mathcal{H}_\rho$, is a function $\tilde{\rho} : G \times \hat{G} \times \bar{\mathbb{Q}} \to PSL_2\mathbb{R} \times L^+$, where

$$\tilde{\rho}(\gamma, t, q) = \rho(\gamma, t) \times h(t, q)$$

with $\rho(\gamma, t) \in Hom(G \times \hat{G}, PSL_2\mathbb{R})$, which satisfies the following conditions:

> Property 4: for each $t \in \hat{G}$, the image $h(t, \bar{\mathbb{Q}}) \subseteq L^+$ is discrete and the center of the horocycle $h(t, q)$ is $\phi_t(q)$, for all $(t, q) \in \hat{G} \times \bar{Q}$ (using here the identification of L^+ with the space of horocycles as in Appendix A);
>
> Property 5: for each $q \in \bar{\mathbb{Q}}$, the restriction $h(\cdot, q) : \hat{G} \to L^+$ is a continuous function from $\hat{G}$ to L^+;
>
> Property 6: $h(t, q)$ is ρ invariant in the sense that
>
> $$\rho(\gamma, t)(h(t, q)) = h(t\gamma^{-1}, \rho(\gamma, t)q).$$

Remark Property 4 implies that $h(t, \bar{\mathbb{Q}})$ is radially dense in L^+ because $\phi_t(\bar{Q})$ is dense in S^1. The continuity from property 5 and the invariance under the covering group from property 6 imply continuity of the map $h(\cdot, \bar{Q})$ from $\hat{G}$ to the space of discrete and radially dense subset of L^+, where the topology is the Hausdorff topology on closed subsets of L^+.

Let $Hom(G \times \hat{G} \times \bar{\mathbb{Q}}, PSL_2\mathbb{R} \times L^+)$ denote the space of all decorated hyperbolic structures satisfying the properties above. We define a topology on $Hom(G \times \hat{G} \times \bar{\mathbb{Q}}, PSL_2\mathbb{R} \times L^+)$. A neighborhood of $\tilde{\rho}(\gamma, t, q) = \rho(\gamma, t) \times h(t, q)$ consists of all $\tilde{\rho}_1(\gamma, t, q) = \rho_1(\gamma, t) \times h_1(t, q)$ such that ρ_1 belongs to a chosen neighborhood of ρ in $Hom(G \times \hat{G}, PSL_2\mathbb{R})$, and the maps $h_1(\cdot, q) : \hat{G} \to L^+$ and $h(\cdot, q) : \hat{G} \to L^+$ are close in the supremum norm, for each $q \in \bar{Q}$. The above condition and the invariance Property 6 implies that the set $h_1(t, \bar{Q})$ is close to the set $h(t, \bar{Q})$ in the Hausdorff metric, for each $t \in \hat{G}$.

We define the *decorated Teichmüller space* as the quotient

$$\tilde{T}(\mathcal{H}_G) = Hom(G \times \hat{G} \times \bar{\mathbb{Q}}, PSL_2\mathbb{R} \times L^+)/Cont(\hat{G}, PSL_2\mathbb{R}),$$

where $\alpha : \hat{G} \to PSL_2\mathbb{R}$ acts on $\tilde{\rho}$ by

$$(\alpha\tilde{\rho})(\gamma, t, q) = \big(\alpha(t\gamma^{-1}) \circ \rho(\gamma, t) \circ \alpha^{-1}(t)\big) \times \big(\alpha(t)h(t, q)\big).$$

The topology on $\tilde{T}(\mathcal{H}_G)$ is the quotient of the topology on $Hom(G \times \hat{G} \times \bar{\mathbb{Q}}, PSL_2\mathbb{R} \times L^+)$. The following is immediate:

Proposition 5.2 *Forgetting decoration induces a continuous surjection $\tilde{T}(\mathcal{H}) \to T(\mathcal{H})$.*

Given a geodesic in $\mathbb{D}$ together with a horocycle centered at each end of the geodesic, the intersection points of the horocycles determine a finite segment of the geodesic. To this geometric data, we assign the square root of the double of the exponential of the signed length of the segment (positive if the segment lies outside both horoballs, otherwise negative) and call it *lambda length.* (For more details see the appendices.) Given $\tilde{\rho}$, to any $e \times t$, for $e \in \tau_*$ and $t \in \hat{G}$, we can assign the lambda length of the image geodesic $\phi_t(e)$ and corresponding horocycles $h(t, q_1)$ and $h(t, q_2)$, where q_1, q_2 are images of the endpoints of e under ϕ_t. Thus, we obtain a natural mapping $\lambda : \tilde{T}(\mathcal{H}) \to (\mathbb{R}^{\tau_*}_{>0})^{\hat{G}}$ which assigns to a function $\tilde{\rho} : G \times \hat{G} \times \bar{\mathbb{Q}} \to PSL_2\mathbb{R} \times L^+$ the lambda lengths corresponding to the edges (e, t), for $e \in \tau_*$ and $t \in \hat{G}$.

We consider $\mathbb{R}^{\tau_*}_{>0}$ with the strong topology induced by taking the supremum over the coordinates. Let $Cont(\hat{G}, \mathbb{R}^{\tau_*}_{>0})$ be the space of all continuous mappings with the compact-open topology. Denote by $Cont^G(\hat{G}, \mathbb{R}^{\tau_*}_{>0})$ the subset of $Cont(\hat{G}, \mathbb{R}^{\tau_*}_{>0})$ which consists of all elements that are invariant with respect to the action of G. In other words, $f \in Cont^G(\hat{G}, \mathbb{R}^{\tau_*}_{>0})$ if $f \in Cont(\hat{G}, \mathbb{R}^{\tau_*}_{>0})$ and

$$f(t\gamma^{-1}, \gamma(e)) = f(t, e)$$

for each $\gamma \in G$ and $e \in \tau^*$.

Theorem 5.3 *The assignment of lambda lengths is a homeomorphism onto*

$$\lambda : \tilde{T}(\mathcal{H}_G) \to Cont^G(\hat{G}, \mathbb{R}^{\tau_*}_{>0});$$

that is, we obtain a parametrization of $\tilde{T}(\mathcal{H}_G)$.

Proof We first show that λ is surjective. If $f \in Cont^G(\hat{G}, \mathbb{R}^{\tau_*}_{>0})$, then we must produce $\tilde{\rho} = \rho \times h$ such that $\lambda(\tilde{\rho}) = f$. Let γ_1 and γ_2 be fixed generating hyperbolic elements in the once punctured torus group G that map $e_{1/2}$ and $e_{2/1}$ onto $e_{-2/1}$ and $e_{-1/2}$, respectively.

By Lemma A.1(i), given $f(t, e_{1/2})$, $f(t, e_{2/1})$ and $f(t, e_0)$ there exists a unique choice of horocycles $h(t, -1)$, $h(t, -i)$ and $h(t, 1)$ based at $(-1, t)$, $(-i, t)$ and $(1, t)$ in $\mathbb{D} \times \hat{G}$ such that induced lambda lengths on $e_{1/2} \times t$, $e_{2/1} \times t$ and $e_0 \times t$ are equal to the above values of f. Further by Lemma A.1(i), given $f(e_{-1/2}, t)$ and $f(e_{-2/1}, t)$ there exists a unique horocycle $h(t, i)$ in $\mathbb{D} \times \hat{G}$ which induces lambda lengths $f(t, e_{-1/2})$ and

$f(t, e_{-2/1})$ on $e_{-1/2} \times t$ and $e_{-2/1} \times t$, respectively. We continue this process inductively and obtain $h(t, e)$ for all $e \in \tau_*$ and $t \in \hat{G}$ such that whenever $p, q \in \bar{\mathbb{Q}}$ are endpoints of an edge $e \in \tau_*$ then the corresponding lambda length for $h(t, p)$ and $h(t, q)$ equals $f(t, e)$.

We define $\phi_t|_{\bar{\mathbb{Q}}}$ by mapping (p, t) onto the center of $h(t, p)$, for each $p \in \bar{\mathbb{Q}}$. We show that ϕ_t extends to a quasiconformal map $\phi_t : \mathbb{D} \times \hat{G} \to \mathbb{D} \times \hat{G}$ which is continuous in $t \in \hat{G}$. By the invariance under G, f descends to a continuous function on the compact set $(\tau_* \times \hat{G})/G$. Therefore, the lambda lengths on each $\mathbb{D} \times \{t\}$ defined by f are pinched (cf. Lemma 6.1 and Appendix A). A theorem of Penner and Sullivan [18] (cf. Theorem A.2) guarantees that ϕ_t extends to a quasiconformal map. The method of [18] shows that ϕ_t varies continuously in t upon replacing ϕ_t with the barycentric extension of its boundary values.

We define $\rho(\gamma, t)$ to be the unique element of $PSL_2\mathbb{R}$ which maps $(-1, t)$, $(-i, t)$ and $(1, t)$ onto $\phi_{t\gamma^{-1}}(\gamma(-1), t\gamma^{-1})$, $\phi_{t\gamma^{-1}}(\gamma(-i), t\gamma^{-1})$ and $\phi_{t\gamma^{-1}}(\gamma(1), t\gamma^{-1})$, where we identify $\mathbb{D} \times \{t\} \equiv \mathbb{D} \times \{t\gamma^{-1}\} \equiv \mathbb{D}$ for $\gamma \in \hat{G}$. We need to show that $\rho(\gamma, t) = \phi_{t\gamma^{-1}} \circ \gamma \circ \phi_t^{-1}$ for all $t \in \hat{G}$, and it is enough to show that the equality holds on the edges of $\phi_t(\tau_*)$. Any $e \in \phi_t(\tau_*)$ is obtained by forming a unique chain of edges from the base triangle with vertices 1, -1 and $-i$ according to the values of f on the edges of τ_*. In the same way, we obtain the edge $\phi_{t\gamma^{-1}} \circ \gamma \circ \phi_t^{-1}(e)$ on $\mathbb{D} \times t\gamma^{-1}$. Since the function f is invariant under the action of G, it follows that the values of f on the chain of edges from the base triangle to e on $\mathbb{D} \times t$ are equal to the values of f on the corresponding edges of the chain from the image of the base triangle under $\phi_{t\gamma^{-1}} \circ \gamma \circ \phi_t^{-1}$ to the edge $\phi_{t\gamma^{-1}} \circ \gamma \circ \phi_t^{-1}(e)$ on $\mathbb{D} \times t\gamma^{-1}$. On the other hand, $\rho(\gamma, t) \in PSL_2\mathbb{R}$ preserves the geometric construction using the horocycles and it agrees with $\phi_{t\gamma^{-1}} \circ \gamma \circ \phi_t^{-1}$ on the base triangle. Thus, $\rho(\gamma, t)$ and $\phi_{t\gamma^{-1}} \circ \gamma \circ \phi_t^{-1}$ agree on e, hence ρ is a representation which automatically satisfies Property 2. Property 3 is also established. Finally, the representation ρ is continuous because the function f is continuous, and Property 1 follows.

Since f is pinched, it follows from [18] (cf. Theorem A.2) that $h(t, \bar{\mathbb{Q}})$ is discrete and radially dense. The continuity of $h(\cdot, q) : \hat{G} \to L^+$ follows by the continuity of f and by continuity of the formulas in Lemma A.1(i). Property 6 follows from invariance of lambda lengths and invariance of f.

It follows that $\lambda(\tilde{\rho}) = f$, and hence λ is surjective. If f and f_1 are close, then the above construction yields $\tilde{\rho}$ and $\tilde{\rho}_1$ which are close in $\tilde{T}(\mathcal{H}_G)$. This holds by the continuity of the construction.

The map λ is continuous because lambda lengths are invariant under G and depend continuously on the representation $\tilde{\rho} = \rho \times h$.

It remains to show that λ is injective. A representation $\tilde{\rho}$ up to conjugation by $\alpha \in Cont(\hat{G}, PSL_2\mathbb{R})$ is determined uniquely by the decorations $h(t, -1)$, $h(t, -i)$, $h(t, 1)$ and $h(t, i)$, for $t \in \hat{G}$. On the other hand, the above decorations and invariance with respect to G uniquely determine the function $f = \lambda(\tilde{\rho}) \in Cont^G(\hat{G}, \mathbb{R}^{\tau_t^*}_{>0})$, whence λ is indeed one-to-one. $\square$

Remark 5.4 One may parametrize other transverse structures on the solenoid in analogy, where conditions other than continuity are imposed on the "lambda functions" $\hat{G} \to \mathbb{R}^{\tau_*}_{>0}$.

The above parametrization of $\tilde{T}(\mathcal{H}_G)$ immediately implies density of TLC decorated structures on $\mathcal{H}$ since $\hat{G}$ is a Cantor set in which G is dense. This is in parallel to Theorem 3.8:

Corollary 5.5. *The union of the natural lifts of the decorated Teichmüller spaces of all finite punctured surfaces is dense in the decorated Teichmüller space $\tilde{T}(\mathcal{H}_G)$.*

6. Convex hull construction for the solenoid

In fact, the results of Appendix A automatically apply to any continuous lambda length function because of the following:

Lemma 6.1. *Continuity of a G-invariant $\lambda : \hat{G} \to \mathbb{R}^{\tau_*}_{>0}$ implies that $\lambda_t : \tau_* \to \mathbb{R}_{>0}$ is pinched for each $t \in \hat{G}$.*

Proof. Continuity of λ means that $\forall K\ \exists N\ \forall e \in \tau_*\ \forall \gamma \in G_N$, we have

$$1 + K^{-1} \leq \frac{\lambda_t(e)}{\lambda_t(\gamma e)} \leq 1 + K.$$

Take $K = 1/2$ and its corresponding N. A fundamental domain for G_N has only a finite collection of lambda lengths, and any other is at most three halves and at least one half times a lambda length in this finite set.□

Thus, if $\lambda : \hat{G} \to \mathbb{R}_{>0}^{\tau_*}$ is continuous, then each $\lambda_t : \tau_* \to \mathbb{R}_{>0}$ is pinched, so in the notation of Appendix A, the corresponding $h_t : \tau_* \to L^+$ has drd image $\mathcal{B}_t$ and closed convex hull C_t, which projects to the cell decomposition τ_t of $\mathbb{D}$. Presumably, cells in the decomposition could be infinite sided. Furthermore, the characteristic map interpolates a quasi-symmetric mapping $\phi_t : S^1 \to S^1$.

Given a tesselation τ of $\mathbb{D}$ with $\tau^\infty = \mathbb{Q} \cup \{\infty\} = \bar{\mathbb{Q}} = \tau_*^\infty$, define

$$\begin{aligned} &\overset{\circ}{\mathcal{C}}(\tau) = \{\lambda_t \in \tilde{T}(\mathcal{H}) : \tau_t = \phi_t(\tau)\} \\ &\ \cap \\ &\mathcal{C}(\tau) = \{\lambda_t \in \tilde{T}(\mathcal{H}) : \tau_t \subseteq \phi_t(\tau)\} \\ &\ \cap \\ &\tilde{T}(\mathcal{H}) \approx Cont^G(\hat{G}, \mathbb{R}_{>0}^{\tau_*}). \end{aligned}$$

Furthermore, define the *classical locus* $\mathcal{L} \subseteq \tilde{T}(\mathcal{H})$ as the subspace consisting of all TLC structures on $\mathcal{H}$, or equivalently, as the union over all finite-index subgroups $K < G$ of the image of $\tilde{T}(\mathbb{D}/K)$ in $\tilde{T}(\mathcal{H})$. The corresponding subspace $\mathcal{L} \subseteq Cont^G(\hat{G}, \mathbb{R}_{>0}^{\tau_*})$ is described by the collection of all TLC lambda length functions $\lambda_t : \tau_* \to \mathbb{R}_{>0}$ where there is some finite-index $K < G$ so that λ_t is K-invariant, for all $t \in \hat{G}$. Define a *TLC tesselation* to be a tesselation τ of $\mathbb{D}$ that is invariant under some finite-index subgroup $K < G$ with ideal points $\tau^\infty = \bar{\mathbb{Q}}$.

For any fixed TLC tesselation τ, a point of $\mathcal{C}(\tau) - \overset{\circ}{\mathcal{C}}(\tau)$ corresponds to $\lambda_t : \tau_* \to \mathbb{R}_{>0}$ whose convex hull C_t has at least one face which is not triangular, i.e., at least four points of $\mathcal{B}_t$ are coplanar; thus, τ_t is not a tesselation, but rather a paving of $\mathbb{D}$. Given an arbitrary point in $\tilde{T}(\mathcal{H}_G)$, it is presumably possible that it does not belong to $\mathcal{C}(\tau)$ for any TLC tesselation τ. Notice, however, that for any point of $\tilde{T}(\mathcal{H}_G)$, we have $\tau_t^\infty \subseteq \bar{\mathbb{Q}}$. On the other hand, we obtain a generically simple picture:

Theorem 6.2. *The subspace $\overset{\circ}{\mathcal{C}}(\tau)$ of $\tilde{T}(\mathcal{H})$ is open for each TLC tesselation τ, and $\cup_\tau\ \overset{\circ}{\mathcal{C}}(\tau)$ is a dense open subset of $\tilde{T}(\mathcal{H})$, where the union is over all TLC tesselations τ of* D.

Proof. Fix $\lambda \in Cont^G(\hat{G}, \mathbb{R}_{>0}^{\tau_*})$ and $\epsilon > 0$. There exists a TLC $\lambda^{'} \in Cont^G(\hat{G}, \mathbb{R}_{>0}^{\tau_*})$, say with corresponding finite-index subgroup $K < G$, so that $\|\lambda - \lambda^{'}\|_\infty < \epsilon/2$ by Corollary 5.5. By the classical theory [19] working on the surface $\mathbb{D}/K$, there exists λ'' which belongs to $\overset{\circ}{\mathcal{C}}(\tau)$ for some TLC tesselation τ and which satisfies $\|\lambda'' - \lambda^{'}\|_\infty < \epsilon/2$. Therefore, the density follows.

Turning to the proof that each $\overset{\circ}{\mathcal{C}}(\tau)$ is open and in order to distinguish coordinates, let us now denote

$$Cont_\lambda(\hat{G}, \mathbb{R}_{>0}^{\tau_*}) = Cont^G(\hat{G}, \mathbb{R}_{>0}^{\tau_*}) \approx \tilde{T}(\mathcal{H}).$$

Let $Cont_\sigma(\hat{G}, \mathbb{R}^{\tau_*})$ denote the abstract space of continuous G-equivariant $\mathbb{R}$-valued function, and define

$$\begin{aligned} \Phi : Cont_\lambda(\hat{G}, \mathbb{R}_{>0}^{\tau_*}) &\to Cont_\sigma(\hat{G}, \mathbb{R}^{\tau}) \\ \lambda_t &\mapsto \sigma_t, \end{aligned}$$

where the "simplicial coordinate function" σ_t is defined for each $t \in \hat{G}$ in accordance with the formula in Lemma A.1(v). By definition of simplicial coordinates and the convex hull construction, in fact we have

$$\begin{aligned} \overset{\circ}{\mathcal{C}}(\tau) &= \Phi^{-1}(Cont_\sigma(\hat{G}, \mathbb{R}_{>0}^{\tau})), \\ \mathcal{C}(\tau) &= \Phi^{-1}(Cont_\sigma(\hat{G}, \mathbb{R}_{\geq 0}^{\tau})), \end{aligned}$$

so the openness assertion of the proposition thus follows from continuity of Φ. □

Remark 6.3 In the classical case [19] (cf. Appendix B), τ is a paving of $\mathbb{D}$ arising as the lift of a paving of some punctured surface $\mathbb{D}/\Gamma$. The intersection $\mathcal{L} \cap \mathcal{C}(\tau)$ is mapped homeomorphically by Φ onto the set of TLC elements of $Cont_\sigma(\hat{G}, \mathbb{R}^\tau_{\geq 0})$ so that there are no (finite) cycles of triangles $t_i, \dots t_{i+k}$ with t_i and t_{i+k} in the same Γ-orbit and with vanishing simplicial coordinates on each $t_i, \dots, t_{i+k-1}$. Each component of $\mathcal{L} \cap \mathcal{C}(\tau)$ is homeomorphic to an open simplex together with certain of its faces.

7. The Modular Group

We define the modular group of the punctured solenoid in analogy to the modular group of the compact universal hyperbolic solenoid [16],[17], [14].

Definition 7.1. The *modular group* $Mod(\mathcal{H}_G)$ of the punctured solenoid $\mathcal{H}_G$ consists of quasiconformal self-maps of $\mathcal{H}_G$ modulo isotopies which are bounded in the hyperbolic metric on leaves. The *baseleaf preserving modular group* $Mod_{BLP}(\mathcal{H}_G)$ is a subgroup of $Mod(\mathcal{H}_G)$ which consists of all isotopy classes of quasiconformal self-maps of $\mathcal{H}_G$ which fix the baseleaf.

Note that a quasiconformal self-map f of $\mathcal{H}_G$ necessarily sends punctures onto punctures on $\mathcal{H}_G$. Indeed, since an arbitrary quasiconformal map of the hyperbolic plane onto the hyperbolic plane is a quasi-isometry [8], a geodesic on a leaf which ends in the puncture is mapped by f onto a quasigeodesic on the image leaf. If the endpoint of the quasigeodesic on the image leaf is not a puncture then it returns to a compact set of $\mathcal{H}_G$ infinitely often, and this contradicts that f is proper.

C. Odden [16], [17] showed that the baseleaf preserving modular group of the compact universal hyperbolic solenoid is isomorphic to the group of virtual automorphisms of the fundamental group of a compact surface with genus greater than one. We use his method to prove a similar statement for $\mathcal{H}_G$.

Theorem 7.2. *The restriction of $Mod_{BLP}(\mathcal{H}_G)$ to the baseleaf is isomorphic to the virtual automorphism group of G.*

Proof. We briefly discuss the extension of Odden's argument (see [17, Theorem 4.6]) to the case of the punctured solenoid. A quasiconformal self-map f of the compact universal solenoid is uniformly continuous, and the compact solenoid admits a finite covering by ϵ balls in the topology coming from the representation $\mathbb{D} \times \hat{G}/G$, for a cocompact Fuchsian G. These are two crucial ingredients in Odden's proof. We replace compactness of the solenoid by considering a compact subset $\tilde{X}$ of the punctured solenoid obtained by lifting of the complement $X \subset \mathbb{D}/G$ of a horoball neighborhood of the cusp on $\mathbb{D}/G$, where G is the punctured torus group. Further, $\tilde{X}_1 = f(\tilde{X})$ is a compact subset of $\mathcal{H}_G$ which is contained in another compact set $\tilde{X}_2$ of the same kind as $\tilde{X}$. In this case, $f : \tilde{X} \to \tilde{X}_1$ is uniformly continuous and both $\tilde{X}, \tilde{X}_1$ can be covered by finitely many ϵ balls as required, and they furthermore admit a local product structure similar to the compact solenoid. Moreover, each closed curve based at a point in $X \subset \mathbb{D}/G$ can be homotoped into a closed curve in X itself. These details and observations allow the further application of Odden's proof to the current setting. $\Box$

For a compact solenoid, the isotropy group of a point in the Teichmüller space in the baseleaf preserving modular group is infinite (see [16], [17] for TLC hyperbolic solenoids and see [14] for non-TLC hyperbolic solenoids). Moreover, each isotropy group contains a subgroup isomorphic to the surface group. There is a countable set of TLC hyperbolic solenoids for which the isotropy group is isomorphic to a dense subgroup of $PSL_2(\mathbb{R})$ (see [16],[17]). These statements hold for $T(\mathcal{H}_G)$ as well by obvious generalizations of the proofs.

We investigate the isotropy subgroups of $Mod_{BLP}(\mathcal{H}_G)$ for a point $\tilde{\mathcal{H}} \in \overset{\circ}{\mathcal{C}}(\tau) \subseteq \tilde{T}(\mathcal{H}_G)$ for τ a TLC tesselation. Note that the pull-back τ_t under the marking map ϕ_t of the extreme edges of the boundary of the convex hull C_t in Minkowski space for a decorated hyperbolic punctured solenoid $\tilde{\mathcal{H}}$ is invariant under the action of $Mod_{BLP}(\mathcal{H}_G)$ by definition. That is, an element $h \in Mod_{BLP}(\mathcal{H}_G)$ which fixes $\tilde{\mathcal{H}} \in \overset{\circ}{\mathcal{C}}(\tau)$ must map τ_t onto itself. Since the projection $\pi : \tilde{T}(\mathcal{H}_G) \to T(\mathcal{H}_G)$ commutes with the action of h, it follows that h fixes $\mathcal{H} = \pi(\tilde{\mathcal{H}}) \in T(\mathcal{H}_G)$, and so h is an isometry of $\mathcal{H}$.

¿From now on, we restrict the action to the baseleaf, which is identified with the unit disk $\mathbb{D}$. A homeomorphism $h : S^1 \to S^1$ extends diagonally to a map of the space of geodesics $\{x \times y \in S^1 \times S^1 : x \neq y\}$. To study the isotropy groups, we need the following simple but important rigidity property of Farey tesselation:

Lemma 7.3. *Assume that $h : S^1 \to S^1$ is a homeomorphism which maps the Farey tesselation τ_* onto itself. Then h is an element of $PSL_2(\mathbb{Z})$.*

Proof. The homeomorphism $h : S^1 \to S^1$ maps $\bar{\mathbb{Q}}$ onto itself because the endpoints of any geodesic in τ_* are mapped onto the endpoints of another geodesic in τ_*. We show that h must belong to $PSL_2\mathbb{Z}$. Fix $a \in \tau_*$ and let $T \subset \mathbb{D} - \tau_*$ be an ideal triangle with a in its frontier. Let γ be the unique element of $PSL_2\mathbb{Z}$ such that $\gamma(a) = h(a)$ and $\gamma(T) \cap h(T) \neq \emptyset$. If $\gamma \neq h$ then there exists $a \in \tau_*$ and a triangle T complementary to τ_* whose frontier contains a such that $\gamma^{-1} \circ h(T) \neq T$. In this case, $\gamma^{-1} \circ h$ maps an edge of T which is in τ_* onto a geodesic not in τ_*. However, both γ and h preserve τ_* which gives a contradiction. $\Box$

Fix a*TLC tesselation* τ of $\mathbb{D}$, i.e. τ is an ideal triangulation of $\mathbb{D}$ which is invariant under a finite index subgroup of $PSL_2\mathbb{Z}$ and has ideal vertices $\tau^\infty = \bar{\mathbb{Q}}$. Denote by $Aut(\tau)$ the subgroup of the baseleaf preserving modular group $Mod_{BLP}(\mathcal{H}_G)$ which fixes τ setwise. In particular, $Aut(\tau_*) = PSL_2\mathbb{Z}$ (by above lemma) acts simply transitively on the oriented edges of τ_* and fixes the basepoint $\mathcal{H}_G \in T(\mathcal{H}_G)$ and the basepoint $\tilde{\mathcal{H}}_G \in \tilde{T}(\mathcal{H}_G)$, where the decoration on $\tilde{\mathcal{H}}_G$ is given by assigning lambda length unity to each edge of τ_*.

Note that if τ is a TLC tesselation invariant under some group K, say without elliptics, then τ/K is a well-defined tesselation of the surface $S = \mathbb{D}/K$. Furthermore, if f is an isotopy class of homeomorphisms of S fixing τ/K setwise, then a lift of f to the universal cover $\mathbb{D}$ fixes the pre-image of τ/K in $\mathbb{D}$. It follows that every automorphism group of every TLC tesselation invariant under a finite index subgroup $K < PSL_2(\mathbb{Z})$ lies as a subgroup of $PSL_2(\mathbb{Z})$. (The same is not true for more general pavings of S as one can easily check directly using simplicial coordinates, cf. Lemma A.1(v).)

We define the *characteristic map* $h = h(\tau, e)$ for a tesselation τ with a distinguished oriented edge and recall that $\tau^\infty \subset S^1$ denotes the set of ideal points of the tesselation τ. Define h to map the initial and final points of the *DOE* of τ_* onto the initial and final points of the *DOE* of τ. Each *DOE* is the common boundary of two complementary triangles, one to the left and one to the right with respect to the orientations on the *DOES*. Map the third vertices of the respective triangles in $\mathbb{D} - \tau_*^\infty$ to the third endpoints of corresponding triangles in $\mathbb{D} - \tau$. Continue in this manner mapping third points of triangles to recursively define $h : \tau_*^\infty \to \tau^\infty$. By construction, h is monotone and hence interpolates a homeomorphism h of S^1. (See Appendix A for more details.)

There is the following immediate corollary of the previous result:

Corollary 7.4 *Suppose that τ is a TLC tesselation of $\mathbb{D}$ with distinguished oriented edge e and corresponding characteristic map $h = h(\tau, e) : S^1 \to S^1$. Then $Aut(\tau) = h \circ PSL_2\mathbb{Z} \circ h^{-1}$. Furthermore, if $\phi = h \circ \gamma \circ h^{-1}$, for $\gamma \in PSL_2\mathbb{Z}$, then $h(\tau, \phi(e)) = h(\tau, e) \circ \gamma$.*

Lemma 7.5. *Let τ be a TLC tesselation of $\mathbb{D}$, i.e., τ is invariant under a finite-index subgroup K of $PSL_2\mathbb{Z}$ with ideal points $\tau^\infty = \bar{\mathbb{Q}}$. Then the characteristic map of τ conjugates a finite-index subgroup H of $PSL_2\mathbb{Z}$ onto K.*

Proof. Let $K^{'}$ be the finite-index subgroup of K which contains no elliptic elements. Choose a fundamental polygon P' for $K^{'}$ whose boundary consists of edges in τ. Thus, the interior of P' is divided into ideal triangles by the edges of τ. To describe subgroup H, consider a combinatorially equivalent polygon P which is comprised of ideal triangles among $\mathbb{D} - \tau_*$. The combinatorial correspondence between P' and P gives the pairing of the boundary of P using the pairing of P' under $K^{'}$. For each pair $a \times b \in \tau_* \times \tau_*$ on the boundary of P, there exists a unique $\gamma_{a,b} \in PSL_2\mathbb{Z}$ such that $\gamma_{a,b}(a) = b$ and $\gamma_{a,b}(P') \cap P' = \{b\}$. By Poincaré's fundamental polygon theorem, the group H generated by all side pairings $\gamma_{a,b}$ of the corresponding boundary edges of P is Fuchsian with a fundamental polygon equal to P. Thus, the Riemann surfaces $\mathbb{D}/K^{'}$ and $\mathbb{D}/H$ are homeomorphic, and we may choose a quasiconformal map between them which maps edges in P' onto the edges of P. This map lifts to $\mathbb{D}$, and its restriction to S^1 is the desired quasisymmetric conjugation h

between $K^{'}$ and H. The map h differs from the characteristic map by a pre-composition with an element of $PSL_2\mathbb{Z}$.

Now, if $K' \neq K$, then we can choose P' such that the orbit of P' under $K^{'}$ is invariant under the full group K. This is equivalent to requiring that a fundamental set for the action of K is contained in P' and finitely many of its translates under some elliptic elements of K cover P'. Any elliptic element $\delta \in K - K^{'}$ is conjugated by h to a finite order quasisymmetric element $h \circ \delta \circ h^{-1}$ of $\mathbb{D}$ which preserves τ_*, whence it must be an element of $PSL_2\mathbb{Z}$ by Lemma 7.3. □

¿From the above we obtain

Theorem 7.6. *$Mod_{BLP}(\mathcal{H}_G)$ acts transitively on $\{\overset{\circ}{\mathcal{C}}(\tau) : \tau$ is $TLC\}$. Furthermore, the isotropy subgroup in $Mod_{BLP}(\mathcal{H}_G)$ of $\overset{\circ}{\mathcal{C}}(\tau)$ is isomorphic to $Aut(\tau)$ and is moreover quasi-conformally conjugate to $PSL_2(\mathbb{Z})$.*

Proof. In Lemma 7.5, we showed that a TLC tesselation τ of $\mathbb{D}$ which is invariant under a finite subgroup K of $PSL_2\mathbb{Z}$ can be mapped to τ_* by a quasisymmetric map h which conjugates K onto a finite-index subgroup of $PSL_2\mathbb{Z}$. Thus, h is a virtual automorphism, and by Theorem 7.2, it defines an element of $Mod_{BLP}(\mathcal{H}_G)$ which sends the cell corresponding to τ_* onto the cell corresponding to τ. Transitivity follows.

The identification of the isotropy group of $\overset{\circ}{\mathcal{C}}(\tau)$ with $Aut(\tau)$ is induced by identifying both these groups with the isometry group of the point of $\tilde{T}(\mathcal{H}_G)$ described via Theorem 5.3 with all lambda lengths constant equal to unity. This point lies in $\overset{\circ}{\mathcal{C}}(\tau)$ (as one checks with simplicial coordinates), and has combinatorial symmetry group given by $Aut(\tau)$ and decorated hyperbolic (or conformal) symmetry group given by the isotropy subgroup. The last part then follows from Corollary 7.4. □

We show that the space $\tilde{T}(\mathcal{H}_G)/Mod_{BLP}(\mathcal{H}_G)$ is "essentially" Hausdorff, more precisely, at least an open dense subset is Hausdorff. This implies that no orbit under $Mod_{BLP}(\mathcal{H}_G)$ is dense. The analogue of the Ehrenpreis conjecture is thus very false for $\tilde{T}(\mathcal{H}_G)$.

Theorem 7.7. *The quotient $\cup_\tau \overset{\circ}{\mathcal{C}}(\tau)/Mod_P(\mathcal{H}_G)$ is Hausdorff, where the union is over all TLC tesselations τ. Moreover, no orbit of a point in $\tilde{T}(\mathcal{H}_G)$ is dense.*

Proof. Fix an identification of the baseleaf of $\mathcal{H}_G$ with the unit disk $\mathbb{D}$. Since the points in $\tilde{T}(\mathcal{H}_G)$ are defined up to post composition by a conformal map, we may identify the image of the baseleaf under $f : \mathcal{H}_G \to \tilde{\mathcal{H}}$ with $\mathbb{D}$ such that f pointwise fixes each of $\pm 1, -i$. Thus, $\tilde{T}(\mathcal{H}_G)/Mod_P(\mathcal{H}_G)$ is mapped to the space $\mathcal{B}_{drd}$ of all discrete, radially dense countable subsets of the light cone L^+ in Minkowski three space containing three points which project to $1, -1, -i$ on S^1. The map $\tilde{T}(\mathcal{H}_G)/Mod_P(\mathcal{H}_G) \to \mathcal{B}_{drd}$ is continuous for the quotient topology on $\tilde{T}(\mathcal{H}_G)/Mod_P(\mathcal{H}_G)$ and the Hausdorff topology on $\mathcal{B}_{drd}$.

We show that the restriction of the map to $\cup_\tau \overset{\circ}{\mathcal{C}}(\tau)/Mod_{BLP}(\mathcal{H}_G)$ is injective which immediately implies that the space is Hausdorff. Let $f \in \overset{\circ}{\mathcal{C}}(\tau)$ and $f_1 \in \overset{\circ}{\mathcal{C}}(\tau_1)$ have the same image β in $\mathcal{B}_{drd}$. The convex hull construction for β yields an ideal triangulation of $\mathbb{D}$ which pulls back to τ and τ_1 by the maps f and f_1, respectively. Since τ and τ_1 are TLC tesselation, there exist $h, h_1 \in Mod_{BLP}(\mathcal{H}_G)$ such that $h(\tau_*) = \tau$ and $h_1(\tau_*) = \tau_1$ by Theorem 7.6. Thus, there exists $\gamma \in PSL_2\mathbb{Z}$ such that $f \circ h \circ \gamma = f_1 \circ h_1$, by Lemma 7.3. Finally, f and f_1 are therefore in the same orbit of $Mod_{BLP}(\mathcal{H}_G)$, and the injectivity follows.

It remains to show that no orbit is dense. An orbit of a point in $\overset{\circ}{\mathcal{C}}(\tau)$ is not dense in $\tilde{T}(\mathcal{H}_G)$ because it is not dense in $\overset{\circ}{\mathcal{C}}(\tau)$ by the above. The orbit of a point outside $\cup_\tau \overset{\circ}{\mathcal{C}}(\tau)$ is not dense because it does not meet this open dense set by modular invariance of the convex hull construction. □

8. Generators of the Modular Group

To describe generators of $Mod_{BLP}(\mathcal{H}_G)$, we shall require certain elementary moves on TLC tesselations as follows.

Definition 8.1. Let K be a finite-index subgroup of G, let τ be a tesselation of $\mathbb{D}$ which is invariant under K with ideal points $\tau^\infty = \bar{\mathbb{Q}}$, and suppose that e is a fixed unoriented edge of τ with distinct endpoints. Define a new tesselation τ' as follows: for each $f \in \tau - Ke$, there is an identical edge $f \in \tau'$; for each $f \in Ke$, consider the quadrilateral P with diagonal f comprised of the triangles on either side of f complementary to τ, and let f' denote the other diagonal of P; for each edge $f \in Ke$, there is a corresponding dual edge $f' \in \tau'$. The resulting tesselation τ' is clearly also invariant under K. We say that τ' arises from τ by the *(K-)equivariant Whitehead move along* $e \in \tau$.

Furthermore, if τ is a tesselation with *DOE* $d \in \tau$, then we may induce a *DOE* d' on τ' as follows: if $d \notin Ke$, then $d' = d$ as oriented edges, while if $d \in Ke$, then there is the unique orientation on d' so that correctly oriented tangent vectors to d, d' give a positive basis for the tangent space at $d \cap d'$.

Thus, for any TLC tesselation τ invariant by a group K, there is a corresponding equivariant Whitehead move for each edge e of τ, and there is exactly one distinct equivariant move for each K-orbit of edges of τ. An equivariant Whitehead move acts not only on invariant tesselations but also on invariant tesselations with *DOE*.

Lemma 8.2 *Suppose that K is a finite-index subgroup of G and τ is an invariant tesselation with DOE d. Perform an equivariant Whitehead move along the edge e to produce the invariant tesselation τ' with DOE d'. Let $h = h(\tau, d), h' = h(\tau', d')$ denote the characteristic maps and define $k = h' \circ h^{-1}$. Then k is a quasi-symmetric map, a virtual automorphism of $PSL_2(\mathbb{Z})$, is independent of the choice of DOE d on τ, and $k(\tau) = \tau'$ and $k(d) = d'$.*

Proof The characteristic maps h, h' are quasisymmetric by Theorem A.1 since lambda lengths for a TLC tesselation are pinched by Lemma 6.1, hence the composition k is quasi-symmetric as well. Likewise, h, h' are virtual automorphims by Lemma 7.5 since τ, τ' are TLC, and hence so too is the composition k.

To prove that k is independent of the choice of *DOE*, let d_1 be another *DOE* on τ. According to Corollary 7.4, the characteristic maps are related by $h(\tau, d_1) = h(\tau, d) \circ \gamma$, for some $\gamma \in PSL_2(\mathbb{Z})$, and in fact for the same γ, we have also $h(\tau', d_1') = h(\tau', d') \circ \gamma$. Thus, $k_1 = h(\tau', d') \circ \gamma \circ \gamma^{-1} \circ h^{-1}(\tau, d) = k$ is indeed invariant.

That k maps τ to τ' and d to d' follow from the definition of characteristic maps, completing the proof. □.

Definition 8.3 We call $k = k(\tau, K, e) \in Mod_{BLP}(\mathcal{H}_G)$ the *Whitehead homeomorphism* associated with the K-equivariant Whitehead move along $e \in \tau$ for any TLC K-invariant tesselation τ. Notice that by definition if $f \in Ke$, then $k(\tau, K, e) = k(\tau, K, f)$, so we regard Whitehead homeomorphisms as indexed by K-orbits of edges Ke rather than by edges e.

In contrast to the case of punctured surfaces where only certain sequences of Whitehead moves give rise to mapping classes (namely, the sequence must begin and end with combinatorially identical ideal triangulations), for the punctured solenoid, each equivariant Whitehead move *does* give rise to a mapping class $k \in Mod_{BLP}$.

As elements in the group $Mod_{BLP}(\mathcal{H}_G)$, any two Whitehead homeomorphisms can be composed, but there is a special case of geometrical significance as follows.

Definition 8.4 Suppose that $G > K_1 > K_2$ are nested subgroups with each finite-index in the next. Perform a K_1-equivariant Whitehead move along some edge e_1 of the K_1-invariant tesselation τ_1 to get τ_2. As was observed before, both τ_1 and τ_2 are K_1-invariant, so in particular, τ_2 is also K_2-invariant. Next perform a K_2-equivariant Whitehead move along some edge e_2 of τ_2, which is a sensible geometric operation on covering spaces, to get another TLC tesselation τ_3. In this case, we have $k(\tau_1, K_1, e_1) \circ k(\tau_2, K_2, e_2) = h_3 \circ h_1^{-1}$, where h_i is the characteristic map of τ_i, for $i = 1, 3$, defined with compatible *DOES*. We will call a composition with this property *geometric*. More generally, if a finite sequence of Whithead homeomorphisms has the same property for each pair of consecutive terms, then it is called *geometric*.

Theorem 8.5 *Any element of the modular group $Mod_{BLP}(\mathcal{H}_G)$ can be written as a composition $w \circ \gamma$, where $\gamma \in PSL_2(\mathbb{Z})$ and w is a geometric composition of K-equivariant Whitehead homeomorphisms for some fixed K. In particular, $Mod_{BLP}(\mathcal{H}_G)$ contains the characteristic map of any TLC tesselation τ with DOE and the automorphism group $Aut(\tau)$.*

Proof We claim that the Farey tesselation τ_* and an arbitrary TLC tesselation τ can be connected by a finite geometric sequence of K-equivariant Whitehead moves for some fixed K. It would follow in particular that $Mod_{BLP}(\mathcal{H}_G)$ and the group generated by all equivariant Whitehead homeomorphisms have the same orbits on the set $\{\overset{\circ}{\mathcal{C}}(\tau) : \tau \text{ is } TLC\}$. Thus, for any $f \in Mod_{BLP}(\mathcal{H}_G)$ and any TLC tesselation τ, there is some geometric word w in equivariant Whitehead homeomorphisms so that $w^{-1} \circ f$ fixes $\overset{\circ}{\mathcal{C}}(\tau_*)$, and hence $w^{-1} \circ f = \gamma \in PSL_2(\mathbb{Z})$ by Theorem 7.6, completing the proof of generation.

In fact, the composition of K_1-equivariant and K_2-equivariant Whitehead moves can be equivalently described as a finite composition of $(K_1 \cap K_2)$-equivariant moves, so by intersecting finite-index subgroups, a single group K suffices as in the statement of the theorem.

It remains to show that geometric sequences of K-equivariant Whitehead moves act transitively on K-invariant tesselations, and this is precisely the statement from punctured surface theory (cf. Corollary B.6) that sequences of Whitehead moves act transitively on the set of all ideal triangulations of the surface $\mathbb{D}/K$.

For the proof of the last sentence, given a TLC tesselation τ with DOE and characteristic map h, Lemma 7.5 states that h conjugates one finite index subgroup of $PSL_2\mathbb{Z}$ onto the other, so h is indeed an element of $Mod_{BLP}(\mathcal{H}_G)$. $\square$

Remark Insofar as the effect of a Whitehead move on lambda lengths is described by a Ptolemy transformation (cf. Lemma A.1(iii)), one derives a real-algebraic representation of $Mod_{BLP}(\mathcal{H}_G)$ in analogy to [19, Section 7].

We next introduce a series of relations satisfied by the generators in Theorem 8.5, where we assume throughout that τ is a K-invariant tesselation of the disk for some finite-index subgroup K of G (and $\tau^\infty = \bar{\mathbb{Q}}$):

1) [Involutivity] If the K-equivariant Whitehead move along $e \in \tau$ produces e' in the resulting tesselation, then $k(\tau, K, e) \circ k(\tau, K, e') = 1$.

2) [Commutativity] If $e \in \tau$ and $f \in \tau$ do not share an ideal endpoint, then $k(\tau, K, e) \circ k(\tau, K, f) = k(\tau, K, f) \circ k(\tau, K, e)$; see Figure 2.

3) [Pentagon Relation] If we adopt the notation for edges in the five tesselations of the pentagon illustrated in Figure 2, then

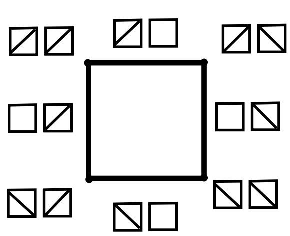
$$k(\tau_1, K, e_1) \circ k(\tau_2, K, f_2) \circ k(\tau_3, K, e_3) \circ k(\tau_4, K, f_4) \circ k(\tau_5, K, e_5) = 1.$$

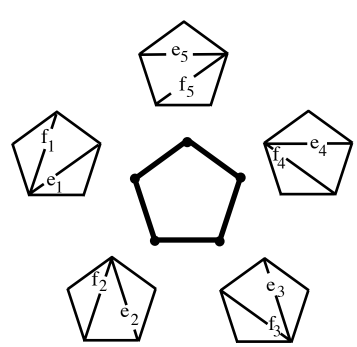
4) [Coset Relation] If H is a finite-index subgroup of K and $f_1, f_2, \ldots, f_\ell$ are representatives for the cosets of H in K, then

$$k(\tau, K, e) = k(\tau, H, f_1) \circ k(\tau, H, f_2) \circ \cdots \circ k(\tau, H, f_\ell),$$

where the order of composition is irrelevant since the individual Whitehead homeomorphisms commute by Relation 2).

Relation 4) holds obviously. To see that the others hold, it is enough to show that the moves leave invariant any starting tesselation with arbitrary DOE since Lemma 8.2 guarantees that the choice of DOE is unimportant. Relations 2) and 3) follow from Figure 2, say if we choose DOE to be on the boundary, and 1) is obvious for any DOE different from the edge e.





Commutativity. Pentagon Relation.

Figure 2 Commutativity and Pentagon Relation.

Definition 8.6. Let ω be a geometric composition of equivariant Whitehead homeomorphisms which are not necessarily equivariant with respect to the same finite index subgroup of $PSL_2(\mathbb{Z})$. Let $\gamma \in PSL_2(\mathbb{Z})$. Any such composition $\omega \circ \gamma$ is called a *normal form* of an element of $Mod_{BLP}(\mathcal{H}_G)$.

We showed in Theorem 8.5 that each element of $Mod_{BLP}(\mathcal{H}_G)$ has a normal form, and we next address its uniqueness.

Theorem 8.7 *Suppose $\gamma_1, \gamma_2 \in PSL_2(\mathbb{Z})$ and ω_1, ω_2 are two geometric Whitehead compositions with $\omega_1 \circ \gamma_1 = \omega_2 \circ \gamma_2$. Then there is some finite index subgroup K of $PSL_2(\mathbb{Z})$ with $S = \mathbb{D}/K$ so that up to Relations 1-4) for K, $\omega_2^{-1} \circ \omega_1 = \gamma_2^{-1} \circ \gamma_1$ is a finite composition $\phi_1 \circ \phi_2 \circ \cdots \circ \phi_k$ of automorphisms $\phi_i \in Aut(\tau_i) < PSL_2(\mathbb{Z})$ of tesselations τ_i of S without DOE, for $i = 1, \ldots, k$.*

Proof Since $\omega_1 \circ \gamma_1 = \omega_2 \circ \gamma_2$ and ω_1, ω_2 are geometric, we conclude that the image tesselations of τ_* under ω_1 and ω_2 are equal when considered as tesselations without *DOE*, i.e. $\omega_1(\tau_*) = \omega_2(\tau_*)$. Thus, $\omega = \omega_2^{-1} \circ \omega_1 = \gamma_2^{-1} \circ \gamma_1$ is a geometric composition of Whitehead homeomorphisms leaving invariant the tesselation τ_* without *DOE*. Let K be the intersection of all finite index subgroups arising in this geometric composition of Whitehead moves. By Relation 4), ω is uniquely equivalent to a geometric composition of Whitehead homeomorphisms invariant under the fixed finite index subgroup K of $PSL_2(\mathbb{Z})$. Arguing as in Theorem B.4 for the surface $\mathbb{D}/K$, we conclude that ω may be reduced using Relations 1-3) to a composition of automorphisms. $\square$

We do not know all the relations among the generators in Theorem 8.5 and have only the weaker statement about normal forms in Theorem 8.7. One difficulty is that the normal form for a non-geometric composition of Whitehead homeomorphisms is not known.

9. Weil-Petersson two form

For an oriented smooth punctured surface of finite type with fixed ideal triangulation τ, it was shown [20, Appendix A] that the Weil-Petersson two form on the Teichmüller space pulls-back to the following form

$$-2 \sum_T (d\log a \wedge d\log b + d\log b \wedge d\log c + d\log c \wedge d\log a), \tag{5}$$

on the decorated Teichmüller space $\tilde{T}(F)$, where the sum is over all triangles T complementary to τ in F and (a, b, c) are the edges of T in the correct counter-clockwise cyclic order determined by the orientation of the surface.

We introduce the Weil-Petersson two form on $\tilde{T}(\mathcal{H}_G)$ by appropriately normalizing the above expression. Recall that τ_* is the Farey tesselation on $\mathbb{D}$. We use the same notation τ_* for the induced tesselation on the baseleaf of $\mathcal{H}_G = \mathbb{D} \times \hat{G}/G$ and its canonical TLC extension to $\mathcal{H}_G$. A fundamental polygon P for the once-punctured torus group G on $\mathbb{D}$ consists of two ideal triangles $T_1 = (e_0, e_{1/2}, e_{2/1})$ and $T_2 = (e_0, e_{-2/1}, e_{-1/2})$.

We define the tangent vectors on $\tilde{T}(\mathcal{H}_G)$ in terms of its lambda length parametrization. Note that $Cont^G(\hat{G}, \mathbb{R}^{\tau_*}_{>0})$ is a subset of vector space $Cont^G(\hat{G}, \mathbb{R}^{\tau_*})$ of all continuous functions from $\hat{G}$ onto $\mathbb{R}^{\tau_*}$ that are invariant under the action of G. The norm on $Cont^G(\hat{G}, \mathbb{R}^{\tau_*})$ is given by

$$\|v\| := \sup_{t \in \hat{G},\ e \in \tau_*} |v(t, e)|,$$

for $v \in Cont^G(\hat{G}, \mathbb{R}^{\tau_*})$. With respect to this norm, $Cont^G(\hat{G}, \mathbb{R}^{\tau_*}_{>0})$ is an open subset of the vector space $Cont^G(\hat{G}, \mathbb{R}^{\tau_*})$. The subspace topology on $Cont^G(\hat{G}, \mathbb{R}^{\tau_*}_{>0})$ induced by the norm on $Cont^G(\hat{G}, \mathbb{R}^{\tau_*})$ coincides with the compact-open topology. Therefore, the tangent space at any point of $\tilde{T}(\mathcal{H}_G)$ is identified with $Cont^G(\hat{G}, \mathbb{R}^{\tau_*})$.

We define the Weil-Petersson two form on the tangent space at an arbitrary point $\lambda \in Cont^G(\hat{G}, \mathbb{R}^{\tau_*}_{>0})$. Let $u, v \in Cont^G(\hat{G}, \mathbb{R}^{\tau_*})$ be two arbitrary vectors at the tangent space at λ. Furthermore, let $\eta^i_{\lambda,u,v}(t)$ be the evaluation of the two form $-2(d\log a\wedge d\log b+d\log b\wedge d\log c+d\log c\wedge d\log a)$ above on the triangle $T_i\times\{t\}$ and vectors $u(t,\cdot)$ and $v(t,\cdot)$, for $i=1,2$ and $t\in\hat{G}$. We define the Weil-Petersson two form by

$$\omega_\lambda(u,v)=\int_{\hat{G}}[\eta^1_{\lambda,u,v}(t)+\eta^2_{\lambda,u,v}(t)]dm(t),$$

where m is the Haar measure on $\hat{G}$. The actual formulas are

$$\eta^1_{\lambda,u,v}(t)=-2\Big(\frac{u(t,e_0)v(t,e_{1/2})}{\lambda(t,e_0)\lambda(t,e_{1/2})}+\frac{u(t,e_{1/2})v(t,e_{2/1})}{\lambda(t,e_{1/2})\lambda(t,e_{2/1})}+\frac{u(t,e_{2/1})v(t,e_0)}{\lambda(t,e_{2/1})\lambda(t,e_0)}\Big),$$

$$\eta^2_{\lambda,u,v}(t)=-2\Big(\frac{u(t,e_0)v(t,e_{-2/1})}{\lambda(t,e_0)\lambda(t,e_{-2/1})}+\frac{u(t,e_{-2/1})v(t,e_{-1/2})}{\lambda(t,e_{-2/1})\lambda(t,e_{-1/2})}+\frac{u(t,e_{-1/2})v(t,e_0)}{\lambda(t,e_{-1/2})\lambda(t,e_0)}\Big).$$

The integral is over $T_i\times\hat{G}$, for $i=1,2$. In the case when λ is a lift of a decoration on the punctured torus $\mathbb{D}/G$ and the tangent vectors represent deformations in $\tilde{T}(\mathbb{D}/G)$, the two form $\omega_\lambda(u,v)$ is equal to the classical Weil-Petersson two form as in [20, Theorem A.2] (cf. Theorem B.2). If λ comes from a lift on some higher genus surface and the tangent vectors represent TLC deformations of the decorations in the decorated Teichmüller space of that surface, then the two form is a certain positive multiple of the classical Weil-Petersson form.

Since $\lambda(t,e)$, for fixed $e\in\tau_*$, is a continuous positive function in $t\in\hat{G}$, it is bounded below away from zero. Moreover, $u(t,e)$ and $v(t,e)$ are continuous in $t\in\hat{G}$, for fixed $e\in\tau_*$. Therefore, $\eta^i_{\lambda,u,v}(t)$ are bounded and continuous real functions in t. Thus, the Weil-Petersson two form $\omega_\lambda(u,v)$ is well-defined, namely, the integral converges.

We restrict attention to the base leaf. Let $\eta_{\lambda,u,v}|_T$ be the evaluation of the two form $-2(d\log a\wedge d\log b+d\log b\wedge d\log c+d\log c\wedge d\log a)$ on the triangle $T=(a,b,c)$ in $\mathbb{D}-\tau_*$, where $d\log a=\dot{a}/a$, etc... are given their corresponding values in terms of $\lambda(id,a)$, $u(id,a)$, and so on, on the baseleaf $\mathbb{D}$. We consider the average evaluation of η on the baseleaf. Consider a sequence of characteristic subgroups G_n (cf. Section 2) of $PSL_2\mathbb{Z}$ such that $\cap_n G_n=\{id\}$. Let P_n be a sequence of fundamental polygons for G_n such that the frontier edges of P_n belong to the Farey tesselation τ_*, and the P_n are nested.

Proposition 9.1. *Let $\lambda\in Cont^G(\hat{G},\mathbb{R}^{\tau_*}_{>0})$ and let $u,v\in Cont^G(\hat{G},\mathbb{R}^{\tau_*})$ be two tangent vectors based at λ. Then*

$$\omega_\lambda(u,v)=\lim_{n\to\infty}\frac{2}{k(n)}\sum_{T\in P_n}\eta_{\lambda,u,v}|_T,$$

where the sum is over all triangles T complementary to τ_* in the fundamental polygon P_n, and $k(n)$ is the number of triangles in P_n.

Proof. We first show that $\lim_{n\to\infty}\frac{2}{k(n)}\sum_{T\in P_n}\eta_{\lambda,u,v}|_T$ does not depend on the choice of the fundamental polygons P_n. Indeed, for n large enough the quantities λ, u and v are almost invariant under the group G_n. Thus, the quantity $\eta_{\lambda,u,v}|_T$ is also almost invariant under G_n. Namely, given $\epsilon>0$ there exists n such that $|\eta_{\lambda,u,v}|_T-\eta_{\lambda,u,v}|_{\gamma T}|<\epsilon$ for all triangles T complementary to τ_* whenever $\gamma\in G_n$. Hence, if the limit exists it does not depend on the choice of P_n.

For a fixed sequence P_n of fundamental polygons, the sum $\frac{2}{k(n)}\sum_{T\in P_n}\eta_{\lambda,u,v}|_T$ is a Cauchy sequence. This immediately follows from the above observation of almost invariance of $\eta_{\lambda,u,v}|_T$.

It remains to show the equality between $\omega_\lambda(u,v)$ and the limit of the sum. In the case when λ, u and v are TLC, the equality follows from the definition of the Haar measure and an observation that our normalization factor $\frac{2}{k(n)}$ is chosen correctly. The general case follows from the uniform continuity of $\eta^i_{\lambda,u,v}(t)$ and the fact that $\eta_{\lambda,u,v}|_{\gamma T}=\eta^i_{\lambda,u,v}(t)$ in the case when $t=\gamma\in\hat{G}$ is an element of $G<\hat{G}$ and T is the corresponding triangle $T_i\times\{\gamma\}$. $\square$

We show that the Weil-Petersson two form pushes forward from $\tilde{T}(\mathcal{H}_G)$ onto the Teichmüller space $T(\mathcal{H}_G)$. Consider the projection map $\pi:\tilde{T}(\mathcal{H}_G)\to T(\mathcal{H}_G)$. To show that there exists a well-defined push-forward of the two form, it is enough to show that $\omega_{\lambda^1}(u^1,v^1)=\omega_{\lambda^2}(u^2,v^2)$ whenever $\pi(\lambda^1)=\pi(\lambda^2)$, $d\pi_{\lambda^1}(u^1)=d\pi_{\lambda^2}(u^2)$ and $d\pi_{\lambda^1}(v^1)=d\pi_{\lambda^2}(v^2)$.

Theorem 9.2. *The Weil-Petersson two form on $\tilde{T}(\mathcal{H}_G)$ projects to a non-degenerate two form on the Teichmüller space $T(\mathcal{H}_G)$.*

Proof. Suppose that λ_i, u_i and v_i, $i=1,2$ are as above. If they are invariant under a finite index subgroup K of G, then $\omega_{\lambda_1}(u_1,v_1)=\omega_{\lambda_2}(u_2,v_2)$. Indeed, since the Weil-Petersson form in this case is a multiple of the Weil-Petersson form on $\tilde{T}(\mathbb{D}/K)$ (which is the lift of the Weil-Petersson form on $T(\mathbb{D}/K)$ by Theorem B.2), it projects to the multiple of the Weil-Petersson form on $T(\mathbb{D}/K)$. Thus, it satisfies $\omega_{\lambda_1}(u_1,v_1)=\omega_{\lambda_2}(u_2,v_2)$. In general, it is enough to show that any λ_i, u_i and v_i, $i=1,2$ can be approximated by TLC objects which satisfy the property in the paragraph above the theorem. Indeed, the union of decorated Teichmüller spaces of all finite surfaces is dense in the decorated Teichmüller space $\tilde{T}(\mathcal{H}_G)$ and the inclusion is smooth. It remains to see that two tangent vectors u_1,u_2 which project to the same vector in the tangent space of $T(\mathcal{H}_G)$ can be approximated by two TLC vectors u_1^c,u_2^c which project to the same TLC vector in the tangent space of $T(\mathcal{H}_G)$. We postpone this to Lemma 9.3. Since the formula for $\omega_\lambda(u,v)$ is continuous in its entries, the limit also satisfies the desired push-forward property. Thus, the Weil-Petersson two form indeed projects to the Teichmüller space $T(\mathcal{H}_G)$.

It remains to show that the push-forward of the Weil-Petersson two form is non-degenerate. To see this, we must show that for an arbitrary vector v at a point $\lambda\in\tilde{T}(\mathcal{H}_G)$ which projects to a non-zero vector on $T(\mathcal{H}_G)$ there exists another vector u such that $\omega_\lambda(u,v)\neq 0$. Note that a vector $v\in Cont^G(\hat{G},\mathbb{R}^{\tau_*})$ projects to a trivial vector on $T(\mathcal{H}_G)$ if and only if it is tangent to a path in $\lambda_s\in Cont^G(\hat{G},\mathbb{R}^{\tau_*}_{>0})$ such that $cr(\lambda_s(t,a),\lambda_s(t,b),\lambda_s(t,c),\lambda_s(t,d))=const$ for all quadrilaterals (a,b,c,d) which are union of two adjacent triangles of the Farey tesselation τ_* and for all $t\in\hat{G}$, where cr denotes the cross ratio. Thus, v projects to a trivial vector on $T(\mathcal{H}_G)$ if and only if there exists a rectangle $Q=(a,b,c,d)$ consisting of two adjacent triangles in $\mathbb{D}-\tau_*$ such that $|v(t,a)/\lambda(t,a)+v(t,c)/\lambda(t,c)-v(t,b)/\lambda(t,b)-v(t,d)/\lambda(t,d)|\geq\epsilon>0$ for $t\in\hat{K}$, where K is a finite index subgroup of G. Thus, either $|v(t,a)/\lambda(t,a)+v(t,c)/\lambda(t,c)|\geq\epsilon/2$ or $|v(t,b)/\lambda(t,b)-v(t,d)/\lambda(t,d)|\geq\epsilon/2$, for $t\in\hat{K}_1$, where K_1 is a finite index subgroup of K. We may assume that they have the same sign for K_1 small enough and that $|v(t,a)/\lambda(t,a)+v(t,c)/\lambda(t,c)|\geq\epsilon/2$, for $t\in\hat{K}_1$. Let $e\in\tau_*$ be the diagonal of Q, and choose u such that $u(t,e)\equiv 1$ for $t\in\hat{K}_1$ and equals zero otherwise.

Since $\omega_\lambda(u,v)=-2\int_{\hat{K}_1}\frac{1}{\lambda(t,e)}(\frac{v(t,a)}{\lambda(t,a)}+\frac{v(t,c)}{\lambda(t,c)})dm(t)$ and either $\frac{1}{\lambda(t,e)}(\frac{v(t,a)}{\lambda(t,a)}+\frac{v(t,c)}{\lambda(t,c)})\geq\epsilon/2$ or $\frac{1}{\lambda(t,e)}(\frac{v(t,a)}{\lambda(t,a)}+\frac{v(t,c)}{\lambda(t,c)})\leq-\epsilon/2$, for all $t\in\hat{K}_1$, we conclude that $\omega_\lambda(u,v)\neq 0$ as required. □

Lemma 9.3. *Let $\epsilon>0$ and let u_1,u_2 be two tangent vectors at λ_1,λ_2, respectively. If $d\pi_{\lambda_1}(u_1)=d\pi_{\lambda_2}(u_2)$ then there exist two TLC points λ_1^c,λ_2^c and two TLC vectors u_1^c,u_2^c such that $d\pi_{\lambda_1^c}(u_1^c)=d\pi_{\lambda_2^c}(u_2^c)$, $\|\lambda_i-\lambda_i^c\|_\infty<\epsilon$ and $\|u_i-u_i^c\|_\infty<\epsilon$, for $i=1,2$.*

Proof. By our assumption, there exist two paths $\lambda_{1,s}$ and $\lambda_{2,s}$ in $\tilde{T}(\mathcal{H}_G)$ such that $\lambda_{1,0}=\lambda_1$, $\lambda_{2,0}=\lambda_2$, $\pi(\lambda_{1,s})=\pi(\lambda_{2,s})$, $\frac{d}{ds}\lambda_{1,s}|_{s=0}=u_1$ and $\frac{d}{ds}\lambda_{2,s}|_{s=0}=u_2$. Let n be an nteger large enough that $\lambda_{1,s}$ and $\lambda_{2,s}$ on the baseleaf are almost invariant under G_n. We form two new paths of TLC lambda lengths which satisfy the required properties. We recall (cf. Lemma A1iv) that the projection $\pi:\tilde{T}(\mathcal{H}_G)\to T(\mathcal{H}_G)$ can be given by the formula

$$cr(Q)=\frac{ac}{bd},\tag{6}$$

where $cr(Q)$ is a cross-ratio of the endpoints of a quadrilateral $Q=T_1\cup T_2$ for two adjacent triangles $T_1,T_2\in\mathbb{D}-\tau_*$ and a,b,c,d are lambda lengths evaluated at consecutive edges of Q. A necessary and sufficient condition that $\pi(\lambda_{1,s})=\pi(\lambda_{2,s})$ is thus to have

$$\frac{\lambda_{1,s}(a)\lambda_{1,s}(c)}{\lambda_{1,s}(b)\lambda_{1,s}(d)}=\frac{\lambda_{2,s}(a)\lambda_{2,s}(c)}{\lambda_{2,s}(b)\lambda_{2,s}(d)}\tag{7}$$

for every such $Q=(a,b,c,d)$. We form new paths:

$$\lambda_{i,s}^c(a)=\lim_{k\to\infty}(\Pi_{\gamma\in H_k\subset G_n}\lambda_{i,s}(\gamma a))^{\frac{1}{k}},$$

where H_k contains k elements of G_n, $H_k \subset H_{k+1}$ and $\cup_{k=1}^{\infty} H_k = G_n$. The limits exist, they do not depend on the choice of H_k and are TLC. Moreover, $\lambda_{i,s}^c$ satisfy (7) because at each finite stage, $(\Pi_{\gamma \in H_k \subset G_n} \lambda_{i,s}(\gamma a))^{\frac{1}{k}}$ satisfy (7). They are close to the original paths for n large, and their tangent vectors at $s = 0$ approximate the original tangent vectors, are also TLC, and their projections agree. The lemma follows. □

Theorem 9.4. *The Weil-Petersson two form on $\tilde{T}(\mathcal{H}_G)$ is invariant under the modular group $Mod_{BLP}(\mathcal{H}_G)$. Therefore, it descends to $\tilde{T}(\mathcal{H}_G)/Mod_{BLP}(\mathcal{H}_G)$ and $T(\mathcal{H}_G)/Mod_{BLP}(\mathcal{H}_G)$.*

Proof. We recall from Theorem 8.5 that the modular group is generated by equivariant Whitehead moves and $PSL_2(\mathbb{Z})$. It is therefore enough to show the invariance under these elements of $Mod_{BLP}(\mathcal{H}_G)$. It is a calculation in [20] (that the reader could provide: show that the two form is algebraically invariant under Ptolemy transformations Lemma A.1(iii)) that a Whitehead move along a single edge does not change the sum $\eta_{\lambda,u,v}|_{T_1} + \eta_{\lambda,u,v}|_{T_2}$ over the two triangles which contain the edge on their boundaries. Since an equivariant Whitehead move is decomposed into Whitehead moves along disjoint edges, the invariance follows. The action of $PSL_2(\mathbb{Z})$ fixes τ_* and changes the marking. It is obvious that the Weil-Petersson two form is independent of the marking. The invariance under the whole $Mod_{BLP}(\mathcal{H}_G)$ follows. □

10. Concluding Remarks and Questions

To what extent is there number theory embedded in our constructions? The "center" of $\mathcal{C}(\tau)$, i.e., a point corresponding to equating all lambda lengths, covers an arithmetic punctured surface for each TLC tesselation τ [18, Section 6], which is evidently related to Grothendieck's dessins d'enfant [12]. In fact, equivariant Whitehead moves were considered before in this context [21]. Is there any connection between the absolute Galois group and the full (non-BLP) mapping class group of the solenoid? Furthermore, just as the Euclidean solenoid [17] is related to the rational adeles, our constructions give an adelic type structure to the punctured solenoid itself. How might the Teichmüller theory developed here, a kind of deformation theory of arithmetic punctured surfaces be manifest algebraically? Furthermore, the calculation of the index of the characterstic subgroups G_n of $G = PSL_2(\mathbb{Z})$ is an open problem in number theory; on one hand, it is evidently related to fatgraph enumeration [20, Appendix B], and on the other, the results of Section 9 suggest connections with Weil-Petersson volumes.

More generally, the regularization of Weil-Petersson form in Section 9 together with the strong topology on the lambda length functions seems to give a much more satisfactory universal geometry than [18]. With this symplectic or Poisson structure, the global lambda length coordinates, and the generators of the modular group for the solenoid, all the ingredients are in place for a Kashaev [11] or Chekhov-Fock type quantization [6] of the decorated Teichmüller theory developed here.

Are $\mathring{\mathcal{C}}(\tau)$ or $\mathcal{C}(\tau)$ contractible for a TLC tesselation or paving τ? The classical tools (from [19] used to prove Lemma B.4) seem to be unavailable here. Which non-TLC pavings of $\mathbb{D}$ arise from the convex hull construction?

We give only an infinite set of generators, and we ask if $Mod_{BLP}(\mathcal{H})$ is finitely generated. Of course, we would also hope to give a presentation of this group, beyond the normal forms presented here; perhaps one can mimic the case of punctured surfaces by first proving contractibility of the $\mathcal{C}(\tau)$. It also seems possible to characterize the mapping class like elements of $Mod_{BLP}(\mathcal{H})$, i.e., those homeomorphisms of $\mathcal{H}$ that arise from lifts of homeomorphisms of punctured surfaces, in terms of our generators, and we wonder if this might be useful to address a question from [17]: do the mapping class like homeomorphisms generate the modular group?

Appendix A-Pinched lambda lengths and the convex hull construction

Let $\mathbb{D}$ denote the unit disk with boundary S^1. A *tesselation* τ of $\mathbb{D}$ is a locally finite decomposition of $\mathbb{D}$ into ideal triangles. We shall think of τ as a set of edges and let $\tau^\infty \subseteq S^1$ denote the collection of ideal vertices of τ. In particular, for the Farey tesselation τ_*, we have $\tau_*^\infty = \mathbb{Q} \cup \{\infty\} = \bar{\mathbb{Q}} \subseteq S^1$. It will be useful sometimes to specify a distinguished oriented edge or *DOE* of a tesselation. In particular, the standard *DOE* of the Farey tesselation is the edge connecting -1 to $+1$.

Define Minkowski three-space to be the vector space $\mathbb{R}^3$ with indefinite pairing $< \cdot, \cdot >$ whose quadratic form is $x^2 + y^2 - z^2$ in the usual coordinates. The upper-sheet of the hyperboloid is $\mathbb{H} = \{w = (x, y, z) :< w, w >= -1,\ z > 0\}$, and the positive light cone in Minkowski space is $L^+ = \{w = (x, y, z) :< w, w >= 0,\ z > 0\}$. The former is a model for the hyperbolic plane, where the distance Δ between two points $u, v \in \mathbb{H}$ is given by $\Delta^2 = \cosh^2 < u, v >$. The latter is identified with the space of all horocycles via the correspondence $L^+ \ni w \mapsto \{u \in \mathbb{H} :< w, u >= -1\} \subseteq \mathbb{H}$, and there is the corresponding identification $L^+/\mathbb{R}_{>0} \equiv S^1$ which maps a horocycle onto its center. The *lambda length* of $u, v \in L^+$ is defined to be $\Lambda(u, v) = \sqrt{- < u, v >}$, and geometrically it is $\sqrt{2 \exp \delta}$, where δ is the signed hyperbolic distance between the corresponding horocycles, taken with positive sign if and only if the horocycles are disjoint. An affine plane in Minkowski space is respectively elliptic, parabolic, hyperbolic if and only if it determines the corresponding conic section, or equivalently if and only if the Minkowski normal v to the plane $\{w :< w, v >= -1\}$ lies interior to, lies on, or lies exterior to the light-cone.

Lemma A.1 *Given pairwise non collinear points $u_1, u_2, u_3, u_4 \in L^+$ so that the rays determined by u_1, u_3 separate those determined by u_2, u_4, set $\lambda_{ij} = \sqrt{- < u_i, u_j >}$.*

(i) [19, Lemmas 2.3, 2.4] *Given three positive real numbers $\ell_{12}, \ell_{13}, \ell_{23} \in \mathbb{R}_{>0}$, there are unique points v_i in the ray in L^+ determined by u_i, for $i = 1, 2, 3$, so that $\ell_{ij} = \lambda_{ij}$. Furthermore, given $v_1, v_2 \in L^+$ so that $\ell_{12} = \lambda_{12}$, there is a unique $v_3 \in L^+$ on either side of the plane through the origin determined by v_1, v_2 so that also $\ell_{13} = \lambda_{13}$ and $\ell_{23} = \lambda_{23}$. In each case, the points v_i depend continuously on the lambda lengths.*

(ii) [19, Lemma 2.2] *The plane spanned by u_1, u_2, u_3 is elliptic if and only if all three strict triangle inequalities hold amongst $\lambda_{12}, \lambda_{13}, \lambda_{31}$, and it is parabolic if and only if a triangle equality holds.*

(iii) [19, Proposition 2.6a] *The* Ptolemy equation *holds:*

$$\lambda_{13}\lambda_{24} = \lambda_{12}\lambda_{34} + \lambda_{14}\lambda_{23}.$$

(iv) [20, Lemma A.2] *Let $\bar{u}_i \in S^1$ denote the projection of $u_i \in L^+$, for $i = 1, 2, 3, 4$. The cross-ratio is given as follows: normalizing so that $\bar{u}_1 \mapsto 1$, $\bar{u}_2 \mapsto 0$, $\bar{u}_4 \mapsto \infty$, then $\bar{u}_3 \mapsto \frac{\lambda_{23}\lambda_{34}}{\lambda_{12}\lambda_{14}}$.*

(v) [19, Proposition 2.6b] *The signed volume of the Euclidean tetrahedron spanned by u_1, u_2, u_3, u_4 is given by $(\lambda_{12}\lambda_{23}\lambda_{34}\lambda_{41})\ \sigma$, where the* simplicial coordinate *σ is defined by:*

$$\sigma = \frac{\lambda_{12}^2 + \lambda_{23}^2 - \lambda_{31}^2}{\lambda_{12}\lambda_{23}\lambda_{31}} + \frac{\lambda_{14}^2 + \lambda_{43}^2 - \lambda_{31}^2}{\lambda_{14}\lambda_{43}\lambda_{31}},$$

and the sign is positive if and only if the edge connecting u_1, u_3 lies below the edge connecting u_2, u_4.

We claim that any function $\Lambda : \tau_* \to \mathbb{R}_{>0}$ gives rise to a unique $h : \tau_*^\infty \to L^+$ realizing the putative lambda lengths in the sense that for all $e \in \tau_*$, we have $\Lambda(e) = \sqrt{- < h(u), h(v) >}$, where u, v are the endpoints of e. To see this, we may uniquely realize the putative lambda lengths on the triangle to the left of the *DOE* by points in the rays in L^+ lying over $\pm 1, -i$ by the first statement in part (i) of Lemma A.1. We may then use the second statement in part (i) to recursively define the required function $h : \tau_*^\infty \to L^+$, where at each stage in the construction in the notation of part (i), v_3 lies on the other side of the plane through the origin containing v_1, v_2 from the triangle to the left of the *DOE*. Post-composing with the projection, we define the "characteristic map" $\bar{h} : \tau_*^\infty \to L^+/\mathbb{R}_{>0} \equiv S^1$ and can ask whether $\bar{h}$ interpolates a homeomorphism $S^1 \to S^1$, and if so, what is the nature of this homeomorphism.

We say that $\Lambda : \tau_* \to \mathbb{R}_{>0}$ is *pinched* if there is a constant $M > 1$ so that for all $e \in \tau_*$, we have $M^{-1} < \Lambda(e) < M$.

Theorem A.2 [18, Theorems 6.3 and 6.4] *If $\Lambda : \tau_* \to \mathbb{R}_{>0}$ is pinched, then $h(\tau_*^\infty) \subseteq L^+$ satisfies the following two properties:*

(1) *$h(\tau_*^\infty)$ is* discrete*, i.e., below any elliptic plane in Minkowski space, there are only finitely many points of $h(\tau_*^\infty)$.*

(2) $h(\tau_*^\infty)$ *is* radially dense*, i.e., in the union of any open set of rays in* L^+*, there is some point of* $h(\tau_*^\infty)$.

Furthermore, $\bar{h} : \tau_*^\infty \to S^1$ *interpolates a homeomorphism* $\phi : S^1 \to S^1$ *which is quasi-symmetric.*

If $\Lambda \in \mathbb{R}_{>0}^{\tau_*}$ is pinched, let $\mathcal{B} = h(\tau_*^\infty) \subseteq L^+$, and let $C \subseteq \mathbb{R}^3$ denote the closed convex hull of $\mathcal{B}$ (in the vector space underlying Minkowski space). We may think of the boundary ∂C of C in Minkowski space as a piecewise-linear approximation to the upper sheet of the unit hyperboloid with its vertices in L^+:

Theorem A.3 [18, Lemma 7.2] *Suppose that* $\mathcal{B} \subseteq L^+$ *is a discrete and radially dense subset with closed convex hull* C*. Then* $C \cap L^+$ *is the set of points of the form* tz*, where* $t \geq 1$ *and* $z \in \mathcal{B}$*; each ray from the origin lying inside* L^+ *meets* ∂C *exactly once. The boundary* ∂C *is the union of* $C \cap L^+$ *and a countable collection of codimension one* faces $F_1, F_2, \ldots$. *Each such face is the convex hull of some coplanar subset* $X \subseteq \mathcal{B}$*. The affine plane containing* X *is either parabolic or elliptic, and if* X *is infinite, then this affine plane is parabolic. The set of faces is locally finite in the interior of* L^+*.*

An edge of ∂C determines a geodesic in $\mathbb{D}$ in the natural way, and we let $\tau \subseteq \mathbb{D}$ denote the set of geodesics corresponding to all the edges of ∂C. According to the previous theorem, τ is a locally finite collection of disjoint geodesic whose complementary regions are either finite-sided polygons (corresponding to elliptic or parabolic support planes) or infinte-sided (corresponding to parabolic support planes), and we call such a decomposition of $\mathbb{D}$ a *paving*. Generically, no four points of $\mathcal{B}$ are coplanar, and the paving τ is a tesselation of $\mathbb{D}$.

We shall adopt this as standard notation: if $\Lambda \in \mathbb{R}_{>0}^{\tau_*}$ is pinched, then the corresponding $h_\Lambda : \tau_*^\infty \to L^+$ has discrete radially dense (drd) image $\mathcal{B}_\Lambda = h_\Lambda(\tau_*^\infty) \subseteq L^+$ and the projection $\bar{h}_\Lambda : \tau_*^\infty \to S^1$ interpolates a quasi-symmetric homeomorphism $\phi_\Lambda : S^1 \to S^1$; we may also sometimes extend ϕ_Λ to a quasi-conformal map $\phi_\Lambda : \mathbb{D} \to \mathbb{D}$ say using [7]. The edges in the boundary of the closed convex hull C_Λ of $\mathcal{B}_\Lambda$ project to a locally finite collection τ_Λ of disjoint geodesics in $\mathbb{D}$.

By the geometric interpretation of simplicial coordinates in Lemma A.1(v), it follows that the simplicial coordinate of (the edge of C_Λ corresponding to) an edge of τ_Λ is a well-defined non-negative real number. In particular in the generic case that τ is a triangulation, all the simplicial coordinates are strictly positive.

Given a tesselation τ and given any unoriented edge f of τ, define the *Whitehead move along* f to be the tesselation $\tau_f = \tau \cup \{g\} - \{f\}$, where f, g are the diagonals of an ideal quadrilateal with frontier in τ. Furthermore, if τ comes equipped with a *DOE* e, then there is an corresponding *DOE* on τ_f, where if $e = f$, then the orientations on e, f in this order is consistent with the orientation of $\mathbb{D}$ itself, and if $e \neq f$, then the *DOE* is unchanged.

Thus, sequences of Whitehead moves act on tesselations or on tesselations with *DOE*. Sequences of Whitehead moves on interior edges also act on triangulations of finite polygons and on triangulations with *DOE* of finite polygons.

Theorem A.4 [18, Lemma 4.4] *Fix a finite-sided connected polygon and consider two triangulations of it with DOE. Then there is a finite sequence of Whitehead moves from one to the other, i.e., sequences of Whitehead moves act transitively on tesselations with DOE of a finite polygon.*

Appendix B-Punctured surfaces

This appendix is intended to recall the principal results regarding punctured surfaces from [19] and to give an explicit presentation for the modular groups of punctured surfaces. Fix a surface F with negative Euler characteristic and $s \geq 1$ many punctures. An "ideal triangulation" of F is the homotopy class of a family τ of disjointly embedded arcs connecting punctures so that each component of $F - \cup\tau$ is a collection of triangles. A "decoration" on F is the specification of one horocycle centered at each puncture of F. The forgetful map $\tilde{T}(F) \to \mathcal{T}(F)$ from decorated to undecorated Teichmüller space of F is a principal $\mathbb{R}_{>0}^s$-bundle, where we may take for instance the hyperbolic lengths of the horocycles as coordinates on $\mathbb{R}_{>0}^s$. As a point of notation, we shall let $\Gamma \in T(F)$ denote the (class of a) Fuchsian group uniformizing a point of $T(F)$, and $\tilde{\Gamma} \in \tilde{T}(F)$ a decorated hyperbolic structure with underlying $\Gamma \in T(F)$.

Theorem B.1 [19, Theorem 3.1] *Fix an ideal triangulation τ of F. Then the natural map $\tilde{T}(F) \to \mathbb{R}^{\tau}_{>0}$, which associates to $\tilde{\Gamma} \in \tilde{T}(F)$ the function that assigns the lambda length of $e \in \tau$ for $\tilde{\Gamma}$, is a homeomorphism onto. That is, lambda lengths on the edges of τ give a parametrization of $\tilde{T}(F)$.*

The proof was sketched after Lemma A.1: use lambda lengths on the lift of τ to the universal cover to recursively define $h : \tau^{\infty}_{*} \to L^{+}$. This characteristic map can easily be shown directly to interpolate a homemorphism of the circle (or alternatively use Theorem A.1 since the finitely many lambda lengths are *a priori* pinched). Thus, there is a corresponding tesselation of $\mathbb{D}$, which is invariant under a Fuchsian group by Poincaré's fundamental polygon theorem. This gives the point $\Gamma \in T(F)$, and the construction furthermore gives a decoration $\tilde{\Gamma} \in \tilde{T}(F)$ on it. This provides the inverse homeomorphism to the map in Theorem B.1.

Theorem B.2 [20, Theorem A.1] *In the notation of Theorem B.1, the Weil-Petersson Kähler two from on $T(F)$ lifts to*

$$-2 \sum \ \mathrm{d}lna \ \wedge \ \mathrm{d}lnb + \ \mathrm{d}lnb \ \wedge \ \mathrm{d}lnc + \ \mathrm{d}lnc \ \wedge \ \mathrm{d}lna$$

on $\tilde{T}(F)$, where the sum is over all triangles complementary to τ in F with oriented boundary having lamba lengths a, b, c in this counter-clockwise cyclic order.

The proof is a calculation in [20, Appendix A] starting from Wolpert's formula.

Turning now to the convex hull construction in this case, the drd set $\mathcal{B}$ consists of $s \geq 1$ many Γ-orbits in the light cone L^{+}. The support planes of the convex hull C of $\mathcal{B}$ are all elliptic (cf. [19, Proposition 4.4]), hence finite sided by discreteness. This is in contrast to Theorem A.2, where parabolic support planes can occur. The extreme edges of the Γ-invariant convex body C project to $\mathbb{D}$ as usual and then to a family $\tau_{\tilde{\Gamma}}$ of arcs connecting punctures in F.

Easy Lemma B.3 [19, Theorem 4.5] *For every $\tilde{\Gamma} \in \tilde{T}(F)$, $\tau_{\tilde{\Gamma}}$ is a paving, i.e., consists of disjointly embedded arcs, no two of which are homotopic, connecting punctures so that each complementary region is a polygon.*

For any paving τ of F, define

$$\begin{aligned} &\overset{\circ}{\mathcal{C}}(\tau) = \{\tilde{\Gamma} \in \tilde{T}(F) : \tau_{\tilde{\Gamma}} = \tau\} \\ &\ \cap \\ &\mathcal{C}(\tau) = \{\tilde{\Gamma} \in \tilde{T}(F) : \tau_{\tilde{\Gamma}} \subseteq \tau\} \\ &\ \cap \\ &\tilde{T}(F). \end{aligned}$$

Hard Lemma B.4 [19, Theorem 5.4] *Each $\mathcal{C}(\tau)$ has the natural structure of an open simplex $\overset{\circ}{\mathcal{C}}(\tau)$ plus certain of its faces.*

Together, the two previous lemmas give:

Theorem B.5 [19, Theorem 5.5] $\{\overset{\circ}{\mathcal{C}}(\tau) :$ τ is a paving of $F\}$ *provides a cell decomposition of $\tilde{T}(F)$ which is invariant under the modular group $Mod(F)$ of F.*

Corollary B.6 [19, Proposition 7.1] *Compositions of Whitehead moves act transitively on the collection of all ideal triangulations of a fixed punctured surface.*

Proof $\overset{\circ}{\mathcal{C}}(\tau) \neq \emptyset$ for any ideal triangulation τ of F; to see this, take all lambda lengths to be unity and compute simplicial coordinates (cf. Lemma A.1(v)). Since $\tilde{T}(F)$ is path connected, there is a path between the top-dimensional cells corresponding to any two ideal triangulations of F. Putting this path in general position with respect to the codimension-one faces of the cell decomposition shows that compositions of Whitehead moves act transitively on ideal triangulations of F. □

Consider the branching locus $\{\tilde{\Gamma} \in \tilde{T}(F) : \exists \phi \in Mod(F)$ with $\phi\ \tilde{\Gamma} = \tilde{\Gamma}\}$ with projection $\tilde{\mathcal{L}} \subseteq \tilde{M} = \tilde{M}(F) = \tilde{T}(F)/Mod(F)$. Recall (cf. [25],[10]) that the "orbifold fundamental group" $Mod(F) = \pi_1^{orb}(\tilde{M})$

is the semi-direct product of the usual topological fundamental group $\pi_1^{top}(\tilde{M} - \tilde{\mathcal{L}})$ with a finite group determined by the branching, i.e., by the symmetry groups of the underlying pavings.

Corollary B.7 *The modular group $Mod(F) = \pi_1^{orb}(\tilde{M}(F))$ of any punctured surface F of negative Euler characteristic is the stabilizer of any object in the groupoid with finite presentation:*

generators are provided by Whitehead moves between ideal triangulations and the union of automorphism groups of all ideal triangulations of F;

relations are given by involutivity, the pentagon and commutativity relations (as in Section 8), and invariance under $\phi \in Mod(F)$, namely, for any ideal triangulation τ of F, we have: i) *the Whitehead move on e in τ is identified with the Whitehead move on $\phi(e)$ in $\phi(\tau)$, and likewise, $Aut(\tau)$ is identified with $Aut(\phi(\tau))$; and* ii) *if w is a sequence of Whitehead moves beginning at τ and ending at $\phi(\tau)$, then up to pre- and post-composition with elements of $Aut(\tau) \approx Aut(\phi(\tau))$, we identify ϕ with w.*

Proof That the putative relations hold amongst the putative generators follows from the discussion in Section 8 plus the obvious invariance i) and ii) under $Mod(F)$. As in [10], Whitehead moves together with the automorphism groups of all pavings of F generates $Mod(F)$ since compositions of Whitehead moves act transitively on ideal triangulations of a polygon (by Theorem A.4). Furthermore, if $\phi \in Aut(\sigma)$ for some paving σ, then we may arbitrarily extend to an ideal triangulation $\tau \supset \sigma$. By Corollary B.6, there is some sequence w of Whitehead moves from $\phi(\tau)$ to τ, and by naturality condition ii), the composition given by first ϕ and then w must lie in $Aut(\tau)$. It follows that Whitehead moves and automorphism groups of ideal triangulations alone indeed generate $Mod(F)$.

As to relations, a homotopy of loops or paths in the orbifold $\tilde{M}$ gives rise to a homotopy also in the underlying space, and this homotopy may be put into general position with the codimension two skeleton of the cell decomposition. The orbifold fundamental group is a semi-direct product of $\pi_1^{top}(\tilde{M} - \tilde{\mathcal{L}})$ with a finite group as above, and this finite group is contained in the span of the generators by the previous paragraph. Up to the action of this finite group, the homotopy is therefore given by (the links of) the intersecting cells of codimension two (namely, the pentagon and commutativity relations) together with the relation of serially and trivially crossing the same face (namely, involutivity). □

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