THE QUASI-ADDITIVITY LAW IN CONFORMAL GEOMETRY

JEREMY KAHN AND MIKHAIL LYUBICH

Abstract. We consider a Riemann surface S of finite type containing a family of N disjoint disks D_i, and prove the following Quasi-Additivity Law: If the total extremal width $\sum \mathcal{W}(S \smallsetminus D_i)$ is big enough (depending on N) then it is comparable with the extremal width $\mathcal{W}(S, \cup D_i)$ (under a certain "separation assumption") .

We also consider a branched covering $f : U \to V$ of degree N between two disks that restricts to a map $\Lambda \to B$ of degree d on some disk $\Lambda \Subset U$. We derive from the Quasi-Additivity Law that if $\operatorname{mod}(U \smallsetminus \Lambda)$ is sufficiently small, then (under a "collar assumption") the modulus is *quasi-invariant* under f, namely $\operatorname{mod}(V \smallsetminus B)$ is comparable with $d^2 \operatorname{mod}(U \smallsetminus \Lambda)$.

This *Covering Lemma* has important consequences in holomorphic dynamics which will be addressed in the forthcoming notes.

1. Introduction

We denote the *extremal length* of a family $\mathcal{G}$ of curves by $\mathcal{L}(\mathcal{G})$, and we let $\mathcal{W}(\mathcal{G}) = \mathcal{L}(\mathcal{G})^{-1}$ be the corresponding *extremal width* (see the Appendix). Let S stand for a compact Riemann surface of finite type with boundary. Given a compact subset $K \subset \operatorname{int} S$, we let $\mathcal{L}(S, K)$ and $\mathcal{W}(S, K)$ be respectively the extremal length and width of the family of curves in $S \smallsetminus K$ connecting ∂S to K.

An open subset $A \Subset \operatorname{int} S$ is called an *archipelago* if its closure is a Riemann surface of finite type (not necessarily connected) with smooth boundary. Let A_j ($j = 1, \dots, N$) be a finite family of archipelagos in S with disjoint closures.

Let us now introduce 3 conformal moduli of our family of archipelagos:

$$X = \mathcal{W}(S, \bigcup_{j=1}^{N} A_j);$$

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$$Y = \sum_{j=1}^{N} \mathcal{W}(S, A_j), \tag{1.1}$$

$$Z = \sum_{j=1}^{N} \mathcal{W}(S \smallsetminus \bigcup_{k\neq j} A_k,\ A_j).$$

It is easy to see that $X \le Y \le Z$. We say that the archipelagos are *ξ-separated* if $Z \le \xi Y$. In this paper we show that in a near degenerate situation (when Y is big), under the separation assumption, the moduli X and Y are comparable:

Quasi-Additivity Law. *Assume that the archipelagos $A_j \Subset \operatorname{int} S$ are ξ-separated. Then there exists K depending only on ξ and the topological complexity of the family of archipelagos*[1] *such that:*
If $Y \ge K$ then $Y \le C\xi X$, where C is an absolute constant.[2]

The general principle behind this result is that families of curves with big width have a small intersection (see §2.4), so that the families of curves connecting ∂S to A are nearly parallel.

In §2.9 we give a variation of the Quasi-Additivity Law adapted to the needs of holomorphic dynamics.

If we have a branched covering $f : U \to V$ of degree N between two disks that restricts to a branched covering $f : \Lambda \to B$ of degree d between smaller disks, then a simple general estimate shows that $\operatorname{mod}(V \smallsetminus B) \le N \operatorname{mod}(U \smallsetminus \Lambda)$. It turns out that given d, in a near degenerate situation the above moduli are, in fact, comparable (under a collar assumption):

Covering Lemma/Quasi-Invariance Law. *Fix some $\eta > 0$. Let $\operatorname{int} U \supset \Lambda' \Supset \Lambda$ and $\operatorname{int} V \supset B' \Supset B$ be two nests of Jordan disks. Let $f : (U, \Lambda', \Lambda) \to (V, B', B)$ be a branched covering between the respective disks, and let $N = \deg(U \to V)$, $d = \deg(\Lambda' \to B')$. Assume the following Collar Property:*

$$\operatorname{mod}(B' \smallsetminus B) > \eta \operatorname{mod}(U \smallsetminus \Lambda).$$

Then there exists an $\varepsilon > 0$ (depending on η and N) such that:
If $\operatorname{mod}(U \smallsetminus \Lambda) < \varepsilon$ then

$$\operatorname{mod}(V \smallsetminus B) < C\eta^{-1} d^2 \operatorname{mod}(U \smallsetminus \Lambda),$$

where C is an absolute constant.

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[1]See §2.3 for the precise definition

[2]One can take $C = 144$.

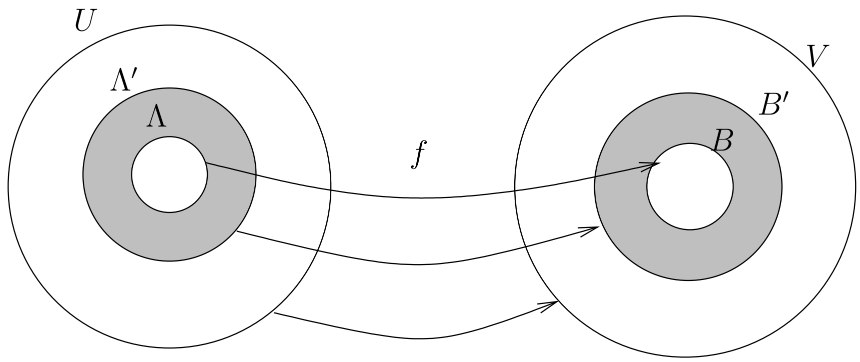

FIGURE 1.1. Covering between two nests of three disks

We derive this lemma from the Quasi-Additivity Law by passing to an appropriate Galois covering of U. For this application, the Quasi-Additivity Law should be applied to a Riemann surface S of finite type in place of the disk U, which is one of the reasons why it is formulated in this generality.

The needed background in the extremal length techniques is summarized in the Appendix.

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2. Quasi-Additivity Law

2.1. **Paths and rectangles.** Let S be a connected Riemann surface with boundary. A curve $\gamma : [0,1] \to S$ is called *proper* if $\gamma\{0,1\} \subset \partial S$. Two proper curves are called *properly homotopic* ($\equiv$ *parallel*) in S if they are homotopic through a family of proper curves. A proper curve is called *trivial* if it is properly homotopic to a curve $[0,1] \to \partial S$. A *path* in S is a curve without self-intersections, i.e., an embedded (oriented) interval $[0,1] \to S$.

A *(topological) rectangle* Π in S is a Jordan disk with four marked points on $\partial\Pi$. The rectangles below are assumed to be closed unless otherwise is explicitly said. In what follows we will usually deal with rectangles with a pair of opposite sides contained in the boundary ∂S. These sides are called *horizontal*, while the other two are called *vertical*. A *vertical path* in P is a path connecting its horizontal sides.

Similarly, a vertical path in an annulus A is a path connecting its boundary components. If we cut the annulus along two disjoint vertical paths, we obtain two rectangles. This situation is special, as only one rectangle would be cut off from any other Riemann surface:

Lemma 2.1. *Assume S is not an annulus. Let $\mathcal{C}^1$ and $\mathcal{C}^2$ be two disjoint properly homotopic non-trivial paths in S such that $\operatorname{int}\mathcal{C}^i \subset \operatorname{int} S$.*

(i) *Then there exist two unique arcs α and ω on the boundary ∂S which together with the paths $\mathcal{C}^i$ bound a rectangle P.*

(ii) *Let $(\mathcal{C}^t)$, $1 \le t \le 2$, be a proper homotopy between the above paths, and let $(e^t) \subset \partial S$ be the corresponding motion of the endpoint e^t of $\mathcal{C}^t$. Then the curve $(e^t)_{1\le t\le 2}$ is homotopic in ∂S rel the endpoints to the arc ω oriented from e^1 to e^2.*

(iii) *Let $\mathcal{C}^3$ be a third path which is disjoint and properly homotopic to the above two. Let P_j, $j = 1, 2, 3$, be the rectangles bounded by the pairs of these three paths. Then one of these rectangles is tiled by the other two.*

Proof. (i) Let us consider the universal covering $\pi : \hat{S} \to S$ of S. It is conformally equivalent to $\bar{\mathbb{D}} \smallsetminus K$, where $\bar{\mathbb{D}}$ is the closed unit disk and $K \subset \mathbb{T}$ is a nowhere dense compact subset of the unit circle (the limit set of the Fuchsian group of deck transformations). Since the paths $\mathcal{C}^i$ are properly homotopic, they lift to (disjoint) properly homotopic paths $\hat{\mathcal{C}}^i$ in $\hat{S}$. Let these lifts begin at points $b^i \in \mathbb{T}$ and end at points $e^i \in \mathbb{T}$. Then b^1 and b^2 (resp., e^1 and e^2) bound an arc $\hat{\alpha} \subset \partial\hat{S}$ (resp. $\hat{\omega} \subset \partial\hat{S}$). These two arcs are disjoint since the paths $\mathcal{C}^i$ are non-trivial. They are also disjoint from the $\operatorname{int}\mathcal{C}^i \subset \operatorname{int}\hat{S}$. Hence the four paths, $\mathcal{C}^1$, $\mathcal{C}^2$, $\hat{\alpha}$ and $\hat{\omega}$, bound a closed rectangle $\hat{P}$ in $\hat{S}$.

Let us consider all the lifts $\hat{\mathcal{C}}^i_j$ of $\mathcal{C}^i$ that cross $\hat{P}$, where $\hat{\mathcal{C}}^i_0 \equiv \hat{\mathcal{C}}^i$. For each $i = 1, 2$, the lifts $\hat{\mathcal{C}}^i_j$ are pairwise disjoint since the paths $\mathcal{C}^i$ do not have self-intersections. Any two paths $\hat{\mathcal{C}}^1_j$ and $\hat{\mathcal{C}}^2_k$ are disjoint as well since $\mathcal{C}^1$ and $\mathcal{C}^2$ do not cross each other. Hence each $\hat{\mathcal{C}}^i_j$ is completely contained in P and moreover, $\partial\hat{\mathcal{C}}^i_j \subset \hat{\alpha} \cup \hat{\omega}$. But $\partial\hat{\mathcal{C}}^i_j$ cannot belong to one horizontal side, α or ω, since the paths $\mathcal{C}^i$ are non-trivial. Thus, we obtain a family of disjoint vertical paths $\hat{\mathcal{C}}^i_j$ in $\hat{P}$.

If one the above curves, say $\mathcal{C}^1$, has more than one lift, then let us consider the lift $\hat{\mathcal{C}}^1_1$ such that there are no other lifts in between $\hat{\mathcal{C}}^1_0$ and $\hat{\mathcal{C}}^1_0$. Then $\hat{\mathcal{C}}^1_0$ and $\hat{\mathcal{C}}^1_1$, together with two subarcs of $\hat{\alpha}$, and $\hat{\omega}$ bound a rectangle $\hat{\Pi}$. The projection of this rectangle to S is a clopen annulus A in S. Since S is connected, $S = A$ contradicting our assumption.

Thus, each curve $\mathcal{C}^i$ has only one lift to $\hat{P}$, so $\hat{P} \cap \pi^{-1}(\mathcal{C}^i) = \hat{\mathcal{C}}^i$. It follows that the paths $\mathcal{C}^i$ lie on the boundary of $P \equiv \pi(\hat{P})$. Hence $\pi(\partial \hat{P}) \subset \partial P$, and the map $\pi : \hat{P} \to P$ is proper. Moreover, it is injective over $\mathcal{C}^i$ and hence has degree 1. Thus, the map $\pi : \hat{P} \to P$ is a homeomorphism.

If there were two rectangles P^1 and P^2 as above then they would be glued along the paths $\mathcal{C}^i$ to form an annulus.

(ii) The homotopy $(\mathcal{C}^t)$ lifts to a proper homotopy $\hat{\mathcal{C}}^t$ on $\hat{S}$ between the lifts $\hat{\mathcal{C}}^i$ considered in (i). The endpoint $\hat{e}^t$ of this lift moves along the component $\hat{\xi}$ of $\partial \hat{S}$. Since $\hat{\xi}$ is an interval, the curve $(\hat{e}^t)$ is homotopic to the arc $\hat{\omega}$ on $\hat{\xi}$ rel the endpoints. Hence (e^t) is homotopic to ω on ∂S rel the endpoints.

(iii) The paths $\mathcal{C}^i$ lift to proper paths $\hat{\mathcal{C}}^i$ in $\hat{S}$ that begin and end on the same component of $\partial \hat{S}$. Then one of the lifted rectangles $\hat{P}_j$ is tiled by the other two. Since $\pi : \hat{P}_j \to P_j$ is a homeomorphism, the same is true for the P_j's.

□

Somewhat loosely, we will say that the above rectangle P is *bounded* by the curves $\mathcal{C}^1$ and $\mathcal{C}^2$. According to our convention, the sides α and ω of P are called horizontal, while the sides $\mathcal{C}^i$ are called vertical. Moreover, we assume that the paths $\mathcal{C}^i$ begin on α and end on ω. Then α is called the *base* of P, while ω is called the *roof*.

Note that P is endowed with a *vertical orientation*, i.e., the orientation of the vertical paths in P consistent with the orientation of its vertical sides (namely, a positively oriented vertical path begins on α and ends on ω.)

To avoid the ambiguity in the choice of the rectangle P, *in what follows we assume that the Riemann surface S under consideration is not an annulus.* This assumption does not reduce generality since we can always puncture the surface at one point and this does not change extremal lengths of paths families under consideration.

Let us consider an archipelago A in S. Given a proper path $\mathcal{C}$ in S that crosses $\bar{A}$, let a be the last point of intersection of $\mathcal{C}$ with $\bar{A}$, and let $\delta \subset S \smallsetminus A$ be the terminal closed segment of $\mathcal{C}$ which connects a to ∂S. Note that $\operatorname{int} \delta \subset \operatorname{int}(S \smallsetminus A)$. If we have several paths $\mathcal{C}^i$ as above, we naturally label the corresponding objects as a^i and δ^i, etc.

Two disjoint proper paths $\mathcal{C}^1$ and $\mathcal{C}^2$ in S that cross $\bar{A}$ are called *roof parallel* (rel A) if:

- $\mathcal{C}^1$ and $\mathcal{C}^2$ are properly homotopic in S, and hence they bound a "big rectangle" P;
- The paths δ^i are properly homotopic in $S \smallsetminus A$, and hence they bound a "terminal little rectangle" $Q \subset S \smallsetminus A$;
- The rectangles P and Q share the roof (Figure 2.1 illustrates that this is not automatic.)

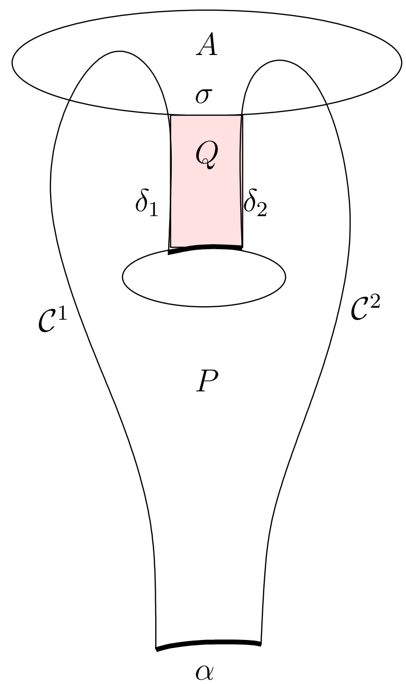

FIGURE 2.1. Strange configuration of rectangles

Two paths are called *base parallel* (rel A) if after reversing orientation they become roof parallel. Initial segments of these paths bound an initial little rectangle $Q_1 \subset S \smallsetminus A$ which shares the base with P. Two paths a called *parallel* if they are roof and base parallel.

We will now formulate several statements about roof parallel paths. The corresponding statements about base parallel paths are obtained by reversing orientation, and the corresponding statements about parallel paths immediately follow.

Lemma 2.2. *Let $\mathcal{C}^1$ and $\mathcal{C}^2$ be two roof parallel (rel A) proper paths in S, and P and Q be the corresponding big and little rectangles. Let $\mathcal{C}$ be a positively oriented vertical path in P which is disjoint from the $\mathcal{C}^i$. Then it is roof parallel to each $\mathcal{C}^i$. Moreover, its terminal segment δ is a vertical path in Q.*

Proof. Any vertical path in P is properly homotopic to the sides $\mathcal{C}^i$. Let P^i be the big rectangles bounded by the paths $\mathcal{C}$ and $\mathcal{C}^i$, and let ω^i be their roofs, $i = 1, 2$. Of course, they tile the roof ω, overlapping at the endpoint e of $\mathcal{C}$.

Let $\mathcal{C}'$ be the path $\mathcal{C}$ with reverse orientation. Since P and Q share the roof, some initial segment of $\mathcal{C}'$ is contained in Q. Since $\mathcal{C}'$ is proper, it must exit Q. Since $\operatorname{int}\mathcal{C}'$ is disjoint from the vertical sides and the roof of Q, it can exit Q only through its base, σ. Let a be the first point of intersection between $\mathcal{C}'$ and σ. Then the terminal segment δ of $\mathcal{C}$ that begins at a is a positively oriented vertical path in Q. Hence it is properly homotopic in $S \smallsetminus A$ to the paths δ^i.

Let $Q^i \subset S \smallsetminus \bar{A}$ be the little rectangles bounded by the paths δ and δ^i, $i = 1, 2$. Since δ is a vertical path in Q ending at e, the arcs ω_i are the roofs of the little rectangles Q^i. Thus, Q_i respectively share the roofs with P_i. □

Lemma 2.3. *Let $\mathcal{C}^i$ be three disjoint properly homotopic paths in S crossing the archipelago $\bar{A}$ in such a way that their terminal segments δ^i are properly homotopic in $S \smallsetminus A$. Then at least two of these paths are roof parallel rel $\bar{A}$.*

Proof. For $i = 1, 2, 3$, let P_i be the big rectangle bounded by the paths $\mathcal{C}^k$ and $\mathcal{C}^l$ with $\{i, k, l\} = \{1, 2, 3\}$, and let Q_i be the corresponding little rectangles. Let ω_i be the roofs of the P_i, and let λ_i be the roofs of the Q_i. We need to show that one of the roofs ω_i coincides with the corresponding λ_i.

Since by Lemma 2.1 (iii) one of the big rectangles, say P_1, is tiled by the other two, the roof ω_1 is tiled by ω_2 and ω_3. Denote the complements of the roofs ω_i by ω_i'. If $\omega_i \neq \lambda_i$ for $i = 2, 3$, then $\lambda_2 = \omega_2' = \omega_1' \cup \omega_3$ and similarly $\lambda_3 = \omega_1' \cup \omega_2$. Hence $\lambda_2 \cup \lambda_3 = \omega_1' \cup \omega_2 \cup \omega_3 = \eta$, where η is the whole component of ∂S containing the endpoints of the paths $\mathcal{C}^i$. But it is impossible since one of the roofs λ_i is tiled by the other two (as one of the little rectangles Q_i is tiled by the other two). □

Corollary 2.4. *Let $\mathcal{C}^i$ be five disjoint properly homotopic paths in S crossing the archipelago $\bar{A}$ in such a way that their terminal and initial segments are (respectively) properly homotopic in $S \smallsetminus A$. Then at least two of these paths are parallel rel $\bar{A}$.*

Let us now enlarge the notion of parallel to an equivalence relation on the class $\mathcal{A}$ of all proper curves $\mathcal{C}$ in S crossing the archipelago $\bar{A}$. Let us say that two curves $\mathcal{C}^1$ and $\mathcal{C}^2$ of class $\mathcal{A}$ are *roof equivalent* if

- They are properly homotopic in $\mathcal{C}$;

- The terminal segments δ^1 and δ^2 are properly homotopic in $S \smallsetminus A$;
- The motions of the endpoints, (e^t) and (q^t), of the above homotopies are homotopic (rel endpoints) curves on ∂S.

The definitions of *base equivalent* and *equivalent* paths are straightforward. Again, we restrict ourselves to a statement concerning roof equivalence only:

Lemma 2.5. *Two disjoint curves $\mathcal{C}^1$ and $\mathcal{C}^2$ of class $\mathcal{A}$ are roof parallel if and only if they are roof equivalent.*

Proof. If $\mathcal{C}^1$ and $\mathcal{C}^2$ are roof parallel then they are homotopic within the big rectangle P in such a way that the endpoint e^t parametrizes the roof ω. Similarly, the curves δ^1 and δ^2 are homotopic in Q in such a way that q^t parametrizes the same roof ω. So, the motions of the endpoints are homotopic.

Vice versa, by Lemma 2.1 (ii), the homotopy class of the endpoint motion determines the roof of the rectangle. □

In what follows, (roof/base) equivalent curves (not necessarily disjoint) will also be called (roof/base) parallel. Also, "parallel in S (rel $\emptyset$)" just means "properly homotopic" in S.

2.2. **Routes and associated rectangles.** Let us now consider a finite family of archipelagos A_j ($j = 1, \dots, N$) in S with disjoint closures. Let us consider a path $\mathcal{C}$ in S that begins at $b \subset \partial S$ and ends at a point e on some archipelago $\bar{A}$. Such a path is called *good* if $\operatorname{int} \mathcal{C}$ does not intersect $\partial S \cup \bar{A}$.

Given a good path $\mathcal{C}$ in S, let $\{A_1, \dots, A_l \equiv A\}$ be the set of *associated* archipelagos whose closures are crossed by $\mathcal{C}$ ordered according to their first appearance.[3] Let e_j be the first point of intersection of $\mathcal{C}$ with A_j, and let $\mathcal{C}_j$ be the segment of $\mathcal{C}$ bounded by $b \equiv e_0$ and e_j. In this way we obtain the *associated sequence*

$$\mathcal{C}_1 \subset \mathcal{C}_2 \subset \dots \mathcal{C}_l \equiv \mathcal{C}$$

of good paths in S. We let $|\mathcal{C}| = l$ and call it the *height* of $\mathcal{C}$.

Let

$$\Lambda_j = \bigcup_{i=j}^{N} A_i, \quad \Omega_j = \bigcup_{i=1}^{j-1} A_i.$$

(Note that $\Omega_1 = \emptyset$. Also, we let $\Lambda \equiv \Lambda_1$ be the union of all archipelagos.) Then $\mathcal{C}_j$ is a proper path in $S \smallsetminus \Lambda_j$, and Ω_j is an archipelago in $S \smallsetminus \Lambda_j$. Let $\boldsymbol{\alpha}_j$ be the class of proper paths in $S \smallsetminus \Lambda_j$ parallel to $\mathcal{C}_j$

[3] We then order the other archipelagos arbitrarily: $A_{l+1}, \dots, A_N$.

rel Ω_j. We say that these paths and classes are *associated* to $\mathcal{C}$. The sequence of the associated parallel classes,

$$\mathcal{R}(\mathcal{C}) = (\boldsymbol{\alpha}_j)_{j=1}^{l},$$

is called the *route* of $\mathcal{C}$. Note that the route determines the base component of ∂S where $\mathcal{C}$ begins, and the components of ∂A_j where the curves $\mathcal{C}_j$ end. Two good paths are called *parallel* rel the family of archipelagos A_i if they have the same route.

Let us consider two disjoint parallel good paths $\mathcal{C}^1$ and $\mathcal{C}^2$ with route of height l. By Lemma 2.1, these two path, together with a base α and a roof ω, bound a *good big rectangle* P. Moreover, for each $j = 1, \dots, l$, the associated good paths $\mathcal{C}_j^1$ and $\mathcal{C}_j^2$, together with a base α_j and a roof ω_j, bound an *associated good big rectangle* $P_j \subset S \setminus \Lambda_j$, where $P_l \equiv P$. In fact, they share the same base, i.e. $\alpha = \alpha_j$, since they share the base with the same *associated initial little rectangle* $Q_1 \equiv P_1$. Furthermore, each rectangle P_j shares the roof with *associated (terminal) little rectangle* Q_j, $j = 2, \dots, l$, bounded by the paths δ_j^1 and δ_j^2, a base σ_j, and the roof ω_j.[4] All the above rectangles are vertically orientated.

We say that a path $\mathcal{C}$ *(positively) vertically overflows* a little rectangle Q_j if $\mathcal{C}$ contains a segment δ which is a (positively oriented) vertical path in Q_j.

Lemma 2.6. *Let $\mathcal{C}^1$ and $\mathcal{C}^2$ be two parallel good paths of height l, and let $P \equiv P_l$ be the corresponding good big rectangle. Let $\mathcal{C}$ be a positively oriented vertical path in a P. Then it is parallel to $\mathcal{C}^1$ and $\mathcal{C}^2$ rel the family (A_i) and, in particular, it has height l. Moreover, $\mathcal{C}$ positively vertically overflows all associated little rectangles Q_j, $j = 1, \dots, l$.*

Proof. Let us begin with the last assertion. For $j = l$ and $j = 1$ it immediately follows from Lemma 2.2 (by reversing orientation for $j = 1$). Let $1 < j < l$. Since P_j has the same base $\alpha \subset \partial S$ as P, a little initial segment of $\mathcal{C}$ is contained in P_j. On the other hand, the endpoint of $\mathcal{C}$ belongs to the archipelago $\bar{A}_l$ which is disjoint from P_j since

$$P_j \subset S \setminus \Lambda_j \subset S \setminus \bar{\Lambda}_l.$$

Hence the curve $\mathcal{C}$ must exit the rectangle P_j. But since $\mathcal{C}$ is a vertical curve in P, it can exit P_j only through the roof ω_j. Let e_j be the first intersection point of $\mathcal{C}$ with this roof. Then the initial segment $\mathcal{C}_j$ of $\mathcal{C}$ with endpoint e_j is a vertical path of P_j. By Lemma 2.2, it positively vertically overflows the little rectangle Q_j. All the more, $\mathcal{C}$ does so.

[4]Note that the little rectangles Q_j are not necessarily contained in the big rectangle P.

Since each P_j is a good big rectangle as well, we can apply to it the previous result and conclude that for any $i \leq j$, $\mathcal{C}_j$ vertically overflows Q_i. In particular it crosses the roof $\omega_i \subset \partial A_i$, and hence $\mathcal{C}_i \subset \mathcal{C}_j$.

Let us show that $\mathcal{C}_1 \subset \cdots \subset \mathcal{C}_l$ is the associated sequence of good paths. Since all the paths $\mathcal{C}_j$ are good initial segments of $\mathcal{C}$, it is part of the associated sequence. Moreover, $\mathcal{C}$ does not contain any other good initial segment since all other archipelagos A_k, $k = l+1, \ldots, N$, are disjoint from P.

In particular, $\mathcal{C}$ has the same height l as $\mathcal{C}^1$. Moreover, by Lemma 2.2, the paths $\mathcal{C}_j$ are parallel to $\mathcal{C}^1_j$ and $\mathcal{C}^2_j$ rel Ω_j. Hence $\mathcal{C}$ is parallel to $\mathcal{C}^1$ and $\mathcal{C}^2$ rel (A_j). □

The previous lemma can be sharpened as follows:

Lemma 2.7. *Let $\mathcal{C}^1$ and $\mathcal{C}^2$ be two parallel good paths of height l, and let $P \equiv P_l$ be the corresponding good big rectangle with base α. Let $\mathcal{C}$ be a disjoint good path which begins on α. Then either the route $\mathcal{R}(\mathcal{C})$ extends $\mathcal{R}(\mathcal{C}^1) = \mathcal{R}(\mathcal{C}^2)$, or the other way around.*

Proof. Assume $\mathcal{C}$ is not contained in the rectangle P. Then it must exit P through the roof ω. Let e be the first point of intersection of $\mathcal{C}$ with ω. Then the initial segment $\mathcal{C}^*$ of $\mathcal{C}$ ending at e is a vertical path in P. By Lemma 2.6, $\mathcal{R}(\mathcal{C}^*) = \mathcal{R}(\mathcal{C}^1)$, so that $\mathcal{R}(\mathcal{C})$ extends $\mathcal{R}(\mathcal{C}^1)$.

Assume now that $\mathcal{C} \subset P$. Let us consider the biggest $j \leq l$ such that $\mathcal{C}$ intersects the roof ω_j of the good big rectangle P_j, and let $e_j \in \mathcal{C} \cap \omega_j$ be the first intersection point. Then the initial segment $\mathcal{C}_j$ of $\mathcal{C}$ with endpoint e_j is a vertical path in P_j. By Lemma 2.6, it has the same route as $\mathcal{C}^1_j$. In particular, it crosses all the archipelagos A_i, $i = 1, \ldots, j$.

But in fact, $\mathcal{C} = \mathcal{C}_j$, for otherwise $\mathcal{C}$ (being good) would end at some archipelago A_i with $i > j$. For $i > l$ it is impossible since those archipelagos are disjoint from P. For $i \in [j+1, l]$ it is impossible for otherwise $\mathcal{C}$ would exit the rectangle int P_i and hence would cross the roof ω_i.

We conclude that $\mathcal{R}(\mathcal{C}^1)$ is an extension of $\mathcal{R}(\mathcal{C}_j) = \mathcal{R}(\mathcal{C})$. □

Let us now consider two disjoint vertical curves Γ^1 and Γ^2 in a good rectangle P. Together with appropriate base and roof arcs, they bound a truncated good rectangle $\tilde{P} \subset P$.

Lemma 2.8. *For the associated sequence of little rectangles, we have: $\tilde{Q}_j \subset Q_j$.*

Proof. By Lemma 2.6, Γ^1 and Γ^2 have the same route as P. Let us consider the associated sequences of good curves Γ^1_j and Γ^2_j, $j = 1, \ldots, l$.

Let $\tilde{\delta}_j^1$ and $\tilde{\delta}_j^2$ be the terminal paths in $S \setminus \cup A_j$ of these curves. By definition, $\tilde{Q}_j$ is the rectangle bounded by these two paths, together with two appropriate horizontal arcs. By Lemma 2.6, the $\tilde{\delta}_j^i$ are vertical paths in the little rectangle Q_j. Hence $\tilde{Q}_j \subset Q_j$. □

Finally, we have the following important disjointness property:

Proposition 2.9. *Let P and P' be two good rectangles with disjoint vertical boundaries. Assume that some associated little rectangles, Q_j and Q'_k, have a non-trivial overlap with matched vertical orientations. Then one of the routes, $\mathcal{R}(P)$ or $\mathcal{R}(P')$, is an extension of the other, and $j = k$.*

Proof. Since the overlapping little rectangles Q_j and Q'_k have disjoint vertical boundaries, one of the vertical boundary components, say $\delta'_k \subset \partial Q'_k$, must be a vertical path in the other rectangle, Q_j. Let $\mathcal{C}'$ be the vertical boundary component of P' containing the path δ'_k, and let $\mathcal{C}'_k$ be the associated good curve ending with the path δ'_k.

Let us consider the associated with P good big rectangle P_j (with the little rectangle Q_j just under its roof ω_j). Since the path δ'_k is positively oriented in Q_j, it ends on the roof ω_j. Thus, the whole curve $\mathcal{C}'_k$ also ends on ω_j. But since $\mathcal{C}'_k$ is good, its interior does not cross ω_j. Neither can it cross the vertical boundary of P_j (by the assumption). Hence $\mathcal{C}'_k$ is trapped in P_j, and must begin on the base α_j of P_j.

Thus, $\mathcal{C}'_k$ is a vertical curve in P_j. By Lemma 2.6, $\mathcal{C}'_k$ and P_j have the same height, so that $k = j$. By Lemma 2.7, the route $\mathcal{R}(\mathcal{C}') = \mathcal{R}(P')$ is either an extension of $\mathcal{R}(P)$, or the other way around. □

2.3. **Vertical foliations.** In what follows, S is a Riemann surface of finite topological type with boundary ∂S and punctures p_k (made in an ambient compact surface).

Let $\mathcal{S}$ stand for the set of all subfamilies of our family of archipelagos A_j, and let $\beta_i(U)$ stand for the Betti numbers of a surface U. We call the number

$$\mathrm{Top} = \mathrm{Top}(S, A_j) = \max_{\mathcal{S}} \, (\beta_0 + \beta_1)(S \setminus \bigcup_{A_j \in \mathcal{S}} A_j)$$

the *topological complexity* of the family of archipelagos.

By making a few artificial punctures (depending only on the topological complexity of the family of archipelagos), we can ensure that *no component of $S \setminus A_j$ is an annulus.* (Note that making extra punctures would not change extremal lengths of the path families in question.)

Let us consider the harmonic measure $\omega_j(z) = \omega_{S \setminus A_j}(\partial A_j, z)$ of ∂A_j in the Riemann surface $S \setminus A_j$ (see [A]). It is a harmonic function on

$\mathrm{int}(S \smallsetminus A_j)$ equal to 1 on ∂A_j and vanishing on ∂S. For instance, if S and A_j are disks, then ω_j is the height function on the annulus $S \smallsetminus A_j$ uniformized by the flat cylinder C_j with height 1 in such a way that ∂S is the base of it.

The *vertical foliation* $\mathcal{F}_j$ on S is the phase portrait of the gradient flow γ_j^t of ω_j. It has finitely many saddle type singularities (with finitely many incoming and outgoing separatrices), where the punctures are considered to be singularities as well. It is oriented according to the direction of the gradient flow. Each non-singular leaf of $\mathcal{F}_j$ begins on ∂S and ends on ∂A_j. In the case of the annulus, $\mathcal{F}_j$ is the actual vertical foliation on the uniformizing cylinder C_j.

Let us remove from $S \smallsetminus A_j$ all separatrices O^k of the foliation $\mathcal{F}_j$ and take the closures of the components of $S \smallsetminus A_j \smallsetminus \cup O_k$. We obtain finitely many (non-closed) rectangles $\Pi = \Pi_j^m$ foliated by the vertical leaves. Indeed, take some component λ of $\partial S \smallsetminus \cup O^k$. The gradient flow brings every point $z \in \lambda$ in time 1 to some archipelago A_j, and these trajectories fill in some component Π of $S \smallsetminus A_j \smallsetminus \cup O^k$. The map

$$(z,t) \to (z, \gamma_j^t(z)), \quad z \in \lambda,\ t \in [0,1],$$

provides us with the rectangular structure on Π. (Since every annuli component of $S \smallsetminus A_j$ contains a puncture, there are no annuli among the Π_i's.)

The conjugate harmonic function ω_j^* induces the natural transverse measure on the Π_j^m. In fact, the map $\omega_j + i\omega_j^*$ provides us with the uniformization of Π_j^m by a standard Cartesian rectangle of height 1.

Every rectangle Π_j^m represents some non-trivial proper homotopy class of paths in $S \smallsetminus A_j$. Moreover, different rectangles represent different classes. Indeed, if two leaves, γ and γ', of $\mathcal{F}_j$ are properly homotopic in $S \smallsetminus A_j$, then by Lemma 2.1 they bound a rectangle Q in $S \smallsetminus A_j$. The conjugate harmonic functions ω_j and ω_j^* are well defined on Q, and ω_j is constant on its horizontal sides, while ω_j^* is constant on the vertical sides. Hence $\omega_j + i\omega_j^*$ is a conformal map of Q onto a Cartesian rectangle, so that neither ω_j nor ω_j^* has critical points in Q. It follows that Q is contained in one of the rectangles Π_j^m.

A *polar rectangle* in S is a subrectangle of some Π_j^m saturated by the leaves of $\mathcal{F}_j$.

Any leaf $\mathcal{C}$ of a vertical foliation $\mathcal{F}_j$ represents a good path in S. Notice that the route $\mathcal{R}(\mathcal{C})$ determines the properly homotopy class of $\mathcal{C}$ in $S \smallsetminus A_j$, and hence determines the foliation $\mathcal{F}_j$ and the rectangle Π_j^m containing $\mathcal{C}$. These remark, together with Lemma 2.6 imply that

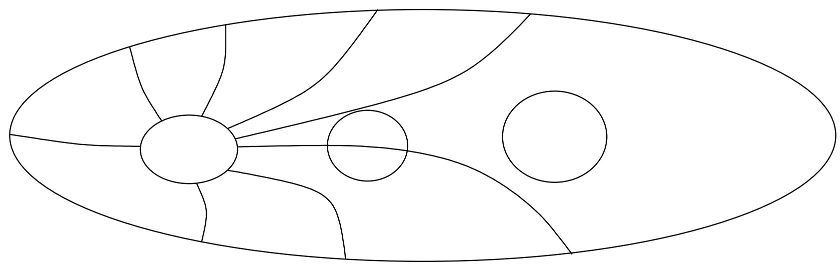

Figure 2.2. Vertical foliation

the leaves with the same route, $\mathcal{R}(\mathcal{C}) = \boldsymbol{\alpha}$, form a (non-closed) polar rectangle $P(\boldsymbol{\alpha})$ in S.

Associated big and little rectangles, $P_j(\boldsymbol{\alpha})$ and $Q_j(\boldsymbol{\alpha})$, $j = 1, \dots l$, come together with any polar rectangle.

Let $\mathcal{P}$ stand for the family of big good rectangles $P(\boldsymbol{\alpha})$. Since the maximal number of possible disjoint routes $\boldsymbol{\alpha}$ is bounded in terms of the topological complexity of our family of archipelagos (taking into account Corollary 2.4), the number of the big good rectangles $P(\boldsymbol{\alpha})$ is bounded in the same way: $|\mathcal{P}| \leq s = s(\mathrm{Top})$.

2.4. **Buffers and the Small-Overlapping Principle.** We are going to make use of an important principle saying that *two wide path families have a relatively small overlap*:

Lemma 2.10. *Let us consider a vertical lamination*[5] Λ *in some annulus or rectangle* A *and a path family* $\mathcal{S}$. *If* $\mathcal{W}(\Lambda) > \kappa$ *and* $\mathcal{W}(\mathcal{S}) > \kappa$, *then there exists a path* $\gamma \in \mathcal{S}$ *that intersects less than* $1/\kappa$ *of the total width of* Λ.[6] *In particular, if* $\kappa = 1$ *then there is a path* $\gamma \in \mathcal{S}$ *that does not cross some leaf of* Λ.

Proof. Assume for definiteness that A is an annulus. Let us uniformize A by a Euclidean cylinder $C = \mathbb{T} \times [0, h]$ such that the projection of Λ onto $\mathbb{T}$ has length κ. Since $\mathcal{W}(\Lambda) \geq \kappa$, we conclude that $h \leq 1$, and thus $\mathrm{area}(\Lambda) \leq \kappa$.

Let us consider the slice of the second family $\mathcal{S}$ by A as it appears on C. If every curve of this slice (perhaps, disconnected) intersected at least $1/\kappa$ of the total width of Λ, then the Euclidean length of these slices on C would be at least 1. Then $\mathcal{W}(\mathcal{S}) \leq \mathrm{area}(C) = \kappa$, contradicting the assumption. $\square$

Take some number $M > 8$. Given a polar rectangle $P(\boldsymbol{\alpha})$ of width greater than M, let us define two *buffers*, $B^l(\boldsymbol{\alpha}) \subset P(\boldsymbol{\alpha})$ and $B^r(\boldsymbol{\alpha}) \subset$

[5] i.e., a sublamination of the vertical foliation.

[6] In the sense of the natural transverse measure of Λ

$P(\boldsymbol{\alpha})$, as polar rectangles of width $M/2$ attached to the vertical sides of $P(\boldsymbol{\alpha})$.

Lemma 2.11. *Let us consider two polar rectangles $P(\boldsymbol{\alpha})$ and $P(\boldsymbol{\beta})$ of width greater than M. Then one can select four disjoint vertical leaves, one from each of the corresponding four buffers.*

Proof. Let Λ be the vertical foliation in $B^l(\boldsymbol{\alpha}) \cup B^r(\boldsymbol{\alpha})$, and let $\mathcal{S}$ be the vertical foliation of $B_l(\boldsymbol{\beta})$. Applying the previous lemma to this data, we conclude that there is a vertical leaf $\Gamma^l(\boldsymbol{\beta})$ in $\mathcal{S}$ that crosses less than $1/4$ of the total width of Λ. Hence it crosses less than $1/2$ of the total width of each $B^l(\boldsymbol{\alpha})$ and $B^r(\boldsymbol{\alpha})$.

Similarly, there is a vertical leaf $\Gamma^r(\boldsymbol{\beta})$ that crosses less than $1/2$ of the total width of each $B^l(\boldsymbol{\alpha})$ and $B^r(\boldsymbol{\alpha})$. Together, $\Gamma^l(\boldsymbol{\beta})$ and $\Gamma^r(\boldsymbol{\beta})$ cross less than full width of each $B^l(\boldsymbol{\alpha})$ and $B^r(\boldsymbol{\alpha})$. Hence each $B^l(\boldsymbol{\alpha})$ and $B^r(\boldsymbol{\alpha})$ contains a vertical leaf, $\Gamma^l(\boldsymbol{\alpha})$ and $\Gamma^r(\boldsymbol{\alpha})$ respectively, disjoint from both $\Gamma^l(\boldsymbol{\beta})$ and $\Gamma^r(\boldsymbol{\beta})$. □

2.5. **Truncated rectangles and Disjointness Property.** Let us remove the buffers from our polar rectangles:

$$\tilde{P}(\boldsymbol{\alpha}) = \mathrm{cl}(P(\boldsymbol{\alpha}) \smallsetminus (B^l(\boldsymbol{\alpha}) \cup B^r(\boldsymbol{\alpha}))).$$

The associated truncated big and little rectangles will be naturally marked with tilde: $\tilde{P}_j(\boldsymbol{\alpha})$ and $\tilde{Q}_j(\boldsymbol{\alpha})$.

We can now formulate the key disjointness property for the truncated rectangles:

Lemma 2.12. *If two associated truncated little rectangles $\tilde{Q}_j(\boldsymbol{\alpha})$ and $\tilde{Q}_k(\boldsymbol{\beta})$ overlap with matched vertical orientation, then one of the routes, $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$, is an extension of the other, and $j = k$.*

Proof. Let us select in the buffers of $P_j(\boldsymbol{\alpha})$ and $P_k(\boldsymbol{\beta})$ two disjoint pairs of leaves (by Lemma 2.11) and consider the rectangles $\mathbf{P}_j(\boldsymbol{\alpha}) \subset P_j(\boldsymbol{\alpha})$ and $\mathbf{P}_k(\boldsymbol{\beta}) \subset P_k(\boldsymbol{\beta})$ bounded by the corresponding pairs. By Lemma 2.8, their associated little rectangles, $\mathbf{Q}_j(\boldsymbol{\alpha})$ and $\mathbf{Q}_k(\boldsymbol{\beta})$, contain the respective little rectangles $\tilde{Q}_j(\boldsymbol{\alpha})$ and $\tilde{Q}_k(\boldsymbol{\beta})$. Hence $\mathbf{Q}_j(\boldsymbol{\alpha})$ and $\mathbf{Q}_k(\boldsymbol{\beta})$ overlap with matched vertical orientation. Since the big rectangles $\mathbf{P}_j(\boldsymbol{\alpha})$ and $\mathbf{P}_k(\boldsymbol{\beta})$ have disjoint vertical boundaries, we can apply Lemma 2.9 and complete the proof. □

Let us now go back to the family $\mathcal{P}$ of rectangles $P(\boldsymbol{\alpha})$ described at the end of §2.3. Truncating these rectangles (by removing the buffers of width $M/2$), we obtain a family $\tilde{\mathcal{P}}$ of rectangles $\tilde{P}(\boldsymbol{\alpha})$. Since the total width of the rectangles $P(\boldsymbol{\alpha})$ is equal to the modulus Y (1.1), while the number of the rectangles if bounded by $s = s(\text{Top})$, we conclude: If $Y > 2Ms$ then

$$\sum_{\tilde{\mathcal{P}}} \mathcal{W}(\tilde{P}(\boldsymbol{\alpha})) > \frac{1}{2}Y. \tag{2.1}$$

2.6. *a*- **and** *b*-**moduli.** Let $S(\alpha)$ be a copy of the Riemann surface S labeled by a route $\boldsymbol{\alpha}$. Let us consider the disjoint union of these copies,

$$\mathbb{S} = \bigsqcup S(\boldsymbol{\alpha}),$$

viewed as a disconnected Riemann surface, and let $\pi : \mathbb{S} \to S$ be the natural projection. This surface is decomposed into subsurfaces

$$\mathbb{S}(k) = \bigsqcup_{|\boldsymbol{\alpha}|=k} S(\boldsymbol{\alpha}),$$

according to the height of the route $\boldsymbol{\alpha}$.

Let us now consider the disjoint union of all rectangles $\tilde{P}(\boldsymbol{\alpha})$ of family $\tilde{\mathcal{P}}$ naturally embedded into $\mathbb{S}$:

$$\mathbb{P} = \bigsqcup_{\tilde{\mathcal{P}}} \tilde{P}(\boldsymbol{\alpha}) \subset \mathbb{S},$$

and decomposed accordingly into *sheets* $\mathbb{P}(k) \subset \mathbb{S}(k)$. Let $\tilde{\mathcal{P}}(k)$ be the family of rectangles $\tilde{P}(\boldsymbol{\alpha}) \in \tilde{\mathcal{P}}$ of height k that assemble the sheet $\mathbb{P}(k)$.

Let us also consider the family $\mathcal{Q}$ of all associated *vertically oriented* little rectangles $Q_i(\boldsymbol{\alpha})$ and their disjoint union:

$$\mathbb{Q} = \bigsqcup_{\mathcal{Q}} Q_i(\boldsymbol{\alpha}) \subset \mathbb{S},$$

decomposed into sheets $\mathbb{Q}(k) \subset \mathbb{S}(k)$. Let $\mathcal{Q}(k)$ stand for the family of little rectangles that assemble $\mathbb{Q}(k)$.

The surface $\mathbb{Q}$ is also decomposed into *layers of height* i:

$$\mathbb{Q}_i = \bigsqcup_{|\boldsymbol{\alpha}|\geq i} Q_i(\boldsymbol{\alpha}).$$

Taking their slices by the sheets, we obtain the decomposition of each layer (and dually: each sheet) $\mathbb{Q}_i$ into *sheet-layers* $\mathbb{Q}_i(k)$. The notations $\mathcal{Q}_i$ and $\mathcal{Q}_i(k)$ are self-explanatory.

Lemma 2.12 implies:

(A1) The projection π embeds the first sheet-layer $\mathbb{Q}_1(k)$ of every sheet $\mathbb{Q}(k)$ into $S \smallsetminus \cup A_j$. Indeed, all little rectangles $Q_1(\boldsymbol{\alpha})$ of the first layer are oriented so that they begin on the base ∂S. Hence if two rectangles $Q_1(\boldsymbol{\alpha})$ and $Q_1(\boldsymbol{\beta})$ overlap, then they have the same orientation. By

Lemma 2.12, the heights of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are different, so that $Q_1(\boldsymbol{\alpha})$ and $Q_1(\boldsymbol{\beta})$ lie on different sheets.

(A2) Let us select one sheet-layer $\mathbb{Q}_i(k_i)$ from every layer $\mathbb{Q}_i$, and let $R = \cup \mathbb{Q}_i(k_i)$. Then the projection $\pi : R \to S \smallsetminus \cup A_j$ is at most $2-to-1$. (In particular, π is at most $2-to-1$ on every sheet $\mathbb{Q}(k)$.) Indeed, if three little rectangles $Q_i(\boldsymbol{\alpha})$ that assemble R overlapped, then two of them would have the same orientation. Then by Lemma 2.12, they would belong to two different sheets of the same layer $\mathbb{Q}_i$, contradicting the structure of R.

Furthermore, the Riemann surface $\mathbb{P}$ is endowed with the vertical path family. Let us consider its width, as well as the width of the individual sheets:

$$a \equiv \mathcal{W}(\mathbb{P}) = \sum_{\tilde{\mathcal{P}}} \mathcal{W}(\tilde{P}(\boldsymbol{\alpha})), \quad a_k \equiv \mathcal{W}(\mathbb{P}(k)) = \sum_{\tilde{\mathcal{P}}(k)} \mathcal{W}(\tilde{P}(\boldsymbol{\alpha}).$$

Of course, $a = \sum a_k$.

Similarly, we have the vertical path family on $\mathbb{Q}$, and we can consider the width of its sheet-layers:

$$\mathcal{W}(\mathbb{Q}_i(k)) = \sum_{\mathcal{Q}(k)} \operatorname{mod} Q_i(\boldsymbol{\alpha}).$$

Let us maximize them over the sheets:

$$b_i = \max_{k \geq i} \mathcal{W}(\mathbb{Q}_i(k)), \quad b = \sum_{i=1}^{l} b_i. \tag{2.2}$$

We let

$$x \oplus y = \frac{1}{\frac{1}{x} + \frac{1}{y}}$$

be the *harmonic sum* of two numbers.

Lemma 2.13. *The a- and b-moduli are related by the Series Inequality:*

$$a_k \leq \bigoplus_{i=1}^{k} b_i.$$

Proof. By Lemma 2.6, every vertical curve of $\mathbb{P}(k)$ overflows each of the sheet-layers $\mathbb{Q}_i(k)$, By the Series Law,

$$a_k = \mathcal{W}(\mathbb{P}(k)) \leq \bigoplus_{i=1}^{k} \mathcal{W}(\mathbb{Q}_i(k)) \leq \bigoplus_{i=1}^{k} b_i.$$

□

Let us now relate the a- and b-moduli to the geometric moduli X,Y and Z in the Quasi-Additivity Law (see the Introduction). By (2.1),

$$a \geq \frac{1}{2}Y. \tag{2.3}$$

Furthermore,

Lemma 2.14. $b_1 \leq X$.

Proof. By property (A1), the first sheet-layer $\mathbb{Q}_1(k)$ of every sheet embeds into $S \setminus \Lambda$, where $\Lambda = \cup A_j$. Moreover, under this embedding the vertical path family of $\mathbb{Q}_1(k)$ is mapped into a vertical path family of $S \setminus \Lambda$ (the latter consists of paths connecting ∂S to $\partial\Lambda$ in $S \setminus \Lambda$). Hence $\mathcal{W}(\mathbb{Q}_1(k)) \leq \mathcal{W}(S, \Lambda)$. Taking maximum over k completes the proof. □

Lemma 2.15. $b \leq 2Z \leq 2\xi Y$.

Proof. The second estimate follows from the Separation Assumption of the Quasi-Additivity Law, so we only need to prove the first one.

Take some height i and find a level $k_i \geq i$ such that $b_i = \mathcal{W}(\mathbb{Q}_i(k_i))$. Letting $R = \cup\, \mathbb{Q}_i(k_i)$, we conclude that $\mathcal{W}(R) = b$.

By property (A2), the map $\pi : R \to S \setminus \Lambda$ is at most $2 - to - 1$. Moreover, every vertical path of R connects some archipelagos $\bar{A}_j$ to $\partial S \cup (\cup_{k\neq j} \partial A_k)$. Hence $\mathcal{W}(R) \leq 2Z$, and we are done. □

2.7. **An arithmetic inequality.**

Lemma 2.16. *Consider two sequences of positive numbers,* $\{a_i\}_{i=1}^n$ *and* $\{b_i\}_{i=1}^n$, *such that* $a_1 = b_1$, $a_{i+1} \leq a_i \oplus b_{i+1}$. *Then*

$$\left(\sum_{i=1}^{n} a_i\right)^2 \leq 18\, b_1 \sum_{i=1}^{n} b_i. \tag{2.4}$$

Proof. By letting $a_{i+1} = a_i \oplus b_{i+1}$, we increase the a_i's without changing the b_i's, so that it is sufficient to consider this extreme case. Also, because the inequality is homogeneous, we can assume that $a_1 = b_1 = 1$.

First note that the (a_i) form a decreasing sequence. For each natural n, let m_n be the least natural number satisfying $a_{m_n} \leq 2^{-n}$. Let $\mathcal{N} \subset \mathbb{N}$ be the set of n for which $m_{n+1} > m_n$, and let $l_n = m_{n+1} - m_n - 1$ whenever $n \in \mathcal{N}$. Then for $n \in \mathcal{N}$, we have

$$2^{-n} \geq a_{m_n} > \ldots > a_{m_n + l_n} > 2^{-(n+1)},$$

and hence

$$a \equiv \sum_{i=1}^{\infty} a_i \le \sum_{n\in\mathcal{N}} (l_n+1)2^{-n} \le 2 + \sum_{n\in\mathcal{N}} l_n 2^{-n}. \tag{2.5}$$

We can also estimate $b \equiv \sum_{i=1}^{\infty} b_i$ from below as follows. Suppose that $l_n > 0$. Then

$$a_{m_n+l_n} = a_{m_n} \oplus \bigoplus_{i=m_n+1}^{m_n+l_n} b_i.$$

Therefore

$$\bigoplus_{i=m_n+1}^{m_n+l_n} b_i > 2^{-n},$$

and it follows from the Arithmetic-Harmonic Mean inequality that

$$\sum_{i=m_n+1}^{m_n+l_n} b_i > l_n^2\, 2^{-n}.$$

All the more,

$$\sum_{i=m_n}^{m_n+l_n} b_i > l_n^2\, 2^{-n},$$

where the last estimate is valid for $l_n = 0$ as well. Hence

$$b > \sum_{n\in\mathcal{N}} l_n^2\, 2^{-n}.$$

By the Cauchy-Schwarz inequality,

$$\left(\sum_{n\in\mathcal{N}} l_n\, 2^{-n}\right)^2 \le \sum_{n\in\mathcal{N}} l_n^2\, 2^{-n} \sum_{n\in\mathcal{N}} 2^{-n} \le 2 \sum_{n\in\mathcal{N}} l_n^2\, 2^{-n} < 2b.$$

Together with (2.5), this yields:

$$(a-2)^2 < 2b,$$

which, along with $b \ge 1$, implies

$$a^2 < 18\, b.$$

□

2.8. **Completion of the proof of the Quasi-Additivity Law.** Let us consider the a- and b-moduli from §2.6. Lemma 2.13 puts us into a position to apply estimate (2.4) to these moduli. Incorporating (2.3) and Lemmas 2.14 and 2.15 into (2.4), we obtain:

$$\frac{1}{4}Y^2 \leq 18 \cdot 2\xi Y X,$$

and we are done. □

2.9. **Variation.** In conclusion, for further reference in holomorphic dynamics, let us formulate a variation of the Quasi-Additivity Law:

Quasi-Additivity Law (Variation). *Fix some $\eta > 0$. Let $W \Subset \operatorname{int} U$ and $D_i' \Subset \operatorname{int} W$, $i = 1, \dots, N$, be topological disks such that the closures of D_i' are pairwise disjoint, and let $D_i \Subset D_i'$ be smaller disks. Then there exists a $\delta_0 > 0$ (depending on η and N) such that:*
If for some $\delta \in (0, \delta_0)$, $\operatorname{mod}(U \smallsetminus D_i) < \delta$, while $\operatorname{mod}(D_i' \smallsetminus D_i) > \eta\delta$ (the Collar Assumption), then

$$\operatorname{mod}(U \smallsetminus W) < \frac{C\eta^{-1}\delta}{N},$$

where C is an absolute constant.

Remark. In the Quasi-Additivity Law, in place of smooth archipelagos we can consider compact sets with finite topology, i.e., sets represented as intersections of nested smooth archipelagos with bounded topological complexity. Similarly, in the Quasi-Invariance Law, in place of smooth disks Λ, B, we can consider arbitrary cellular (i.e, connected full compact) sets.

3. Proof of the Covering Lemma.

We will make use of the following well-known result:

Proposition 3.1. *Let $f : U \to V$ be a branched cover of Riemann surfaces of degree N. Then there is a Galois branched cover $g : S \to V$ of degree at most $N!$ that factors as $g = f \circ h$ for some $h : S \to U$. Moreover, g is ramified only over critical values of f.*

The proof uses a lemma that is a simple exercise in group theory:

Lemma 3.2. *Suppose that H is a subgroup of a group G, and $[G : H] = N$. Then there is a normal subgroup L of G such that $L < H$, and $[G : L] \leq N!$.*

Proof. The coset action of G on G/H provides a homomorphism from G to the group of permutations of G/H, which has order at most $N!$. We let L be the kernel of this homomorphism; it has the desired properties. □

Proof of Proposition 3.1. Let X be the set of critical values of f, and let $E = f^{-1}(X)$. Then $f : U \setminus E \to V \setminus X$ is an unbranched cover of degree N. Hence $f_* \pi_1(U \setminus E)$ has index N in $\pi_1(V \setminus X)$, so by Lemma 3.2 we can find a subgroup of $f_* \pi_1(U \setminus E)$ that is a normal subgroup of $\pi_1(V \setminus X)$ of degree at most $N!$. There is then the corresponding cover $g : S' \to V \setminus X$ which we can complete to a branched cover $g : S \to V$ with the desired properties. □

Now we are ready to give a proof of the Covering Lemma.

First of all, we will consider a more general case when Λ is an archipelago rather than a single domain. Second, we can assume without loss of generality that $\Lambda = (f|\,\Lambda')^{-1}(B)$.

Let $X \subset V$ be the set of critical values of f, and let $E = f^{-1}(X) \subset U$. By Proposition 3.1, there exists a branched covering $h : S \to U$ of degree at most $(N-1)!$ with critical values in E such that $g = f \circ h : S \to U$ is a Galois branched covering, i.e., the fibers of g are orbits of a group Γ of conformal symmetries of S.

Let $A'_j \subset S$ be the connected components of $g^{-1}(B')$, $j = 1, \dots, L$, labeled in such a way that $h(A'_1) = \Lambda'$; we let $A' \equiv A'_1$. These components are transitively permuted by Γ. Let us also consider the corresponding archipelagos $A_j = (g|\,A'_j)^{-1}(B)$ in S, and let $A \equiv A_1$.

We now let X, Y and Z be the moduli from the Introduction associated with this family of archipelagos. By Lemma 4.4 from the Appendix, we have:

$$X = |\Gamma|\,\mathcal{W}(V, B). \tag{3.1}$$

Let $l = \deg(A \to \Lambda)$. Then A consists of ld islands, and $|\Gamma| = ldL$. Since all the archipelagos are symmetric in S, we have:

$$Y = L\,\mathcal{W}(S, A) \geq Ll\,\mathcal{W}(U, \Lambda) = \frac{|\Gamma|}{d}\mathcal{W}(U, \Lambda), \tag{3.2}$$

where the inequality follows from Lemma 4.3 from the Appendix.

Using symmetry of the archipelagos and Lemma 4.4 once again, we obtain:

$$Z \leq L\,\mathcal{W}(A', A) = |\Gamma|\,\mathcal{W}(B', B). \tag{3.3}$$

Putting (3.2) and (3.3) together with the Collar Assumption, we obtain the separation property for our archipelagos:

$$Z \le |\Gamma|\, \mathcal{W}(B', B) \le \eta^{-1} |\Gamma|\, \mathcal{W}(U, \Lambda) \le \eta^{-1} d\, Y.$$

We are now in the position to apply the Quasi-Additivity Law. Together with (3.1) and (3.2), it implies the desired estimate:

$$\frac{|\Gamma|}{d} \mathcal{W}(U, \Lambda) \le Y \le C\eta^{-1} d\, X = C\eta^{-1} d\, |\Gamma|\, \mathcal{W}(V, B).$$

□

Remark. The Covering Lemma can be also proved directly, along the lines of the proof of the Quasi-Additivity Law. (It gives a better bound on $\varepsilon(N)$.)

4. Appendix: Extremal length and width

There is a worth of sources containing background material on extremal length, see, e.g., the book of Ahlfors [A]. We will briefly summarize the necessary minimum.

4.1. **Definitions.** Let $\mathcal{G}$ be a family of curves on a Riemann surface U. Given a (measurable) conformal metric $\mu = \mu(z)|dz|$ on U, let

$$\mu(\mathcal{G}) = \inf_{\gamma \in \mathcal{G}} \mu(\gamma).$$

where $\mu(\gamma)$ stands for the μ-length of γ. The length of $\mathcal{G}$ with respect to μ is defined as

$$\mathcal{L}_\mu(\mathcal{G}) = \frac{\mu(\mathcal{G})^2}{\mu^2(U)},$$

where $\mu^2 = \mu(z)^2 dz \wedge \bar{d}z$ is an area form of μ. Taking the supremum over all conformal metrics μ, we obtain the *extremal length* $\mathcal{L}(\mathcal{G})$ of the family $\mathcal{G}$.

The *extremal width* is the inverse of the extremal length:

$$\mathcal{W}(\mathcal{G}) = \mathcal{L}^{-1}(\mathcal{G}).$$

It can be also defined as follows. Consider all conformal metrics μ such that $\mu(\gamma) \ge 1$ for any $\gamma \in \mathcal{G}$. Then $\mathcal{W}(\mathcal{G})$ is the infimum of the areas $\mu^2(U)$ of all such metrics.

4.2. **Electric circuits laws.** We say that a family $\mathcal{G}$ of curves *overflows* a family $\mathcal{H}$ if any curve of $\mathcal{G}$ contains some curve of $\mathcal{H}$. We say that two families, $\mathcal{G}_1$ and $\mathcal{G}_2$, are disjoint if any two curves, $\gamma_1 \in \mathcal{G}_1$ and $\gamma_2 \in \mathcal{G}_2$, are disjoint.

We let $x \oplus y = (x^{-1} + y^{-1})^{-1}$ be the *harmonic sum* of x and y (it is conjugate to the usual sum by the inversion map $x \mapsto x^{-1}$).

The following crucial properties of the extremal length and width show that the former behaves like the resistance in electric circuits, while the latter behaves like conductance.

Series Law/Grötzsch Inequality. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two disjoint families of curves, and let $\mathcal{G}$ be a third family that overflows both $\mathcal{G}_1$ and $\mathcal{G}_2$. Then*

$$\mathcal{L}(\mathcal{G}) \geq \mathcal{L}(\mathcal{G}_1) + \mathcal{L}(\mathcal{G}_2),$$

or equivalently,

$$\mathcal{W}(\mathcal{G}) \leq \mathcal{W}(\mathcal{G}_1) \bigoplus \mathcal{W}(\mathcal{G}_2).$$

Parallel Law. *For any two families $\mathcal{G}_1$ and $\mathcal{G}_2$ of curves we have:*

$$\mathcal{W}(\mathcal{G}_1 \cup \mathcal{G}_2) \leq \mathcal{W}(\mathcal{G}_1) + \mathcal{W}(\mathcal{G}_2).$$

If $\mathcal{G}_1$ and $\mathcal{G}_2$ are disjoint then

$$\mathcal{W}(\mathcal{G}_1 \cup \mathcal{G}_2) = \mathcal{W}(\mathcal{G}_1) + \mathcal{W}(\mathcal{G}_2)$$

Note that the Parallel Law inequality implies the estimate $X \leq Y$ between the moduli from the Introduction.

4.3. **Transformation rules.** Both extremal length and extremal width are conformal invariants. More generally, we have:

Lemma 4.1. *Let $f : U \to V$ be a holomorphic map between two Riemann surfaces, and let $\mathcal{G}$ be a family of curves on U. Then*

$$\mathcal{L}(f(\mathcal{G})) \geq \mathcal{L}(\mathcal{G}).$$

Moreover, if f is at most $d - to - 1$, then

$$\mathcal{L}(f(G)) \leq d \cdot \mathcal{L}(\mathcal{G}).$$

Proof. Let μ be a conformal metric on U. Let us push-forward the area form μ^2 by f. We obtain the area form $\nu^2 = f_*(\mu^2)$ of some conformal metric ν on V. Then $\nu^2(V) = \mu^2(U)$ and $f^*(\nu) \geq \mu$. It follows that

$$\mathcal{L}_\mu(\mathcal{G}) \leq \mathcal{L}_\nu(f(\mathcal{G})) \leq \mathcal{L}(f(\mathcal{G})).$$

Taking the supremum over μ completes the proof of the first assertion.

For the second assertion, let us consider a conformal metric ν on V and pull it back to U, $\mu = f^*\nu$. Then $\mu(\gamma) = \nu(f(\gamma))$ for any $\gamma \in \mathcal{G}$, while $\mu^2(U) \le d \cdot \nu^2(V)$. Hence

$$\mathcal{L}(\mathcal{G}) \ge \mathcal{L}_\mu(\mathcal{G}) \ge \frac{1}{d}\mathcal{L}_\nu(f(\mathcal{G})),$$

and taking the supremum over ν completes the proof. □

4.4. **Extremal distance and Dirichlet integral.** Given a compact subset $K \subset \operatorname{int} U$, the *extremal distance*

$$\mathcal{L}(U, K) \equiv \operatorname{mod}(U, K)$$

(between ∂U and K) is defined as $\mathcal{L}(\mathcal{G})$, where $\mathcal{G}$ is the family of curves connecting ∂U and K. In case when U is a topological disk and K is connected, we obtain the usual modulus $\operatorname{mod}(U \smallsetminus K)$ of the annulus $U \smallsetminus K$.

Remark. $\mathcal{L}(U, K)$ can also be defined as $\mathcal{L}(\mathcal{G}')$ where $\mathcal{G}'$ is the family of curves *in* $U \smallsetminus K$ connecting ∂U to K. Indeed, since $\mathcal{G} \supset \mathcal{G}'$, $\mathcal{L}(\mathcal{G}) \le \mathcal{L}(\mathcal{G}')$. Since each curve of $\mathcal{G}$ overflows some curve of $\mathcal{G}'$, $\mathcal{L}(\mathcal{G}) \ge \mathcal{L}(\mathcal{G}')$. One can also make a compromise and use the intermediate family of curves in U connecting ∂U to K.

We let $\mathcal{W}(U, K) = \mathcal{L}^{-1}(U, K)$.

Lemma 4.2. *Let $f : U \to V$ be a branched covering of degree N between two compact Riemann surfaces with boundary. Let A be a compact subset of* $\operatorname{int} U$ *and let $B = f(A)$. Then*

$$\operatorname{mod}(U, A) \le \operatorname{mod}(V, B) \le N \operatorname{mod}(U, A).$$

Proof. Let $\mathcal{G}$ be the family of curves in U connecting ∂U to A, and let $\mathcal{H}$ be the similar family in V. Notice that every curve $\gamma \in \mathcal{H}$ lifts to a curve in $\mathcal{G}$: begin the lifting on A, and it must end on ∂U since $f : U \to V$ is proper. Thus, $\mathcal{H} = f(\mathcal{G})$, and Lemma 4.1 completes the proof. □

Extremal width $\mathcal{W}(U, A)$ can be explicitly expressed as the *Dirichlet integral of the harmonic measure* (see [A, §4-9]):

$$\mathcal{W}(U, A) = 4 \int_{U \smallsetminus A} |\partial h|^2$$

where $h : V \smallsetminus B \to \mathbb{R}$ is the harmonic function equal to 1 on ∂B and vanishing on ∂U, and $|\partial h|^2$ is the area form associated with the holomorphic differential $\partial h = (\partial h / \partial z) dz$.

4.5. **More transformation rules.** The Dirichlet integral formulation allows us to sharpen the lower bound in Lemma 4.2:

Lemma 4.3. *Let $f : U \to V$ be a branched covering between two compact Riemann surfaces with boundary. Let A be an archipelago in U, $B = f(A)$, and assume that $f : A \to B$ is a branched covering of degree d. Then*

$$\mathrm{mod}(V, B) \geq d\ \mathrm{mod}(U, A).$$

Proof. The Riemann surface $V \smallsetminus B$ is decomposed into finitely many rectangles saturated by the leaves of the harmonic flow (see §2.3). Slit these rectangles by the leaves containing the critical values of f. We obtain finitely many foliated rectangles Π_i such that

$$\sum \mathcal{W}(\Pi_i) = \mathcal{W}(V, B).$$

Each of these rectangles lifts to d properly embedded rectangles P_i^j in $U \smallsetminus A$ (with the horizontal sides on ∂U and ∂A). Moreover, $\mathcal{W}(P_i^j) = \mathcal{W}(\Pi_i)$. Hence

$$\mathcal{W}(U, A) \geq \sum \mathcal{W}(P_i^j) = d\,\mathcal{W}(V, B).$$

□

Remark. A similar estimate is still valid for an arbitrary compact set A, and can be proved by approximating A by archipelagos.

Putting the above two lemmas together (or using directly that the Dirichlet integral is transformed as the area under branched coverings) we obtain:

Lemma 4.4. *Let (U, A) and (V, B) be as above, and let $f : U \smallsetminus A \to V \smallsetminus B$ be a branched covering of degree N. Then*

$$\mathrm{mod}(V, B) = N\ \mathrm{mod}(U, A).$$

References

[A] L. Ahlfors. Conformal invariants: Topics in geometric function theory. McGraw Hill, 1973.

Stony Brook University and University of Toronto