EXTERNAL RAYS AND THE REAL SLICE OF THE MANDELBROT SET

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Abstract. This paper investigates the set of angles of the parameter rays which land on the real slice $[-2, 1/4]$ of the Mandelbrot set. We prove that this set has zero length but Hausdorff dimension 1. We obtain the corresponding results for the tuned images of the real slice. Applications of these estimates in the study of critically non-recurrent real quadratics as well as biaccessible points of quadratic Julia sets are given.

Contents

1. Introduction

The *Mandelbrot set* M is the connectedness locus of the family $Q_c : z \mapsto z^2 + c$ of normalized complex quadratic polynomials:

$$M := \{c \in \mathbb{C} : \text{The Julia set of } Q_c \text{ is connected}\}.$$

It is a compact, full, and connected subset of the plane, with a tremendously intricate structure near the boundary (see Fig. 1). In recent years, great deal of research has gone towards understanding the topology, geometry, and combinatorics of M, as M and its higher degree cousins are the universal objects which appear in the bifurcation locus of any holomorphic family of rational maps [**Mc**].

The normalized Riemann mapping $\Phi : \widehat{\mathbb{C}} \smallsetminus M \xrightarrow{\cong} \widehat{\mathbb{C}} \smallsetminus \overline{\mathbb{D}}$ which satisfies $\Phi(\infty) = \infty$ and $\Phi'(\infty) = 1$ plays a special role in the study of the quadratic family and has a dynamical meaning: $\Phi(c)$ is the conformal position of the escaping critical value c in the basin of attraction of infinity for Q_c. The normalized Lebesgue measure m on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z} \cong \partial\mathbb{D}$ pulls back by Φ to the *harmonic measure* μ_M supported in ∂M. More precisely, define the *parameter ray* $R_M(t)$ of *external angle* $t \in \mathbb{T}$ as the Φ-preimage of the radial line $\{re^{2\pi i t} : r > 1\}$. We say that $R_M(t)$ *lands* at $c \in \partial M$ if $\lim_{r \to 1} \Phi^{-1}(re^{2\pi i t}) = c$. It follows from a classical theorem of A. Beurling (see for example [**P**]) that every parameter ray lands at a well-defined

Date: August 20, 2002.

Stony Brook IMS Preprint #2002/02
August 2002

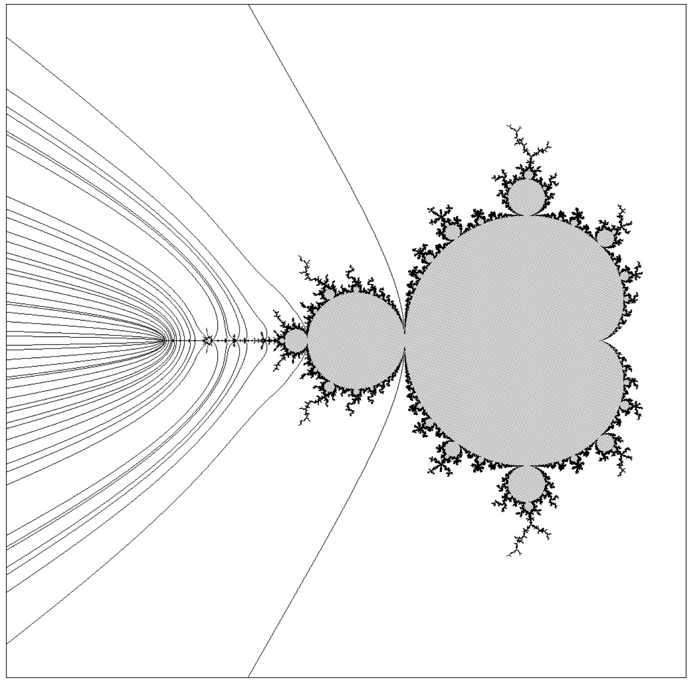

FIGURE 1. *The Mandelbrot set M and some parameter rays landing on its real slice.*

point of ∂M, except possibly for a set of external angles of capacity zero (conjecturally empty in this case). For a Borel measurable set $S \subset M$, the harmonic measure is given by

$$\mu_M(S) := m\{t \in \mathbb{T} : R_M(t) \text{ lands at a point of } S\}.$$

In other words, $\mu_M(S)$ is the probability that a Poincaré geodesic in $\widehat{\mathbb{C}} \smallsetminus M$ emanated from infinity in a random direction hits S. The great complexity of M near each of its boundary points suggests that the harmonic measure of embedded arcs in M is zero so that they are almost invisible from infinity. For example, all the hyperbolic components of M (the "bulbs" in Fig. 1) have piecewise analytic boundary and a direct computation shows that the harmonic measure of each of these boundary arcs is zero [**D1**]. On the other hand, M contains many other essentially different embedded arcs, the basic example of which is the *real slice* $M \cap \mathbb{R} = [-2, 1/4]$. By tuning (see §2 below), one obtains countably many distinct embedded arcs in M which are all images of the real slice under embeddings $M \hookrightarrow M$. The harmonic measure of each tuned image of the real slice is zero since it can be shown using the tuning algorithm that the corresponding angles are contained in a self-similar Cantor set (see [**Mn**] and compare §2). However, this question for the real slice itself is non-trivial:

Theorem 1.1. *The harmonic measure of the real slice of the Mandelbrot set is zero.*

Naturally, one is led to consider a finer tool to measure the size of the real slice as seen from infinity. In contrast to Theorem 1.1, we show that

Theorem 1.2. *The set of external angles $t \in \mathbb{T}$ for which the parameter ray $R_M(t)$ lands at a point of* $[-2, 1/4]$ *has Hausdorff dimension* 1.

The main ingredient of the proofs of these theorems is an explicit description for the set $\mathcal{R}$ of all angles $t \in [0, 1/2]$ such that the prime-end impression of the ray $R_M(t)$ in ∂M

intersects the real line (see §3). Conjecturally, this set coincides with the set of angles of the parameter rays which land on the real slice, but whether or not this is true is irrelevant here, as the difference between the two sets has zero capacity by Beurling's theorem. The description of $\mathcal{R}$ is obtained by establishing the existence of a canonical homeomorphism $c \mapsto \tau(c)$ between $\partial M \cap \mathbb{R}$ and $\mathcal{R}$, with the property that the prime-end impression of the parameter ray at angle $\tau(c)$ intersects $\mathbb{R}$ precisely at $c \in \partial M$; alternatively, the associated dynamic ray at angle $\tau(c)$ lands at the critical value c in the Julia set of Q_c (see Theorem 3.3 and Lemma 3.4).

Once the appropriate description of $\mathcal{R}$ is in hand, Theorem 1.1 follows easily from ergodicity of the doubling map $\mathbf{d} : t \mapsto 2t \pmod 1$ on the circle. For the Hausdorff dimension question, we introduce a one-parameter family of compact sets $\{\mathcal{K}_\sigma\}_{\sigma>0}$ in §4 whose dimension is estimated from below by an application of Frostman's Lemma. The close relation between the family $\{\mathcal{K}_\sigma\}_{\sigma>0}$ and the set $\mathcal{R}$ allows us to use these estimates and prove Theorem 1.2.

The discussion in §4 concludes with a generalization of Theorem 1.2 to all tuned images of the real slice:

Theorem 1.3. *Let H be a hyperbolic component of M of period $p > 1$ and let $\eta_H \subset M$ be the corresponding tuned image of the real slice $[-2, 1/4]$. Then, the set of external angles $t \in \mathbb{T}$ for which $R_M(t)$ lands at a point of η_H has Hausdorff dimension $1/p$.*

Combining Theorem 1.2 with standard dimension theorems in conformal mapping theory leads to dimension estimates in the parameter space of real quadratic polynomials. As an example, we show in §5 that

Theorem 1.4. *The set of parameters $c \in \partial M \cap \mathbb{R}$ for which the quadratic Q_c is critically non-recurrent has Hausdorff dimension 1.*

Note that this set has Lebesgue measure zero by a theorem of D. Sands, although the full set $\partial M \cap \mathbb{R}$ has positive Lebesgue measure according to Jakobson (compare [**Sa**] and [**J**]).

Other applications of these estimates will be discussed in §6. Let B_c denote the set of angles of dynamic rays which land on the *biaccessible* points in the Julia set of the quadratic polynomial Q_c. In other words, $t \in B_c$ if there exists an $s \neq t$ such that the dynamic rays at angles t and s land at a common point of the Julia set.

Theorem 1.5. *For $-2 < c \leq -1.75$,*

$$0 < \ell(c) \leq \dim_{\mathrm{H}}(B_c) < 1,$$

where $\ell(c)$ is an explicit constant which tends to 1 as c tends to -2. In particular,

$$\lim_{c \searrow -2} \dim_{\mathrm{H}}(B_c) = \dim_{\mathrm{H}}(B_{-2}) = 1.$$

The function $c \mapsto \dim_{\mathrm{H}}(B_c)$ is monotonically decreasing on $[-2, 1/4]$ and vanishes for $c_{\mathrm{Feig}} < c \leq 1/4$, where $c_{\mathrm{Feig}} \approx -1.401155$ is the Feigenbaum value (see §6). I do not know what exactly happens for $-1.75 < c \leq c_{\mathrm{Feig}}$.

The explicit form of $\ell(c)$ in Theorem 1.5 will be given in §6. It is interesting to contrast this theorem with the fact that the measure of B_c is zero for all complex parameters $c \neq -2$ (see [**Sm**], [**Za**], [**Zd**]). The statement that B_c has positive Hausdorff dimension has been shown by S. Smirnov for Collet-Eckmann real quadratics by a very different argument [**Sm**].

Acknowledgment. I would like to thank Peter Jones who, some time ago, suggested the study of the parameter rays landing on the real slice. I am grateful to Stas Smirnov who brought [**Sa**] to my attention and kindly pointed out that a more careful application of the argument in my first draft gives the sharper result of Theorem 1.4. My further thanks are due to John Milnor and Carsten L. Petersen for their useful comments on earlier drafts of this paper. The pictures of the Julia sets and the Mandelbrot set with its parameter rays are created by John Milnor's program `polyjul` and Scott Sutherland's adapted version `manray`.

2. Background material

We collect a few basic facts about quadratic Julia sets, hyperbolic components of the Mandelbrot set, rational parameter rays, and the tuning algorithm. For details, see [**D2**], [**DH**], [**H**], and [**M**].

Quadratic Julia sets. Fix a parameter $c \in \mathbb{C}$ and consider the quadratic polynomial $Q_c : z \mapsto z^2 + c$. The *filled Julia set* of Q_c, denoted by K_c, is the set of all points in the plane with bounded forward orbit under Q_c. The topological boundary $J_c := \partial K_c$ is called the *Julia set* of Q_c. Both sets are non-empty, compact, and totally-invariant. Moreover, K_c is always full, in the sense that $\mathbb{C} \smallsetminus K_c$ is connected. The domain $\mathbb{C} \smallsetminus K_c$ is called the *basin of attraction of infinity*; it consists of all points with forward orbit tending to ∞. The components of the interior of K_c, if any, are called the *bounded Fatou components* of Q_c.

When $c \in M$, the filled Julia set K_c is connected, so there exists a unique conformal isomorphism $\varphi_c : \widehat{\mathbb{C}} \smallsetminus K_c \xrightarrow{\cong} \widehat{\mathbb{C}} \smallsetminus \overline{\mathbb{D}}$ which satisfies $\varphi_c(\infty) = \infty$ and $\varphi_c'(\infty) = 1$. It conjugates the dynamics of Q_c to the squaring map so that $\varphi_c(z^2 + c) = (\varphi_c(z))^2$ for all z in the basin of attraction of infinity. The analytic curve $R_c(t) := \varphi_c^{-1}\{re^{2\pi i t} : r > 1\}$ is called the *dynamic ray* at angle $t \in \mathbb{T}$. One immediately obtains $Q_c(R_c(t)) = R_c(\mathbf{d}(t))$, where $\mathbf{d} : t \mapsto 2t \pmod 1$ is the doubling map on $\mathbb{T}$. We say $R_c(t)$ *lands* at $z \in J_c$ if $\lim_{r \to 1} \varphi_c^{-1}(re^{2\pi i t}) = z$.

By a *cycle* of Q_c we simply mean a periodic orbit $z \mapsto Q_c(z) \mapsto \cdots \mapsto Q_c^{\circ n}(z) = z$. The quantity $\lambda := (Q_c^{\circ n})'(z)$ is called the *multiplier* of this cycle. The cycle is *attracting*, *repelling*, or *indifferent* if $|\lambda| < 1$, $|\lambda| > 1$, or $|\lambda| = 1$, respectively. An indifferent cycle is *parabolic* if its multiplier is a root of unity. A quadratic polynomial has always infinitely many cycles in the plane, but at most one of them can be non-repelling.

A point $c \in \mathbb{C}$ is called a *hyperbolic* parameter if the sequence $\{Q_c^{\circ n}(0)\}_{n \geq 0}$ tends to ∞ or to a necessarily unique attracting cycle in $\mathbb{C}$. It is called a *parabolic* parameter if Q_c has a necessarily unique parabolic cycle. Finally, c is called a *Misiurewicz* parameter if the critical point 0 of Q_c is preperiodic, i.e., if 0 has a finite forward orbit but is not periodic.

Hyperbolic components of M. We now turn to the parameter space. Recall that the Mandelbrot set M is the set of parameters c for which the (filled) Julia set of Q_c is connected. Equivalently, $c \in M$ if and only if $0 \in K_c$. Thus $Q_c^{\circ n}(0) \to \infty$ if $c \notin M$, and it follows that all parameters outside M are hyperbolic. The hyperbolic parameters in M form an open set; in fact, each connected component of this set is a connected component of the interior of M. As such, it is called a *hyperbolic component* of M. The main hyperbolic component H_0 containing $c = 0$ is the prominently visible cardioid in any picture of M. It consists of all c for which Q_c has an attracting fixed point in $\mathbb{C}$. The *period* of a hyperbolic component H, denoted by $\mathrm{per}(H)$, is the length of the unique attracting cycle of Q_c for any $c \in H$. There is a canonical conformal isomorphism $\lambda_H : H \xrightarrow{\cong} \mathbb{D}$ which assigns to each $c \in H$ the multiplier

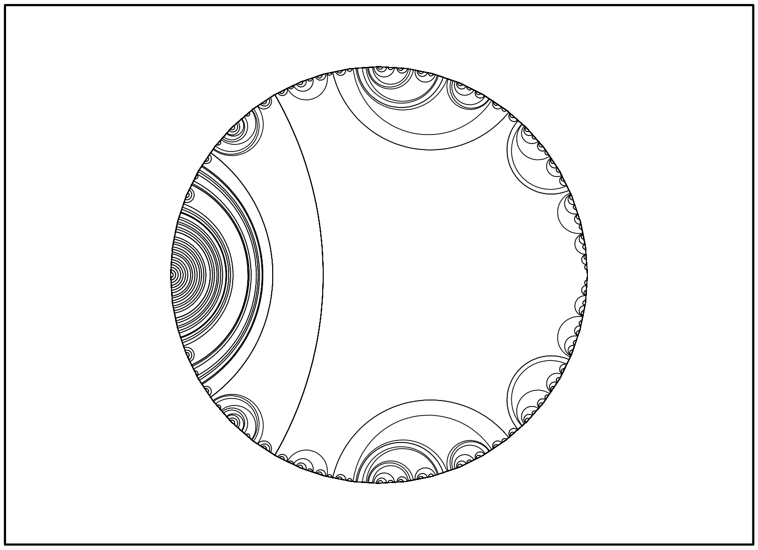

FIGURE 2. *The rational equivalence classes of $\simeq$ used to define the abstract Mandelbrot set M_{abs}. The set $\mathcal{R}$ (see §3) corresponds to the closure of the Poincaré geodesics which are symmetric with respect to the real line, or equivalently, the ones which separate points 1 and -1 on the boundary circle.*

of its attracting cycle. The map λ_H extends to a homeomorphism $\overline{H} \xrightarrow{\cong} \overline{\mathbb{D}}$. The *center* and *root* of H are by definition the points $\lambda_H^{-1}(0)$ and $\lambda_H^{-1}(1)$, respectively. The root of H is a parabolic parameter; in fact every parabolic parameter is realized as the root of a unique hyperbolic component. There are exactly two parameter rays of angles $\theta_-(H) < \theta_+(H)$ landing at the root of H which are rationals of the form $n/(2^p - 1)$, where $p = \text{per}(H)$. (When $H = H_0$, the two angles $\theta_-(H_0) = 0$ and $\theta_+(H_0) = 1$ coincide, and only one ray $R_M(0) = R_M(1)$ lands at the root point $c = 1/4$.) It follows that these two angles have binary expansions of the form

$$\theta_-(H) = 0.\,\overline{\theta_0} \quad \text{and} \quad \theta_+(H) = 0.\,\overline{\theta_1}, \tag{2.1}$$

where θ_0 and θ_1 are binary words of length p, and the bars indicate infinite repetition as usual. Conversely, every parameter ray $R_M(t)$ for which t is rational of odd denominator lands at the root of a unique hyperbolic component.

Given any hyperbolic component H and any irreducible fraction $0 < p/q < 1$, there exists a unique hyperbolic component W which satisfies

$$\overline{H} \cap \overline{W} = \lambda_H^{-1}(e^{2\pi i p/q}) = \lambda_W^{-1}(1).$$

This W is usually called the *p/q-satellite* of H.

The abstract Mandelbrot set. Define an equivalence relation $\simeq$ on $\mathbb{Q}/\mathbb{Z}$ by setting $t \simeq s$ if and only if $R_M(t)$ and $R_M(s)$ land at a common point. This can be extended to an equivalence relation on $\mathbb{T}$ by taking closure, and then to the closed disk $\overline{\mathbb{D}}$ by taking the Poincaré convex hulls of the equivalence classes on the boundary. We still denote this equivalence relation by $\simeq$ (see Fig. 2). The *abstract Mandelbrot set* is by definition the quotient $M_{\text{abs}} := \overline{\mathbb{D}}/\simeq$. It is a compact, full, connected, and locally-connected space [**D2**].

The *MLC conjecture* asserts that the Mandelbrot set is locally-connected, which, if true, would allow a complete topological description of M. The celebrated *density of hyperbolicity conjecture* asserts that all the interior components of M are hyperbolic, so that every quadratic polynomial can be approximated by a sequence of hyperbolic quadratics. Douady and Hubbard have shown that MLC implies density of hyperbolicity. In fact, they construct a continuous surjection $\chi : M \to M_{\text{abs}}$ whose fibers $\chi^{-1}(\text{point})$ are reduced to points if and only if MLC holds. In this case, the combinatorial model M_{abs} is actually homeomorphic to M via χ. Density of hyperbolicity is equivalent to the weaker statement that the fibers of χ have empty interior (see [**D2**], [**DH**], or [**Sc**]).

The tuning operation. For every hyperbolic component H, denote by $\iota_H : M \hookrightarrow M$ the Douady-Hubbard's *tuning* map (see [**DH**], [**H**], and [**M**]). Then ι_H is a topological embedding which maps H_0 onto H and respects centers and roots of hyperbolic components. The image $\iota_H(M)$ is what is often called the *small copy of M growing from H*. For a hyperbolic component W, the image $\iota_H(W)$ is a hyperbolic component of period $\text{per}(H) \cdot \text{per}(W)$, which is called *$H$ tuned by W*. One has the relation $\lambda_W(c) = \lambda_{\iota_H(W)}(\iota_H(c))$ for every $c \in W$. The binary operation $(H, W) \mapsto \iota_H(W)$ makes the set of hyperbolic components into a free semigroup with H_0 as the two sided identity.

The effect of this tuning map on Julia sets can be roughly described as follows. Let H be a hyperbolic component, $c_1 \in M$ and $c_2 := \iota_H(c_1)$. Take the filled Julia set K_c for any $c \in H$ and replace each bounded component of $\mathbb{C} \smallsetminus K_c$ by a copy of the filled Julia set K_{c_1} by appropriately identifying their Carathéodory loops. The resulting compact set is homeomorphic to the filled Julia set K_{c_2}; see [**H**] for details.

The tuning operation also acts on external angles. Let $\theta_-(H) = 0.\overline{\theta_0} < \theta_+(H) = 0.\overline{\theta_1}$ be the angles of the two parameter rays landing at the root of a hyperbolic component $H \neq H_0$ as in (2.1). Take an angle $t \in \mathbb{T}$ with the binary expansion $0.t_1t_2t_3\cdots$. Define the tuned angle

$$A_H(t) := 0.\theta_{t_1}\theta_{t_2}\theta_{t_3}\cdots \tag{2.2}$$

obtained by concatenating blocks of words of length $p = \text{per}(H) > 1$. Note that under this *tuning algorithm* on angles, a dyadic rational has two distinct images since it has two different binary representations. It can be shown that if $c \in M$ is the landing point of $R_M(t)$, then the tuned point $\iota_H(c)$ is the landing point of $R_M(A_H(t))$. The image $A_H(\mathbb{T})$ is a self-similar Cantor set. In fact, for $i = 0, 1$ let $\Lambda_i : \mathbb{T} \to \mathbb{T}$ be the map defined by $\Lambda_i(0.t_1t_2t_3\cdots) := 0.\theta_i t_1t_2t_3\cdots$. Then Λ_i is an affine contraction by a factor 2^{-p} and the image $A_H(\mathbb{T})$ is precisely the invariant set generated by (Λ_0, Λ_1) [**Mn**]. A standard computation then shows that $A_H(\mathbb{T})$ has Hausdorff dimension $1/p < 1$ and hence measure zero (see for example [**Mt**]).

Real hyperbolic components. Let $\mathcal{H}$ denote the collection of all hyperbolic components of M which intersect the real line. Every $H \in \mathcal{H}$ is invariant under the conjugation $c \mapsto \overline{c}$, and has its center on the real line. If $H \cap \mathbb{R} =]c', c[$, then $c = \lambda_H^{-1}(1)$ is the root of H and $c' = \lambda_H^{-1}(-1)$. It follows that c is the landing point of two parameter rays at angles

$$\theta_-(H) = \frac{n}{2^p - 1} \quad \text{and} \quad \theta_+(H) = 1 - \theta_-(H),$$

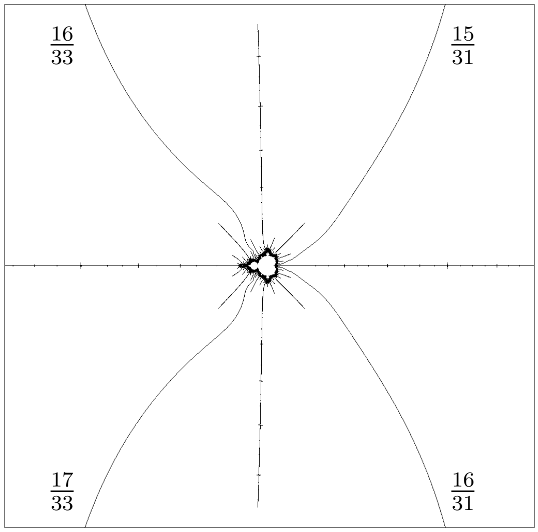

FIGURE 3. *A real period 5 hyperbolic component H whose center is approximately located at -1.985424253. Here $\theta_-(H) = 15/31$ and $\omega_-(H) = 16/33$. By definition, the opening of this component is the interval $O(H) =]15/31, 16/33[$.*

where $p = \mathrm{per}(H)$ and n is an integer which satisfies $0 \leq n \leq 2^{p-1} - 1$. A brief computation using the tuning algorithm shows that c' is the landing point of the parameter rays at angles

$$\omega_-(H) = \frac{n+1}{2^p+1} \quad \text{and} \quad \omega_+(H) = 1 - \omega_-(H).$$

The open interval $O(H) :=]\theta_-(H), \omega_-(H)[\subset [0, 1/2]$ is called the *opening* of H (see Fig. 3). For example, $O(H_0) =]0, 1/3[$.

The following long-standing conjecture of Fatou, which has been proved rather recently, will be used repeatedly in the next section (see [**Ly1**] and [**GS**]):

Theorem 2.1 (Density of real hyperbolics)**.** *The set of all real hyperbolic parameters is dense in $M \cap \mathbb{R} = [-2, 1/4]$. In particular, every component of the interior of M which meets the real line is hyperbolic.*

3. Real quadratics and the set $\mathcal{R}$

The τ-function. Suppose $c \in \partial M \cap \mathbb{R}$. By definition, the *dynamic root* r_c of Q_c is the critical value c if c is not a parabolic parameter so that the Julia set J_c is full. On the other hand, when c is a parabolic parameter, the dynamic root r_c is the unique point of the parabolic cycle which is on the boundary of the bounded Fatou component containing c. In this case, $c < r_c$ and the open interval $]c, r_c[$ does not intersect J_c.

Recall that the *(prime-end) impression* of a parameter ray $R_M(t)$ is the set of all $c \in \partial M$ for which there is a sequence $\{w_n\}$ such that $|w_n| > 1$, $w_n \to e^{2\pi i t}$, and $\Phi^{-1}(w_n) \to c$. We denote the impression of $R_M(t)$ by $\widehat{R}_M(t)$. It is a non-empty, compact, connected subset of ∂M. Every point of ∂M belongs to the impression of at least one parameter ray. Conjecturally, every parameter ray $R_M(t)$ lands at a well-defined point $c(t) \in \partial M$ and $\widehat{R}_M(t) = \{c(t)\}$.

According to Douady-Hubbard and also Tan Lei, this certainly holds for every rational angle t and the landing point is either parabolic or Misiurewicz depending on whether t has odd or even denominator (compare [**DH**], [**T1**] and [**T2**]). Moreover, one can describe which rational external rays land at parabolic and Misiurewicz parameters. In the special case of real quadratics, their result yields the following

Theorem 3.1 (Douady-Hubbard-Tan Lei). *Let $c \in \partial M \cap \mathbb{R}$ be parabolic or Misiurewicz. Then there exists a unique angle $\tau(c) \in [0, 1/2]$ such that the dynamic rays $R_c(\pm\tau(c))$ land at the dynamic root r_c of Q_c. In the parameter plane, the two rays $R_M(\pm\tau(c))$ land at c; in fact $\widehat{R}_M(\pm\tau(c)) = \{c\}$ and no other parameter ray can have c in its impression.*

Note that the cases $c = 1/4$ and $c = -2$ are special since $\tau(1/4) = 0$ and $\tau(-2) = 1/2$ and the two rays given by the theorem coincide. The following statement is immediate:

Corollary 3.2. *If t belongs to the opening $O(H)$ of some real hyperbolic component H, then the impression $\widehat{R}_M(t)$ of the parameter ray at angle t does not intersect the real line.*

It is not hard to partially generalize Theorem 3.1 to all real quadratics with connected Julia sets. The dynamic part of the following proof which uses harmonic measure on the Julia set is inspired by the more general combinatorial arguments in [**Za**].

Theorem 3.3. *Let $c \in \partial M \cap \mathbb{R}$. Then there exists a unique angle $\tau(c) \in [0, 1/2]$ such that the dynamic rays $R_c(\pm\tau(c))$ land at the dynamic root r_c of Q_c. In the parameter plane, the two rays $R_M(\pm\tau(c))$, and only these rays, contain c in their impression.*

Fig. 4 illustrates the content of this theorem.

Proof. In view of Theorem 3.1 we may assume that c is neither parabolic nor Misiurewicz. First consider the dynamic plane. According to [**LS**], the Julia set J_c is locally-connected. In particular, by the theorem of Carathéodory, all dynamic rays land. By real symmetry, there exists at least one angle $0 < t < 1/2$ such that the two dynamic rays $R_c(\pm t)$ land at $r_c = c$. Assume by way of contradiction that there are two angles $0 < s < t < 1/2$ such that $R_c(s)$ and $R_c(t)$ both land at c. Consider the component W of $\mathbb{C} \smallsetminus \overline{R_c(s) \cup R_c(t)}$ which does not intersect the real line, and set $L_0 := \overline{W} \cap J_c$. By the theorem of F. and M. Riesz, L_0 is a non-degenerate continuum of positive harmonic measure $\mu(L_0) = t - s$. Set $L_n := Q_c^{\circ n}(L_0)$ and $c_n := Q_c^{\circ n}(c)$.

It is easy to see that $J_c \cap \mathbb{R} = [-\beta, \beta]$, where the fixed point $\beta > 0$ is the landing point of $R_c(0)$. Note that

$$Q_c^{-1}[-\beta, \beta] = [-\beta, \beta] \cup [i\xi, -i\xi],$$

where $\xi = \sqrt{\beta + c} > 0$ is the positive root of the equation $Q_c(\pm i\xi) = -\beta$. We claim that

$$L_n \cap \mathbb{R} = \{c_n\} \quad \text{for all } n \geq 0. \tag{3.1}$$

Otherwise, take the smallest integer $n \geq 1$ for which this is false. Then, since $L_{n-1} \cap \mathbb{R} = \{c_{n-1}\}$, L_{n-1} must intersect $[i\xi, -i\xi]$ at some point z. Since L_{n-1} is path-connected, it follows that it contains the unique arc in J_c which joins z to $c_{n-1} \in \mathbb{R} \smallsetminus \{0\}$. Evidently this arc has to intersect $\mathbb{R}$ along a non-degenerate interval, contradicting $L_{n-1} \cap \mathbb{R} = \{c_{n-1}\}$. Thus (3.1) holds.

Now it easily follows that each map $Q_c : L_n \mapsto L_{n+1}$ is injective. In fact, if this were not true for some $n \geq 0$, then L_n would have to contain a pair of symmetric points $\pm z$ in J_c

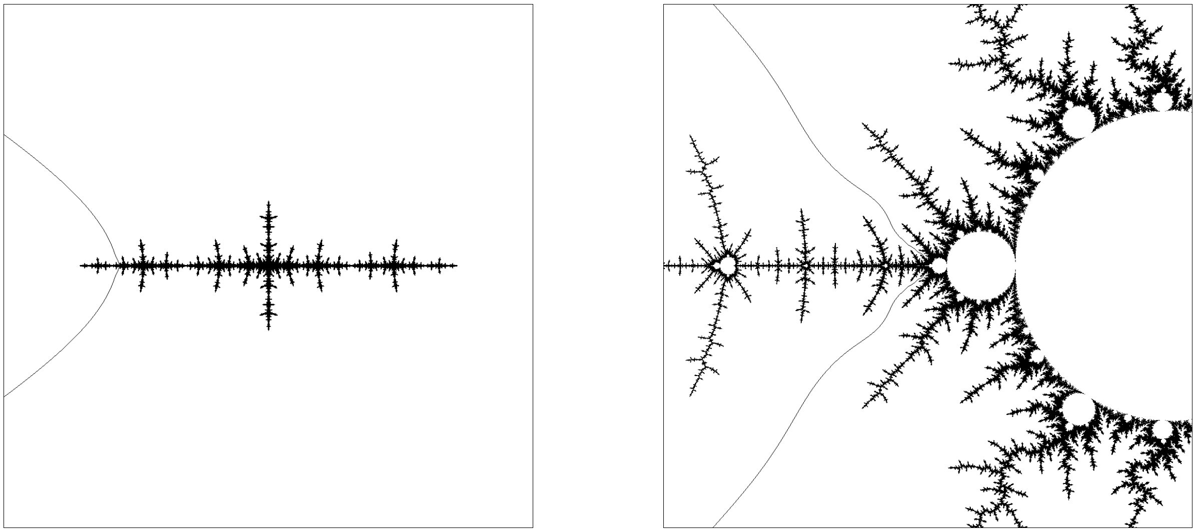

FIGURE 4. *Left: The Julia set of the main Feigenbaum quadratic $z \mapsto z^2 + c_{\text{Feig}}$, where $c_{\text{Feig}} \approx -1.401155$, with the dynamic rays at angles $\pm\tau(c_{\text{Feig}})$ landing at its critical value. Computation gives $\tau(c_{\text{Feig}}) \approx 0.412454$. Right: The corresponding parameter rays outside the Mandelbrot set.*

and hence the critical point 0. Since $L_n \cap \mathbb{R} = \{c_n\}$, this would mean $c_n = 0$, which would be impossible. Thus, in the cascade of injective maps $L_0 \mapsto L_1 \mapsto \cdots \mapsto L_n \mapsto \cdots$, the harmonic measure μ doubles at each step. However, $\mu(L_0) > 0$ and $\mu(L_n) \leq \mu(J_c) = 1$ for all n. The contradiction shows there is a unique $t = \tau(c) \in [0, 1/2]$ such that $R_c(t)$ lands at c, and the proof in the dynamic plane is complete.

To prove the result in the parameter plane, one possible approach is to use the following standard construction: For any $c \in \partial M \cap \mathbb{R}$, the *itinerary* of the dynamic root r_c is the infinite sequence $\mathbf{itin}(r_c) := (\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots)$ of signs $\pm$ determined by $\varepsilon_j = \operatorname{sgn}(Q_c^{\circ j}(r_c))$. It is easy to see that the binary expansion $0.\, t_0\, t_1\, t_2 \cdots$ of the angle $\tau(c)$ constructed above is uniquely determined by $\mathbf{itin}(r_c)$. In fact, $t_0 = 0$ and a brief computation shows that for $j \geq 0$,

$$t_{j+1} = \begin{cases} t_j & \text{if} \quad \varepsilon_j = + \\ 1 - t_j & \text{if} \quad \varepsilon_j = - \end{cases}$$

It follows in particular that if $\mathbf{itin}(r_c)$ and $\mathbf{itin}(r_{c'})$ coincide up to the first n signs, then $|\tau(c) - \tau(c')| \leq 2^{-(n+1)}$.

Now consider a parameter $c \in \partial M \cap \mathbb{R}$ which is neither parabolic nor Misiurewicz, and let $0 < s < 1/2$ be any angle for which $c \in \widehat{R}_M(s)$. Choose a decreasing (resp. increasing) sequence $\{a_n\}$ (resp. $\{b_n\}$) of real parabolic parameters converging to c (the existence of such sequences is guaranteed by Theorem 2.1). It follows from Theorem 3.1 that

$$\tau(a_n) < \tau(a_{n+1}) < s < \tau(b_{n+1}) < \tau(b_n) \tag{3.2}$$

for all n. It is not hard to see that the sequences $\mathbf{itin}(r_{a_n})$ and $\mathbf{itin}(r_{b_n})$ converge to $\mathbf{itin}(c)$ in the 2-adic metric. Hence both $\tau(a_n)$ and $\tau(b_n)$ converge to $\tau(c)$ and it follows from (3.2) that $s = \tau(c)$. Since c must belong to the impression of at least one parameter ray, it follows that $c \in \widehat{R}_M(\tau(c))$ and this completes the proof. $\square$

The set $\mathcal{R}$. We now consider the set of external angles in $[0, 1/2]$ of the parameter rays whose impressions intersect the real line:

$$\mathcal{R} := \{t \in [0, 1/2] : \widehat{R}_M(t) \cap \mathbb{R} \neq \emptyset\}. \tag{3.3}$$

Lemma 3.4. *The mapping $\tau : \partial M \cap \mathbb{R} \to \mathcal{R}$ given by Theorem 3.3 is a homeomorphism. Its inverse $\pi = \tau^{-1}$ is determined by $\{\pi(t)\} = \widehat{R}_M(t) \cap \mathbb{R}$.*

Proof. Given $t \in \mathcal{R}$, let us show that the impression $\widehat{R}_M(t)$ intersects the real line at a unique point. By Theorem 3.1 this is true if $t = 0$ or $1/2$, so we may assume $0 < t < 1/2$. Let $c, c' \in \partial M \cap \mathbb{R}$ both belong to $\widehat{R}_M(t)$, and $c' < c$. By symmetry, c and c' belong to the impression $\widehat{R}_M(-t)$ also. By Theorem 2.1, there exists a real hyperbolic component H such that $H \cap \mathbb{R} \subset]c', c[$. It follows that the union $\widehat{R}_M(t) \cup \widehat{R}_M(-t)$ separates the plane and H is contained in a bounded component W of $\mathbb{C} \smallsetminus (\widehat{R}_M(t) \cup \widehat{R}_M(-t))$. Now $\partial W \subset \widehat{R}_M(t) \cup \widehat{R}_M(-t) \subset \partial M$, so W must be a component of the interior of M, implying $W = H$. In particular, the root of H belongs to $\widehat{R}_M(t)$. This, by Theorem 3.1, implies $\widehat{R}_M(t)$ is a singleton, which contradicts our assumption.

Thus the map $\pi : \mathcal{R} \to \partial M \cap \mathbb{R}$ given by $\widehat{R}_M(t) \cap \mathbb{R} = \{\pi(t)\}$ is well-defined. The relations $\pi \circ \tau = \mathrm{id}$ and $\tau \circ \pi = \mathrm{id}$ follow easily from Theorem 3.3. In particular, τ is both injective and surjective.

It remains to prove continuity of τ, or equivalently π. Clearly π is monotone. Assume by way of contradiction that $\{t_n\}$ is an increasing sequence in $\mathcal{R}$ converging to $t \in \mathcal{R}$ such that $c^* := \lim \pi(t_n) > c := \pi(t)$. If there exists some $c' \in \partial M \cap \mathbb{R}$ in the interval $]c, c^*[$, then $\tau(\pi(t_n)) < \tau(c') < \tau(c)$ or $t_n < \tau(c') < t$ for all n, which is impossible. Hence $]c, c^*[$ is a subset of the interior of the Mandelbrot set. By the density of real hyperbolics and the fact that $c, c^* \in \partial M$, there exists a component $H \in \mathcal{H}$ such that $H \cap \mathbb{R} =]c, c^*[$. Take, for example, the root point of the $1/3$-satellite of H which is the landing point of two rational parameter rays at angles $\alpha < \beta$. Then $t_n < \alpha < \beta < t$ for all n, which again is a contradiction. This proves that π is left-continuous. The proof of right-continuity is similar. $\square$

Lemma 3.5. *The union $\bigcup_{H \in \mathcal{H}} O(H)$ of the openings of real hyperbolic components is dense in $[0, 1/2]$.*

Proof. Let E be the set of endpoints of the openings $O(H)$ for $H \in \mathcal{H}$. Evidently, $E \subset \mathcal{R}$. Since $\mathcal{R}$ is closed by Lemma 3.4, we actually have $\overline{E} \subset \mathcal{R}$.

Assuming the lemma is false, let $[t, t']$ be a maximal non-degenerate interval in the complement of the union $\bigcup_{H \in \mathcal{H}} O(H)$. Then $t, t' \in \overline{E}$, so both t and t' are angles in $\mathcal{R}$. Applying the homeomorphism π, we obtain $c' := \pi(t') < c := \pi(t)$. Density of real hyperbolics now implies the existence of some $H \in \mathcal{H}$ with $H \cap \mathbb{R} \subset]c', c[$. It follows that $]t, t'[$ must contain the opening $O(H)$, which is a contradiction. $\square$

Corollary 3.6. $\mathcal{R} = [0, 1/2] \smallsetminus \bigcup_{H \in \mathcal{H}} O(H)$.

Proof. The inclusion $\subset$ follows from Corollary 3.2. To see the inclusion $\supset$, note that by Lemma 3.5, any $t \in [0, 1/2]$ outside the union $\bigcup_{H \in \mathcal{H}} O(H)$ belongs to $\overline{E}$, which is a subset of $\mathcal{R}$. $\square$

The definition of $\mathcal{R}$ is simple but rather hard to work with. In fact, by Corollary 3.6 the explicit construction of $\mathcal{R}$ boils down to deciding which rational angles form the endpoints of the openings of real hyperbolic components. There is a combinatorial algorithm due to P. Lavaurs which describes the equivalence relation $\simeq$ on $\mathbb{Q}/\mathbb{Z}$ used in the definition of M_{abs} in §2 [**La**]. In other words, it tells which rational angles t, s satisfy $t \simeq s$, or equivalently, which rational ray pairs $(R_M(t), R_M(s))$ land at a common point. The rational parameter rays landing on the real line correspond to the angles t in $[0, 1/2]$ for which $t \simeq -t$. The set $\mathcal{R}$ is obtained by taking the closure of the set of such rational angles. Thus, *in principle*, one should be able to determine the set $\mathcal{R}$ using Lavaurs' algorithm. However, this algorithm is not quite suitable for the purpose of computing measure and dimension of $\mathcal{R}$. To circumvent this problem, we found an alternative description of $\mathcal{R}$, much easier to work with since it is purely given by the dynamics of the doubling map $\mathbf{d} : t \mapsto 2t \pmod 1$ on the circle. As it turned out, this description was not new: In [**D3**], Douady gives a description for the dynamic rays which land on the spine $[-\beta, \beta]$ of the Julia set of a real quadratic polynomial (compare equation (6.3) below). When transferred to the parameter plane, it would give the same alternative description of $\mathcal{R}$.

Theorem 3.7. $\mathcal{R} = \{t \in [0, 1/2] : \mathbf{d}^{\circ n}(t) \notin \,]t, 1-t[\text{ for all } n \geq 1\}$.

Proof. Suppose that $t \in \mathcal{R}$ and $c = \pi(t)$, or equivalently $t = \tau(c)$ as in Theorem 3.3. This means the dynamic rays $R_c(\pm t)$ land at the dynamic root r_c of Q_c, and hence the dynamic rays $R_c(\mathbf{d}^{\circ n}(\pm t))$ land at $Q_c^{\circ n}(r_c)$ for all $n \geq 1$. Note that since c and r_c are real, $Q_c^{\circ n}(r_c) \geq c$ for all $n \geq 1$. Moreover, since $J_c \cap \,]c, r_c[= \emptyset$, we actually have $Q_c^{\circ n}(r_c) \geq r_c$. It easily follows that $\mathbf{d}^{\circ n}(t) \notin \,]t, 1-t[$ for all $n \geq 1$.

Now let $t \notin \mathcal{R}$. By Corollary 3.6, this means $t \in O(H)$ for a real hyperbolic component H of some period $p \geq 1$, so that

$$\frac{n}{2^p - 1} = \theta_-(H) < t < \omega_-(H) = \frac{n+1}{2^p + 1}.$$

Thus $t < 2^p t - n < 1 - t$, or $\mathbf{d}^{\circ p}(t) \in \,]t, 1-t[$. □

4. Measure and dimension of $\mathcal{R}$

The alternative description of $\mathcal{R}$ given by Theorem 3.7 makes the question of measure of $\mathcal{R}$ almost trivial:

Lemma 4.1. *$\mathcal{R}$ is a set of Lebesgue measure zero.*

Proof. Choose a nested sequence $I_1 \supset I_2 \supset \cdots$ of open intervals centered at $1/2$ such that $\bigcap_{n \geq 1} I_n = \{1/2\}$. Since $\mathbf{d}$ is ergodic with respect to Lebesgue measure, for each n there exists a set $X_n \subset \mathbb{T}$ with $m(X_n) = 0$ such that the forward orbit of t under $\mathbf{d}$ hits I_n whenever $t \in \mathbb{T} \smallsetminus X_n$. Taking $X := \bigcup_{n \geq 1} X_n$, it follows that the forward orbit of every $t \in \mathbb{T} \smallsetminus X$ hits every I_n. Since for every $t \in [0, 1/2[$ there exists a large n with $I_n \subset \,]t, 1-t[$, it follows that the forward orbit of every $t \in \mathbb{T} \smallsetminus X$ hits $]t, 1-t[$. By Theorem 3.7, we must have $\mathcal{R} \subset X$, which proves the claim. □

Proof of Theorem 1.1. The set of external angles of the parameter rays which land on the real slice is evidently contained in $\mathcal{R} \cup -\mathcal{R}$, and thus has measure zero by Lemma 4.1. It follows that $\mu_M[-2, 1/4] = 0$. □

Using Corollary 3.6, we obtain

Corollary 4.2. *The sum of the lengths of the openings of all real hyperbolic components is* $1/2$.

We now turn to the question of dimension of $\mathcal{R}$. First let us introduce some notation. By a *dyadic rational of generation* n in $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ we mean a number of the form $p/2^n$ modulo $\mathbb{Z}$, where p is an odd integer. The set of all such numbers will be denoted by $\mathcal{D}_n$. For $t \in \mathbb{T}$ and $n \geq 1$, the notation $\|t\|_n$ will be used for the distance from t to the closest dyadic rational of generation n:

$$\|t\|_n := \inf_{x \in \mathcal{D}_n} |t - x|.$$

Clearly $0 \leq \|t\|_n \leq 2^{-n}$ so that $\|t\|_n \to 0$ as $n \to \infty$. For $0 < \sigma < 1$ and $n \geq 2$, we consider the nested sequence of non-empty compact sets

$$\mathcal{K}_\sigma^n := \{t \in \mathbb{T} : \|t\|_k \geq \sigma\, 2^{-k} \ \text{ for all } \ 2 \leq k \leq n\},$$

and we define

$$\mathcal{K}_\sigma := \bigcap_{n \geq 2} \mathcal{K}_\sigma^n = \{t \in \mathbb{T} : \|t\|_n \geq \sigma\, 2^{-n} \ \text{ for all } \ n \geq 2\}. \tag{4.1}$$

The proof of Theorem 1.2 will depend on the following

Lemma 4.3.

$$\lim_{\sigma \to 0} \ \dim_{\mathrm{H}}(\mathcal{K}_\sigma) = \lim_{\sigma \to 0} \ \dim_{\mathrm{H}} \left(\mathcal{K}_\sigma \cap \left[\frac{1-\sigma}{2}, \frac{1}{2} \right] \right) = 1.$$

Before proving this lemma, let us show how Theorem 1.2 would follow.

Proof of Theorem 1.2 (assuming Lemma 4.3). First observe that for all small $\sigma > 0$,

$$\mathcal{R} \supset \mathcal{K}_\sigma \cap \left[\frac{1-\sigma}{2}, \frac{1}{2} \right]. \tag{4.2}$$

In fact, if $t \in \mathcal{K}_\sigma$, then $\|t\|_{n+1} \geq \sigma 2^{-(n+1)}$ for all $n \geq 1$. Applying the iterate $\mathbf{d}^{\circ n}$, it follows that $|\mathbf{d}^{\circ n}(t) - 1/2| \geq \sigma/2$ for all $n \geq 1$. If in addition $t \in [(1-\sigma)/2, 1/2]$, then $\sigma/2 \geq 1/2 - t$, so that $|\mathbf{d}^{\circ n}(t) - 1/2| \geq 1/2 - t$ for all $n \geq 1$. By Theorem 3.7, this implies $t \in \mathcal{R}$, which proves (4.2). Thus,

$$1 \geq \dim_{\mathrm{H}}(\mathcal{R}) \geq \dim_{\mathrm{H}} \left(\mathcal{K}_\sigma \cap \left[\frac{1-\sigma}{2}, \frac{1}{2} \right] \right).$$

Taking the limit as $\sigma \to 0$, we obtain $\dim_{\mathrm{H}}(\mathcal{R}) = 1$. But by the theorem of Beurling, after removing a set of zero capacity (and hence zero Hausdorff dimension) from $\mathcal{R}$, we may assume that all the remaining rays in $\mathcal{R}$ land at a real parameter. This completes the proof of Theorem 1.2. □

The proof of Lemma 4.3 will be based on the following two lemmas. The first one describes the metric structure of the compact sets $\mathcal{K}_\sigma^n$, and the second one uses this structure to estimate the dimension of $\mathcal{K}_\sigma$. For simplicity we state these lemmas in the case σ is a negative power of 2.

Lemma 4.4. *Fix a parameter $\sigma = 2^{-p}$ where $p \geq 2$ is an integer, and let $n \geq 2$. Then*

(i) *$\mathcal{K}_\sigma^n$ is the disjoint union of a finite collection $|\mathcal{K}_\sigma^n|$ of closed non-degenerate intervals in $\mathbb{T}$ whose endpoints are dyadic rationals in $\bigcup_{k=2}^n \mathcal{D}_{k+p}$.*

(ii) *There are two distinguished intervals* $I_0^n, I_{1/2}^n \in |\mathcal{K}_\sigma^n|$ *centered at* 0 *and* $1/2$*, with* $m(I_0^n) = m(I_{1/2}^n) = 2^{-n+1}(1-\sigma)$.

(iii) *Every* $I \in |\mathcal{K}_\sigma^n|$ *contains at least one and at most three elements of* $|\mathcal{K}_\sigma^{n+1}|$*. If* $I = I_0^n$ *or* $I_{1/2}^n$*, then it contains exactly 3 elements of* $|\mathcal{K}_\sigma^{n+1}|$.

(iv) *For every* $I \in |\mathcal{K}_\sigma^n|$, $0 < m(I) \leq 2^{-n+1}(1-\sigma)$.

(v) *For every* $I \in |\mathcal{K}_\sigma^n|$,

$$\frac{m(I \cap \mathcal{K}_\sigma^{n+1})}{m(I)} \geq \frac{3-8\sigma}{3-4\sigma}.$$

Proof. For convenience, let us introduce the sets

$$\mathcal{E}_\sigma^k := \{t \in \mathbb{T} : \|t\|_k \geq \sigma 2^{-k}\}, \qquad k = 2, 3, \ldots$$

Evidently, each $\mathcal{E}_\sigma^k$ is the union of a collection $|\mathcal{E}_\sigma^k|$ of 2^{k-1} disjoint closed intervals of length $2^{-k+1}(1-\sigma)$ whose endpoints belong to $\mathcal{D}_{k+p}$.

To prove (i), note that

$$\mathcal{K}_\sigma^n = \mathcal{E}_\sigma^2 \cap \mathcal{E}_\sigma^3 \cap \cdots \cap \mathcal{E}_\sigma^n,$$

so the statement about the endpoints of the intervals in $|\mathcal{K}_\sigma^k|$ follows from the corresponding statement for $|\mathcal{E}_\sigma^k|$. If some $I \in |\mathcal{K}_\sigma^n|$ degenerate to a singleton $\{x\}$, then x must be the common endpoint of two intervals in $|\mathcal{E}_\sigma^k|$ and $|\mathcal{E}_\sigma^s|$ for some $2 \leq k < s \leq n$. This is clearly impossible since $\mathcal{D}_{k+p} \cap \mathcal{D}_{s+p} = \emptyset$.

The statements (ii) and (iii) are trivial. The statement (iv) follows from the fact that every interval $I \in |\mathcal{K}_\sigma^n|$ is contained in an interval $J \in |\mathcal{E}_\sigma^n|$, so that $m(I) \leq m(J) \leq 2^{-n+1}(1-\sigma)$.

It remains to prove (v). If $I = I_0^n$ or $I_{1/2}^n$, a brief computation shows that

$$\frac{m(I \cap \mathcal{K}_\sigma^{n+1})}{m(I)} = \frac{1-2\sigma}{1-\sigma} \geq \frac{3-8\sigma}{3-4\sigma}.$$

So take an $I \in |\mathcal{K}_\sigma^n|$ such that $I \neq I_0^n$ and $I \neq I_{1/2}^n$. Let

$$[x + \sigma 2^{-n}, x + 2^{-n+1} - \sigma 2^{-n}] = [x + 2^{-(n+p)}, x + 2^{-n+1} - 2^{-(n+p)}]$$

be the unique interval in $|\mathcal{E}_\sigma^n|$ which contains I, where $x \in \mathcal{D}_n$. Let

$$t := x + 2^{-(n+1)}, \ y := x + 2^{-n}.$$

Clearly $t \in \mathcal{D}_{n+1}$ and $y \in \mathcal{D}_m$ for some $m < n$. It is easy to see that one of the following two cases must occur:

Case 1. $m + p = n + 1$ and $I = [x + 2^{-(n+p)}, y - 2^{-(n+1)}] = [x + 2^{-(n+p)}, t]$. In this case

$$\frac{m(I \cap \mathcal{K}_\sigma^{n+1})}{m(I)} = 1 - \frac{2^{-(n+p+1)}}{2^{-(n+1)} - 2^{-(n+p)}} = \frac{1-3\sigma}{1-2\sigma} \geq \frac{3-8\sigma}{3-4\sigma}.$$

Case 2. $m + p > n + 1$ and $I = [x + 2^{-(n+p)}, y - 2^{-(m+p)}]$. In this case, t belongs to the interior of I and we have

$$\frac{m(I \cap \mathcal{K}_\sigma^{n+1})}{m(I)} = 1 - \frac{2^{-(n+p)}}{2^{-n} - 2^{-(n+p)} - 2^{-(m+p)}} \geq 1 - \frac{2^{-(n+p)}}{2^{-n} - 2^{-(n+p)} - 2^{-(n+2)}} = \frac{3-8\sigma}{3-4\sigma}.$$

In either case, we obtain the lower bound in (v). □

Lemma 4.5. *Fix a parameter $\sigma = 2^{-p}$ where $p \geq 2$ is an integer, and let*

$$1 < \lambda := \frac{3-4\sigma}{3-8\sigma} \leq 2.$$

Then

$$\dim_{\mathrm{H}}(\mathcal{K}_\sigma) \geq \dim_{\mathrm{H}}\left(\mathcal{K}_\sigma \cap \left[\frac{1-\sigma}{2}, \frac{1+\sigma}{2}\right]\right) \geq 1 - \frac{\log\lambda}{\log 2}.$$

Note that Lemma 4.3 follows immediately, since by symmetry

$$\dim_{\mathrm{H}}\left(\mathcal{K}_\sigma \cap \left[\frac{1-\sigma}{2}, \frac{1+\sigma}{2}\right]\right) = \dim_{\mathrm{H}}\left(\mathcal{K}_\sigma \cap \left[\frac{1-\sigma}{2}, \frac{1}{2}\right]\right)$$

and $\lambda \to 1$ as $\sigma = 2^{-p} \to 0$.

Proof. This follows from Lemma 4.4(v) by a standard mass distribution argument. Define for each n the probability measure μ_n supported in $\mathcal{K}_\sigma^n$, with uniform density on each interval in $|\mathcal{K}_\sigma^n|$, as follows: For $I \in |\mathcal{K}_\sigma^2|$, set $\mu_2(I) := m(I)/m(\mathcal{K}_\sigma^2)$. When $n \geq 2$ and $I \in |\mathcal{K}_\sigma^{n+1}|$, let J be the unique interval in $|\mathcal{K}_\sigma^n|$ that contains I, and set

$$\mu_{n+1}(I) := \frac{m(I)}{m(J \cap \mathcal{K}_\sigma^{n+1})}\, \mu_n(J).$$

By Lemma 4.4(v),

$$\frac{\mu_{n+1}(I)}{m(I)} = \frac{m(J)}{m(J \cap \mathcal{K}_\sigma^{n+1})} \cdot \frac{\mu_n(J)}{m(J)} \leq \lambda\, \frac{\mu_n(J)}{m(J)}.$$

Continuing inductively, it follows that

$$\frac{\mu_{n+1}(I)}{m(I)} \leq (\text{const.})\, \lambda^n.$$

Now let μ^* be the weak limit of the sequence $\{\mu_n\}$. Then μ^* is supported in $\mathcal{K}_\sigma$ and we have

$$\mu^*(I) \leq (\text{const.})\, \lambda^n\, m(I) \quad \text{for all} \quad I \in |\mathcal{K}_\sigma^n|. \tag{4.3}$$

To estimate the μ^*-measure of an arbitrary interval T of length $\varepsilon = m(T) > 0$, choose n so that $2^{-(n+1)} < \varepsilon \leq 2^{-n}$, and consider the union $\mathcal{J}$ of all the intervals in $|\mathcal{K}_\sigma^n|$ which intersect T. Then, by (4.3) and Lemma 4.4(iv),

$$\begin{aligned} \mu^*(T) \leq \mu^*(\mathcal{J}) \leq\ & (\text{const.})\, \lambda^n\, m(\mathcal{J}) \\ \leq\ & (\text{const.})\, \lambda^n(\varepsilon + 2 \cdot 2^{-n+1}) \\ \leq\ & (\text{const.})\, \lambda^n\, \varepsilon \\ \leq\ & (\text{const.})\, \varepsilon^s(\lambda^n 2^{-n(1-s)}) \end{aligned}$$

for all $s > 0$. If $0 < s < 1 - (\log\lambda/\log 2)$, we have $\lambda 2^{s-1} < 1$ and hence

$$\mu^*(T) \leq (\text{const.})\, m(T)^s. \tag{4.4}$$

To finish the argument, let us recall the following (see for example [**Mt**]):

> **Frostman's Lemma.** *A Borel set $X \subset \mathbb{R}^d$ satisfies $\dim_{\mathrm{H}}(X) \geq s$ if and only if there exists a finite Borel measure μ supported in X and a constant $C > 0$ such that $\mu(B(x,r)) \leq Cr^s$ for all $x \in \mathbb{R}^d$ and all $r > 0$.*

Applying this lemma to (4.4), we obtain the inequality

$$\dim_{\mathrm{H}}(\mathcal{K}_\sigma) \geq 1 - \frac{\log \lambda}{\log 2}. \tag{4.5}$$

The claim for $\dim_{\mathrm{H}}(\mathcal{K}_\sigma \cap [\,(1-\sigma)/2, (1+\sigma)/2\,])$ follows from the same argument: Just start with the natural probability measure on $\mathcal{K}_\sigma^{p+1} \cap [\,(1-\sigma)/2, (1+\sigma)/2\,]$, and inductively define the sequence of measures μ_n supported in $\mathcal{K}_\sigma^n \cap [\,(1-\sigma)/2, (1+\sigma)/2\,]$ for all $n > p+1$. $\square$

The following observation will be used in the proof of Theorem 1.5 in §6. Recall that a compact set K of real numbers is *porous* if there exists an $0 < \varepsilon < 1$ such that every interval I has a subinterval J disjoint from K, with $m(J) > \varepsilon\, m(I)$. It is an easy exercise to show that a porous set has Hausdorff dimension less than 1.

Lemma 4.6. *For every* $0 < \sigma < 1$*, the compact set* $\mathcal{K}_\sigma$ *of (4.1) is porous, hence* $\dim_{\mathrm{H}}(\mathcal{K}_\sigma) < 1$.

Proof. Given an interval I of length $m(I) > 0$, choose n so that $2^{-(n-1)} < m(I) \leq 2^{-(n-2)}$ and consider a subinterval $]a/2^n, (a+1)/2^n[\, \subset I$, with $a \in \mathbb{Z}$. Then the midpoint $(2a+1)/2^{n+1}$ is dyadic of generation $n+1$, so the open interval

$$J := \left] \frac{2a+1-\sigma}{2^{n+1}}, \frac{2a+1+\sigma}{2^{n+1}} \right[\subset I$$

is disjoint from $\mathcal{K}_\sigma^{n+1}$ hence from $\mathcal{K}_\sigma$. Moreover,

$$m(J) = \sigma\, 2^{-n} = \frac{\sigma}{4} 2^{-(n-2)} \geq \frac{\sigma}{4} m(I).$$

This proves $\mathcal{K}_\sigma$ is porous. $\square$

Remark 4.7. When $\sigma = 2^{-p}$, one can interpret $\mathcal{K}_\sigma$ symbolically as the set of all binary angles $0.t_1t_2t_3\cdots$ in $\mathbb{T}$ such that if $t_jt_{j+1}\cdots t_{j+p-1}$ is the first occurrence of p consecutive 0's or 1's, then $j = 1$ or $j = 2$. This description allows a more combinatorial approach to the fact that the Hausdorff dimension of $\mathcal{K}_\sigma$ tends to 1 as σ tends to 0.

The rest of this section will be devoted to the proof of Theorem 1.3, which generalizes Theorem 1.2 to all tuned images of the real slice. Consider a hyperbolic component $H \neq H_0$ with the associated tuning maps $\iota_H : M \hookrightarrow M$ on the parameter plane and $A_H : \mathbb{T} \to \mathbb{T}$ on the external angles (see §2 and recall that A_H is 2-valued at every dyadic rational). Let $\eta_H := \iota_H[-2, 1/4]$ be the tuned image of the real slice of M. Then, if $R_M(t)$ lands at $c \in [-2, 1/4]$, $R_M(A_H(t))$ lands at $\iota_H(c) \in \eta_H$. Of course, not every parameter ray landing on η_H comes from tuning. Along η_H there are many "branch points" at which more than two parameter rays land. All such branch points are Misiurewicz parameters and hence the external angles of the rays landing on them are rational with even denominator (a countable set). It follows that such rays can be completely ignored when it comes to computing Hausdorff dimensions.

Recall from §2 that the image $K = A_H(\mathbb{T})$ is a Cantor set of Hausdorff dimension $1/\operatorname{per}(H)$ which is invariant under the two contractions Λ_0 and Λ_1. It is easy to see that $A_H : [0,1] \to [\theta_-(H), \theta_+(H)]$ is the right inverse of a "devil's staircase." More precisely, there exists a continuous non-decreasing map $\psi_H : [\theta_-(H), \theta_+(H)] \to [0,1]$ which maps every gap of K to a well-defined dyadic rational, mapping K onto $[0,1]$, such that $\psi_H \circ A_H(t) = t$ for all t.

The proof of Theorem 1.3 will depend on the following

Lemma 4.8. *The map $\psi_H : [\theta_-(H), \theta_+(H)] \to [0,1]$ is Hölder continuous of exponent $1/\operatorname{per}(H)$.*

Proof. Let $p = \operatorname{per}(H)$, $\theta_-(H) = 0.\overline{\theta_0}$, and $\theta_+(H) = 0.\overline{\theta_1}$ as in (2.1). Pick any two points $a < b$ in $[\theta_-(H), \theta_+(H)]$. We may assume that $\psi_H(a) < \psi_H(b)$ since otherwise $\psi_H(a) = \psi_H(b)$ and there is nothing to prove. Let $a' := \inf(K \cap [a, \theta_+(H)])$ and similarly define $b' := \sup(K \cap [\theta_-(H), b])$, and note that $a \le a' < b' \le b$. Moreover, since ψ_H is constant on each gap of K, we have $\psi_H(a') = \psi_H(a)$ and $\psi_H(b') = \psi_H(b)$. Expand a' and b' in base 2 as

$$a' = 0.\theta_{t_1}\theta_{t_2}\cdots \quad \text{and} \quad b' = 0.\theta_{s_1}\theta_{s_2}\cdots ,$$

where $t_i, s_i \in \{0,1\}$ are uniquely determined by a' and b'. Set $t := 0.t_1t_2\cdots$ and $s := 0.s_1s_2\cdots$ in base 2, so that $t = \psi_H(a')$ and $s = \psi_H(b')$. Note that by the choice of a', if t happens to be a dyadic rational, then the binary expansion $0.t_1t_2\cdots$ is the one which terminates with a string of 0's. Similarly, by the choice of b', if s is dyadic, then the binary expansion $0.s_1s_2\cdots$ is the one which terminates with a string of 1's. With this observation in mind, let $j \ge 1$ be the smallest integer such that $t_j \neq s_j$. Then,

$$\psi_H(b) - \psi_H(a) = s - t \le (\text{const.})2^{-j}. \tag{4.6}$$

On the other hand, the open interval $]t, s[$ contains the dyadic point $r = 0.t_1\cdots t_{j-1}1\overline{0}$, so $]a', b'[$ contains the gap

$$\big]0.\theta_{t_1}\cdots\theta_{t_{j-1}}\theta_0\overline{\theta_1}\,,\, 0.\theta_{t_1}\cdots\theta_{t_{j-1}}\theta_1\overline{\theta_0}\big[$$

whose endpoints are the two values of $A_H(r)$. Since the length of this gap is bounded below by $(\text{const.})2^{-jp}$, it follows that

$$b - a \ge b' - a' \ge (\text{const.})2^{-jp}. \tag{4.7}$$

Combining the two inequalities (4.6) and (4.7), we obtain

$$\psi_H(b) - \psi_H(a) \le (\text{const.})(b-a)^{1/p},$$

which proves the lemma. □

Proof of Theorem 1.3. Let $F \subset \mathcal{R}$ denote the (conjecturally empty) set of angles whose corresponding parameter rays do not land. We prove that the set $A_H(\mathcal{R} \smallsetminus F) \subset K$ has Hausdorff dimension $1/p$. Theorem 1.3 would follow since any other ray landing on η_H which is not in $A_H(\mathcal{R} \smallsetminus F)$ must be rational of even denominator and there are countably many such rays (compare the discussion before Lemma 4.8).

Since $\dim_H(\mathcal{R} \smallsetminus F) = 1$ by Theorem 1.2, Frostman's Lemma shows that for any $0 < \delta < 1$ there exists a Borel probability measure μ supported in $\mathcal{R} \smallsetminus F$ such that $\mu(I) \le (\text{const.})\, m(I)^\delta$ for all intervals I. Let $\nu := (A_H)_*\mu$ be the push-forward measure supported in $A_H(\mathcal{R} \smallsetminus F)$. Take any interval $J \subset [\theta_-(H), \theta_+(H)]$ and let $I = \psi_H(J)$. Then $\nu(J) = \mu(I) \le (\text{const.})\, m(I)^\delta$. By Lemma 4.8, $m(I) \le (\text{const.})\, m(J)^{1/p}$. It follows that $\nu(J) \le (\text{const.})\, m(J)^{\delta/p}$. By another application of Frostman's Lemma, we conclude that

$$\frac{1}{p} = \dim_{\mathrm{H}}(A_H(\mathbb{T})) \ge \dim_{\mathrm{H}}(A_H(\mathcal{R} \smallsetminus F)) \ge \frac{\delta}{p}\,.$$

Letting $\delta \to 1$, we obtain the result. □

5. Critically non-recurrent real quadratics

The results of the preceding section can be used to obtain, with minimal effort, dimension estimates in the parameter space of real quadratic polynomials. As an example, a difficult theorem of Jakobson [**J**] asserts that the nowhere dense set $\partial M \cap \mathbb{R}$ has positive linear measure, hence full Hausdorff dimension 1. It is interesting to see that this last statement also follows from Theorem 1.2. To this end, let us recall the following result from the theory of univalent maps (see [**Mk**] or [**P**] for a proof):

Makarov Dimension Theorem. *For any univalent map $\phi : \mathbb{D} \to \mathbb{C}$ and any Borel set $X \subset \mathbb{T}$ with $\dim_{\mathrm{H}}(X) = \delta$, we have the estimates*

$$\dim_{\mathrm{H}}(\phi(X)) > \begin{cases} \dfrac{\delta}{2} & \text{if} \quad 0 < \delta < \dfrac{11}{12} \\[2ex] \dfrac{\delta}{1+\sqrt{12(1-\delta)}} & \text{if} \quad \dfrac{11}{12} < \delta < 1 \end{cases}$$

In our case, after removing a set F of capacity zero (conjecturally empty) from $\mathcal{R}$, we can assume that all the parameter rays with angles in $\mathcal{R} \smallsetminus F$ land at a point of $\partial M \cap \mathbb{R}$. The set $\mathcal{R} \smallsetminus F$ still has Hausdorff dimension 1, so by Makarov Dimension Theorem the image $\pi(\mathcal{R} \smallsetminus F) \subset \partial M \cap \mathbb{R}$ has dimension at least 1. It follows that $\dim_{\mathrm{H}}(\partial M \cap \mathbb{R}) = 1$.

It was pointed out to me by S. Smirnov that a more careful application of the above argument, combined with the dimension estimates in §4, shows the existence of a set of full Hausdorff dimension in $\partial M \cap \mathbb{R}$ which consists only of critically non-recurrent quadratics. Recall that $c \in \mathbb{C}$ is called a *critically non-recurrent* parameter if the critical point 0 of the quadratic Q_c does not belong to the closure of its forward orbit $\{Q_c^{\circ n}(0)\}_{n \geq 1}$. (Caution: In the theory of interval maps, the term "Misiurewicz" is often used for "critically non-recurrent non-hyperbolic." Following the standard terminology of complex dynamics, we have used the term "Misiurewicz" in the more restricted sense of "critically finite non-hyperbolic.") It has been shown by D. Sands that the set of critically non-recurrent parameters in $\partial M \cap \mathbb{R}$ has linear measure zero [**Sa**]. Here we prove a complement to his result by showing that this set has full Hausdorff dimension.

Proof of Theorem 1.4. Fix a small $\sigma > 0$ such that $(1-\sigma)/2 \notin \mathcal{R}$, and consider the compact set $\mathcal{K}_\sigma$ defined in (4.1). By (4.2), $\mathcal{K}_\sigma \cap [(1-\sigma)/2, 1/2] \subset \mathcal{R}$, so $\mathcal{N}_\sigma := \pi(\mathcal{K}_\sigma \cap [(1-\sigma)/2, 1/2])$ is a well-defined subset of $\partial M \cap \mathbb{R}$. We claim that every parameter $c \in \mathcal{N}_\sigma$ is critically non-recurrent. Assuming this is false, take a critically recurrent parameter c in $\mathcal{N}_\sigma$ and a sequence $n_j \to \infty$ such that $Q_c^{\circ n_j}(c) \to c$ as $j \to \infty$. By Theorem 3.3 the critical value c of Q_c is the landing point of the two dynamic rays at angles $\pm\tau(c)$, so $Q_c^{\circ n_j}(c)$ is the landing point of the rays at angles $\pm\mathbf{d}^{\circ n_j}(\tau(c))$. Since $Q_c^{\circ n_j}(c) \to c$, an easy exercise shows that $\mathbf{d}^{\circ n_j}(\tau(c)) \to \pm\tau(c)$. In particular,

$$\lim_{j\to\infty} \left| \mathbf{d}^{\circ n_j}(\tau(c)) - \frac{1}{2} \right| = \frac{1}{2} - \tau(c). \tag{5.1}$$

On the other hand, the definition of $\mathcal{N}_\sigma$ and the fact that $\tau = \pi^{-1}$ shows that $\tau(c) \in \mathcal{K}_\sigma \cap [(1-\sigma)/2, 1/2]$. As in the proof of Theorem 1.2, it follows that

$$\left| \mathbf{d}^{\circ n}(\tau(c)) - \frac{1}{2} \right| \geq \frac{\sigma}{2} \geq \frac{1}{2} - \tau(c) \tag{5.2}$$

for all $n \geq 1$. Comparing (5.1) and (5.2), we obtain $\tau(c) = (1-\sigma)/2$. This is a contradiction since $\tau(c) \in \mathcal{R}$ and $(1-\sigma)/2 \notin \mathcal{R}$.

Now choose a sequence $\sigma_n \to 0$ subject to the condition $(1-\sigma_n)/2 \notin \mathcal{R}$; this is possible since $\mathcal{R}$ is nowhere dense. Use Lemma 4.3 and Makarov Dimension Theorem to deduce $\lim_{n\to\infty} \dim_{\mathrm{H}} \mathcal{N}_{\sigma_n} = 1$. This proves Theorem 1.4 since by the above argument the set of critically non-recurrent parameters in $\partial M \cap \mathbb{R}$ contains $\mathcal{N}_{\sigma_n}$ for all n. □

6. Biaccessibility in real quadratics

Let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial with connected Julia set $J(P)$. A point $z \in J(P)$ is called *biaccessible* if there are two or more dynamic rays landing at z. We denote the set of all such points by $\tilde{J}(P)$. It is known that the harmonic measure of $\tilde{J}(P)$ is zero unless P is affinely conjugate to a Chebyshev polynomial for which $J(P)$ is an interval and hence $\tilde{J}(P)$ has full harmonic measure 1. This was proved by the author for all locally-connected and some non locally-connected quadratic Julia sets [**Za**]. Proofs for the general case were later found independently by S. Smirnov [**Sm**] and A. Zdunik [**Zd**].

Naturally, one would like to know about the Hausdorff dimension of the set of angles landing at biaccessible points of a non-Chebyshev polynomial. In this section we prove Theorem 1.5 which shows how this dimension can be effectively estimated, at least for real quadratics. Let $c \in [-2, 1/4]$, $J_c = J(Q_c)$, $\tilde{J}_c = \tilde{J}(Q_c)$, and let $\beta > 0$ be the fixed point of Q_c at which the dynamic ray $R_c(0)$ lands. Define

$$\begin{aligned} S_c &:= \{t \in \mathbb{T} : R_c(t) \text{ lands on } [-\beta, \beta]\}, \\ B_c &:= \{t \in \mathbb{T} : R_c(t) \text{ lands at a point of } \tilde{J}_c\}. \end{aligned}$$

It is not hard to prove that

$$\tilde{J}_c = \bigcup_{n \geq 0} Q_c^{-n}(J_c \cap]-\beta, \beta[).$$

(see [**Za**]), so that

$$\dim_{\mathrm{H}}(\tilde{J}_c) = \dim_{\mathrm{H}}(J_c \cap]-\beta, \beta[). \tag{6.1}$$

By ignoring the countable set of angles of dynamic rays landing at β and its iterated preimages, it also follows that

$$\dim_{\mathrm{H}}(B_c) = \dim_{\mathrm{H}}(S_c). \tag{6.2}$$

In order to prove Theorem 1.5, we link the set S_c to the family of compact sets $\{\mathcal{K}_\sigma\}_{\sigma > 0}$ introduced in §4. Once this connection is found, we simply use the dimension estimates in §4 to find bounds on the dimension of S_c.

Lemma 6.1 (Douady)**.** *If $c \in \partial M \cap \mathbb{R}$, then*

$$S_c = \{t \in \mathbb{T} : \mathbf{d}^{\circ n}(t) \notin]\tau(c), 1-\tau(c)[\text{ for all } n \geq 1\}. \tag{6.3}$$

Proof. We follow the argument in [**D3**]. Let E denote the closed set defined by the right side of (6.3). If $t \in S_c$, then $R_c(t)$ lands at some $z \in [-\beta, \beta]$, and so for $n \geq 1$ the dynamic ray $R_c(\mathbf{d}^{\circ n}(t))$ lands at $Q_c^{\circ n}(z)$. Evidently, $Q_c^{\circ n}(z) \geq c$, and since $J_c \cap]c, r_c[= \emptyset$, we actually have the stronger inequality $Q_c^{\circ n}(z) \geq r_c$. Since r_c is the landing point of the rays $R_c(\pm\tau(c))$, it follows that $\mathbf{d}^{\circ n}(t) \notin]\tau(c), 1-\tau(c)[$ for all $n \geq 1$, which implies $t \in E$. This proves $S_c \subset E$. On the other hand, let $]s, t[$ be a connected component of $\mathbb{T} \smallsetminus E$. Then there exists an $n \geq 1$ so that the iterate $\mathbf{d}^{\circ n}$ maps $]s, t[$ homeomorphically to $]\tau(c), 1-\tau(c)[$, but $\mathbf{d}^{\circ j}(]s,t[) \not\subset]\tau(c), 1-\tau(c)[$ if $1 \leq j \leq n-1$. It easily follows by a recursive argument that for every $0 \leq j \leq n$ the dynamic rays $R_c(\mathbf{d}^{\circ j}(s))$ and $R_c(\mathbf{d}^{\circ j}(t))$ land on $[-\beta, \beta]$. In particular, $s, t \in S_c$. Since E has no interior, the set of all such s, t is dense in E, proving $E \subset S_c$. □

Corollary 6.2. *For every $c \in \partial M \cap \mathbb{R}$, we have $S_c = \mathcal{K}_\sigma$, where $\sigma = 1 - 2\tau(c)$ and where $\mathcal{K}_\sigma$ is the compact set defined in (4.1).*

Proof. This simply follows from the previous lemma and the definition of $\mathcal{K}_\sigma$. □

Corollary 6.3. *For $c \in \partial M \cap [-2, -1.75]$, we have*

$$\dim_{\mathrm{H}}(B_c) \geq 1 - \frac{1}{\log 2} \log \left(\frac{16\tau(c) - 5}{32\tau(c) - 13} \right)$$

Observe that $c = -1.75$ is the landing point of the parameter ray $R_M(3/7)$, i.e., the root point of the unique real hyperbolic component of period 3.

Proof. If $c = -2$, then $\tau(c) = 1/2$ and the inequality is evident. So assume $3/7 \leq \tau(c) < 1/2$, and choose a dyadic $\sigma = 2^{-p} \leq 1/4$ such that $\sigma/2 < 1 - 2\tau(c) \leq \sigma$. Then, by Corollary 6.2, $S_c \supset \mathcal{K}_\sigma$. By Lemma 4.5,

$$\begin{aligned} \dim_{\mathrm{H}}(S_c) \geq \dim_{\mathrm{H}}(\mathcal{K}_\sigma) &\geq 1 - \frac{1}{\log 2} \log \left(\frac{3 - 4\sigma}{3 - 8\sigma} \right) \\ &\geq 1 - \frac{1}{\log 2} \log \left(\frac{16\tau(c) - 5}{32\tau(c) - 13} \right). \end{aligned}$$

Since $\dim_{\mathrm{H}}(S_c) = \dim_{\mathrm{H}}(B_c)$ by (6.2), we obtain the result. □

We can improve the above estimate and at the same time extend it to all real parameters in $[-2, -1.75]$. To this end, we need a few preliminary remarks.

Recall that $\mathcal{H}$ is the collection of all real hyperbolic components of M. Let $H \in \mathcal{H}$ and let H_1 be the 1/2-satellite of H (see §2 for the definition). Then $H_1 \in \mathcal{H}$ and in the passage from H to H_1 the attracting cycle of Q_c undergoes a period-doubling bifurcation. We express the relationship between H_1 and H by writing $H_1 \lhd H$.

Lemma 6.4. *Let $H, H_1 \in \mathcal{H}$ have root points c and c_1, and assume $H_1 \lhd H$. Consider any real parameter $c_0 \in H$. Then*

(i) *$S_{c_0} = S_c$ and $B_{c_0} = B_c$.*
(ii) *$S_{c_1} \smallsetminus S_c$ is countably infinite.*

Proof. Statement (i) follows from the well-known fact that the quadratics Q_c and Q_{c_0} are combinatorially equivalent, so that $R_c(t)$ and $R_c(s)$ land at a common point in the dynamic plane of Q_c if and only if $R_{c_0}(t)$ and $R_{c_0}(s)$ land at a common point in the dynamic plane of Q_{c_0} (see for example [**M**]). To see (ii), first note $\tau(c_1) > \tau(c)$ so that $S_{c_1} \supset S_c$ by Lemma 6.1. Also note that $c_1 = \iota_H(-0.75)$, where -0.75 is the root point of the 1/2-satellite of the main cardioid. Hence, by the discussion of tuning in §2, the filled Julia set K_{c_1} is obtained from K_{c_0} by replacing each bounded Fatou component of Q_{c_0} by a copy of the filled Julia set $K_{-0.75}$. A brief inspection shows that

$$S_{-0.75} = \{0, 1/2\} \cup \left\{ \pm \frac{1}{6 \cdot 2^n} \right\}_{n \geq 0} \cup \left\{ \frac{1}{2} \pm \frac{1}{6 \cdot 2^n} \right\}_{n \geq 0},$$

so the set $S_{-0.75}$ is countably infinite. Thus, the process of constructing K_{c_1} from K_{c_0} adds only countably many new angles to $S_{c_0} = S_c$, and we obtain the result. The details are straightforward and will be left to the reader. $\square$

From Lemma 6.1 and Lemma 6.4(i) one immediately obtains the monotonicity of the family S_c. In particular,

Corollary 6.5. *The dimension function* $c \mapsto \dim_{\mathrm{H}}(B_c) = \dim_{\mathrm{H}}(S_c)$ *is monotonically decreasing on* $[-2, 1/4]$.

Given any $H \in \mathcal{H}$, one can form the infinite cascade

$$\cdots \lhd H_n \lhd \cdots \lhd H_2 \lhd H_1 \lhd H \tag{6.4}$$

of period-doubling bifurcation components originating from H. Then the root c_n of H_n converges exponentially fast to a limit $c^*(H) \in \partial M \cap \mathbb{R}$ as $n \to \infty$ (compare [**Ly2**]).

Definition 6.6. We call the parameter $c^*(H)$ *the main Feigenbaum point associated to* H.

As an example, the much studied main Feigenbaum point associated to the main cardiod H_0, which we denote by c_{Feig}, is approximately located at -1.401155 (compare Fig. 4).

It is not hard to see using the properties of the tuning map that for every $H \in \mathcal{H}$,

$$c^*(H) = \iota_H(c_{\mathrm{Feig}}).$$

Note that the assignment $H \mapsto c^*(H)$ is far from being one-to-one. In fact, $c^*(H) = c^*(H_1)$ whenever $H_1 \lhd H$, and so all the components in (6.4) have the same main Feigenbaum point associated to them.

Using the fact that $S_{1/4} = \{0, 1/2\}$ and applying Lemma 6.4 repeatedly, we obtain

Corollary 6.7. *For* $c_{\mathrm{Feig}} < c \leq 1/4$, $\dim_{\mathrm{H}}(B_c) = \dim_{\mathrm{H}}(S_c) = 0$.

The following definition will be used in the proof of Theorem 1.5.

Definition 6.8. For $c \in [-2, 1/4]$, we define

$$\rho(c) := \begin{cases} \tau(c^*(H)) & \text{if } c \in \overline{H} \text{ for some } H \in \mathcal{H} \\ \tau(c) & \text{otherwise} \end{cases}$$

In other words, if c is not hyperbolic or parabolic, then $\rho(c) = \tau(c)$, but if c is hyperbolic or parabolic, then $\rho(c)$ is the external angle of the associated main Feigenbaum point. Clearly $\rho(c) \geq \tau(c)$ whenever $c \in \partial M \cap \mathbb{R}$.

We are now ready to prove Theorem 1.5 cited in the introduction.

Proof of Theorem 1.5. For $-2 < c \leq -1.75$, define

$$\ell(c) := 1 - \frac{1}{\log 2} \log \left(\frac{16\rho(c) - 5}{32\rho(c) - 13} \right). \tag{6.5}$$

If c does not belong to the closure of any real hyperbolic component, then $\ell(c)$ coincides with the lower bound in Corollary 6.3, so $\dim_{\mathrm{H}}(B_c) \geq \ell(c)$. If, on the other hand, $c \in \overline{H}$ for some $H \in \mathcal{H}$, then we may consider the cascade of period doubling bifurcations (6.4), with the root points c_n. An inductive application of Lemma 6.4 then shows that $S_{c_n} \smallsetminus S_c$ is countably infinite for every n. Hence, by Corollary 6.3,

$$\dim_{\mathrm{H}}(B_c) = \dim_{\mathrm{H}}(S_c) = \dim_{\mathrm{H}}(S_{c_n}) = \dim_{\mathrm{H}}(B_{c_n}) \geq 1 - \frac{1}{\log 2} \log\left(\frac{16\tau(c_n) - 5}{32\tau(c_n) - 13}\right).$$

But $c_n \to c^*(H)$ implies $\tau(c_n) \to \tau(c^*(H)) = \rho(c)$ by Lemma 3.4, so that the right side of the above inequality tends to $\ell(c)$ as $n \to \infty$, implying $\dim_{\mathrm{H}}(B_c) \geq \ell(c)$. By Lemma 4.6 and Corollary 6.2, $\dim_{\mathrm{H}}(S_c) < 1$. Combining these two inequalities, we obtain

$$\ell(c) \leq \dim_{\mathrm{H}}(B_c) < 1.$$

The fact that $-2 < c \leq -1.75$ implies $4/9 < \rho(c) < 1/2$, and a brief computation shows that $\ell(c) > 0$. That $\lim_{c \searrow -2} \ell(c) = 1$ follows from $\lim_{c \searrow -2} \rho(c) = 1/2$. □

Remark 6.9. I do not know if $\dim_{\mathrm{H}}(B_{c_{\mathrm{Feig}}}) > 0$. More generally, I do not know how to obtain explicit lower bounds for $\dim_{\mathrm{H}}(B_c)$ for $-1.75 < c \leq c_{\mathrm{Feig}}$. On the other hand, the preceding arguments show $\dim_{\mathrm{H}}(B_c) < 1$ for $-2 < c \leq 1/4$ and I do not know if this phenomenon is general in the quadratic family. In other words, does there exist a complex parameter $c \neq -2$ for which $\dim_{\mathrm{H}}(B_c) = 1$?

Finally, it is quite easy to prove an analogue of Theorem 1.5 for the set of biaccessible points in the Julia set itself.

Corollary 6.10. *Let $-2 < c \leq 1/4$ and let $\tilde{J}_c$ denote the set of biaccessible points in the Julia set J_c. If J_c is full, then $\dim_{\mathrm{H}}(\tilde{J}_c) = 1$. On the other hand, if J_c is not full and $c \leq -1.75$, then*

$$0 < \ell'(c) \leq \dim_{\mathrm{H}}(\tilde{J}_c) \leq 1. \tag{6.6}$$

where $\ell'(c)$ is an explicit constant which tends to 1 as c tends to -2. In particular,

$$\lim_{c \searrow -2} \dim_{\mathrm{H}}(\tilde{J}_c) = \dim_{\mathrm{H}}(\tilde{J}_{-2}) = 1. \tag{6.7}$$

Proof. If J_c is full, then $J_c \supset [-\beta, \beta]$ and hence $\dim_{\mathrm{H}}(\tilde{J}_c) = 1$ by (6.1). If J_c is not full and $-2 < c \leq -1.75$, then we can take $\ell'(c) := \lambda(\ell(c))$, where $\ell(c)$ is given by (6.5) and $\lambda = \lambda(\delta)$ is the lower bound function given by Makarov Dimension Theorem. The upper bound in (6.6) follows from (6.1). Finally, $\lim_{c \searrow -2} \dim_{\mathrm{H}}(B_c) = 1$ together with Makarov Dimension Theorem shows that $\liminf_{c \searrow -2} \dim_{\mathrm{H}}(\tilde{J}_c) \geq 1$, which combined with $\dim_{\mathrm{H}}(\tilde{J}_c) \leq 1$ proves (6.7). □

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