Abstract

In this work we deal with non-discrete subgroups of $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$, the group of orientation-preserving analytic diffeomorphisms of the circle. If Γ is such a group, we consider its natural diagonal action $\widetilde{\Gamma}$ on the $n-$dimensional torus $\mathbb{T}^n$. It is then obtained a complete characterization of these groups Γ whose corresponding $\widetilde{\Gamma}-$action on $\mathbb{T}^n$ is not piecewise ergodic (cf. Introduction) for all $n \in \mathbb{N}$ (cf. Theorem A). Theorem A can also be interpreted as an extension of Lie's classification of Lie algebras on $\mathbb{S}^1$ to general non-discrete subgroups of $\mathbb{S}^1$.

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The multiple ergodicity of non-discrete subgroups of $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$

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1 Introduction

Consider a finite dimensional Lie group G and denote by Γ a finite generated subgroup of G. The subgroups Γ as above are naturally divided into two categories according to whether or not they are discrete. If Γ is not discrete, then its topological closure is again a Lie group with non-trivial Lie algebra. Furthermore, when G acts on some manifold M, by restricting this action, we obtain a natural action of the subgroup Γ on M and the fact that the Lie algebra of the closure of Γ may be non-trivial has nice consequences on the dynamics of the $\Gamma-$action.

We shall be concerned with the group $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$ of orientation-preserving real analytic diffeomorphisms of the circle $\mathbb{S}^1$ (however the results presented in this paper can easily be generalized to the group of analytic diffeomorphisms of $\mathbb{S}^1$ including orientation-reversing diffeomorphisms). Equipped with its natural analytic topology (cf. Section 2.1), $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$ becomes an infinite dimensional topological group. Nonetheless there is evidence that $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$ shares dynamical properties with finite dimensional Lie groups and it is natural to consider discrete and non-discrete subgroups of $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$.

A subgroup of $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$ is said to be non-discrete if it contains a non-trivial sequence of elements converging to the identity. While conjugacies of Fuchsian groups constitute the main source of examples of discrete subgroups of $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$, groups admitting a finite generating set close to the identity furnish several examples of non-discrete subgroups of $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$. Precisely a non-solvable group possessing a set of generators close to the identity is non-discrete (cf. Theorem (2.1) for an accurate statement). Because these groups provide most examples of non-discrete subgroups of $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$, we shall restrict our attention to them. Thus the main purpose of this article is to completely describe the multiple ergodicity of subgroups of $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$ possessing a finite generating set close to the identity. Our description also encompasses the analogous "topological" case, namely it characterizes the structure of the minimal sets for the natural diagonal action of the subgroup in question on the torus of dimension $n \in \mathbb{N}$ (cf.

below). The last section however gives further comments and extensions of the main results to the general case of non-discrete subgroups of $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$.

Before stating our main result, we need to recall some classical concepts used in Ergodic Theory as well as to introduce some terminology.

Given a manifold M, let μ be the *normalized* Lebesgue measure on M. Suppose that Γ is a group of measurable transformations acting on M and preserving the class of μ, that is, if γ belongs to Γ, then $\gamma^*\mu$ and μ have the same sets of measure *zero* (we say that μ is quasi-invariant under Γ). The group Γ is called ergodic if every Borel set $\mathcal{B}$ $\mu-$a.e. invariant under Γ satisfies $\mu(\mathcal{B}) = 0$ or $\mu(\mathcal{B}) = 1$.

To abridge notations, we shall introduce the following definition:

Definition *Consider a group Γ acting on a manifold M endowed with the normalized Lebesgue measure μ. Γ will be called piecewise ergodic if and only if:*

a) There is a finite number of open sets $U_1, \ldots, U_l$ which are invariant under Γ and such that the union $\bigcup_{i=1}^{l} U_i$ has total $\mu-$measure.

b) The restriction of the action of Γ to each U_i is ergodic (w.r.t. the restriction of μ) and minimal (i.e. all orbits are dense). Furthermore each U_i has a finite number of connected components which are permuted by the action of Γ.

Next consider the direct product $M^{\times n} = M \times \cdots \times M$ of $n \in \mathbb{N}$ copies of M ($M^{\times 1} = M$). Since Γ acts on M, it also acts on $M^{\times n}$ through the natural *diagonal action* and we denote by $\widetilde{\Gamma}$ the corresponding group of transformations acting diagonally on $M^{\times n}$. The *multiple ergodicity* of Γ is a classical object of Ergodic Theory which consists of analysing the ergodicity of $\widetilde{\Gamma}$ on $M^{\times n}$ for all $n \in \mathbb{N}$ (cf. for instance [C-S-V]).

Finally notice that there are two well-known distinguished subgroups of $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$, namely tha affine group $\mathrm{Aff}(\mathbb{R})$ and the projective group $\mathrm{PSL}(2, \mathbb{R})$. Furthermore we also have "finite coverings" of these groups whcih are defined as follows. Let $\pi_{k_0} : \mathbb{S}^1 \to \mathbb{S}^1$ be a covering map of degree $k_0 \in \mathbb{N}$. Every diffeomorphism f in $\mathrm{Aff}(\mathbb{R})$ (resp. $\mathrm{PSL}(2, \mathbb{R})$) induces other diffeomorphisms f_{π_k} of $\mathbb{S}^1$ through the equation $f \circ \pi_{k_0} = \pi_{k_0} \circ f_{\pi_k}$. Fixed π_{k_0}, the set of diffeomorphisms f_{π_k}, $f \in \mathrm{Aff}(\mathbb{R})$ (resp. $\mathrm{PSL}(2, \mathbb{R})$), obtained as above constitutes a subgroup of $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$ which will be denoted by $\mathrm{Aff}_{k_0}(\mathbb{R})$ (resp. $\mathrm{PSL}_{k_0}(2, \mathbb{R})$). The groups $\mathrm{Aff}_{k_0}(\mathbb{R})$ (resp. $\mathrm{PSL}_{k_0}(2, \mathbb{R})$), $k_0 \in \mathbb{N}$, will be called the *finite coverings* of $\mathrm{Aff}(\mathbb{R})$ (resp. $\mathrm{PSL}(2, \mathbb{R})$). Besides a subgroup of $\mathrm{Aff}_{k_0}(\mathbb{R})$ (resp. $\mathrm{PSL}_{k_0}(2, \mathbb{R})$) will be referred to as *a finite covering of a subgroup* of $\mathrm{Aff}(\mathbb{R})$ (resp. $\mathrm{PSL}(2, \mathbb{R})$).

Using this convention we shall prove the following result:

Theorem A *There exists an open neighborhood $\mathcal{U}$ of the identity in* $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$ *such that, if* $\Gamma \subset \mathrm{Diff}^{\omega}(\mathbb{S}^1)$ *is an infinite group generated by a finte set of diffeomorphisms S with $S \subset \mathcal{U} \subset$* $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$, *then one has:*

1. *Γ is piecewise ergodic unless Γ is a finite extension of $\mathbb{Z}$.*

2. *If Γ is not a finite extension of $\mathbb{Z}$, then Γ is piecewise ergodic on $\mathbb{T}^2$ unless Γ is Abelian.*

3. *If Γ is not Abelian, then Γ is piecewise ergodic on $\mathbb{T}^3$ unless Γ is conjugate to a finite covering of a subgroup of the affine group* $\text{Aff}(\mathbb{R})$.
4. *If Γ is not conjugate to a finite covering of a subgroup of the affine group* $\text{Aff}(\mathbb{R})$*, then Γ is piecewise ergodic on $\mathbb{T}^4$ unless Γ is conjugate to a finite covering of a subgroup of* $\text{PSL}(2,\mathbb{R})$
5. *If Γ is not conjugate to a finite covering of a subgroup of* $\text{PSL}(2,\mathbb{R})$ *then Γ is piecewise ergodic on $\mathbb{T}^n$ for all $n \in \mathbb{N}$.*

Remark It will be seen that, indeed, the complement of the open sets U_i's involved in the definition of piecewise ergodic is a finite union of "properly embedded" open submanifolds, having also the local behavior of an analytic set (cf. Theorem (6.1) and Lemma (6.3)). These sets and the open sets U_i's are invariant under $\widetilde{\Gamma}$. Moreover they are minimal in the sense that their points have dense orbits.

The minimal sets of the action of a group Γ as above on $\mathbb{S}^1$ were characterized by E. Ghys who proved an "analytic version" of an unpublished theorem due to Duminy (cf. [Gh]). The corresponding structure of the ergodic components was settled later in [Reb1].

The characterization presented in the theorem above is clearly sharp: a finite extension of a "south pole - north pole" diffeomorphism cannot be piecewise ergodic; the group of rotations is ergodic, Abelian but not piecewise ergodic on $\mathbb{T}^2$; the affine group is not piecewise ergodic on $\mathbb{T}^3$ since it preserves the level sets of the function from $\mathbb{T}^3$ to $\mathbb{S}^1$ obtained through the cross-ratio where the "infinity" is fixed; finally a subgroup of $\text{PSL}(2,\mathbb{R})$ cannot be piecewise ergodic on $\mathbb{T}^4$ since it preserves the level sets of the cross-ratio viewed as a function from $\mathbb{T}^4$ to $\mathbb{S}^1$.

We mention that results on multiple ergodicity for subgroups of $\text{Diff}^{\omega}(\mathbb{S}^1)$ are likely to have interesting applications. For instance already the piecewise ergodicity on $\mathbb{T}^2$ is intimately related to rigidity theorems for these groups (cf. [Reb2] for subgroups generated by diffeomorphisms close to the identity and [Tu] for Fuchsian groups).

Note also that the preceding theorem gives the first "large" class, for instance contained open sets, of finitely generated subgroups of $\text{Diff}^{\omega}(\mathbb{S}^1)$ known to be piecewise ergodic on $\mathbb{T}^n$ for all n. Obviously Fuchsian groups cannot satisfies such a property since they cannot be piecewise ergodic on $\mathbb{T}^4$. Furthermore, since there are non-Abelian free Fuchsian groups which are structurally stable, we see that the assumption on the existence of a generating set close to the identity cannot be dropped even "generically".

Notice that a subgroup of $\text{PSL}(2,\mathbb{R})$ having a common fixed point is necessarily conjugate to $\text{Aff}(\mathbb{R})$ and thus solvable. Hence we obtain the following consequence:

Corollary B *Assume Γ is as in Theorem A. Suppose in addition that Γ is non-solvable and possesses a fixed point p. Then Γ is piecewise ergodic on $\mathbb{T}^n$ for all $n \in \mathbb{N}$.*

This last result might be compared to the Theorem 1 of [B-L-L]. In that paper the authors studied the local pseudo-action of a non-solvable subgroup of $\text{Diff}(\mathbb{C},0)$ on the corresponding jet bundle so as to answer in the negative some questions concerning a certain type of differential equations.

Finally we would like to make some analogies between Theorem A and Lie's classification of finite-dimensional Lie algebras contained in the algebra of analytic vector fields on $\mathbb{S}^1$ (as well as the corresponding Lie groups). Recall that the Zassenhaus Lemma ensures the existence of a non-trivial Lie algebra for the topological closure of every non-nilpotent subgroup $\Gamma \subset G$ of a finite-dimensional Lie group G provided that Γ possesses a generating set close to the identity. If $\Gamma \subset G$ acts on the circle, then Lie's classification allows to recognize the correponding algebra and therefore the nature of the group Γ. Theorem A then shows that the infinite-dimensional topological group $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$ behaves in a coherent way as a Lie group from the above viewpoint. Actually, if $\Gamma \subset \mathrm{Diff}^{\omega}(\mathbb{S}^1)$ is generated by elements close to the identity, then either Γ is associated which one of the possible finite-dimensional Lie algebras (and hence with the corresponding Lie group) or "it is associate to an infinite-dimensional Lie algebra" which is reflected by the piecewise ergodicity of $\widetilde{\Gamma}$ on the product of n copies of $\mathbb{S}^1$ (for all $n \in \mathbb{N}$). In other words, thinking of piecewise ergodicity for all n as an analog of infinite-dimensional Lie algebra, Lie's classification may be interpreted as an "infinitesimal" version of Theorem A. In the case in which Γ is, in fact, related to a finite-dimensional Lie algebra, this algebra can be detected by considering the action on at most four copies of $\mathbb{S}^1$. This analogy is discussed further in the last section.

We finish this introduction by giving a brief outline of the structure of this paper. Let us begin by explaining the typical example of a piecewise ergodic action which was our initial motivation.

Notice that, unless otherwise stated, all groups considered through sections 3, 4, 5 and 6 are *non-solvable*. Actually *solvable* groups are treated only in Section 7 which can be read independently of the preceding sections.

First consider a non-countable group G of diffeomorphisms of a compact manifold M containing the flows of n analytic vector fields $X^1, \ldots, X^n$ defined on M. Suppose in addition that the dimension of M is exactly n and the vector fields $X^1, \ldots, X^n$ are linearly independent at generic points. The dynamics of G is therefore "almost transitive" in the sense that it is transitive restricted to a finite number of open sets (which are the open sets consisting of the points of M at which the $X^1, \ldots, X^n$ are linearly independent). Denoting by U_i these sets, it results that the action of G restricted to a certain U_i is transitive and thus minimal and ergodic. Next consider a *countable* subgroup Γ of G which is *dense* in G for the $C^{\infty}-$topology. Obviously Γ cannot be transitive on the U_i's since it is countable, however we can ask about properties like ergodicity and density of orbits which make sense for countable groups. From the transitivity of G on U_i is very easy to deduce that Γ is ergodic and minimal on U_i.

Now suppose we are given a subgroup Γ of $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$ and consider the corresponding group $\widetilde{\Gamma}$ acting on the torus $\mathbb{T}^n$. Suppose that $\widetilde{\Gamma}$ contains the flow of a vector field X defined on $\mathbb{T}^n$ (so that $\widetilde{\Gamma}$ is not countable). At least under "generic" assumptions, it is reasonable to wonder that the action of $\widetilde{\Gamma}$ on X will produce n vector fields linearly independent at generic points of $\mathbb{T}^n$. Actually we may expect that, if we cannot produce these n vector fields, then the action of $\widetilde{\Gamma}$ has some kind of "symmetry" (or "infinitesimal symmetry"). It is also reasonable to expect that these "symmetric" cases can be well understood.

Finally considering the general case of a group Γ as in the statement of Theorem A, it is possible to show that the action of $\widetilde{\Gamma}$ can be *locally* approximated by local vector fields around any point in the complement of a certain set $\mathrm{P}(\widetilde{\Gamma})$ of $\mathbb{T}^n$ (cf. Section 2.3). In the possible absence of a global vector field, we are led to work out with these local ones. The fact that our vector fields are local rather than global, leads of course to a number of additional analytic difficulties. In fact our first conclusions will be of local nature and therefore they will require further work to imply global conclusions. Also an asymptotic analyse of the behavior of the dynamics when one approaches $\mathrm{P}(\widetilde{\Gamma})$ will be needed in Section 6.

The plan of this article is as follows. Section 2 contains the basic definitions as well as a brief summary of some general results which are often used in the course of the work.

In Section 3 we begin to discuss the local action of $\widetilde{\Gamma}$ on local vector fields. Recall that we expect to produce other vector fields linearly independent with the initial one. Since our diagonal action is "$x-$component-wise", in order to obtain non-trivial relations it is important to ensure that the initial vector field has all components different from *zero*. In the presence of periodic points for the group, the last assertion follows from Theorem (3.4) which is the first main result of this section. The second main result of Section 3, namely Theorem (3.8), characterizes the local nature of a group which fails to have n linearly independent local vector fields (this is done under the assumption that there is a vector field whith all coordinates different from *zero*).

At the beginning of Section 4 we state the Proposition (4.4) which completely clarifies the meaning of the characterization provided by Theorem (3.8). The proof of Proposition (4.4) is however deferred to Section 5. In the rest of Section 4 we analyse the global behavior on $\mathbb{T}^n \setminus \mathrm{P}(\widetilde{\Gamma})$ of some invariant sets introduced in Section 3. These sets are natural candidates to be the complement of the sets U_i's involved in the definition of piecewise ergodicity. The main result is that, apart from one perfectly determined case, the sets in question are locally analytic submanifolds of $\mathbb{T}^n \setminus \mathrm{P}(\widetilde{\Gamma})$.

In Section 5 we give the proof of Proposition (4.4).

After Section 5 we have a clear picture about the dynamics of $\widetilde{\Gamma}$ for a non-solvable group. Basically it only remains to verify the finiteness of the number of sets U_i (in which the action is ergodic and minimal). This verification is easier when $\mathrm{P}(\widetilde{\Gamma})$ is empty. Indeed, if $\mathrm{P}(\widetilde{\Gamma}) = \emptyset$, then it will follow from an argument similar to those employed in Section 4. However, at this point, this argument will be more immediate since the basic results of the section in question will be already settled. On the other hand, in the case of $\mathrm{P}(\widetilde{\Gamma}) \neq \emptyset$, it is needed to control the dynamics of vector fields when the orbits go to $\mathrm{P}(\widetilde{\Gamma})$. Note that $\mathrm{P}(\widetilde{\Gamma})$ is empty if and only if Γ has *no finite orbits*.

Section 7 is independent of the other sections and devoted to study solvable groups. Here the techniques involved are very different and rely on the existence of an invariant measure for the group in question. The paper ends with a section discussing a notion of dicreteness motivated by the point of view followed in this work. In particular we shall extend Theorem A to the case of these more general "non-discrete" groups.

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Main Notations: a subgroup of $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$ is always denoted by Γ. The letter G is reserved for other types of groups such as Lie groups or subgroups (and pseudogroups) of $\mathrm{Diff}(\mathbb{C}, 0)$.

Diffeomorphisms of $\mathbb{S}^1$ (i.e. elements of Γ) are denoted by the letters f, h (as well as the variants $\mathbf{f}, \mathbf{h}$, etc). Notice however that in Section 2.2, the letters f, h stand for local diffeomorphisms of $\mathbb{C}$ leaving $\mathbb{R}$ invariant. This notation is consistent since the only local diffeomorphisms entering into our discussion will be given as restrictions of elements in $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$ possessing fixed points. Very often we shall be concerned with diffeomorphisms $\tilde{f}, \tilde{h}$ of $\mathbb{T}^n$, these have components which are referred to as $\tilde{f}_i, \tilde{h}_i$. Of course $\tilde{f}_i, \tilde{h}_i$ are elements of $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$. In a very few cases, we may be involved with a sequence of distinct diffeomorphisms as well as components of a same diffeomorphism. In any case, we shall use *superscripts* to denote the sequence of distinct diffeomorphism (while subscripts stand for the components of a same diffeomorphism). Hence, for instance, $\tilde{\mathbf{h}}^1, \tilde{\mathbf{h}}^2, \ldots$ is a sequence of diffeomorphisms while $\tilde{\mathbf{h}}^1_1, \tilde{\mathbf{h}}^1_2, \ldots$ are the components of $\tilde{\mathbf{h}}^1$.

Vector fields are always denoted by uppercases X, Y, Z. They are always defined on some open set of a torus (the only exceptions being in Section 2.2 and Lemma (3.1) where the vector fields $X, Y, \ldots$ in question are defined on open sets of $\mathbb{C}$ or $\mathbb{R}$, this, however, cannot lead to any misunderstanding). In the case of vector fields, subscripts $X_1, X_2, \ldots$ always denote the components of a same vector field X defined on a torus of appropriate dimension. On the other hand, superscripts $X^1, X^2, \ldots$ will stand for different vector fields.

Finally gothic uppercase is basically reserved for Lie algebras of vector fields. The gothic character is used to refer to an ideal of functions.

2 Preliminaries

In this section we collect some preliminary results and definitions which are going to be used throughout this work.

Let us denote by μ the normalized Lebesgue measure of $\mathbb{S}^1$ or more generally on $\mathbb{T}^n$. A transformation $\psi : \mathbb{S}^1 \to \mathbb{S}^1$ (or from $\mathbb{T}^n$ to $\mathbb{T}^n$) is called *measurable* if the pre-image of every Borel set is again a Borel set.

In general, a group of measurable transformations of a measurable space is called *ergodic* if the only Borel sets $\nu-$a.e. invariant necessarily have $\nu-$measure total or zero. If the group is not ergodic, then, under very mild assumptions, we can consider the ergodic components of the action. Nonetheless, strictly speaking, this notion of "ergodic decomposition" will not be necessary in this work. In fact we shall be involved only with the easy case in which the number of ergodic components is finite, more precisely our groups will be piecewise ergodic in the sense of the Introduction.

In this work, we shall deal with the measurable space consisting of a torus $\mathbb{T}^n$ ($\mathbb{T}^1 = \mathbb{S}^1$), the $\sigma-$algebra of Borel and the normalized Lebesgue measure. Furthermore our measurable transformations will always be analytic diffeomorphisms.

2.1 Analytic Topology and convergence of commutators

It is known that $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$ can be endowed with a natural topology called the real Analytic Topology (also referred to as the Analytic Topology or the $C^{\omega}-$topology).

The $C^{\omega}-$topology has some remarkable features, but the main use of it in the present work can be summarized by a theorem of Ghys (Theorem (2.1)) stated below. For the convenience of the reader which is not familiar with this topology, we shall present this theorem in a "extrinsic way" (which will eventually turn out to be "intrinsic") so as to avoid any explicit reference to the $C^{\omega}-$topology.

Given two diffeomorphisms f, h of some manifold, denote by $[f, h]$ the *commutator* of these diffeomorphisms defined as $[f, h] = f \circ h \circ f^{-1} \circ h^{-1}$.

Let Γ be a subgroup of $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$ generated by a finite set S of diffeomorphisms in $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$ (notation: $\Gamma =< S >$). Following [Gh], let us associate to S a sequence of sets $S(m)$, $m = 1, 2, \ldots$ as follows:

$\imath$) $S(0) = S$

$\imath\imath$) $S(m+1)$ is the set whose elements can be written in the form $[f^{\pm 1}, h^{\pm 1}]$ where f belongs to $S(m)$ and h belongs to $S(m) \cup S(m-1)$ ($h \in S(0)$ if $m = 0$)

Consider $\mathbb{S}^1$ as the unit circle in $\mathbb{C}$. For $\tau \in \mathbb{R}$ verifying $0 < \tau < 1/4$, let $A_{2\tau}$ denote the annulus $A_{2\tau} = \{z \in \mathbb{C} \, ; \, 1 - 2\tau <\| z \|< 1 + 2\tau\}$. Next notice that any f belonging to $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$ admits a unique holomorphic extension to some neighborhood of $\mathbb{S}^1$ in $\mathbb{C}$. Suppose we are given $\varepsilon > 0$ and $0 < \tau < 1/4$. Let $U_{2\tau}^{\varepsilon} \subset \mathrm{Diff}^{\omega}(\mathbb{S}^1)$ be the set whose elements f are characterized by:

1. f admits a holomorphic extension f_A to $A_{2\tau}$.
2. $\sup_{x \in A_{2\tau}} \| f_A(x) - x \|< \varepsilon$.

With the preceding notations, one has:

Theorem 2.1 (**Ghys [Gh]**) *For any given τ with $0 < \tau < 1/4$, there is $\varepsilon > 0$ so that, if $S \subset \mathrm{Diff}^{\omega}(\mathbb{S}^1)$ is a finite set of diffeomorphisms belonging to $U_{\varepsilon}^{2\tau}$ and $\Gamma =< S >$ is a non-solvable group, then the following holds:*

1. *$S(m)$ does not degenerate into $\{id\}$ for every $m \geq 0$;*
2. *Any $h \in S(m)$ has a holomorphic extension h_A to the annulus A_{τ};*
3. *If $h \in S(m)$ then $\sup_{x \in A_{\tau}} \| h_A(x) - x \|< 2^{-m}\varepsilon$.*

As mentioned, the reader may consult [Reb1] or [Gh] for a definition of the $C^{\omega}-$topology on $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$. Basically the definition is as follows. We consider a *complexification* of $\mathbb{S}^1$, i.e. an open Riemann Surface $\widetilde{\mathbb{S}}$ containing $\mathbb{S}^1$ and such that any analytic function defined on $\mathbb{S}^1$ possesses a holomorphic extension to some neighborhood of $\mathbb{S}^1$ in $\widetilde{\mathbb{S}}$. Fixed such a complexification $\widetilde{\mathbb{S}}$, a basis of neighborhood for the identity is defined by the corresponding collection of sets U_τ^ε, $\tau, \varepsilon > 0$. A basis of neighborhoods for an arbitrary element $f \in \mathrm{Diff}^\omega(\mathbb{S}^1)$ is obtained by translating the basis for the identity. It also turns out that the resulting topology does not depend on the chosen complexification so that it is actually an intrinsic topology on $\mathrm{Diff}^\omega(\mathbb{S}^1)$.

For the purposes of the present article, it is enough to know that the set $U_{2\tau}^\varepsilon \subset \mathrm{Diff}^\omega(\mathbb{S}^1)$ obtained by means of Theorem (2.1) does contain a neighborhood of the identity in the $C^\omega-$topology. In other words, there exists a neighborhood $\mathcal{U}$ of the identity in the $C^\omega-$topology such that, if $S \subset \mathcal{U}$ is a finite set of diffeomorphisms generating a non-solvable group Γ, then the conclusions of Theorem (2.1) do hold for some $\tau, \varepsilon > 0$.

2.2 Germs of holomorphic diffeomorphism of $(\mathbb{C}, 0)$

Here we shall discuss a result which illustrates how interesting may be the local dynamics arising from a non-solvable subgroup of the group of germs of analytic diffeomorphisms of $(\mathbb{C}, 0)$ denoted by $\mathrm{Diff}(\mathbb{C}, 0)$. This group is related to our context because the subgroup of $\mathrm{Diff}^\omega(\mathbb{S}^1)$ fixing a given point p in $\mathbb{S}^1$ projects injectively on a subgroup of $\mathrm{Diff}(\mathbb{C}, 0)$ well-defined up to conjugacies. In fact such subgroup is contained in $\mathrm{Diff}_{\mathbb{R}}(\mathbb{C}, 0)$, the subgroup of $\mathrm{Diff}(\mathbb{C}, 0)$ whose elements preserve the real line (or equivalently have real coefficients).

Given a finitely generated subgroup G of $\mathrm{Diff}(\mathbb{C}, 0)$, we choose a set S of representatives for a generating set of G and consider the pseudogroup generated by the local diffeomorphims of $\mathbb{C}$ contained in S. This pseudogroup will be denoted by G as well. Furthermore both an element of $\mathrm{Diff}(\mathbb{C}, 0)$ as well as a local diffeomorphisms fixing $0 \in \mathbb{C}$ and representing this element will be denoted by the same letter (say f).

In practice we shall work with two local diffeomorphisms, say f, h, which can be written as

$$f(z) = z + az^{r+1} + \cdots \quad \text{and} \quad h(z) = z + bz^{s+1} + \cdots$$

(where $a, b \in \mathbb{C}^*$ and $0 < r < s$). Indeed the assumption that the pseudogroup G is not solvable is required only to ensure that G contains elements f, h as above.

Consider $g \in \mathrm{Diff}_{\mathbb{R}}(\mathbb{C}, 0)$ and denote by $\mathbf{bas}\,(g)$, the basin of g, that is, the set of points $z \in \mathbb{C}$ for which $g^k(z)$ is defined for all $k \in N$ and converges to 0 when k goes to infinity. It is well known that $\mathbf{bas}\,(g) \cup \mathbf{bas}\,(g^{-1})$ is a neighborhood of $0 \in \mathbb{C}$ provided that $g'(0) = 1$ and $g \neq id$.

The following theorem is due to Nakai.

Theorem 2.2 (**Nakai [Na]**) *Let f and h be as above. Then there is a real analytic vector field X defined on $\mathbf{bas}\,(f) \setminus \{0\}$ which satisfies:*

1. *X has no singularities.*

2. *Let V be a relatively compact open subset of $\mathbf{bas}\,(f) \setminus \{0\}$ and let $t_0 > 0$ be so small that the induced local flow of X, Φ_X^t, is defined on V whenever $0 \leq t \leq t_0$ (i.e. $\Phi_X^t(V) \subset$*

bas $(f)\backslash\{0\}$ *for* $0 \leq t \leq t_0$*). Then the sequence of mappings* $\{f^{-k} \circ h^{v(k)} \circ f^k\}$ *converges uniformly on* V *to* $\Phi_X^{t_0}$ *when* k *goes to infinity provided that* $v(k)k^{(r-s)/r} \rightarrow t_0$.

In addition, it is known that the vector field X in question is asymptotic to the formal vector field

$$\widehat{X} = (az^{r+1} + \cdots)\frac{\partial}{\partial z},$$

where $\widehat{X}$ is the unique formal vector field having the property that f is the time-one mapping induced by its formal flow. This asymptotic behavior will be useful in Section 5 and mainly in Section 6. Recall that a formal series $\mathbf{s} = \sum_{j=1}^{\infty} \alpha_j z^j$ is called the *asymptotic development* (or the asymptotic expansion) of X if and only if there are constants $Const_k$ such that for every $k \in \mathbb{N}$ one has $\| X(z) - \sum_{j=1}^{k} \alpha_j z^j \| < Const_k \| z \|^{k+1}$ as long as $z \in \mathbb{R}$ is sufficiently small.

Consider a vector field X as above which is defined and analytic in an interval $(0, \varepsilon)$. Assume that X possesses a non-trivial asymptotic development at 0. It should be observed that the definition of asymptotic development above immediately implies that X has only a finite number of singularities in $(0, \varepsilon)$. This last remark will be rather useful in Section 6.

2.3 The diagonal action

In the sequel we shall recall the main analytic result used in the proof of Theorem A, such result establishes the existence of local vector fields approximating the action in question.

A diffeomorphisms f of $\text{Diff}^{\omega}(\mathbb{S}^1)$ naturally induces a diffeomorphism $\tilde{f}$ of the torus $\mathbb{T}^n$ by letting

$$\tilde{f}(x_1, \ldots, x_n) = (f(x_1), \ldots, f(x_n)),$$

where $(x_1, \ldots, x_n)$ belongs to $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ ($n-$times).

Along this work the letter n will always denote the dimension of $\mathbb{T}^n$.

Consider a subgroup Γ of $\text{Diff}^{\omega}(\mathbb{S}^1)$. For a fixed $n \in \mathbb{N}$, Γ induces a subgroup $\widetilde{\Gamma}$ of $\text{Diff}^{\omega}(\mathbb{T}^n)$ by means of the above formula. The action of $\widetilde{\Gamma}$ on $\mathbb{T}^n$ is called the *diagonal action* of Γ on $\mathbb{T}^n$. Given $f \in \Gamma$, we denote $\tilde{f}$ the element of $\widetilde{\Gamma}$ associated to f in the obvious way indicated above. Of course, if $n = 1$, $\widetilde{\Gamma}$ is nothing but Γ itself.

Notice that $\tilde{f}$ has n components which are going to be referred to as $\tilde{f}_i$ ($i = 1, \ldots, n$). Clearly each $\tilde{f}_i$ globally agrees with f itself, but in local coordinates around a point $p = (p_1, \ldots, p_n) \in \mathbb{T}^n$ the components $\tilde{f}_i$ may differ.

For the subgroup $\widetilde{\Gamma}$ of $\text{Diff}^{\omega}(\mathbb{T}^n)$ obtained through $\Gamma \subset \text{Diff}^{\omega}(\mathbb{S}^1)$, the following terminology will often be used.

$$\text{Per}\,(\widetilde{\Gamma}) = \{x \in \mathbb{T}^n \,;\, x \text{ is a periodic point of } \widetilde{\Gamma}\};$$

$$\text{P}\,(\widetilde{\Gamma}) = \{x = (x_1, \ldots, x_n) \in \mathbb{T}^n \,;\, x_i \in \text{Per}\,(\Gamma) \text{ for some } i \in \{1, ..., n\}\};$$

$$\mathrm{Fix}\,(\widetilde{\Gamma}) = \{x \in \mathbb{T}^n \; ; \; \tilde{f}(x) = x\}.$$

Naturally in the one-dimensional case (i.e. $n = 1$,$\widetilde{\Gamma} = \Gamma$) one has $\mathrm{Per}\,(\widetilde{\Gamma}) = \mathrm{P}\,(\widetilde{\Gamma})$. A substantial part of our strategy to prove Theorem A relies on replacing the complicated analyse of orbits of a subgroup $\widetilde{\Gamma}$ as above by the study of orbits of local vector fields which approximate the action of $\widetilde{\Gamma}$, these are the so called *vector fields in the closure of* $\widetilde{\Gamma}$. In order to make this idea precise, we recall the following definition:

Definition 2.3 *Let $\widetilde{\Gamma}$ be a subgroup of $\mathrm{Diff}^{\omega}(\mathbb{T}^n)$ and X a C^{∞} vector field defined on a open set $U \subseteq \mathbb{T}^n$. We say that X is in the $C^{\infty}-$closure of $\widetilde{\Gamma}$ relative to U if, for any given relatively compact open set V of U and $t_0 > 0$ so small that Φ_X^t is defined on $[0, t_0] \times V$, there is a sequence of diffeomorphisms in $\widetilde{\Gamma}$ such that the restriction of these diffeomorphisms to V converges C^{∞} to $\Phi_X^{t_0} : V \longrightarrow \Phi_X^{t_0}(V)$ (where Φ_X stands for the local flow associated to X).*

Because $\mathrm{P}\,(\widetilde{\Gamma})$ and $\mathrm{Per}\,(\widetilde{\Gamma})$ are invariant sets, every vector field in the closure of $\widetilde{\Gamma}$ will leave these sets invariant as well. Thus it seems resonable to try to construct non-trivial vector fields in the closure of $\widetilde{\Gamma}$ only on the complement of the sets in question. In this direction we have.

Theorem 2.4 (**Rebelo [Reb2]**) *Let $\mathcal{U}$ be the neighborhood of the identity in $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$ furnished by Theorem (2.1) (in its intrinsic version). If S is a finite subset of $\mathcal{U}$ and $\Gamma =< S >$ is a non-solvable group, then for any point p in $\mathbb{T}^n \backslash \mathrm{P}\,(\widetilde{\Gamma})$, there is a neighborhood $V \subset \mathbb{T}^n$ of p, equipped with a C^{ω} non-singular vector field X in the $C^{\infty}-$closure of $\widetilde{\Gamma}$ relative to V.*

3 Local dynamics

Throughout this section, we shall deal with a non-solvable subgroup Γ of $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$ admitting a finite set of generators S contained in the neighborhood of the identity map furnished by Theorem (2.1) (in its intrinsic version). In the sequel, using the notion of local vector fields in the closure of $\widetilde{\Gamma}$, we shall construct certain invariant sets of $\widetilde{\Gamma}$. This construction will require in particular a version of Nakai's theorem appropriate to our context. The first part of this section is devoted to prove Theorem (3.4) which corresponds to the desired extension of Nakai's theorem.

In the remaining part of the section, we shall consider certain distributions of planes whose importance to our problem is apparent. In fact they lead to a natural decomposition of $\mathbb{T}^n$ into invariant sets and a detailed study of this decomposition will be the object of Section 4.

Let us begin with a well-known result.

Lemma 3.1 *Assume we are given $f(z) = z + az^{r+1} + \cdots$ and $h(z) = z + bz^{s+1} + \cdots$, with $s > r \geq 1$ and $ab \neq 0$. Then the commutator $[f, h] = f \circ h \circ f^{-1} \circ h^{-1}$ can be written as*

$$[f, h](z) = z + (s - r)abz^{r+s} + \cdots .$$

Proof. There are unique formal vector fields at the origin, X and Y, so that $f(z) = \exp(X)z$ and $h(z) = \exp(Y)z$, where $\exp(X)$ denotes the formal flow of X at time one. Indeed one has $X = (az^{r+1} + \cdots)\partial/\partial z$ and $Y = (bz^{s+1} + \cdots)\partial/\partial z$. Thus Campbell-Hausdorff formula shows that

$$f \circ h = \exp(X)\exp(Y) = \exp(X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] + \cdots)$$

and

$$f^{-1} \circ h^{-1} = \exp(-X)\exp(-Y) = \exp(-X - Y + \frac{1}{2}[-X,-Y] + \frac{1}{12}[-X,[-X,-Y]] + \cdots).$$

Next we set

$$Z^{+} = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] + \ldots \quad \text{and}$$

$$Z^{-} = -X - Y + \frac{1}{2}[-X,-Y] + \frac{1}{12}[-X,[-X,-Y]] + \cdots \quad .$$

It then follows that

$$[f,h] = \exp(Z^{+})\exp(Z^{-}) = \exp(Z^{+} + Z^{-} + \frac{1}{2}[Z^{+},Z^{-}] + \frac{1}{12}[Z^{+},[Z^{+},Z^{-}]] + \cdots).$$

Because the last expression is clearly reduced to $\exp([X,Y] + \cdots)$, we obtain

$$[f,h](z) = \exp([X,Y] + \cdots)z = z + (s-r)abz^{r+s+1} + \cdots .$$

□

For each $m \in N$, denote by $\Gamma^{(m)}$ the subgroup of Γ generated by $H(m) = S(m-2) \cup S(m-1) \cup S(m)$. Note that $\Gamma^{(m)}$ is not solvable. In fact, the second set associated to $H(m)$ by way of the definition of Section 2.1 clearly contains $S(m+2)$ which does not degenerate into $\{id\}$ (see Theorem (2.1)). Thus $\Gamma^{(m)}$ is not a metabelian group and therefore it is not solvable as well [Gh].

As mentioned, local diffeomorphisms f, h as above will arise from considering elements of Γ possessing a common fixed point. Let us denote by $\widetilde{\Gamma}^{(m)}$ the subgroup of $\mathrm{Diff}^{\omega}(\mathbb{T}^n)$ which represents the diagonal action of $\Gamma^{(m)}$ on $\mathbb{T}^n$.

Now fix a point $p \in \mathrm{Per}\,(\widetilde{\Gamma}^{(m_0)})$. If m_0 is large enough then the set $\mathrm{Per}\,(\widetilde{\Gamma}^{(m_0)})$ actually coincides with $\mathrm{Fix}\,(\widetilde{\Gamma}^{(m_0)})$ (because all diffeomorphisms are orientation preserving). Indeed one has $p_i \in \mathrm{Fix}\,(\Gamma^{(m_0)})$, where p_i is the $i^{\text{th}}-$coordinate of p, $i = 1 \ldots n$. Using a coordinate system z_i around $p_i \in \mathbb{S}^1$ with $z_i(p_i) = 0$ we project $\Gamma^{(m_0)}$ on $\mathrm{Diff}_{\mathbb{R}}(\mathbb{C}, 0)$. Given $\tilde{h} \in \widetilde{\Gamma}^{(m_0)}$, let us denote by $\tilde{h}_i$ the $i^{\text{th}}-$component of $\tilde{h}$ written in the coordinate z_i $(i = 1, \ldots, n)$. Since $\widetilde{\Gamma}^{(m_0)}$ is a non-solvable group, there are $\tilde{f}$ and $\tilde{h}$ in $\widetilde{\Gamma}^{(m_0)}$ such that

$$\tilde{f}_1(z_1) = z_1 + a_1 z_1^{r_1+1} + \cdots \quad \text{and} \quad \tilde{h}_1(z_1) = z_1 + b_1 z_i^{s_1+1} + \cdots \qquad (\text{with } a_1 b_1 \neq 0, s_1 > r_1).$$

Recall that f and h are global diffeomorphisms of the circle, thus f_i and h_i are different from the identity of $\mathrm{Diff}_{\mathbb{R}}(\mathbb{C}, 0)$, $i = 1, \ldots, n$. Furthermore, modulo considering appropriate commutators, we can suppose that

$$\tilde{f}_i(z_i) = z_i + a_i z_i^{s_i+1} + \cdots \quad \text{and} \quad \tilde{h}_i(z_i) = z_i + b_i z^{s_i+1} + \cdots$$

(with $a_ib_i \neq 0$, $s_i > r_i$ for every $i = 1, \dots, n$).

The following elementary formulas will be useful in the proof of Lemma (3.2), they can immediately be deduced from Lemma (3.1):

$$[\tilde{f}_i, \tilde{h}_i](z_i) = z_i + (s_i - r_i)a_ib_iz_i^{r_i+s_i+1} + \cdots , \tag{1}$$

$$[\tilde{f}_i, [\tilde{f}_i, \tilde{h}_i]](z_i) = z_i + s_i(s_i - r_i)a_ib_i^2 z_i^{2r_i+s_i+1} + \cdots , \tag{2}$$

$$[\tilde{f}_i, [\tilde{h}_i, [\tilde{f}_i, \tilde{h}_i]]](z_i) = z_i + 2r_is_i(s_i - r_i)a_i^2b_i^2 z_i^{2(r_i+s_i)+1} + \cdots . \tag{3}$$

Lemma 3.2 *Assume that p belongs to* $\mathrm{Fix}\,(\widetilde{\Gamma}^{(m_0)})$. *It is always possible to choose $\tilde{f}, \tilde{h}$ as before so as to fulfil the additional conditions:*

$$a_i < 0, \;\; b_i > 0 \;\; \textit{and} \;\; (r_i - s_i)/r_i = -1, \;\; i = 1, \dots, n \, .$$

In particular $s_i > r_i$.

Proof Assume that $\tilde{f}_i, \tilde{h}_i$ are such that the product a_ib_i is less than *zero* and $s_i > r_i$ for all i. Considering the formulas (1) and (3), we see that

$$(s_i - r_i)a_ib_i < 0 \;\; , \;\; r_is_i(s_i - r_i)a_i^2b_i^2 > 0 \;\; \text{and} \;\; \frac{r_i + s_i - 2(r_i + s_i)}{r_i + s_i} = -1 \, .$$

In other words, maybe replacing $\tilde{f}$ by $[\tilde{f}, \tilde{h}]$ and $\tilde{h}$ by $[\tilde{f}, [\tilde{h}, [\tilde{f}, \tilde{h}]]]$ we obtain the desired conditions.

Therefore to prove the lemma it suffices to find elements $\tilde{\mathbf{f}}, \tilde{\mathbf{h}}$, belonging to the subgroup generated by $\tilde{f}, \tilde{h}$, whose components $\tilde{\mathbf{f}}_i, \tilde{\mathbf{h}}_i$ have Taylor expansions given by

$$\tilde{\mathbf{f}}_i = z + a_i' z_i^{r_i'} + \cdots \;\; \text{and} \;\; \tilde{\mathbf{h}}_i = z + b_i' z_i^{s_i'} + \cdots$$

where $a_i'b_i' < 0$ and $s_i' > r_i'$. Formulas (3) and (2) then show that it is enough to define $\tilde{\mathbf{f}} = ([\tilde{f}, [\tilde{h}, [\tilde{f}, \tilde{h}]]])^{-1}$ and $\tilde{\mathbf{h}} = [\tilde{h}, [\tilde{h}, \tilde{\mathbf{f}}^{-1}]]$. This proves the lemma. □

Definition 3.3 *Consider a connected component $\mathcal{K}$ of $\mathbb{T}^n \setminus \mathrm{P}\,(\widetilde{\Gamma})$. A point q in* $\mathrm{Per}\,(\widetilde{\Gamma})$ *which also belongs to the boundary of $\mathcal{K}$ will be called a corner point of $\mathcal{K}$.*

Considering $\mathbb{S}^1$ as the unit circle in $\mathbb{C}$, we have a natural coordinate system taking values in $\mathbb{T}^n$, namely the coordinate system given by

$$z(x_1, \dots, x_n) = (z_1, \dots, z_n) = (e^{2\pi\sqrt{-1}(u_1+\alpha_1x_1)}, \dots, e^{2\pi\sqrt{-1}(u_n+\alpha_nx_n)}) \, ,$$

where $e^{2\pi\sqrt{-1}u_i} = q_i$ and $\alpha_i \in \{-1, +1\}, i = 1, \dots, n$ which can be defined so that its image in $\mathbb{T}^n$ contains a neighborhood of any chosen point $q = (q_1, \dots, q_n)$ in $\mathrm{Per}\,(\widetilde{\Gamma})$. Of course, if q is a corner point of $\mathcal{K}$ (a connected component of $\mathbb{T}^n \setminus \mathrm{P}\,(\widetilde{\Gamma})$) then there is just one such coordinate system realizing a correspondence between the intersection of small neighborhoods of 0 with the set $\{x = (x_1, \dots, x_n) \in \mathbb{R}^n \; ; \; x_i \geq 0, i = 1, \dots, n\}$ and the intersection of small neighborhoods of q with the set $\mathcal{K}$. Actually it is sufficient to coherently choose the α_i's, that is to choose a convenient sense of running each factor in $\mathbb{T}^n = \mathbb{S}^1 \times \ldots \times \mathbb{S}^1$ ($n-$times). Thus for ε sufficiently small and suitable α_i's, the set $\prod_{i=1}^n (0, \varepsilon)$ corresponds to a open subset of $\mathcal{K}$. By a small abuse of notation this last set will be denoted by $U(\varepsilon) = \prod_{i=1}^n (q_i, q_i + \varepsilon)$.

It is now easy to establish the previously mentioned Theorem (3.4).

Theorem 3.4 *Let q be a corner point of $\mathcal{K}$, a connected component of $\mathbb{T}^n \setminus \mathrm{Per}\,(\widetilde{\Gamma})$ and choose $\alpha_1, \ldots, \alpha_n$ so that $U(\varepsilon) = \prod_{i=1}^n (q_i, q_i + \varepsilon)$ is contained in $\mathcal{K}$ for ε sufficiently small. Then, reducing ε if needed, there is a real analytic vector field X defined on $U(\varepsilon)$ and satisfying:*

1. *If $X = (X_1, \ldots, X_n)$ then X_i never vanishes on $(q_i, q_i + \varepsilon)$, $i = 1, \ldots, n$.*
2. *X is in the closure of $\widetilde{\Gamma}$ relative to $U(\varepsilon)$*

Proof The preceding discussion allows us to find elements $\tilde{f}, \tilde{h}$ in $\widetilde{\Gamma}$, fixing q, whose projections on $\mathrm{Diff}_{\mathbb{R}}(\mathbb{C}, 0)$ through $z_i(x_i) = e^{2\pi\sqrt{-1}(u_i + \alpha_i x i)}$ have the forms

$$\tilde{f}_i(x_i) = x_i + a_i x_i^{r_i+1} + \cdots \quad \text{and} \quad \tilde{h}_i(x_i) = x_i + b_i x_i^{s_i+1} + \cdots$$

(where $s_i > r_i$, $a_i < 0$ and $(r_i - s_i)/r_i = -1$, $i = 1, \ldots, n$).

Let us consider the Nakai's vector field Y_i associated to $\tilde{f}_i$ and $\tilde{h}_i$ and defined on $\mathbf{bas}\,(\tilde{f}_i) \setminus \{0\}$.

According to Theorem (2.2) and because $(r_i - s_i)/r_i = -1$ does not depend on i, the sequence of local diffeomorphims $\{\tilde{f}_i^{-k} \circ \tilde{h}_i^{[t_0 k]} \circ \tilde{f}_i^{k}\}$ converges uniformly on compact subsets of $\mathbf{bas}\,(\tilde{f}_i) \setminus \{0\}$ to $\Phi_{Y_i}^{t_0}$, provided that t_0 is sufficiently small (where $[\,.\,]$ stands for the integral part of a real number).

Recall that $a_i < 0$, $i = 1, \ldots, n$, thus for small $\varepsilon > 0$ the interval $(0, \varepsilon)$ is contained in $\mathbf{bas}\,(\tilde{f}_i) \setminus \{0\}$ $(i = 1, \ldots, n)$. Hence $Y = (Y_1, \ldots, Y_n)$ is defined on $\prod_{i=1}^n (0, \varepsilon)$ and $X = z^* Y$ is the desired vector field. □

The construction above possesses a consequence (Corollary (3.5)) which will be very useful in Section 6. Even though this corollary will not be used until Section 6, let us present it before continuing our discussion since here seems to be the natural place.

Consider the elements f and h in Γ satisfying Lemma (3.2) (i.e. their natural images $\tilde{f}, \tilde{h}$ in $\mathrm{Diff}^{\omega}(\mathbb{T}^n)$ satisfy Lemma (3.2)) which were employed in the construction of the vector fields X of Theorem (3.4). In this case X_i, the $i^{\text{th}}-$vector field corresponding to the $i^{\text{th}}-$coordinate of X is asymptotic to (cf. Section 2.2)

$$\widehat{X}_i = (a_i z_i^{r_i+1} + \ldots)\frac{\partial}{\partial z_i}, \; i = 1, \ldots, n\,.$$

Now notice that, beginning with the pair $\tilde{h}, [\tilde{f}, \tilde{h}]$, we can apply the procedure of Lemma (3.2) in order to obtain a new pair $\tilde{\mathbf{f}}, \tilde{\mathbf{h}}$ of elements in $\widetilde{\Gamma}$ whose germs $\tilde{\mathbf{f}}_i, \tilde{\mathbf{h}}_i$ at p_i are given in the coordinate z_i used in Theorem (3.4) by ($i = 1, \ldots, n$, $p = (p_1, \ldots, n)$)

$$\tilde{\mathbf{f}}_i(z_i) = z_i + c_i z_i^{t_i+1} + \ldots \quad \text{and} \quad \tilde{\mathbf{h}}_i(z_i) = z_i + d_i z_i^{u_i+1} + \ldots$$

where one has the following relations:

$$a_i b_i \neq 0,\ \ (t_i - u_i)/t_i = -1,\ \ c_i < 0 \ \text{and}\ d_i > 0 \ \text{for all}\ i = 1, \ldots, n.$$

Furthermore $t_i > s_i$, $i = 1, \ldots, n$ since, in the procedure used to prove Lemma (3.2), the order of tangency to the identity of the local diffeomorphisms in question never decreases.

Modulo reducing ε, we can therefore construct another vector field Y in the closure of $\widetilde{\Gamma}$ relative to $U(\varepsilon)$ but using the pair $\tilde{\mathbf{f}}, \tilde{\mathbf{h}}$ rather than $\tilde{f}, \tilde{h}$. Analogously to the vector field X, this new vector field has its $i^{\text{th}}-$vector field coordinates asymptotic to

$$\widehat{Y}_i = (c_i z_i^{t_i+1} + \ldots)\frac{\partial}{\partial z_i},\ i = 1, \ldots, n\,.$$

In other words, the vanishing order of $\widehat{Y}_i$ is greater than $\widehat{X}_i$ (recall that $t_i > s_i > r_i$, $i = 1, \ldots, n$). To abridge notation we will refer to this fact by writing $X \prec Y$. Now Corollary (3.5) can be stated as

Corollary 3.5 *Given a sufficiently small $\varepsilon > 0$, the open cube $U(\varepsilon)$ of Theorem (3.4) can be equipped with two analytic vector fields $X \prec Y$ which also satify conditions 1 and 2 of Theorem (3.4).* □

We can now go further into the analyse of the action of $\widetilde{\Gamma}$. Since the dynamics of vector fields in the closure of a group represents a relevant part of the dynamics associated to the group itself, it is natural to look for a description of these vector fields. Locally the dynamics of all the vector fields in question is encoded in the definition below:

Definition 3.6 *Fixed $p \in \mathbb{T}^n \setminus \mathrm{P}\,(\widetilde{\Gamma})$, we say that p has rank k, if and only if there are k vector fields linearly independent at p and contained in the closure of $\widetilde{\Gamma}$ relative to some neighborhood of p. Moreover there are not $k+1$ vector fields satisfying these conditions.*

By Theorem (2.4), for every point $p \in \mathbb{T}^n \setminus \mathrm{P}\,(\widetilde{\Gamma})$, there is a nowhere vanishing vector field in the closure of $\widetilde{\Gamma}$ relative to some neighborhood of p. In other words, p has rank at least one. On the other hand, the rank of a point is at most n. Therefore $\mathbb{T}^n \setminus \mathrm{P}\,(\widetilde{\Gamma}) = \bigcup_{k=1}^{n} \mathbb{A}^k$, where $\mathbb{A}^k = \{p \in \mathbb{T}^n \setminus \mathrm{P}\,(\widetilde{\Gamma})\ \ ;\ \ p \text{ has rank } k\}$. The rest of this section as well as the next section is mainly devoted to describe this decomposition.

In the definition above we may assume that all the vector fields considered have the following special form: $X(p) = (X_1(p_1), \ldots, X_n(p_n))$, where X_i (resp. p_i) denotes the $\imath^{\text{th}}-$coordinate of X (resp. p) for $i = 1, \ldots, n$ (i.e. X_i depends only on the $\imath^{\text{th}}-$variable, $i = 1, \ldots, n$). Indeed, fixed a vector field X in the closure of $\widetilde{\Gamma}$ and $t \in \mathbb{R}$, it follows that the corresponding local flow Φ_X^t is uniformly approximated by elements of $\widetilde{\Gamma}$ on compact subsets of its domain provided that t is sufficiently small. Thus $\partial \Phi_{Xj}^t / \partial z_i = 0$, whenever $i \neq j$, for $i, j = 1, \ldots, n$, where Φ_{Xj} denotes the $\jmath^{\text{th}}-$coordinate of Φ_X^t (since a similar property is clear for the elements of $\widetilde{\Gamma}$). Therefore

$$\frac{\partial X_j}{\partial z_i} = \frac{\partial}{\partial t}\frac{\partial}{\partial z_i}\Phi_{Xj}^t = 0\,,$$

whenever $i \neq j$ $(i, j = 1, \ldots, n)$. This remark shows that, if a local vector field X is in the closure of $\widetilde{\Gamma}$ and furthermore X has some coordinate i_0 equal to zero at a point $p = (p_1, \ldots, p_n)$ then the hypersurface $[z_{i_0} = p_{i_0}] = \{z = (z_1, \ldots, z_n) \in \mathbb{T}^n \; ; \; z_{i_0} = p_{i_0}\}$ is invariant by X. If at a point p every vector field in the closure of $\widetilde{\Gamma}$ leaves this hypersurface invariant, then the associated dynamics does not seem to be ergodic. Thus it is important to exclude this possibility which motivates us to introduce the following definition.

Definition 3.7 *Given a point $p \in \mathbb{T}^n \setminus \mathrm{P}\,(\widetilde{\Gamma})$, we say that p is $i_1 < i_2 < \cdots < i_l-$null, where $i_s \in \{1, \ldots, n\}$, $s = 1, \ldots, l$, if:*

$\imath$) Every vector field in the closure of $\widetilde{\Gamma}$ relative to some neighborhood of p has its $i_1, i_2, \ldots, i_l$ coordinates equal to zero at p.

$\imath\imath$) The set $\{i_1, i_2, \ldots, i_l\}$ is the maximal set of indices having the property $\imath$) above.

Naturally, if a point p is $i_1 < \ldots < i_l-$null, its rank is at most $n - l$ and if its rank is r, being $i_1 < \cdots < i_l$, then necessarily $l \leq n - r$.

Let us denote by V_p the set whose elements are all the vectors $X(p)$, where X is a local vector field in the closure of $\widetilde{\Gamma}$ relative to some neighborhood of p. Since linear combinations as well as Lie brackets between two vector fields X, Y as above yield a vector field which still belongs to the closure of $\widetilde{\Gamma}$ relative to a smaller domain, it follows that V_p is a vector space and, in fact, a Lie algebra. We are interested in the distribution o planes $p \longmapsto V_p$ and in its integrability. Precisely the next section is mainly devoted to prove that the sets $\mathbb{A}^k$ are locally analytic sets of dimension k satisfying $T_p\mathbb{A}^k = V_p$ for $p \in \mathbb{A}^k$ apart from a specific case (cf. Corollary (4.3) in the general case and Lemma (4.1) for the exception).

Let us close this section by proving Theorem (3.8) below which will be an essential tool for the discussion carried out in Section 4.

Denote by H^ε the set $\{h \in \Gamma \; ; \; || \; h - id \; || < \varepsilon\}$, where $|| \; ||$ stands for the sup norm on $\mathbb{S}^1$ ($\mathbb{S}^1$ being considered as the unit circle of $\mathbb{C}$). Let $p = (p_1, \ldots, p_n)$ be a chosen point of $\mathbb{T}^n \setminus \mathrm{P}\,(\widetilde{\Gamma})$ and suppose that $X^1, \ldots, X^k$ are k vector fields in the closure of $\widetilde{\Gamma}$ relative to a neighborhood U of p. Suppose also that $X^1, \ldots, X^k$ are linearly independent at p and that X^1 *has all coordinates different from zero on U*. Finally, given $\varepsilon > 0$ so small that $\tilde{h}(U)$ is still a neighborhood of p whenever h belongs to H^ε (recall that $\tilde{h} = (\tilde{h}_1, \ldots, \tilde{h}_n) = (h, \ldots, h)$), we define $B(X^1, \ldots, X^k, \varepsilon)$ as the set formed by the following local vector fields:

$$[X^i, X^j] \; ; \; 1 \leq i, j \leq k \quad \text{and} \quad \tilde{h}_* X^i \; ; \; 1 \leq i \leq k \, , \, h \in H^\varepsilon \; .$$

Of course vector fields of the form $\tilde{h}_* X^i$ (resp. $[X^i, X^j]$) are defined on $U \cap \tilde{h}(U)$ (resp. U).

Recall that every local vector field X in the closure of $\widetilde{\Gamma}$ is such that its $l^{\text{th}}-$coordinate depends only of the $l^{\text{th}}-$variable. Thus its $l^{\text{th}}-$coordinate can be thought of as an one-dimensional vector field defined on an open subset of $\mathbb{S}^1$. In the sequel X_l will denote the $l^{\text{th}}-$coordinate of X considered as an one-dimensional vector field defined on an interval of $\mathbb{S}^1$ (notation: in what follows subscripts $X_1, X_2, \ldots$ stands for different components of a same vector field X while superscripts $X^1, X^2, \ldots$ denote different vector fields). With this notation we can state the

Theorem 3.8 *Suppose that k is smaller than n (the dimension of $\mathbb{T}^n$). Then given a vector field $Z \in B(X^1, \ldots, X^k, \varepsilon)$ and indices $1 \leq i_1 < \ldots < i_{k+1} \leq n$, it makes sense to consider the function*

$$\det \begin{pmatrix} X^1_{i_1} & \cdots & X^k_{i_1} & Z_{i_1} \\ X^1_{i_2} & \cdots & X^k_{i_2} & Z_{i_2} \\ \vdots & \ddots & \vdots & \vdots \\ X^1_{i_{k+1}} & \cdots & X^k_{i_{k+1}} & Z_{i_{k+1}} \end{pmatrix}.$$

Furthermore, if we always obtain a function which vanishes identically, then for some index (i.e. component) $i_0 \in \{1, \ldots, n\}$, every element of H^ε is given in the coordinate system $\phi(t) = \Phi^t_{X^1_{i_0}}(p_{i_0})$ as a suitable restriction of an element of $\mathrm{PSL}(2, \mathbb{R})$. In this case k is necessarily greater than or equal to three.

Proof Since $X^1, \ldots, X^k$ are linearly independent at p, modulo a rearrangement of indices, we can suppose that the function

$$\det \begin{pmatrix} X^1_2 & \ldots & X^k_2 \\ \vdots & \ddots & \vdots \\ X^1_{k+1} & \ldots & X^k_{k+1} \end{pmatrix}$$

is not equal to zero at p. Shrinking U if needed, we can assume that this function does not vanish on U.

By assumption for every local vector field Z in $B(X^1, \ldots, X^k, \varepsilon)$ there are scalar functions $\alpha_1, \ldots, \alpha_k$ verifying the equation

$$\alpha_1 X^1 + \alpha_2 X^2 + \cdots + \alpha_k X^k = Z$$

at any point of the domain of Z. In particular one has

$$\begin{pmatrix} X^1_2 & \ldots & X^k_2 \\ \vdots & \ddots & \vdots \\ X^1_{k+1} & \ldots & X^k_{k+1} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{pmatrix} = \begin{pmatrix} Z_2 \\ \vdots \\ Z_{k+1} \end{pmatrix}$$

on the domain of Z. Because the first matrix on the left is invertible and its entries depend only on the variables $x_2, \ldots, x_{k+1}$, as well as the entries of the matrix at the right, it results that also the α_l's depend only on the variables $x_2, \ldots, x_{k+1}$. Thus, fixing $x_2 = p_2, \ldots, x_{k+1} = p_{k+1}$, we obtain the

Claim: For every $Z \in B(X^1, \ldots, X^k, \varepsilon)$ there are constants $\beta_1, \ldots, \beta_k$ ($\beta_l = \alpha_l(p_2, \ldots, p_{k+1})$) such that

$$\beta_1 X^1_1 + \cdots + \beta_k X^k_1 = Z_1$$

on a small neighborhood $U_1 \subset \mathbb{S}^1$ of $p_1 \in \mathbb{S}^1$. □

Let us denote by   the vector space generated by the germs at $p_1 \in \mathbb{S}^1$ of the following one-dimensional vector fields $X^1_1, \ldots, X^k_1$. Note that   is a subspace of the infinite dimensional space consisting of all germs of analytic vector fields at $(\mathbb{R}, 0)$ $(p_1 = 0)$ and not a subspace of the tangent line to $\mathbb{S}^1$ at p_1. By definition is of finite dimension, furthermore, since $[X^i, X^j]$ is in $B(X^1, \ldots, X^k, \varepsilon)$, $1 \leq i, j \leq k$ and $[X^i, X^j]_1 = [X_1^i, X_1^j]$, $1 \leq i, j \leq k$, we conclude that is actually a finite dimensional Lie algebra. According to Lie's Theorem (see [Lie]), via the coordinate system $\phi(t) = \Phi^t_{X_1^1}(p_1)$, becomes a sub-Lie algebra of the Lie algebra generated by $\partial/\partial t$, $t\partial/\partial t$ and $t^2\partial/\partial t$. Therefore there are only three possibilities for .

Case I: is generated by $\partial/\partial t$.

Consider $h \in H^\varepsilon$, by the claim above $h_* X_1^1$ belongs to which implies that $h_*\partial/\partial t = c\partial/\partial t$ for some constant c. In other words $h'(t) = c$ for every t sufficiently small. Thus $h(t) = ct + d$ for some constant d. Choose m_0 so large that $H(m_0)$ is a subset of H^ε (see Theorem (2.1)). Because $h(t) = ct + d$, the sequence of subsets of Γ associated to $H(m_0)$ through the definition in Section 2.1 degenerates into $\{id\}$ (in fact recall that $\text{Aff}(\mathbb{R})$ is a solvable group). However $\Gamma^{(m_0)}$, the subgroup generated by $H(m_0)$, is not solvable because Γ is not solvable, thus there is a contradiction with Theorem (2.1). Therefore Case I cannot be produced.

Case II: is generated by $\partial/\partial t$ and $t\partial/\partial t$.

Again consider $h \in H^\varepsilon$, as before we see that $h_*(X_1^1), h_*(X_1^2), \ldots, h_*(X_1^k)$ are elements of , indeed $h_* \ \subset$. Thus there are constants C_1, C_2, C_3 and C_4 such that.

$$\begin{cases} h_*(\frac{\partial}{\partial t}) = C_1\frac{\partial}{\partial t} + C_2 t\frac{\partial}{\partial t} \\ \\ h_*(t\frac{\partial}{\partial t}) = C_3\frac{\partial}{\partial t} + C_4 t\frac{\partial}{\partial t} \end{cases} \quad \text{i.e.} \quad \begin{cases} h'(t) = C_1 + C_2 h(t) \\ th'(t) = C_3 + C_4 h(t) \end{cases} .$$

Writing the Taylor series of h as $h(t) = \sum_{i=o}^{\infty} c_i t^i$, for some constants c_i's, the equation $h'(t)t = C_3 + C_4 h(t)$ implies that

$$lc_l = C_4 c_l, \quad l = 1, 2, \ldots \quad .$$

Thus for $l = 1$ we obtain $C_4 = 1$ (recall that h is a diffeomorphism) and for $l \geq 2$ we get $c_l = 0$, $l = 2, 3, \ldots$. Therefore $h(t) = c_0 + c_1 t$ and it results a contradiction just as in Case I. Hence Case II cannot be produced as well.

Case III: is generated by $\partial/\partial t, t\partial/\partial t$ and $t^2\partial/\partial t$.

Similarly to Case II, if $h \in H^\varepsilon$, then $h_* \ \subset$, so $h_*(\partial/\partial t), h_*(t\partial/\partial t)$ and $h_*(t^2\partial/\partial t)$ belong to . Furthermore these elements form a basis of . Hence there are constants c_{ij} $(i, j = 1, 2, 3)$ such that

$$\begin{cases} h_*(\frac{\partial}{\partial t}) = c_{11}\frac{\partial}{\partial t} + c_{12}t\frac{\partial}{\partial t} + c_{13}t^2\frac{\partial}{\partial t} \\ \\ h_*(\frac{t\partial}{\partial t}) = c_{21}\frac{\partial}{\partial t} + c_{22}t\frac{\partial}{\partial t} + c_{23}t^2\frac{\partial}{\partial t} \\ \\ h_*(t^2\frac{\partial}{\partial t}) = c_{31}\frac{\partial}{\partial t} + c_{32}t\frac{\partial}{\partial t} + c_{33}t^2\frac{\partial}{\partial t} \end{cases} \quad \text{i.e.} \quad \begin{cases} h'(t)\frac{\partial}{\partial t} = c_{11}\frac{\partial}{\partial t} + c_{12}h(t)\frac{\partial}{\partial t} + c_{13}h(t)^2\frac{\partial}{\partial t} \\ \\ th'(t)\frac{\partial}{\partial t} = c_{21}\frac{\partial}{\partial t} + c_{22}h(t)\frac{\partial}{\partial t} + c_{23}h(t)^2\frac{\partial}{\partial t} \\ \\ t^2h'(t)\frac{\partial}{\partial t} = c_{31}\frac{\partial}{\partial t} + c_{32}h(t)\frac{\partial}{\partial t} + c_{33}h(t)^2\frac{\partial}{\partial t} \end{cases} .$$

Let $\{c'_{ij}\}_{i,j=1,2,3}$ be the inverse matrix of $\{c_{ij}\}_{i,j=1,2,3}$. Then the preceding equations can be reduced to

$$
\begin{aligned}
(c'_{11} + c'_{12}t + c'_{13}t^2)h'(t) &= 1\,, & (4)\\
(c'_{21} + c'_{22}t + c'_{23}t^2)h'(t) &= h(t)\,, & (5)\\
(c'_{31} + c'_{32}t + c'_{33}t^2)h'(t) &= h(t)^2\,. & (6)
\end{aligned}
$$

When $h(0) = 0$, equations (5) and (6) allow us to conclude that $c'_{21} = 0$, $c'_{31} = 0$ and $h(t) = (c'_{22} + c'_{23}t)/(c'_{32} + c'_{33}t)$. On the other hand, if $h(0) \neq 0$, one has two cases:

ı) $c'_{13} = c'_{23} = 0$. In this case $h(t) = (c'_{11} + c'_{12}t)/(c'_{21} + c'_{22}t)$ (see equations (4) and (5) above).

ıı) $c'_{13} \neq 0$ or $c'_{23} \neq 0$.

In the case ıı) above, the polynomials $c'_{11} + c'_{12}t + c'_{13}t^2$ and $c'_{21} + c'_{22}t + c'_{23}t^2$ have a non-trivial common factor. Indeed, otherwise $(c'_{21} + c'_{22}t + c'_{23}t^2)^2/(c'_{11} + c'_{12}t + c'_{13}t^2)^2$ defines a function on $\hat{\mathbb{C}}$ of degree 4. However for t sufficiently small and real the following equation does hold

$$\left(\frac{c'_{21} + c'_{22}t + c'_{23}t^2}{c'_{11} + c'_{12}t + c'_{13}t^2}\right)^2 = \frac{c'_{31} + c'_{32}t + c'_{33}t^2}{c'_{11} + c'_{12}t + c'_{13}t^2} = h(t)^2\ .$$

Since the coefficients involved are real, the equation above actually holds for $t \in \widehat{\mathbb{C}}$. However note that the degree of the function on the right is at most two which is a contradiction. Therefore, cancelling this common factor, we write h as $(C_5 + C_6t)/(C_7 + C_8t)$. Moreover, the constants C_5, C_6, C_7, C_8 can be chosen in $\mathbb{R}$ since h, viewed in the local coordinate ϕ, preserves the real line. In other words, in this local coordinate, h agrees with an element of $\mathrm{PSL}(2,\mathbb{R})$.

Concluding, we have seen that the only possible case for   is the third one. In this case, the dimension of   is exactly three so that k is therefore at least three. Furthermore, through the coordinate ϕ, every element of H^{ε} is given as a suitable restriction of an element of $\mathrm{PSL}(2,\mathbb{R})$. The proof of our theorem is finished. □

Remark 3.9 Some words about the particular case in which Γ is a subgroup of a finite cover of $\mathrm{PSL}(2,\mathbb{R})$.

Recall the well-known fact that any element of $\mathrm{PSL}(2,\mathbb{R})$ preserves the cross-ratio. In other words, the representation of $\mathrm{PSL}(2,\mathbb{R})$ in $\mathrm{Diff}^{\omega}(\mathbb{T}^4)$ obtained through the the diagonal action of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathbb{T}^4$ preserves the foliation whose leaves are the level sets of the function

$$\mathcal{C}R\,(x_1, x_2, x_3, x_4) = (x_1 : x_2 : x_3 : x_4)\ \ ,\ \ (x_1, x_2, x_3, x_4) \in \mathbb{T}^4\ ,$$

where $(x_1 : x_2 : x_3 : x_4)$ stands for the cross-ratio of x_1, x_2, x_3 and x_4. If $\mathrm{PSL}_{k_0}(2,\mathbb{R})$ denotes the cover of $\mathrm{PSL}(2,\mathbb{R})$ with k_0-sheets, it follows that $\mathrm{PSL}_{k_0}(2,\mathbb{R})$ cannot be piecewise ergodic on $\mathbb{T}^4$ in the sense of Theorem A. Indeed, let $\pi_{k_0} : \mathbb{S}^1 \to \mathbb{S}^1$ be the covering map of degree k_0 used in the definition of $\mathrm{PSL}_{k_0}(2,\mathbb{R})$. Now notice that the representation of $\mathrm{PSL}_{k_0}(2,\mathbb{R})$ in

Diff($\mathbb{T}^4$) obtained through the diagonal action preserves the foliation defined by the level sets of the function

$$\mathcal{C}R_{\pi}\,(x_1, x_2, x_3, x_4) = \mathcal{C}R\,(\pi_{k_0}(x_1), \pi_{k_0}(x_2), \pi_{k_0}(x_3), \pi_{k_0}(x_4)), \quad (x_1, x_2, x_3, x_4) \in \mathbb{T}^n \ .$$

We immediately conclude that $\mathrm{PSL}_{k_0}(2, \mathbb{R})$ is not piecewise ergodic on $\mathbb{T}^4$ as desired.

Remark 3.10 The reader has certainly noticed that the Lie algebra generated by $\partial/\partial t, t\partial/\partial t$ and $t^2\partial/\partial t$ corresponds to the representation of $(2, \mathbb{R})$ (the Lie algebra of $\mathrm{PSL}(2, \mathbb{R})$) in $^{\omega}(\mathbb{S}^1)$, the infinite dimensional algebra of all analytic vector fields defined on $\mathbb{S}^1$. In the next section, the image of this representation will be denoted by .

It also follows from the proof of the above theorem that every diffeomorphism h (resp. local diffeomorphism) of $\mathbb{S}^1$ preserving (i.e. such that $h_* \subset$) is contained (resp. local contained) in $\mathrm{PSL}(2, \mathbb{R})$.

4 Invariant sets

Here, we shall still work under the same assumptions of the preceeding section, so Γ will satisfy all the conditions of Theorem (2.1).

Throughout this section we study of the invariant sets $\mathbb{A}^k$. The results and constructions of the previous section are going to be largely used in our discussion.

Let us begin with the rather easy case in which Γ is $C^{\omega}-$conjugate to a subgroup of a finite cover of $\mathrm{PSL}(2, \mathbb{R})$

Lemma 4.1 *Suppose that Γ is $C^{\omega}-$conjugate to a subgroup of a finite cover of* $\mathrm{PSL}(2, \mathbb{R})$. *Then the rank of a point p of $\mathbb{T}^n$ relative to the action of $\widetilde{\Gamma}$ is at most three. Besides, denoting by l the maximal rank attained by a point of $\mathbb{T}^n$ under the $\widetilde{\Gamma}-$action, one has*

$\imath$) *$l = 3$ for $n \geq 3$.*

$\imath\imath$) *All points of $\mathbb{T}^n$ but a non-trivial analytic set belong to $\mathbb{A}^l$.*

Proof Let us first suppose that Γ is a subgroup of $\mathrm{PSL}(2, \mathbb{R})$. Since Γ satisfies the conditions of Theorem (2.1), Γ is non-discrete. Thus its closure $\overline{\Gamma}$ is a Lie group contained in $\mathrm{PSL}(2, \mathbb{R})$ and having a non-trivial Lie algebra.

Choose a non-trivial element X in the representation of the Lie algebra of $\overline{\Gamma}$ in $^{\omega}(\mathbb{S}^1)$. Next define the vector field Y^1 on $\mathbb{T}^n$ by letting $Y^1 = (X, \ldots, X)$. Obviously Y^1 is singular only on $\mathrm{Sing}(Y^1) = \{(x_1, \ldots, x_n) \in \mathbb{T}^n \ ; \ x_i$ is a singularity of X for every $i = 1, \ldots, n\,\}$

If $n \geq 2$, then Theorem (3.8) can be applied in the complement of $\mathrm{Sing}(Y^1)$[1] to ensure the existence of another vector field Y^2 in the closure of $\widetilde{\Gamma}$ and linearly independent with Y^1 away

[1]since we are working on a Lie group, Theorem (3.8) can in fact be applied to the entire $\mathbb{T}^n$

from a non-trivial analytic set A^1. Again, if $n \geq 3$, Theorem (3.8) guarantees the existence of a third vector field Y^3 in the closure of $\widetilde{\Gamma}$ such that Y^1, Y^2, and Y^3 are linearly independent away from a non-trivial analytic set A^2.

We now consider the action of $\widetilde{\Gamma}$ on $\mathbb{T}^n$ for $n \geq 4$. Assume that $l \in \mathbb{N}$ is the maximal rank attained by a point of $\mathbb{T}^n$ under the corresponding action. Thus there are l vector fields $X^1, \ldots, X^l$ in the closure of $\widetilde{\Gamma}$ which are linearly independent at some point $p \in \mathbb{T}^n$. If l were strictly greater than 3, then we could find indices $1 \leq i_1 < i_2 < i_3 < i_4 \leq n$ such that the projection of the vector fields $X^1, \ldots X^l$ on the corresponding factors of $\mathbb{T}^n$ would give us 4 vector fields linearly independent at generic points of $\mathbb{T}^4$. Nonetheless this action should preserve a codimension 1 foliation (cf. Remark (3.9)). The resulting contradiction shows that $l = 3$ proving the lemma in this first case.

In the general case, let $\phi : \mathbb{S}^1 \to \mathbb{S}^1$ be the covering map of degree k used in the construction of Γ, i.e. there is a subgroup Γ_ϕ of $\mathrm{PSL}(2, \mathbb{R})$ such that

$$f \in \Gamma \text{ if and only if } F \circ \phi = \phi \circ f, \text{ for some } F \in \Gamma_\phi$$

The previous proof does apply to Γ_ϕ and, using the function $\Phi = (\phi, \ldots, \phi)$ from $\mathbb{T}^n$ to itself, we obtain the desired conclusions about Γ. □

In view of the preceeding lemma we shall suppose in the sequel that Γ is not $C^\omega-$conjugate to a subgroup of $\mathrm{PSL}(2, \mathbb{R})$ whenever $n \geq 4$ (actually this assumption is not needed for $n = 1, 2, 3$).

Recall that we are interested in the study of the ergodicity of $\widetilde{\Gamma}$. So we would like to show that every point has maximal rank n, which would imply that the action is locally ergodic (see the proof of Theorem A at the end of Section 6). To show the existence of points having rank as large as possible, our strategy consists of letting $\widetilde{\Gamma}$ act on the local vector fields contained in its closure. We already know that the closure of $\widetilde{\Gamma}$ does contain non-trivial local vector fields, however we are going to need vector fields such that none of their coordinates vanishes at a "generic" fixed point $p \in \mathbb{T}^n$. The existence of these vector fields is the content of the main result of this section, namely

Theorem 4.2 *If $p \in \mathbb{T}^n$ does not belong to $\mathrm{P}(\widetilde{\Gamma})$, then p is not $i_1 < i_2 < \ldots < i_s-$null for any set of indices $\{i_1, i_2, \ldots, i_s\}$.*

Obviously Theorem (3.4) will be helpful to establish the above theorem when Γ has finite orbits. Furthermore the proof of this theorem is a delicate argument in which Theorem (3.8) itself will be employed whenever it is possible.

Of much importance to us is the following corollary which will be derived at the end of the section. We say that a set $\mathcal{D}$ of $\mathbb{T}^n$ is *locally analytic* if any point p in $\mathcal{D}$ possesses a neighborhood $U \subset \mathbb{T}^n$ such that $\mathcal{D} \cap U$ is the intersection of the *zero* sets of a family of analytic functions defined on U.

Corollary 4.3 *$\mathbb{A}^k$ is a smooth submanifold of $\mathbb{T}^n \setminus \mathrm{P}(\widetilde{\Gamma})$, satisfying $T_p\mathbb{A}^k = V_p$ for every $p \in \mathbb{A}^k \subset \mathbb{T}^n \setminus \mathrm{P}(\widetilde{\Gamma})$ and $k = 1, 2, \ldots, n-1$. In fact $\mathbb{A}^k$ is also a locally analytic set of $\mathbb{T}^n \setminus \mathrm{P}(\widetilde{\Gamma})$ of dimension k.*

By virtue of the repceding results, we can conclude that $\mathbb{A}^n$ has total measure and possesses at most a countable number of connected components. Actually in Section 6 we shall see that there are only a finite number of such connected components which will constitute the ergodic components of the action (see the end of Section 6).

To abridge notations, we shall use the proposition below which clarifies the conclusions of Theorem (3.8). The proof of Proposition (4.4) will be however deferred to the next section since it involves ideas very different from those employed in the sequel.

Proposition 4.4 *Assume that $\Gamma \subset \mathrm{Diff}^{\omega}(\mathbb{S}^1)$ is a non-solvable group as in the statement of Theorem (3.8). Assume also that, for some $\varepsilon > 0$, there is a local coordinate ϕ, defined by means of a vector field in the closure of Γ, on an interval I of $\mathbb{S}^1$ in which the restriction of all elements of H^{ε} coincides with the restriction of an element of $\mathrm{PSL}(2,\mathbb{R})$. Then Γ is $C^{\omega}-$conjugate to a subgroup of a finite covering of $\mathrm{PSL}(2,\mathbb{R})$.*

The reader has certainly noted that the above proposition nicely complements Theorem (3.8). We often say that Γ is a finite cover of $\mathrm{PSL}(2,\mathbb{R})$ rather than saying that Γ is $C^{\omega}-$conjugate to such a group.

The remainder of this section is devoted to the proof of Theorem (4.2).

First let us recall a result due to Sussmann which gives a local description of orbits obtained through a family of local vector fields.

Let . (M) denote the set of all C^{∞} vector fields defined on M, where M is a manifold. A collection of real Lie algebras $\mathcal{G}(V)$ indexed by open sets of M which is stable under restriction of the domain will be called a pseudo-Lie algebra on M.

Theorem 4.5 (**Sussmann's Lemma [Su]**) *Suppose that M is a C^{∞} manifold and $X = (X_{\imath})_{\imath \in \mathcal{I}}$ is a family of C^{∞} local vector fields on M. Denote by $\mathcal{H}$ the pseudogroup generated on M by the local flows associated to all the $X_{\imath}$, $\imath \in \mathcal{I}$. Then any orbit of $\mathcal{H}$ is locally a smooth submanifold of M. Furthermore, if $\mathcal{G}$ is the pseudo-Lie algebra generated by all the $X_{\imath}$'s, then the tangent space of the orbit $\mathcal{O}_p$ of a point $p \in M$ under $\mathcal{H}$ is just the tangent space of $\mathcal{G}$ at p, that is*

$$T_p\mathcal{O}_p = \{X(q)\ ;\ X \in \mathcal{G}\},\ \ p \in M\ .$$

In our application of Sussmann's Lemma, M will be a connected component of $\mathbb{T}^n \setminus \mathrm{P}\,(\widetilde{\Gamma})$ and the corresponding collection of vector fields $(X^{\imath})_{\imath \in \mathcal{I}}$ will consist of all local vector fields defined on open sets of $\mathbb{T}^n$ contained in the closure of $\widetilde{\Gamma}$. The first result of global nature in this section concerns the orbit $\mathcal{O}_p$ of a point p (as in Sussmann's Lemma) under the previously mentioned family of local vector fields.

Proposition 4.6 *Let $\mathcal{K}$ be a connected component of $\mathbb{T}^n \setminus \mathrm{P}\,(\widetilde{\Gamma})$ and suppose that $p \in \mathcal{K}$ is a point $i_1 < \ldots < i_r-$null of rank k. If q is a point $i_1 < \ldots < i_r-$null of rank k which is accumulated by $\mathcal{O}_p$ (the orbit of p as in Sussmann's Lemma) then q in fact belongs to $\mathcal{O}_p$.*

Since the statement is clear for $k = n$, we consider only the case $k < n$. Up to a rearragement of indices, we may assume that $i_1 = n - r + 1, \dots, i_r = n$ without loss of generality. Hence V_p admits the decomposition

$$V_p = V_p' \oplus 0 \, ,$$

where V_p' is a subspace of $\mathbb{R}^s$, $s = n - r$ and 0 stands for the null vector of $\mathbb{R}^r$. Furthermore s is obviously greater than or equal to k.

Since p is $n - r + 1 < \dots < n-$null, there is a vector $\vec{u} \in V_p$ having its s first coordinates different from *zero*. Next we choose a vector field X^1 in the closure of $\widetilde{\Gamma}$ relative to some neighborhood of p with $X^1(p) = \vec{u}$

Consider vector fields $X^1, X^2, \dots, X^k$ in the closure of $\widetilde{\Gamma}$ relative to some neighborhood U of p and linearly independend at p. Modulo shirinking U, we can suppose that the s first coordinates of $X^1(q)$ are different from *zero* for every q in U (recall that $X^1(p) = \vec{u}$) and moreover that $X^1, X^2, \dots, X^k$ are linearly independent on U.

Denote by $\mathcal{C}$ the collection of all germs of analytic sets A at p such that, for some neighborhood $\mathcal{N}$ of p, one has $A \cap \mathcal{N} \supset \mathbb{A}^k \cap \mathcal{N}$.

Let us define as the ideal generated by all analytic functions vanishing on some analytic set A of the collection $\mathcal{C}$. Because the set of all germs of analytic functions at p is a noetherian ring, there are l germs of analytic functions $_1, \dots, _l$ in , such that $Z(\ ) = \bigcap_{d=1}^{l} [\ _d = 0]$, where $Z(\ )$ stands for the zero set of .

Again modulo shrinking U, we can suppose $_d$ defined on U $d = 1, \dots, l$ which clearly implies that $U \cap Z(\ ) \supset U \cap \mathbb{A}^k$. Next choose $\epsilon_0 > 0$ and $\alpha_0 > 0$ so that $\Phi^t_{X^d}(B_{\epsilon_0}(p))$ is contained in U provided that $\mid t \mid < \alpha_0$ for $d = 1, \dots, k$, where $B_{\epsilon_0}(p)$ stands for the open ball centered at p of radius ϵ_0. Choose also δ_0 such that $\| \Phi^t_{X^d}(x) - x \|_U < \epsilon_0/9$, for $\mid t \mid < \delta_0$, $x \in B_{\epsilon_0}(p)$ and all indices $d = 1, \dots, k$, where $\| \ \|_U$ stands for the sup norm on U. Using these constants we shall establish the following lemma.

Lemma 4.7 *The local flow associated to X^d, $\Phi^t_{X^d}$, preserves $Z(\ )$ for all indices $d = 1, \dots, k$.*

Proof Fix t such that $\mid t \mid < \delta_0$ and d in $\{1, \dots, k\}$. Let $\lambda = \epsilon_0/9$. Next observe that $\mathbb{A}^k \cap B_\lambda(p)$ is contained in $\Phi^t_{X^d}(Z(\ ) \cap B_{2\lambda}(p))$. In fact, if x belongs to $\mathbb{A}^k \cap B_\lambda(p)$, then certainly the point $\Phi^{-t}_{X^d}(x)$, $x \in \mathbb{A}^k \cap B_\lambda(p)$, has rank k and belongs to the open ball $B_{2\lambda}(p)$. Because $\mathbb{A}^k \subset Z(\ )$, it results that $\Phi^{-t}_{X^d}(x)$ is in $Z(\ ) \cap B_{2\lambda}(p)$ and the image of this point by $\Phi^t_{X^d}$ is clearly x.

By the observation above, the analytic set $\Phi^{-t}_{X^d}(Z(\ ) \cap B_\lambda(p))$ is contained in $\mathcal{C}$. Since $\Phi^{-t}_{X^d}(Z(\ )) = \bigcap_{j=1}^{n} [\ _j \circ \Phi^t_{X^d} = 0]$ for some neighborhood of p, it follows that $_j \circ \Phi^t_{X^d}$ belongs to , $j = 1, \dots, n$, so that $_j \circ \Phi^t_{X^d}$ vanishes identically on $Z(\ )$. Therefore the relation

$$\Phi^{-t}_{X^d}(Z(\ )) \supset Z(\ )$$

holds whenever both members are defined. Applying the local diffeomorphism $\Phi^t_{X^d}$ to the above equation we obtain

$$Z(\ ) \supset \Phi^t_{X^d}(Z(\ )) \, ,$$

provided that both members are defined. □

The next lemma determines the dimension of $Z(\ )$.

Lemma 4.8 *The dimension of* $\mathcal{V} = Z(\ \ ) \cap \{\bigcap_{j=n-r+1}^{n} [x_j = p_j]\}$ *is exactly* k.

Proof Suppose for a contradiction that the statement is false. Notice that any point of $\mathcal{O}_p$ is "linked" to p by a local diffeomorphism which can be uniformly approximated by elements of $\widetilde{\Gamma}$. Therefore this diffeomorphism preserves both the rank and the notion of points $i_1 < \ldots < i_r$-nulls. Hence we see that

$$\mathcal{O}_p \subset \mathcal{V} \, .$$

Sussmann Lemma then shows that the dimension of $\mathcal{V}$ is an integer m, with $m \geq k$.

Now let q be a regular point of $\mathcal{V}$. Up to rearrangement of indices we can write

$$\mathcal{V} \cap B_\epsilon(q) = \{(x_1, \ldots, x_m, F(x_1, \ldots, x_m)) \; ; \; (x_1, \ldots, x_m) \in U'\} \, ,$$

for suitable $\epsilon > 0$, a neighborhhod U' of $q' = (q_1, \ldots, q_m) \in \mathbb{R}^m$ and an analytic function F defined on U' and having the form

$$F = (F_{m+1}, \ldots, F_s, p_{s+1}, \ldots, p_n) \, .$$

Note that the last $n - s$ coordinates of F are constants equal to the corresponding coordinates of p.

Defining $Z^1 = (X_1^1, \ldots, X_m^1), \ldots, Z^k = (X_1^k, \ldots, X_m^k)$, there are k vector fields in the closure of $\widetilde{\Gamma}$ acting on $\mathbb{T}^m$ (i.e. at this point $\widetilde{\Gamma}$ is viewed as a subgroup of $\mathrm{Diff}^\omega(\mathbb{T}^m)$) and linearly independent on U'. Actually note that the projection of $T_q\mathcal{V}$ onto its m first coordinates is an isomorphism and, after Lemma (4.7), X^d belongs to $T_q\mathcal{V}$, $d = 1, \ldots, k$. Furthermore none of the coordinates of Z^1 has a *zero* on U'.

Therefore Theorem (3.8) does apply to conclude that there is $Z \in B(Z^1, \ldots, Z^k, \alpha)$ (provided that α is small enough) as well as indices $1 \leq j_1 < \ldots < j_{k+1} \leq m$ such that the function

$$\det \{X, Z, j, k\} = \det \begin{pmatrix} X_{j_1}^1 & \ldots & X_{j_1}^k & Z_{j_1} \\ \vdots & \ddots & \vdots & \vdots \\ X_{j_{k+1}}^1 & \ldots & X_{j_{k+1}}^k & Z_{j_{k+1}} \end{pmatrix}$$

does not vanish identically (here we are using the fact that Γ is not contained in a finite cover of $\mathrm{PSL}(2, \mathbb{R})$).

On the other hand the zero set of this function clearly contains all points of U with rank k. Thus $\det \{X, Z, j, k\}$ necessarily belongs to . However the last assertion implies that $\det \{X, Z, j, k\}$ vanishes identically on $\mathcal{V}$. Since $\det \{X, Z, j, k\}$ depends only on the m first coordinates, we conclude that it is identically *zero*. The resulting contradiction proves the lemma. □

We are finally ready to prove the Proposition (4.6).

Proof of Proposition (4.6) Keeping the preceding notation, we decompose $\mathcal{V} = \mathcal{V}_1 \cup \cdots \cup \mathcal{V}_r$ into irreducible components passing through p.

Consider a regular point q of $\mathcal{V}_j$, $j \in \{1, \ldots, r\}$ and an open ball $B_\gamma(q)$ so small that $\mathcal{V}_j \cap B_\gamma(q) = \mathcal{V} \cap B_\gamma(q)$. Given $d \in \{1, \ldots, k\}$, if $|\, t \,|$ is very small, it follows from Lemma (4.7) that $\Phi^t_{X^d}$ preserves $\mathcal{V}_j \cap B_{\frac{\gamma}{2}}(q)$. Finally the same argument employed in the proof of Lemma (4.7) allows us to conclude that each $\mathcal{V}_j$ is invariant under $\Phi^t_{X^d}$, $d = 1, \ldots, k$.

Define $\mathbf{F}(t_1, \ldots, t_k) = \Phi^{t_k}_{X^k} \circ \ldots \circ \Phi^{t_1}_{X^1}(p)$. Since $X^1(p), \ldots, X^k(p)$ are linearly independent, the derivative $D_0\mathbf{F}$ of $\mathbf{F}$ at the origin is non-singular so that $\mathbf{F}(t_1, \ldots, t_k)$ is an immersion provided that $\mid t_1 \mid < \delta_1, \ldots, \mid t_k \mid < \delta_k$, for suitable positive constants $\delta_1, \ldots, \delta_k$. Up to a rearrangement of indices, we then obtain

$$\mathbf{F}((-\delta_1, \delta_1) \times \ldots \times (-\delta_k, \delta_k)) = \{(x_1, \ldots, x_k, F(x_1, \ldots, x_k)) \; ; \; (x_1, \ldots, x_k) \in U'\} ,$$

where U' is a small neighborhood of $p' = (p_1, \ldots, p_k)$ in $\mathbb{R}^k$ and $F = (F_{k+1}, \ldots, F_s, p_{s+1}, \ldots, p_n)$ is an analytic function on U'.

If $\mathcal{V}_{j_1} \neq \mathcal{V}_{j_2}$ for some indices $j_1 \neq j_2$, then $\mathcal{V}' = \mathbf{F}((-\delta_1, \delta_1) \times \ldots \times (-\delta_k, \delta_k))$ is an analytic set of dimension k contained in $\mathcal{V}_{j_1} \bigcap \mathcal{V}_{j_2}$. However the last set has dimension at most $k-1$ which is impossible. Thus $\mathcal{V}$ is a irreducible analytic set which implies that $\mathcal{V}' = \mathcal{V}$ and therefore $\mathbf{F}$ is a local parametrization of $\mathcal{V}$ at p.

The following sequence of inclusions is clear for sufficiently small $\epsilon > 0$

$$\mathcal{O}_p \cap B_\epsilon(p) \subset \mathbb{A}^k \cap \{ \bigcap_{j=n-r+1}^{n} [x_j = p_j]\} \cap B_\epsilon(p) \subset \mathcal{V} \cap B_\epsilon(p) .$$

Reducing ϵ if needed, we have that the first and third set in the above sequence are smooth manifolds of dimension k so that they actually coincide. Thus the second set of this sequence is a smooth manifold of dimension k as well (in fact analytic). This remark immediately yields

$$\mathbb{A}^k \cap \{ \bigcap_{j=n-r+1}^{n} [x_j = p_j]\} \cap B_\epsilon(p) = \mathcal{O}_p \cap B_\epsilon(p) ,$$

provided that ϵ is very small.

Finally let us consider a point p_0 $(n-r+1) < \ldots < n$-null, having rank k and being approximated by $\mathcal{O}_p$. It follows from the above considerations that, for some $\epsilon > 0$, the points of $\mathcal{O}_{p_0}$ and the points of $B_\epsilon(p_0)$ which have rank k and are $n-r+1 < \ldots < n-$null actually define the same set. Obviously there is a point p' of $\mathcal{O}_p$ in this open ball. Because p' is $n-r+1 < \ldots < n-$null and has rank k, we conclude that p' belongs to $\mathcal{O}_{p_0}$. Thus $\mathcal{O}_{p_0} = \mathcal{O}_{p'} = \mathcal{O}_p$, i.e. p_0 belongs to $\mathcal{O}_p$ as desired. □

Suppose for a moment that Γ has no periodic point and that $\widetilde{\Gamma}$ acts on $\mathbb{T}^2$. Using the above result, it is easy to see that the orbit $\mathcal{O}_p$ of a point $1-$null must cut the diagonal of $\mathbb{T}^2$. However no point in this diagonal may be $1-$null. As a consequence, it follows that there is no $1-$null points in the torus $\mathbb{T}^2$. The next lemma is a suitable generalization of this idea to higher dimensions.

Lemma 4.9 *Assume that* $\mathrm{Per}\,(\widetilde{\Gamma})$ *is empty,* $p \in \mathbb{T}^n$ *is* $i_1 < \ldots < i_s$*-null and* $\mathcal{O}_p$ *is a closed submanifold of* $\mathbb{T}^n$*. Then the projection on the* l^{th} *coordinate* π_l *of* $\mathcal{O}_p$ *is the whole* $\mathbb{S}^1$ *provided that* l *does not belong to* $\{i_1, \ldots, i_s\}$*.*

Proof Fix an index l in the complement of $\{i_1, \ldots, i_s\}$. We claim that

$\imath$) $\pi_l(\mathcal{O}_p)$ is a non-empty subset of $\mathbb{S}^1$;

$\imath\imath$) $\pi_l(\mathcal{O}_p)$ is an open subset of $\mathbb{S}^1$;

$\imath\imath\imath$) $\pi_l(\mathcal{O}_p)$ is a closed subset of $\mathbb{S}^1$.

The items $\imath$) and $\imath\imath\imath$) are clear. To check $\imath\imath$), consider a given point $q_l \in \pi_l(\mathcal{O}_p)$. There is $q \in \mathcal{O}_p$ with $\pi_l(q) = q_l$. Because l does not belong to $\{i_1, \ldots, i_s\}$, there exists a vector field $X = (X_1, \ldots, X_n)$ in the closure of $\widetilde{\Gamma}$ relative to some neighborhood of q with $X_l(q_l) \neq 0$. Thus a small neighborhood of q_l is contained in $\pi_l(\mathcal{O}_p)$. Indeed the following relation holds for $\mid t \mid$ sufficiently small:

$$\pi_l(\Phi_X^t(q)) = \Phi_{X_l}^t(q_l) \ .$$

From $\imath, \imath$) and $\imath\imath\imath$), it follows that $\pi_l(\mathcal{O}_p) = \mathbb{S}^1$. □

On the other hand, when $\mathrm{Per}\,(\widetilde{\Gamma})$ is not empty, the above argument does not apply *a priori* since there may exist invariant open sets of $\mathbb{T}^n$ which do not meet the diagonal.

The next proposition will provide us with a diffeomorphism taking an arbitrary given point $p \in \mathbb{T}^n \setminus \mathrm{P}\,(\widetilde{\Gamma})$ to a small neighborhood $U(\varepsilon)$ (as in Theorem (3.4)). The neighborhood $U(\varepsilon)$ contains several vector fields having non-singular coordinates and this will help us to deal with these points p. Actually the existence of this diffeomorphism is a very useful fact which will also be exploited in section 6. Before stating this proposition we need to establish a preliminary lemma.

Lemma 4.10 *Assume that* $\mathrm{Per}(\Gamma)$ *is not empty and that* p *is point of* $\mathcal{K}$ *(a connected component of* $\mathbb{T}^n \setminus \mathrm{P}\,(\widetilde{\Gamma})$*). Then the closure of* $\mathcal{O}_p$ *in* $\mathbb{T}^n$ *intersects* $\partial\mathcal{K}$ *(the boundary of* $\mathcal{K}$*), provided that* $\mathcal{O}_p$ *is a closed submanifold of* $\mathcal{K}$*.*

Proof Suppose for a contradiction that the lemma is false. In this case there is a compact subset $\mathcal{K}'$ of $\mathcal{K}$ which contains $\mathcal{O}_p$. Now let us consider an extremity c_1 of the interval $\pi_1(\mathcal{O}_p)$. By our assumption on the closedness of $\mathcal{O}_p$, there is $q^1 \in \mathcal{O}_p$ satisfying $\pi_1(q^1) = c_1$. From the definition of c_1, it follows that $\pi_1(T_{q^1}\mathcal{O}_p) = \{0\}$, so $\mathcal{O}_p \subset [x_1 = q_1^1]$. In the same way as before, we obtain the corresponding equations for the indices $2, \ldots, n$. Hence one has $\mathcal{O}_p = \{p\}$ which is a contradiction. □

Proposition 4.11 *Let* p *be a point in* $\mathcal{K}$ *(a connected component of* $\mathbb{T}^n \setminus \mathrm{P}\,(\widetilde{\Gamma})$*, here we also suppose that* $\mathrm{P}\,(\widetilde{\Gamma}) \neq \emptyset$*). There is a corner point* q *of* $\mathcal{K}$ *with the following property:*

($\diamond$) *any open cube* $U(\epsilon) = \Pi_{j=1}^n (q_i, q_i + \epsilon)$ *(as in Theorem(3.4)) contained in* $\mathcal{K}$ *contains the image of* p *under a suitable diffeomorphism* $\tilde{\mathbf{h}} \in \widetilde{\Gamma}$*.*

Remark 4.12 Alternatively we could say that, given p and $U(\epsilon)$ as above, there is a diffeomorphism $\tilde{\mathbf{h}} \in \widetilde{\Gamma}$ such that $\tilde{\mathbf{h}}(U(\epsilon))$ is a neighborhood of p. Note that $\tilde{\mathbf{h}}(U(\epsilon))$ is still asymptotic to q. This point of view will be adopted in Lemma (6.5).

The proof of this proposition is a bit long because it involves the analyse of several possibilities.

First note that $\Gamma^{(m)}$, the group generated by $H(m)$, fix each connected component of $\mathbb{S}^1 \setminus \mathrm{Per}\,(\Gamma)$, provided that m is sufficiently large. In fact just recall that $\mathrm{Per}\,(\Gamma^{(m)})$ is a finite subset of $\mathbb{S}^1$, so that, after Theorem(2.1), $\mathrm{Fix}\,(\Gamma^{(m)})$ is a non-empty set as long as m is large enough. The last assertion implies that $\mathrm{Fix}\,(\Gamma^{(m)}) = \mathrm{Per}\,(\Gamma^{(m)})$ since Γ is a group of orientation-preserving diffeomorphisms of $\mathbb{S}^1$. Thus for m sufficiently large, $\Gamma^{(m)}$ must fix each connected component of $\mathbb{S}^1 \setminus \mathrm{Per}\,(\Gamma^{(m)})$.

Therefore replacing Γ by some group $\Gamma^{(m)}$ if needed, we can suppose that Γ fix each connected component of $\mathbb{S}^1 \setminus \mathrm{Per}\,(\Gamma)$.

Proof of Proposition (4.11) We shall argue inductively on the dimension n of $\mathbb{T}^n$. For $n = 1$, any point $p \in \mathbb{S}^1 \setminus \mathrm{Per}\,(\Gamma)$ has its $\Gamma-$orbit dense in the connected componnent I of $\mathbb{S}^1 \setminus \mathrm{Per}(\Gamma)$ containing p (see [Gh]). Consider an extremity q of I and an interval $U(\epsilon) = (q, q+\epsilon)$ contained in I. Then there exists $\mathbf{h} \in \Gamma$ such that $\mathbf{h}(p)$ belongs to $U(\epsilon)$ which proves the proposition in this case.

Suppose now that the propositon holds for $1, \ldots, n-1$. It only remains to deduce that it also holds for n.

Let us consider a point $p \in \mathcal{K}$. Firstly we shall suppose that p is $i_1 < i_2 < \ldots < i_s-$null. Considering the number s, we are led to discuss different possibilities.

• ($s = n-1$). Without loss of generality we can suppose $i_1 = 1, i_2 = 2, \ldots, i_{n-1} = n-1$. In this case the rank of p is necessarily one. By Proposition (4.6), $\mathcal{O}_p$ is a closed submanifold of $\mathcal{K}$. Thus there is a point p' of $\mathbb{T}^n$ in the intersection of closure of $\mathcal{O}_p$ with the boundary of $\mathcal{K}$ (see Lemma (4.10)). Naturally the $(n-1)-$first coordinates of p and p' are equal, that is $q' = (p_1, p_2, \ldots, p_{n-1}, p'_n)$. According to the induction assumption, there is a corner point $q' = (q_1, \ldots, q_{n-1})$ of $\mathcal{K}' = \pi_{1,\ldots,n-1}(\mathcal{K})$ (where $\pi_{1,\ldots,n-1}$ stands for the projection on the $(n-1)-$first coordinates) having the desired property ($\diamond$), namely, to any open cube $U'(\epsilon) = \Pi_{j=1}^{n-1}(q_j, q_j + \epsilon)$ contained in $\mathcal{K}'$, it corresponds a diffeomorphisms $\tilde{\mathbf{h}}$ of $\widetilde{\Gamma}$ (considered as a subgroup of $\mathrm{Diff}^{\omega}(\mathbb{T}^{n-1})$) whose *inverse* takes $(p_1, \ldots, p_{n-1})$ to $U(\epsilon)$.

Next let $q = (q_1, \ldots, q_{n-1}, p'_n)$ and note that q is a corner point of $\mathcal{K}$. Since $\widetilde{\Gamma}$ preserves every connected component of $\mathbb{T}^n \setminus \mathrm{P}\,(\widetilde{\Gamma})$, it follows that the closure of the image of $U(\epsilon) = U' \times (p'_n, p'_n + \epsilon)$ under $\tilde{\mathbf{h}}$ is a neighborhood of p' in the closure of $\mathcal{K}$ (provided that $U(\epsilon)$ is contained in $\mathcal{K}$). Because p' is a point in the closure of $\mathcal{O}_p$, there is a point p'' of $\mathcal{O}_p$ belonging to $\tilde{\mathbf{h}}(U(\epsilon))$. Clearly there is a local diffeomorphism $\phi : R \longrightarrow R'$, where R is a neighborhood of p'' contained in $\tilde{\mathbf{h}}(U(\epsilon))$ and R' is a neighborhood of p, which is uniformly approximated by elements of $\widetilde{\Gamma}$. In particular there is an element $\tilde{\mathbf{f}}$ of $\widetilde{\Gamma}$ such that $\tilde{\mathbf{f}}(R)$ becomes a neighborhood of p. Composing the diffeomorphisms $\tilde{\mathbf{h}}$, $\tilde{\mathbf{f}}$ we obtain the desired diffeomorphism.

The next step of our proof is to analyse the case $s = n-2$.

• $s = n-2$. If p is a point $i_1 < i_2 < \ldots < i_{n-2}-$null, then its rank r is at most two. Without loss of generality, we can suppose that $i_1 = 1, i_2 = 2, \ldots, i_{n-2} = n-2$. Considering the rank r of p, we have two sub-cases.

$(s = n-2,\ r = 1)$. We still divide this case according to whether or not the closure of $\mathcal{O}_p$ contains a point in $\mathcal{K}$ which is $i_1 < i_2 < \ldots < i_{n-1}$−null.

$(s = n-2,\ r = 1,\ a))$. The closure of $\mathcal{O}_p$ contains a point $p' \in \mathcal{K}$ which is $i_1 < i_2 < \ldots < i_{n-1}$−null.

Replacing p by p', the item $(s = n-1)$ allows us to conclude the existence of a corner point q of $\mathcal{K}$ having property $(\diamond)$. Denote by $\tilde{\mathbf{h}}$ a diffeomorphism of $\widetilde{\Gamma}$ whose *inverse* takes p to a suitable open cube $U(\epsilon) \subset \mathcal{K}$. Because p' is in the closure of $\mathcal{O}_p$, there is a point p'' of $\mathcal{O}_p$ inside $\tilde{\mathbf{h}}(U(\epsilon))$. Let us consider a local diffeomorphism $\phi : R \longrightarrow R'$ which is a uniform limit of elements in $\widetilde{\Gamma}$ (where R and R' are open cubes containing p'' and p respectively and R is contained in $\tilde{\mathbf{h}}(U(\epsilon))$). Thus there is an element $\tilde{\mathbf{f}}$ of $\widetilde{\Gamma}$, such that $\tilde{\mathbf{f}}(R)$ is a neighborhood of p. A suitable composition of $\tilde{\mathbf{h}}$ and $\tilde{\mathbf{f}}$ satisfies the required property.

$(s = n-2,\ r = 1,\ b))$. The closure of $\mathcal{O}_p$ does not contain points $i_1 < i_2 < \ldots < i_{n-1}$−null in $\mathcal{K}$.

In this case Proposition (4.6) ensures that $\mathcal{O}_p$ is a closed submanifold of $\mathcal{K}$. So there is a point p' in the intersection of the boundary of $\mathcal{K}$ with the closure of $\mathcal{O}_p$ (see Lemma (4.9)). Now the argument is as in item $(s = n-1)$.

$(s = n-2,\ r = 2)$. We will divide this case according to whether or not the closure of $\mathcal{O}_p$ contains a point in $\mathcal{K}$ of smaller rank.

$(s = n-2,\ r = 2,\ a))$. The closure of $\mathcal{O}_p$ contains a point $p' \in \mathcal{K}$ of smaller rank.

Replacing p by p', item $(s = n-2, r = 1)$ enables us to conclude the existence of a corner point q of $\mathcal{K}$ having property $(\diamond)$ and now the argument is as in the second paragraph of item $(s = n-2, r = 1, a))$.

$(s = n-2,\ r = 2,\ b))$ The closure of $\mathcal{O}_p$ does not contain points of smaller rank in $\mathcal{K}$.

In this case Proposition (4.6) ensures that $\mathcal{O}_p$ is a closed submanifold of $\mathcal{K}$. So there is some point in the intersection of the closure of $\mathcal{O}_p$ with the boundary of $\mathcal{K}$. The argument now is as in item $(s = n-1)$.

Continuing in this way, we shall eventually arrive to $s = 1$.

- $s = 1$. Without loss of generality we may assume that p is 1−null. Since the rank of p is at most $n-1$, we have all the following possibilities to discuss.

$(s = n-2,\ r = 1)$. We will divide this case according to whether or not the closure of $\mathcal{O}_p$ contains a point $i_1 < \ldots < i_s$−null in $\mathcal{K}$, with $s \geq 2$.

$(s = 1,\ r = 1,\ a))$. The closure of $\mathcal{O}_p$ contains a point p' $i_1 < \ldots < i_s$−null, with $s \geq 2$.

Replacing p by p', the previous cases enable us to conclude the existence of a corner point q of $\mathcal{K}$ having property $(\diamond)$. Now the argument is as in the second paragraph of item $(s = n-2, r = 1, a))$.

$(s = 1,\ r = 1,\ b))$. The closure of $\mathcal{O}_p$ does not contain points $i_1 < \ldots < i_s$−null with $s \geq 2$.

In this case Proposition (4.6) ensures that $\mathcal{O}_p$ is a closed submanifold of $\mathcal{K}$. Then there exists a point p' in the intersection of the closure of $\mathcal{O}_p$ with the boundary of $\mathcal{K}$ (see Lemma (4.9)).

The result now follows as in item ($s = n-1$).

Proceeding in the same way with $r = 2, 3, \ldots, n-2$ we arrive to $r = n-1$.

($s = 1$, $r = n-1$).We will divide this case according to whether or not the closure of $\mathcal{O}_p$ contains points of smaller rank in $\mathcal{K}$.

($s = 1$, $r = n-1$, a)). The closure of $\mathcal{O}_p$ contains a point $p' \in \mathcal{K}$ of smaller rank.

The previous items ensure that replacing p by p', there exists a corner point q having property ($\diamond$) and now the argument is as in item ($s = n-2, r = 1, a$)).

($s = 1$, $r = n-1$, b)). The closure of $\mathcal{O}_p$ does not contain points of smaller rank in $\mathcal{K}$.

By Proposition (4.6) $\mathcal{O}_p$ is a closed submanifold of $\mathcal{K}$. So there is a point p' in the intersection of the closure of $\mathcal{O}_p$ with the boundary of $\mathcal{K}$ and now the argument is as in item ($s = n-1$).

Finally for points which are not $i_1 < i_2 < \ldots < i_s-$null, arguing with the possible ranks $1, 2, \ldots, n$ as before we shall establish the existence of a corner point q satisfying the property ($\diamond$) in all the cases. The proof of the proposition is over. □

Now the proof of Theorem (4.2) is nothing but a recurrent argument based on the previous ideas.

Proof of Theorem (4.2) First let us suppose that $\mathrm{P}(\widetilde{\Gamma})$ is not empty. Consider a point p in a connected component $\mathcal{K}$ of $\mathbb{T}^n \setminus \mathrm{P}(\widetilde{\Gamma})$. Proposition (4.11) asserts the existence of a corner point q of $\mathcal{K}$ with the property that, if the open cube $U(\epsilon)$ is contained in $\mathcal{K}$, then this cube contains the image of p via a suitable element $\tilde{\mathbf{h}}$ of $\widetilde{\Gamma}$. For $\epsilon > 0$ sufficiently small, Theorem (3.4) shows that $U(\epsilon)$ can be equipped with a vector field X in the closure of $\widetilde{\Gamma}$ such that no component of X has a *zero* on $U(\epsilon)$. Hence $(\tilde{\mathbf{h}}^{-1})^* X$ give us a vector field around p having all the components without *zero*. This proves the theorem in this first case.

We now suppose that $\mathrm{P}(\widetilde{\Gamma})$ is empty. Again we shall be led to analyse different possibilities. Recall that p is supposed to be $1 \leq i_1 < \cdots < i_s \leq n-$null, thus we have the following cases:

• ($s = n-1$). Without loss of generality we can suppose that $p \in \mathbb{T}^n$ is $1 < \ldots < (n-1)-$null. Then it follows from Proposition (4.6) that the orbit $\mathcal{O}_p$ is a one-dimensional closed submanifold of $\mathbb{T}^n$. Thus $\mathcal{O}_p = (p_1, p_2, \ldots, p_{n-1}) \times \mathbb{S}^1$, where p_j stands for the j^{th} coordinate of p, $j = 1, \ldots n-1$ (see Lemma (4.9)). Since the point $p' = (p_1, p_2, \ldots, p_{n-1}, p_1)$ belongs to $\mathcal{O}_p$ and is $1 < 2 < \ldots < (n-1)-$null, there is a vector field $X = (X_1, X_2, \ldots, X_n)$ in the closure of $\widetilde{\Gamma}$ relative to a neighborhood U of p' such that the component X_n does not vanish on U.

Because X is in the closure of $\widetilde{\Gamma}$, for $t_0 > 0$ sufficiently small, $\Phi_X^{t_0}$ is the uniform limit on a suitable neighborhood U' of p' of a sequence $\{\tilde{\mathbf{h}}^{(j)}\}_{j\in\mathbb{N}}$ of elements of $\widetilde{\Gamma}$. However, since the components of $\tilde{\mathbf{h}}^{(j)}$ are *globally* the same, we have $\tilde{\mathbf{h}}_1^{(j)} = \tilde{\mathbf{h}}_n^{(j)}$ around p' for every j (where $\tilde{\mathbf{h}}_1^{(j)}$, $\tilde{\mathbf{h}}_n^{(j)}$ stand respectively for the first and last component of $\tilde{\mathbf{h}}^{(j)}$). In particular the first and the last components of $\{\tilde{\mathbf{h}}^{(j)}(p')\}$ converge simultaneously to $\Phi_{X_n}^{t_0}(p_1)$ and to $\Phi_{X_1}^{t_0}(p_1)$, so

$$\Phi_{X_1}^{t_0} = p_1 = \Phi_{X_n}^{t_0} .$$

The preceding equation implies that X_n has a singularity at p_1, which is a contradiction. Therefore this case cannot occur.

• $(s = n-2)$. There is some point $p \in \mathbb{T}^n$, $i_1 < i_2 < \ldots < i_{n-2}$−null. Without loss of generality, we suppose that $i_1 = 1, i_2 = 2, \ldots, i_{n-2} = n-2$. In this case the rank r of p is at most two. Hence we have two sub-cases to treat.

$(s = n-2,\, r = 1)$. By Proposition (4.6) and item $(s = n-1)$, $\mathcal{O}_p$ is a one-dimensional closed submanifold of $\mathbb{T}^n$. Thus Lemma (4.9) ensures that $\pi_n(\mathcal{O}_p) = \mathbb{S}^1$ (where π_n stands for the projection in the n^{th} component). We conclude that $\mathcal{O}_p$ contains a point p' having the form $p' = (p_1, \ldots, p_{n-1}, p_1)$. After the preceding case, there exists a vector field $X = (X_1, X_2, \ldots, X_n)$ in the closure of $\widetilde{\Gamma}$ relative to some neighborhood U of p' such that X_n has no singularity on U. Now the argument is as in item $(s = n-1)$ and the conclusion is that this situation is impossible.

$(s = n-2,\, r = 2)$. By Propostion (4.6) and items $(s = n-1)$, $(s = n-2,\, r = 1)$, $\mathcal{O}_p$ is a closed submanifold of $\mathbb{T}^n$. Thus $\pi_n(\mathcal{O}_p) = \mathbb{S}^1$. Next take a point p' of $\mathcal{O}_p$ with the first and last coordinate equal to p_1. After the preceding items, there is a vector field in the closure of $\widetilde{\Gamma}$ with non-trivial last vector coordinate at p_1. It is now enough to argue as in item $(s = n-1)$ to conclude that this case cannot be produced.

Proceeding in this way, we shall eventually arrive to the case $s = 1$.

• $(s = 1)$. This time we can suppose without loss of generality that there is a point $p \in \mathbb{T}^n$ which is 1−null. The rank of p may vary from 1 to $n-1$. Let us consider each case.

$(s = 1,\, r = 1)$. The cases $(s = n-1)$, $(s = n-2)$, …,$(s = 2)$ and Proposition (4.6) shows that $\mathcal{O}_p$ is a one-dimensional closed submanifold of $\mathbb{T}^n$. Thus $\pi_n(\mathcal{O}_p) = \mathbb{S}^1$. Choose p' a point of $\mathcal{O}_p$ with the first and last coordinate equal to p_1 (the first coordinate of p). After the preceding items, there is a vector field X in the closure of $\widetilde{\Gamma}$ whose last coordinate is not zero at p_1. Now, as in item $(s = n-1)$, we conclude that this case cannot occur.

Continuing our recurrent argument, we shall arrive to $r = n-1$.

$(s = 1,\, r = n-1)$. By Proposition (4.6) and the preceding items, we know that $\mathcal{O}_p$ is a closed submanifold of $\mathbb{T}^n$. Thus $\pi_1(\mathcal{O}_p) = \mathbb{S}^1$. Similarly to the other cases, we finally conclude that this last possibility cannot be produced as well.

The theorem is proved. □

We finish this section by deducing the important Corollary (4.3).

Proof of Corollary (4.3) Fix a point p in $\mathbb{T}^n \setminus \mathrm{P}(\widetilde{\Gamma})$ having rank k. The combination of Theorem (4.2) and the proof of Proposition (4.6) ensures the existence of $\epsilon > 0$ such that

$$\mathbb{A}^k \cap \mathrm{B}_\epsilon(p) = \mathcal{O}_p \cap B_\epsilon(p)\,.$$

Furthermore we have seen in the proof of Proposition (4.6) that $\mathcal{O}_p$ is an analytic submanifold (recall that we construct a regular local parametrization for this set). Hence the same applies to $\mathbb{A}^k$. In addition, Lemma (4.8) shows that $\mathbb{A}^k$ is contained in $\mathcal{V}$ which is an (smooth) analytic set of a neighborhood of p having dimension k. The rest of the statement is immediate. □

5 Analytic continuation

This section is devoted to the proof of Proposition (4.4). We shall keep the terminology and notations of the preceding section.

In what follows, we assume that $\Gamma \subset \mathrm{Diff}^{\omega}(\mathbb{S}^1)$ is a subgroup as in Section 3. Furthermore there are $\varepsilon > 0$, a point $p \in \mathbb{S}^1$, an open interval $I \subset \mathbb{S}^1$ containing p and vector field X in the closure of Γ defined on I such that the following holds: if ϕ is a local coordinate defined through the local flow of X then, in this coordinate, the restriction to I of every element of H^{ε} coincides with the restriction of an element of $\mathrm{PSL}(2,\mathbb{R})$.

The proof of Proposition (4.4) consists of showing that the local coordinate ϕ considered above, admits an analytic extension to the whole circle $\mathbb{S}^1$. However the extension of ϕ need not be a diffeomorphism, actually it may be a non-trivial finite covering map of $\mathbb{S}^1$. If k is the degree of this extension, then we can construct a group $C^{\omega}-$conjugacy between Γ and a subgroup of the cover of $\mathrm{PSL}(2,\mathbb{R})$ with $k-$sheets.

We consider $\mathbb{S}^1$ as the unit circle in $\mathbb{C}$ equipped with the Euclidean metric. Let us identify an element of $\mathrm{PSL}(2,\mathbb{R})$ with the corresponding diffeomorphism of $\mathbb{S}^1$ induced by the projective action of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathbb{S}^1$. Similarly we consider the Lie algebra of $\mathrm{PSL}(2,\mathbb{R})$, $(2,\mathbb{R})$, identified with its image $\subset$ ${}^{\omega}(\mathbb{S}^1)$ by the representation in question.

Our first two lemmas are elementary. We shall give brief proofs since it may be difficult to find them explicitly stated in the literature.

Lemma 5.1 *Given a non-empty open interval $J \subset \mathbb{S}^1$ and a neighborhood U of the identity in* $\mathrm{PSL}(2,\mathbb{R})$*, there exists a constant $\delta > 0$, such that every diffeomorphism f in* $\mathrm{PSL}(2,\mathbb{R})$ *satisfying $\| f(z) - z \|_J < \delta$ necessarily belongs to U (where $\| \; \|$ stands for the distance on $\mathbb{S}^1$).*

Proof Let I_1, I_2 be the intervals of $\mathbb{S}^1$ defined by

$$I_1 = \{ z \in \mathbb{C} \, ; \, \| z \| = 1 \text{ and } -\pi/4 \leq \arg z \leq 5\pi/4 \} \text{ and}$$

$$I_2 = \{ z \in \mathbb{C} \, ; \, \| z \| = 1 \text{ and } 0 \leq \arg z \leq \pi/4 \text{ or } 3\pi/4 \leq \arg z \leq 2\pi \} \, .$$

Without loss of generality we can suppose that $J \subset I_1$, furthermore we also consider an auxiliar non-empty open interval $J_{\cap}$ contained in $I_1 \cap I_2$.

Now consider the projections $\mathcal{P}_n$ and $\mathcal{P}_s$ respectively from the north-pole and the south-pole. Because in these coordinates f is a Mœbius transformations and the interval $\mathcal{P}_n(I_2) \subset \mathbb{R}$ is compact, it follows from an elementary calculation that f is close to the identity on $\mathcal{P}_n(I_2)$ provided that it is sufficiently close to the identity on $\mathcal{P}_n(J) \subset \mathcal{P}_n(I_2)$. In particular f is close to the identity on $J_{\cap} \subset \mathcal{P}_s(I_1)$ and the same argument shows that f is close to the identity in the whole $\mathcal{P}_s(I_1)$. This proves the lemma. □

Lemma (5.2) below is quite simple as well.

Lemma 5.2 *Suppose we are given a C^{∞} vector field $X : J \subset \mathbb{S}^1 \longrightarrow TJ$, where J is a non-empty open interval of* $\mathbb{S}^1$*. If X is in the closure of* $\mathrm{PSL}(2,\mathbb{R})$ *(in the sense of Definition (2.3)), then X belongs to* .

Proof We want to check the existence of some vector $\vec{v} \in \quad (2, \mathbb{R}) = \quad$ such that

$$X(z) = \frac{d}{d\,t} \left(\exp(t.\vec{v}) \right) .z \, |_{t=0} \ .$$

Choose a neighborhood U of the identity in $\mathrm{PSL}(2, \mathbb{R})$ in which the inverse functions of the exponential map $\exp : \quad \rightarrow \mathrm{PSL}(2, \mathbb{R})$ and of the map Θ given by $\Theta : A \mapsto A^2$, $A \in \mathrm{PSL}(2, \mathbb{R})$, are well defined functions. Note that the derivatives of exp and Θ are non-singular at the origin and at the identity respectively. Next consider a non-empty open interval J' contained in J and fix a $\gamma > 0$ so small that Φ_X^t is defined in J' whenever $0 \leq t \leq \gamma$ (where Φ_X^t stands for the local flow of X). Using Lemma (5.1) we choose $\delta > 0$ so that any element T of $\mathrm{PSL}(2, \mathbb{R})$ satisfying $|| \, T(z) - z \, ||_{J'}$ belongs to U. Finally take $\epsilon > 0$ ($\epsilon < \gamma$) such that $|| \, \Phi_X^t(z) - z \, ||_{J'} < \delta$ provided that $0 \leq t \leq \epsilon$.

Because $\mathrm{PSL}(2, \mathbb{R})$ is closed as subgroup of $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$, we conclude that, for every $0 \leq t \leq \epsilon$, Φ_X^t is the restriction of an element of $\mathrm{PSL}(2, \mathbb{R})$ (since otherwise it could not be approximated by elements in $\mathrm{PSL}(2, \mathbb{R})$ in the sense of Definition (2.3)). Furthermore, for $0 \leq t \leq \epsilon$, Φ_X^t actually coincides with an element of $U \subset \mathrm{PSL}(2, \mathbb{R})$. Thus we can consider $A = \Phi_X^{\epsilon/2}$ and $\vec{v} = 2 \exp^{-1}(A)/\epsilon$. Therefore

$$[\Phi_X^{\epsilon/4}]^2 = [\exp(\epsilon/4.\vec{v})]^2 \quad (= A) \, .$$

Since Θ is one-to-one, it results that $\Phi_X^{\epsilon/4} = \exp(\epsilon/4.\vec{v})$. An easy inductive argument shows that $\Phi_X^{j\epsilon/2^i} = \exp(j\epsilon/2^i.\vec{v})$, for $i, j \in \mathbb{N}$, $j \leq 2^i$. Using the density of the numbers of the form $j\epsilon/2^i$ (i, j as above) in $(0, \epsilon/2)$, we conclude that $\Phi_X^t = \exp(t.\vec{v})$ for $t \in (0, \epsilon/2)$. In other words $X(z) = \frac{d}{dt} exp(t.\vec{v}).z|_{t=o}$. □

Proposition 5.3 *Assume that p and X are as before. Then $\phi(t)$, the inverse function of $t \mapsto \Phi_X^t(p)$, has an analytic extension to I, where I is the connected component of $\mathbb{S}^1 \setminus \mathrm{Per}\,(\Gamma)$ containing p. Furthermore such extension is a local diffeomorphism.*

Fix an orientation of $\mathbb{S}^1$. If $\mathrm{Per}\,(\Gamma) \neq \emptyset$ denote by q the first point of $\mathrm{Per}\,(\Gamma)$ following the positive orientation of $\mathbb{S}^1$ from p. Otherwise set $q = p$. Now consider

$$\mathcal{E} = \{ x \in [p, q) \ ; \ \phi \text{ has an analytic extension as a local diffeomorphism to } [p, x) \} \, .$$

At this point the proof of Proposition (5.3) is reduced to prove that the supremum of $\mathcal{E}$ (following the positive orientation of $\mathbb{S}^1$) is q. We therefore assume for a contradiction that the supremum of $\mathcal{E}$, denoted by q_0, is "smaller" than q. The following lemma gives us some technical information which will be useful later.

Lemma 5.4 *If q_0 is smaller than q then, for any $\varepsilon > 0$, there is $h \in H^{\varepsilon} \setminus \{id\}$ verifying the conditions below:*

1. *q_0 does not belong to $\mathrm{Per}\,(h)$ (where $\mathrm{Per}\,(h)$ stands for the periodic points of h);*

2. $h(p) > p$;

3. *the interval* $[p, h(p)]$ *is contained in the domain of definition of* X.

Proof Let us first verify item *1*. Suppose for a contradiction that, for some $\varepsilon > 0$, every element in H^{ε} fixes q_0. Recall that, fixed $\varepsilon > 0$, the set H^{ε} does contain the set $H(m) = S(m-2) \cup S(m-1) \cup S(m)$ of generators of $\Gamma^{(m)}$ provided that m is large enough. According to our assumption, q_0 belongs to the set of periodic points of every diffeomorphism in H^{ε}. It results that q_0 belongs to $\mathrm{Fix}\,(\Gamma^{(m)})$ for some m large enough. Indeed, choose m_1 such that $H(m_1) \subset H^{\varepsilon/2}$ and consider a generator $\mathbf{h} \neq id$ of $\Gamma^{(m_1)}$. Of course q_0 belongs to the set of periodic point of $\mathbf{h}$, $\mathrm{Per}\,(\mathbf{h})$, which is a finite set. Hence if $\mathbf{f}$ is very close to the identity $\mathbf{f}(q_0)$ does not belong to $\mathrm{Per}\,(\mathbf{h})$ unless q_0 is fixed by $\mathbf{f}$. However, if q_0 is not fixed by one such $\mathbf{f}$, then q_0 is not a periodic point of $(\mathbf{f})^{-1} \circ \mathbf{h} \circ \mathbf{f}$ which belongs to H^{ε} and thus condition *1* is satisfied.

Therefore we can suppose that $q_0 \in \mathrm{Fix}\,(\Gamma^{(m)})$ for some m sufficiently large.

Since $\Gamma^{(m)}$ is a non-solvable group, it results the existence of elements, still denoted $\mathbf{h}, \mathbf{f}$, in $\Gamma^{(m)}$ whose germs $\mathbf{h}_{q_0}$ and $\mathbf{f}_{q_0}$ at q_0 can be written in a (orientation-reversing) coordinate system z with $z(q_0) = 0$ as

$$\mathbf{h}_{q_0}(z) = z + az^{s+1} + \ldots \ \text{ and } \ \mathbf{f}_{q_0}(z) = z + bz^{r+1} + \ldots\,, \ \ (\text{where } \ a < 0, b < 0 \ \text{ and } \ s > r)\,.$$

By Nakai's theorem, there is an one-sided neighborhood I' of q_0 in $[p, q_0]$ such that $I' \setminus \{q_0\}$ is equipped with two vector fields X and Y which are in the closure of Γ and also asymptotic respectively to the vector fields $\widehat{X} = (at^{r+1} + \ldots)\frac{\partial}{\partial t}$ and $\widehat{Y} = (bt^{s+1} + \ldots)\frac{\partial}{\partial t}$. In fact, to check this claim, it suffices to apply Theorem (2.2) to the pairs $\mathbf{f}, \mathbf{h}$ and $\mathbf{h}, [\mathbf{f}, \mathbf{h}]$.

Thanks to the equation $[(at^{r+1} + \ldots)\frac{\partial}{\partial t}, (bt^{s+1} + \ldots)\frac{\partial}{\partial t}] = (s-r)a.bt^{s+r} + \ldots$, the Lie algebra generated by X, Y is infinite-dimensional. Actually the vector fields $X, Y, [X, Y], [Y, [X, Y]], \ldots$ have a strictly increasing asymptotic order of "flatness" at the origin which ensures that they are all linearly independent. However, in the coordinate ϕ, the above mentioned vector fields are in the closure of a subgroup of $\mathrm{PSL}(2, \mathbb{R})$ and, because of Lemma (5.2), they are in fact contained in which has dimension 3. The resulting contradiction proves the existence of some $h \in H^{\varepsilon} \setminus \{id\}$ for which q_0 is not a periodic point.

Let $h \in H^{\varepsilon} \setminus \{id\}$ be such that $q_0 \notin \mathrm{Per}\,(h)$. We have to check that it is possible to find such a h for which $h(p) > p$. Modulo replacing h by h^{-1} it suffices to check that $h(p) \neq p$. Note that, thanks to the above argument, for every $m \in \mathbb{N}$, p cannot be simultaneously fixed by all the generators of $\Gamma^{(m)}$. Hence, if $h(p) = p$, let $\mathbf{h}^{(1)} \in \Gamma^{(m_1)}$ be such that $\mathbf{h}^{(1)}(p) \neq p$. Modulo choosing m_1 very large, it follows that $\mathbf{h}^{(1)}(p)$ is not fixed by h and $\mathbf{h}^{(1)}(q_0)$ is not periodic for h as well. Thus $(\mathbf{h}^{(1)})^{-1} \circ h \circ \mathbf{h}^{(1)}$ is $\varepsilon-$close to the identity and satisfy conditions *1, 2*. Finally for condition *3*, it is enough to reduce ε. The lemma is proved. □

Let us now consider the sequence of fundamental domains $[p, h(p)], [h(p), h^2(p)], \ldots$ where $h \neq id$ is as in the above lemma.

Lemma 5.5 *The above sequence is entirely contained in* $[p, q_0]$.

Proof Suppose for a contradiction that the statement is false. Denote by k_0 the first natural number for which $[h^{k_0}(p), h^{k_0+1}(p)]$ is not contained in $[p, q_0]$. Because of the definition of k_0, the vector field $Z = (h^{k_0})_* X$ is defined on a neighborhood of q_0. Moreover $\phi_* Z$ is a vector field in the closure of $\mathrm{PSL}(2,\mathbb{R})$ (where ϕ is the local coordinate considered at the beginning of this section). Actually $\phi_* Z$ belongs to and one has

$$\phi(\Phi_Z^t(x)) = \Phi_{\phi_* Z}^t(x) \ ,$$

for any t, x such that both members are defined. This equation gives an analytic extension of ϕ beyond q_0 as a local diffeomorphism. The resulting contradiction establishes the lemma. $\square$

Lemma (5.5) above implies that the sequence $\{h^k(p)\}_{k\in\mathbb{N}}$ converges to a point $p_1 \in [p, q_0)$ with $h(p_1) = p_1$ (in particular $p_1 \neq q_0$ since $h(q_0) \neq q_0$). The next step will be to equip points p_1' ($p_1' > p_1$) near p_1 with a special non-zero local vector field in the closure of Γ relative to some neighborhood of p_1'. First note that there is $f \in \Gamma$ which coincides in the coordinate ϕ with a suitable restriction of an element of $\mathrm{PSL}(2,\mathbb{R})$ and satisfies:

i) $f(p_1) = p_1$ and $f'(p_1) > 0$;

ii) $f(p_1') < p_1'$ for $p_1' > p_1$ and p_1' sufficiently close to p_1.

Indeed just begin with $f = h$ and, if needed, consider h^{-1} (recall that our group preserves the orientation).

Proof of Proposition (5.3) Recall that we have supposed for a contradiction that $q_0 < q$. Furthermore the fixed diffeomorphism h has a fixed point p_1 lying on $[p, q_0)$. To achieve the expected contradiction, we are going to prove that h has in fact an infinite number of fixed points. This is obviously impossible since h is analytic and $h \neq id$.

We shall keep the preceding notations. Consider a fixed $\varepsilon > 0$. Observe that p_1 is not fixed by all diffeomorphisms in H^{ε}. Indeed, otherwise the argument of Lemma (5.4) would apply to show that the Lie algebra generated by local vector fields in the closure of Γ and defined around a point of the domain of ϕ is infinite-dimensional. This is of course impossible since it was seen that this Lie algebra has dimension at most three.

In view of the above paragraph, there is a sequence $\{\mathbf{h}^{(i)}\}$ of elements in Γ converging uniformly to the identity and satisfying $\mathbf{h}^{(i)}(p_1) > p_1$, $i \in \mathbb{N}$. Following [Reb1], we can construct a vector field X^1 defined on $[f(p_1'), p_1']$, where p_1' sufficiently close to p_1 is such that $p_1' > p_1$, possessing the property below:

($*$) For appropriate exponents $s(t_0)$ and l_i, both in $\mathbb{N}$, the sequence $\{(f)^{-l_i} \circ (\mathbf{h}^{(i)})^{s(t_0)} \circ (f)^{k_i}\}_{i\in\mathbb{N}}$ converges uniformly to $\Phi_{X^1}^{t_0}$ on U, as long as $\Phi_{X^1}^{t_0}(U) \subset [f(p_1), p_1]$, for $0 \leq t \leq t_0$.

Now consider $p_1' > p_1$ very close to p_1. Since we do not need anymore the fact that $h(p) > p$, we are able to replace h by h^{-1} so as to ensure that $h(p_1') > p_1'$. From now on, p_1' will be fixed.

Again we consider the sequence of fundamental domains $[p_1', h(p_1')]$, $[h(p_1'), h^2(p_1')], \ldots$. The claim below is very similar to Lemma (5.5).

Claim: The above sequence is entirely contained in $[p_1, q_0)$.

Proof of the Claim Suppose for a contradiction that the claim is false. Let k_1 be the first natural number such that $[h^{k_1}(p_1'), h^{k_1+1}(p_1')]$ is not contained in $[p_1, q_0]$. Because of the definition of k_1 and since $h(q_0) \neq q_0$, the vector field $Z^1 = (h^{n_1})_* X^1$ is a vector field in the closure of Γ which is defined around q_0. By the construction of X^1, it follows that $\phi_* Z^1$ belongs to the closure of PSL$(2, \mathbb{R})$ and therefore to . Thus one has

$$\phi(\Phi^t_{Z^1}(x)) = \Phi^t_{\phi_* Z^1}(x)\,,$$

for any t, x such that both members are defined. This equation gives an analytic extension of ϕ beyond q_0 which is a local diffeomorphism. The resulting contradiction proves the claim. □

The above claim ensures that the sequence $\{h^k(p_1')\}$ converges to a point $p_2 \in (p_1, q_0)$ which is fixed by h. Continuing this procedure we can produce an infinite sequence of points $p_1 < p_2 < \cdots < q_0$ which are fixed by h. Thus we shall conclude that $h = id$ which is the final contradiction. The proposition is proved. □

The following result is a by-product of our discussion. Recall that we are working under the assumption made at the beginning of this section (which corresponds to the assumption of Proposition (4.4)).

Corollary 5.6 *The group* Γ *does not admit periodic points, i.e.* $\mathrm{Per}\,(\Gamma) = \emptyset$.

Proof Suppose for a contradiction that $\mathrm{Per}\,(\Gamma) \neq \emptyset$. Now consider p, I, q and ϕ as in Proposition (5.3). Recall that q is in $\mathrm{Fix}(\Gamma^{(m)})$ for m sufficiently large and $\Gamma^{(m)}$ is non-solvable. Then the argument of Lemma (5.4) guarantees that the Lie algebra generated by the Nakai vector fields in an interval of the form $[p_1, q)$ cannot have a finite dimension. This is of course impossible since this Lie algebra is conjugate to a sub-Lie algebra of . □

We are almost ready to prove Proposition (4.4). Notice that the combination of Proposition (5.3) and Corollary (5.6) implies that ϕ admits an analytic extension as a local diffeomorphism to the whole $\mathbb{S}^1$ apart from the fact that this extension does not need to be univalued (in particular $\lim_{x \to p^+} \phi(x)$ may differ from $\phi(p)$ where $x \to p^+$ is to be understood as convergence of x to p after going around $\mathbb{S}^1$). To take care of this difficulty, one last lemma will be needed.

Lemma 5.7 *Assume we are given a non-empty open interval* U *of* $\mathbb{S}^1$ *and a diffeomorphism* $\psi : U \longrightarrow \psi(U)$ *such that the equation*

$$\psi \circ T \circ \psi^{-1} = T\,, \tag{7}$$

holds for every $T \in \mathrm{PSL}(2, \mathbb{R})$ *as long as both members are defined. Then* $\psi = id$.

Proof Consider $\partial/\partial t$ as an element of $\subset {}^{\omega}(\mathbb{S}^1)$. Equation (7) then gives

$$\psi_*(\partial/\partial t) = \partial/\partial t\,.$$

Hence $\phi = t + c$, for some constant $c \in \mathbb{R}$. Now letting $T(t) = 2t$, the same equation shows that $\psi \circ T \circ \psi^{-1}(t) = 2t - c = 2t$, for $t \in U$. Thus $c = 0$ and $\psi = id$ □

Proof of Proposition (4.4) Let $\check{\phi}$ be the analytic continuation of ϕ to p, obtained after one tour around $\mathbb{S}^1$ with starting point at p. Choose a neighborhood U of p such that both ϕ and $\check{\phi}$ define diffeomorphisms from U to their respective images. Setting $U' = \check{\phi}(U)$, $U'' = \phi(U)$ we obtain a diffeomorphism $\psi = \check{\phi} \circ \phi^{-1}$ from U'' to its image in U'. The following equations do hold for ε small:

$$\phi \circ f \circ \phi^{-1} = T \text{ on } U'', \text{ where } f \in H^{\varepsilon} \text{ and for some } T \text{ in PSL}(2,\mathbb{R}) \, ; \tag{8}$$

$$\check{\phi} \circ f \circ \check{\phi} = T \text{ on } U'', \text{ where } f \text{ and } T \text{ are related as in equation (8)} \, ; \tag{9}$$

$$T = \psi \circ T \circ \psi^{-1} \text{ on } U'', \text{ for any } T \text{ as in equation (8)} \, . \tag{10}$$

Actually equation (8) is obvious while equation (9) occurs because $\check{\phi}$ is an analytic continuation of ϕ. Finally equation (10) follows from the simple calculation on U''

$$T = \check{\phi} \circ f \circ \check{\phi}^{-1} = \psi \circ \phi \circ f \circ \phi^{-1} \circ \psi^{-1} = \psi \circ T \circ \psi^{1} \, .$$

Observe that equation (10) holds for a set of T's in $\text{PSL}(2,\mathbb{R})$ which generates a non-solvable subgroup Γ' of $\text{PSL}(2,\mathbb{R})$. In fact, to check this, it is enough to recall that $\Gamma^{(m)}$ is a non-solvable group which is contained in H^{ε} provided that m is very large. Since Γ' is not discrete (for instance because it has generators close to the identity), it follows that the closure of Γ' is the whole $\text{PSL}(2,\mathbb{R})$: indeed every non-discrete and non-solvable subgroup of $\text{PSL}(2,\mathbb{R})$ must be dense. Therefore equation (10) actually holds for the whole $\text{PSL}(2,\mathbb{R})$ as long as both members are defined. Hence Lemma (5.7) implies that $\psi = id$, i.e. ϕ coincides with $\check{\phi}$. Thus ϕ has an analytic extension to $\mathbb{S}^1$ as a local diffeomorphism. Furthermore ϕ must be a finite covering of $\mathbb{S}^1$. Denote by $k_0 \in \mathbb{N}$ the degree of ϕ.

Using ϕ, we construct a finite covering $\text{PSL}_{k_0}(2,\mathbb{R})$ of $\text{PSL}(2,\mathbb{R})$, which contains H^{ε} for ε small enough (see equation (8)). It remains to verify that this covering contains the whole Γ.

Consider a generating set for Γ, $< f^1, \ldots, f^l >$, satisfying our standard assumptions. Notice that for m_0 suficiently large, both $\mathbf{h}$ and $h^i = f^i \circ \mathbf{h} \circ (f^i)^{-1}$ lies in H^{ε} provided that $\mathbf{h} \in H(m_0)$, $i = 1, 2, \ldots, l$. Next we fix $i \in \{1, 2, \ldots, l\}$ and observe that the following equation does hold for suitable branches of ϕ^{-1}:

$$\phi \circ h^i \circ \phi^{-1} = \phi \circ f^i \circ \phi^{-1} \circ \phi \circ \mathbf{h} \circ \phi^{-1} \circ \phi \circ (f^i)^{-1} \circ \phi^{-1} \, . \tag{11}$$

It should be noted that for any branch of ϕ^{-1}, the map $\phi \circ \mathbf{h} \circ \phi^{-1}$ $\mathbf{h} \in H^{\varepsilon}$ is the restriction of an element $T^{\mathbf{h}}$ of $\text{PSL}(2,\mathbb{R})$. Now note that the collection of all $T^{\mathbf{h}}$'s as before generates a non-solvable subgroup Γ'' of $\text{PSL}(2,\mathbb{R})$. Letting $T^i = \phi \circ f^i \circ \phi^{-1}$, the equation (11) allows to conclude that

$$T^i \circ \Gamma'' \circ (T^i)^{-1} \subset \text{PSL}(2,\mathbb{R}) \, .$$

Passing to the closure of Γ'', one has

$$T^i \circ \text{PSL}(2,\mathbb{R}) \circ (T^i)^{-1} \subset \text{PSL}(2,\mathbb{R}) \, .$$

Hence $(T^i)_* \quad \subset \quad$, which implies that T^i belongs to $\text{PSL}(2,\mathbb{R})$ (see Remark (3.10)), in other words $f^i \in \text{PSL}_{k_0}(2,\mathbb{R})$. Letting i vary from 1 to l, we finally conclude that Γ is contained in $\text{PSL}_{k_0}(2,\mathbb{R})$. This accomplishes the proof. □

6 Finiteness of the number of connected components

In this section we shall complete the proof of Theorem A in the non-solvable case. This will be accomplished by proving that the sets $\mathbb{A}^k$ have only a finite number of connected components (cf. Theorem (6.1)). Throughout this section, unless we explicitly mention the contrary, we shall work under two basic assumptions, namely:

1. Γ is a non-solvable group (generated by a finite number of elements close to the identity).
2. If $n \geq 4$, then Γ is not a finite covering of a subgroup of $\mathrm{PSL}(2, \mathbb{R})$.

The main result of this section is:

Theorem 6.1 *The number of connected components of $\mathbb{A}^k$ is finite for all $k = 1, \ldots, n-1$.*

Because $\mathbb{A}^n = \mathbb{T}^n \setminus [\mathrm{P}\,(\widetilde{\Gamma}) \cup \bigcup_{k=1}^{n-1} \mathbb{A}^k]$, we immediately obtain the following consequence.

Corollary 6.2 *$\mathbb{A}^n$ has a finite number of connected components.* □

Let us prove the theorem above firstly in the case of $\mathrm{P}\,(\widetilde{\Gamma}) = \emptyset$. This case will follow from the lemma below which slightly generalizes Corollary (4.3). The argument is now simpler since the existence of vector fields in the closure of $\widetilde{\Gamma}$ and having all coordinates without singularity in a neighborhood of a point of $\mathbb{T}^n \setminus \mathrm{P}\,(\widetilde{\Gamma})$ was already settled.

Lemma 6.3 *Suppose that $p \in \mathbb{T}^n \setminus \mathrm{P}\,(\widetilde{\Gamma})$ is a point of rank s which is accumulated by points of $\mathbb{A}^k$, with $n > k \geq s$. Then there is a neighborhood U of p and an analytic set $\mathcal{A}$ of dimension k such that*

$$\overline{\mathbb{A}^k} \cap U \subset \mathcal{A} \cap U \,,$$

where $\overline{\mathbb{A}^k}$ stands for the closure of $\mathbb{A}^k$.

Proof Corollary (4.3) implies the statement for points $p \in \mathbb{A}^k$. Hence we just need to check the case of points $p \in \overline{\mathbb{A}^k} \setminus \mathbb{A}^k$.

Consider a small neighborhood U of p and let $\mathcal{A}$ be the smallest analytic set of U containing $\overline{\mathbb{A}}^k$. The set $\mathcal{A}$ is clearly invariant under all vector fields defined around p and contained in the closure of $\widetilde{\Gamma}$.

Suppose for a contradiction that the dimension of $\mathcal{A}$ is $m > k$. Because p has rank s, there are s linearly independent vector fieds in the closure of $\widetilde{\Gamma}$ relative to U (modulo shrinking U if needed). Furthermore, thanks to Theorem (4.2), there is no loss of generality in supposing that all the coordinates of X^1 are non-singular in U.

Choosing a regular point q of $\mathcal{A}$ and modulo a rearrangement of indices, we can suppose that

$$\mathcal{A} \cap \mathrm{B}_\epsilon(q) = \{(x_1, \ldots, x_m, F(x_1, \ldots, x_m) \;\; ; \;\; (x_1, \ldots, x_m) \in U'\} \,, \tag{12}$$

for some $\epsilon > 0$, where U' is a suitable neighborhood of $q' = (q_1, \ldots, q_m)$ and F is an analytic function. Letting $Z^1 = (X_1^1, \ldots, X_m^1), \ldots, Z^s = (X_1^s, \ldots, X_m^s)$, we see that these vector fields are linearly independent on U'.

Thus applying successively Theorem (3.8) $k-s+1$ times, we obtain a non-constant function D which depends on the variables $x_1, \ldots, x_m$ and vanishes on $\overline{\mathbb{A}}^k$ (since points in $\overline{\mathbb{A}}^k \setminus \mathbb{A}^k$ have rank less than k). In fact D is given by an appropriate determinant (here we are using the fact that Γ is not contained in a finite cover of $\mathrm{PSL}(2, \mathbb{R})$). Because $\mathcal{A}$ is the smallest analytic set containing $\overline{\mathbb{A}}^k$, it follows that D vanishes on $\mathcal{A}$. However, since D depends only on the $x_1, \ldots, x_m$ variables, the local expression of $\mathcal{A}$ given by equation (12), implies that D vanishes everywhere. The resulting contradiction proves the lemma. □

Remark 6.4 The proof above actually shows that, apart from an analytic subset strictly contained in $\mathcal{A}$, all the points in $\mathcal{A}$ have rank exactly k.

Proof of Theorem (6.1) in the case $\mathrm{P}(\widetilde{\Gamma}) = \emptyset$ Consider a point $p \in \mathbb{T}^n$ and let U be a small neighborhood of p. According to Lemma (6.3), modulo reducing U, either $\mathbb{A}^k$ does not intersect U or there is an analytic subset $\mathcal{A}$ of U having dimension k and containing all points of $\overline{\mathbb{A}}^k \cap U$ (provided that $k < n$).

From the previous remark, it results that all the irreducible components $\mathcal{A}'$ of $\mathcal{A}$ contain only a finite number of connected components of $\mathbb{A}^k$. Indeed every point of $\mathcal{A}'$ has rank k apart from a non-trivial analytic set. Since $\mathcal{A}$ has only a finite number of irreducible components, we obtain that only a finite number of connected components of $\mathbb{A}^k$ intersects U. Now the proof follows from the compactness of $\mathbb{T}^n$. □

From now on, let us suppose that $\mathrm{P}\,(\widetilde{\Gamma})$ is a non-empty set. We also fix a connected component $\mathcal{K}$ of $\mathbb{T}^n \setminus \mathrm{P}\,(\widetilde{\Gamma})$.

By a small abuse of language, given a point $p \in \partial\mathcal{K}$, every set which is the intersection of a neighborhood of p in $\mathbb{T}^n$ with $\mathcal{K}$ will be referred to as a neighborhood of p in $\mathcal{K}$.

Our method to prove Theorem (6.1) in the case under consideration is based on the existence of vector fields as those of Theorem (3.4) around every point of $\overline{\mathcal{K}}$. Precisely, we have

Lemma 6.5 *Let p be a point of $\overline{\mathcal{K}}$. There is a corner point q of $\mathcal{K}$ with the following properties:*

ı) For any open cube $U(\epsilon) = \prod_{j=1}^n (q_i, q_i + \epsilon)$ (as in Theorem (3.4)), there is an element $\tilde{f} \in \widetilde{\Gamma}$, such that $\tilde{f}(U(\epsilon))$ becomes a neighborhood of p.

ıı) The neighborhood of p obtained in item ı is equipped with a sequence of vector fields $X \prec Y \prec \ldots$ contained in the closure of $\widetilde{\Gamma}$. Moreover we may suppose that X has all coordinates without singularities on this neighborhood.

Proof Let us first check item ı. If p belongs to $\mathcal{K}$ or to $\mathrm{Fix}\,(\widetilde{\Gamma})$, the statement is clear. Thus suppose that p is not as before. Up to a rearrangement of indices, we can write $p =$

$(p_1, \dots, p_r, p_{r+1}, \dots, p_n)$, where $p_1, \dots, p_r$ do not belong to Fix(Γ) and $p_{r+1}, \dots, p_n$ are fixed points of Γ.

Thanks to Proposition (4.11), there is a corner point $q' = (q_1, \dots, q_r)$ of $\mathcal{K}' = \prod_{1,\dots,r}(\mathcal{K})$ (where $\prod_{1,\dots,r}$ stands for the projection of $\mathcal{K}$ onto the $r-$first coordinates), such that, for any open cube $U'(\epsilon) = \prod_{j=1}^{r}(q_j, q_j+\epsilon)$ contained in $\mathcal{K}'$, there is an element $\tilde{f}$ of $\widetilde{\Gamma}$ for which $\tilde{f}(U'(\epsilon))$ is a neighborhood of $(p_1, \dots, p_r)$ in $\mathcal{K}'$.

Next define $q = (q_1, \dots, q_r, p_{r+1}, \dots, p_n)$. Clearly this is a corner point of $\mathcal{K}$. Moreover, if $U(\epsilon) = U'(\epsilon) \times \prod_{j=r+1}^{n}$ is contained in $\mathcal{K}$, then $\tilde{f}(U(\epsilon))$ is a neighborhood of p in $\mathcal{K}$ (cf. the comment before the proof of Proposition (4.11)).

Let us now check item $\imath\imath$. As long as ϵ is small, $U(\epsilon)$ may be equipped with a pair of vector fields $X \prec Y$, contained in the closure of $\widetilde{\Gamma}$ (see Corollary (3.5)). We may also suppose that all the coordinates of X have no singularities of $U(\epsilon)$ (modulo reducing ϵ).

Therefore the image $\tilde{f}(U(\epsilon))$ of $U(\epsilon)$ may be equipped with the pair of vector fields $\tilde{f}_*X \prec \tilde{f}_*Y$ contained in the closure of $\widetilde{\Gamma}$ and such that $\tilde{f}_*X$ has all coordinates without singularities in $\tilde{f}(U(\epsilon))$. Finally considering the infinite sequence

$$\tilde{f}_*X, \tilde{f}_*Y, [\tilde{f}_*X, \tilde{f}_*Y], [\tilde{f}_*X, [\tilde{f}_*X, \tilde{f}_*Y]], \dots ,$$

a simple calculation shows that this sequence fulfils all the required conditions (see Section 3).
□

Lemma 6.6 *Assume we are given two vector fields $X \prec Y$ in the closure of $\widetilde{\Gamma}$ relative to a neighborhood $\tilde{f}(U(\epsilon))$ as before. Then, for any pair of indices $1 \leq i_1 < i_2 \leq n$, the vector fields $X' = (X_{i_1}, X_{i_2})$ and $Y' = (Y_{i_1}, Y_{i_2})$ are linearly independent away from a non-trivial analytic subset of the plane.*

Proof Suppose for a contradiction that the expression $X_{i_1}Y_{i_2} - Y_{i_1}X_{i_2}$ vanishes identically. Hence, for x_{i_1}, x_{i_2} in $(0, \varepsilon)$ (ε small), we have

$$\frac{X_{i_1}}{Y_{i_1}}(x_{i_1}) = \frac{X_{i_2}}{Y_{i_2}}(x_{i_2})\,. \tag{13}$$

Therefore equation (13) defines a constant function λ. Thus one has $X_{i_1} = \lambda Y_{i_1}$, for x_{i_1} in $(0, \varepsilon)$. however this contradicts our previous assumption about $X \prec Y$. □

In the sequel, it is sometimes convenient to think of a vector field X defined of an open set U as a vector field defined also on certain points of the boundary of U. Precisely we simply set $X(p) = \lim_{q \to p} X(q)$, where $q \in U$ and p belongs to the boundary ∂U of U provided that this limit makes sense. In particular a point of ∂U may or may not be singular.

Also we say that an open cube $U \subset \mathcal{K}$ is greater than another open cube $V \subset \mathcal{K}$, if and only if, there are open cubes U' and V' of $\mathbb{T}^n$, with $\overline{V'} \subset U'$ and $U = U' \cap \mathcal{K}$, $V = V' \cap \mathcal{K}$.

Hereafter these conventions will be assumed without further comments.

Proof of Theorem (6.1) in the case $\mathrm{P}(\widetilde{\Gamma}) \neq \emptyset$ Fix a natural number k with $1 \leq k \leq n-1$ and suppose for a contradiction that $\mathbb{A}^k$ has an infinite number of connected components C_j, $j = 1, 2, \dots$ in $\mathcal{K}$.

To achieve the desired contradiction, we shall reduce the dimension of $\mathbb{A}^k$ up to zero. In each step, the submanifold of reduced dimension will be invariant by a sequence of vector fields $X \prec Y \prec \ldots$ in the closure of $\widetilde{\Gamma}$. In the last step, we shall end-up with an infinite number of singularities for a sequence of vector fields as before. This will imply the final contradiction.

Step 1: For each $j \in \mathbb{N}$, choose a point p_j^k of C_j. We can suppose that all these points have distance to all corner points bounded from below by a positive constant α as it will follow from the next lemma.

Lemma 6.7 *There exists a positive number α such that, for every $j \in \mathbb{N}$, there is a point of C_j whose distance to all corner points is not less than α.*

Proof Recall that each corner point q may be equipped with a vector field X in the closure of $\widetilde{\Gamma}$ relative to $U(\epsilon) = \prod_{i=1}^{n}$. Furthermore q is the unique singularity of X on $\overline{\mathcal{K}}$ (for a suitable $\epsilon > 0$). We can suppose that all these neighborhoods are pairwise disjoint, and in fact, that the distance between two of them is not less than a fixed $\beta > 0$ (otherwise we reduce ϵ).

Because q is the unique singularity of X, there is a positive number $\alpha(q)$, such that, to any point $q' \in \mathrm{B}_{\alpha(q)}(q) \cap \mathcal{K}$, it corresponds a time $t_{q'}$ satisfying $\|\Phi_X^{t_{q'}}(q') - q\| \geq \alpha(q)$.

Given a point p_j of C_j whose distance to a corner point is less than $\alpha(q)$, the point $\Phi_X^{t_{p_j}}(p_j)$ is again a point of C_j. Actually X leaves C_j invariant. Moreover the distance between $\Phi_X^{t_{p_j}}(p_j)$ and q is at least $\alpha(q)$. Therefore defining $\alpha = \min\{\beta, \alpha(q) \ ; \ q \text{ is a corner point}\}$, the result follows at once. □

Passing to a subsequence if needed, we can suppose that $\{p^{k,j}\}_{j\in\mathbb{N}}$ converges to some point p^k of $\overline{\mathcal{K}}$. After Lemma (6.5), p^k is contained in a neighborhood U^k of the form $\tilde{f}(U(\epsilon))$, for appropriate $\tilde{f}$ and $U(\epsilon)$, which is equipped with a sequence of vector fields $X \prec Y \prec \ldots$ contained in the closure of $\widetilde{\Gamma}$. Moreover no coordinate of X has a singularity on U^k. Because p^k is not a corner point, for some index $i_k \in \{1, \ldots, n\}$ we have $X_{i_k}(p_{i_k}^k) \neq 0$. Let us define

$$L^k = [x_{i_k} = p_{i_k}] \cap \mathcal{K}\,.$$

On the other hand, one has

Lemma 6.8 *There is a sequence of vector fields $X^k \prec Y^k \prec \ldots$ in the closure of $\widetilde{\Gamma}$ relative to U^k leaving L^k invariant.*

Proof Let $p' \in L^k$ be a point at which X and Y are linearly independent. Note that such a point always exists after Lemma (6.6). Choosing a suitable linear combination X^k between X and Y, it is possible to obtain $X^k(p') = 0$. Hence L^k is invariant under X^k.

Applying this procedure to every consecutive pairs of the sequence $X \prec Y \prec \ldots$, we obtain a new sequence of vector fields $X^k \prec Y^k \prec \ldots$ as desired. □

For every point p' in $\mathrm{B}_\delta(p^k) \bigcap \mathcal{K}$, we consider the image $\Phi_X^t(p')$ of p' under the local flow of X. Since $X_{i_k}^k(p_{i_k}) \neq 0$, there is a positive number δ such that the projection of the local orbit of p' through X on the $i_k^{\mathrm{th}}-$variable covers the interval $(p_{i_k} - \delta, p_{i_k} + \delta)$.

We conclude that all but a finite number of the sets C^j's must intersect L^k. Indeed for every j sufficiently large p^j belongs to $\mathrm{B}_\delta(p^k) \cap \mathcal{K}$ and X leaves C^j invariant. Therefore we may suppose without loss of generality that all the C^j's intersect L^k $(j = 1, 2, \ldots)$. Furthermore we have:

Lemma 6.9 *In small neighborhoods of* p^k, $C^j \cap L^k$ *is a submanifold of dimension* $k-1$ $(j = 1, 2, \ldots)$.

Proof Let W be a sufficiently small neighborhood of p. Since $X_{i_k}(p_{i_k}) \neq 0$, we see that $C^j \cap W$ is transverse to $L^k \cap W$. The result follows at once. □

Finally consider two open cubes V^k, W^k containing p^k such that V^k is smaller than U^k and W^k is smaller than V^k. We also suppose that V^k is so small that Lemma (6.9) holds in V^k. Define $C^{(k-1),j} = C^j \cap L^k \cap V^k$. Without loss of generality we may suppose that $C^{(k-1),j}$ is a connected manifold (otherwise we just have to "forget" other connected components). Since $C^{(k-1),j}$ is the intersection of sets which are invariant under the vector fields in the sequence $X^k \prec Y^k \prec \ldots$, it follows that $C^{(k-1),j}$ is invariant under these vector fields as well.

Step 2: For each $j \in \mathbb{N}$, choose a point $p^{(k-1),j}$of $C^{(k-1),j} \cap W^k$. Arguing as in Lemma (6.7), we can suppose that all these points have distance to all singularities of X^k bounded from below by a positive constant.

Passing to a subsequence if needed, we may assume that $\{p^{(k-1),j}\}_{j \in \mathbb{N}}$ converges to some point p^{k-1} of $\overline{W^k}$. From the choice of $p^{(k-1),j}$ $(j = 1, 2, \ldots)$, it follows the existence of an index i_{k-1}, such that $X_{i_{k-1}}(p^{k-1}_{i_{k-1}}) \neq 0$. Let us define.

$$L^{k-1} = [x_{i_{k-1}} = p^{k-1}_{i_{k-1}}] \cap [x_{i_k} = p_{i_k}] \cap \mathcal{K}\,.$$

As in **Step 1**, we can obtain a sequence of vector fields $X^{k-1} \prec Y^{k-1} \prec \ldots$, in the closure of $\widetilde{\Gamma}$ relative to U^k, which leave L^{k-1} invariant.

Again as in **Step 1**, there are open cubes V^{k-1}, W^{k-1} containing p^{k-1} with V^{k-1} smaller than V^k and W^{k-1} smaller than V^{k-1}. Furthermore, choosing these cubes small enough, $C^{(k-2),j} = C^{(k-1),j} \cap L^{k-1} \cap V^{k-1}$ is a smooth analytic manifold of dimension $k-2$.

Without loss of generality we may suppose that $C^{(k-2),j}$ is connected for $j = 1, 2, \ldots$. Finally all the vector fields of the sequence $X^{k-1} \prec Y^{k-1} \prec \ldots$ leave these sets invariant.

Continuing this inductive argument, we eventually arrive to **Step k**.

Step k: Just as before, we obtain the following objects:

A point p^1, an index i_1 and a set $L^1 = [x_{i_k} = p^k_{i_k}] \cap \ldots \cap [x_{i_1} = p^1_{i_1}] \cap \mathcal{K}$. Moreover there is a sequence of vector fields $X^1 \prec Y^1 \prec \ldots$ in the closure of $\widetilde{\Gamma}$ relative to U^k leaving L^1 invariant.

Besides, we have open cubes containing p^1, V^1 smaller than V^k and W^1 smaller than V^1, such that $C^{0,j} = C^{1,j} \cap L^1 \cap W^1$ is a set of points $(j = 1, 2, \ldots)$.

In particular all the vector fields of the sequence $X^1 \prec Y^1 \prec \ldots$ leave $C^{0,j}$ invariant. In other words $C^{0,j}$ is constituted by singularities of X^1 for $j = 1, 2, \ldots$. However X^1 is an analytic vector field on V^k possessing non-trivial asymptotic development at every point of the boundary of V^k. Actually, to check this assertion it suffices to recall that X^1 is obtained as a linear combination of vector fields whose asymptotic "flatness" order is strictly increasing. Because this observation is verified for every coordinate of X^1, it results that none of these coordinates has a trivial asymptotic development at a point of the boundary of V^k. In particular none of these coordinates vanishes identically on V^k.

On the other hand, since X^1 has a infinite number of singularities on V^k, it follows that one of its coordinates either vanishes identically or at least has a trivial asymptotic development at some point in the boundary of V^k. The resulting contradiction proves the theorem. □

We finish this section by proving the most difficult part of Theorem A namely, the case of non-solvable groups. Recall that solvable groups will be treated independently in the next section.

Proof of Theorem A in the non-solvable case We consider the connected components of the sets $\mathbb{A}^n$. It was seen that there are only a finite number of these components. Furthermore the union of these components is an open dense set having in addition total Lebesgue measure of $\mathbb{T}^n$, unless Γ is conjugate to a subgroup of a finite covering of $\mathrm{PSL}(2,\mathbb{R})$ and $n \geq 4$. The case $n \geq 4$ and Γ conjugate to a finite subgroup of $\mathrm{PSL}(2,\mathbb{R})$ was already discussed in Remark (3.9) and Lemma (4.1).

In the sequel we suppose that Γ is not conjugate to a subgroup of $\mathrm{PSL}(2,\mathbb{R})$ whenever $n \geq 4$.

Fix a connected component U of $\mathbb{A}^n$. First suppose that U is invariant under $\widetilde{\Gamma}$. Let us to check that the restriction of the $\widetilde{\Gamma}-$action to U is ergodic.

Suppose for a contradiction that the statement is false. So there is a $\widetilde{\Gamma}-$invariant Borel set $\mathcal{B} \subset U$ whose measure $\mu(\mathcal{B})$ is strictly comprised between 0 and $\mu(U)$. Hence there are Lebesgue density points p for $\mathcal{B}$ and q for $U \setminus \mathcal{B}$. Now notice that the orbit $\mathcal{O}_p$ of p under the pseudo-Lie algebra of vector fields in the closure of $\widetilde{\Gamma}$ (as in Sussmann's Lemma) must be the whole U since that U connected. In particular $q \in \mathcal{O}_p$. Therefore there is a local diffeomorphism ψ defined around p and satisfying $\psi(p) = q$. Besides ψ can be $C^\infty-$aproximated by elements of $\widetilde{\Gamma}$, i.e. there is a sequence $\tilde{\mathbf{h}}^i$ of elements in $\widetilde{\Gamma}$ converging C^∞ to ψ on a small neighborhhod of p.

Finally consider a sequence of balls $B(\epsilon_j)$ centered at p whose diameters ϵ_j are going to *zero*. Because p is a Lebesgue density point of $\mathcal{B}$, it results that

$$\lim_{j\to\infty} \frac{\mu(\mathcal{B} \cap B(\epsilon_j))}{\mu(B(\epsilon_j))} = 1\,.$$

For each j, let $i(j)$ be such that $\tilde{\mathbf{h}}^{i(j)}(B(\epsilon_j))$ is a neighborhood of q. Since the sequence $\tilde{\mathbf{h}}^i$ is C^1-uniformly bounded, one has

$$\frac{\mu(\tilde{\mathbf{h}}^{i(j)}(\mathcal{B} \cap B(\epsilon_j))}{\mu(\tilde{\mathbf{h}}^{i(j)}(B(\epsilon_j)))} \geq Const \frac{\mu(\mathcal{B} \cap B(\epsilon_j))}{\mu(B(\epsilon_j))}\,,$$

for some positive constant $Const$. Using the $\mu-$a.e. invariance of $\mathcal{B}$, we deduce that

$$\lim_{j\to\infty} \frac{\mu(\mathcal{B} \cap \tilde{\mathbf{h}}^{i(j)}(B(\epsilon_j)))}{\mu(\tilde{\mathbf{h}}^{i(j)}(B(\epsilon_j))} \geq \lim_{j\to\infty} \frac{\mu(\tilde{\mathbf{h}}^{i(j)}(\mathcal{B} \cap B(\epsilon_j))}{\mu(B(\epsilon_j))} \geq Const \frac{\mu(\mathcal{B} \cap B(\epsilon_j))}{\mu(B(\epsilon_j))} > 0$$

which contradicts the fact that q is a Lebesgue density point for $U \setminus \mathcal{B}$. This concludes the proof in this first case.

In general case, the connected components $U_1, \ldots, U_l$ of $\mathbb{A}^n$ do not need to be invariant under $\widetilde{\Gamma}$. However the set $\{1, \ldots, l\}$ can be partitioned into "cycles" (i.e. subsets) whose corresponding connected components are permuted by $\widetilde{\Gamma}$. For each of these cycles, the corresponding union of the U_i's constitutes an ergodic component of $\widetilde{\Gamma}$. □

It follows directly from the proof that that each ergodic component U of $\widetilde{\Gamma}$ has all $\widetilde{\Gamma}-$orbit dense in U. Of course the same argument may be employed for the connected components of $\mathbb{A}^k$, for $n = 1, \ldots, n-1$, which give us a complete description of the structure of the minimal sets.

7 Solvable groups

This section is intended to treat the case of solvable subgroups of $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$ so as to complete the proof of Theorem A. We shall deal with an infinite solvable subgroup Γ of $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$ possessing a finite set of generators.

Of course there is much information on solvable subgroups of both $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$ and $\mathrm{Diff}_{\mathbb{R}}(\mathbb{C}, 0)$ available. This will make our task reasonably easy. These results are folkloric and collected in several papers (e.g. [Gh], [Na], [EISV]). We mention here only those which will be useful to us.

The study of solvable subgroups Γ of $\mathrm{Diff}^{\omega}(\mathbb{S}^1)$ relies on the probability measure ν preserved by Γ (cf. for instance [C-S-V]). The support of ν, $\mathrm{supp}\,(\nu)$, is a closed subset of $\mathbb{S}^1$ which is invariant under Γ. A *priori* there are three possibilities for $\mathrm{supp}\,(\nu)$, namely:

1. the whole $\mathbb{S}^1$;
2. a Cantor set;
3. a finite union of points.

However the item 2 cannot occur: according to a well-known theorem due to Sacksteder (cf. [Sa]) there would be a diffeomorphism $f \in \Gamma$ having a hyperbolic fixed point. This is obviously incompatible with the existence of the invariant measure ν. Furthermore, if $\mathrm{supp}\,(\nu) = \mathbb{S}^1$, then Γ is Abelian. To verify this assertion, it suffices to parametrize $\mathbb{S}^1$ by means of ν so as to obtain a conjugacy between Γ and a group of rotations.

Finally, if Γ has a finite orbit $\{p_1, \ldots, p_l\}$, then the orbit of any point in $\mathbb{S}^1 \setminus \{p_1, \ldots, p_l\}$ is dense in the connected component of $\{p_1, \ldots, p_l\}$ containing this point.

Next let $\widehat{\mathrm{Diff}}(\mathbb{R}, 0)$ denote the group of formal diffeomorphisms of $(\mathbb{R}, 0)$ which are tangent to the identity. Let $\hat{f} \neq id$ be an element of $\widehat{\mathrm{Diff}}(\mathbb{R}, 0)$. It is known that f is the time-one map induced by a formal vector field $\widehat{X}$. Furthermore the centralizer of $\hat{f}$ in $\widehat{\mathrm{Diff}}(\mathbb{R}, 0)$ consists precisely of the formal maps $\exp(t\widehat{X})$ for $t \in \mathbb{R}$.

Consider an element f of $\mathrm{Diff}_{\mathbb{R}}(\mathbb{C}, 0)$ such that $f'(0) > 0$ (recall that our diffeomorphisms preserve the orientation). If $f'(0) \neq 1$ (i.e. f is hyperbolic) then there is an analytic coordinate in which f can be written as a homothety $x \mapsto \lambda x$, $\lambda = f'(0)$. In particular all diffeomorphisms different from the identity and commuting with f are also hyperbolic (and given by homotheties in the same coordinate).

On the other hand, suppose that $f'(0) = 1$ but $f \neq id$. As mentioned above the centralizer of f (i.e. the elements of $\text{Diff}_{\mathbb{R}}(\mathbb{C}, 0)$ which commute with f) consists of diffeomorphisms tangent to the identity. Indeed the centralizer of f is contained in the one-parameter group $\exp(t\widehat{X})$, $t \in \mathbb{R}$ of formal diffeomorphisms (where $\widehat{X}$ stands for the formal vector field whose time-one induced map agrees with f). In addition this centralizer is isomorphic either to $\mathbb{R}$ or to $c\mathbb{Z}$ (for some $c \in \mathbb{R}$). When this centralizer is isomorphic to $\mathbb{R}$ then the formal vector field $\widehat{X}$ is *convergent* i.e. X actually defines an analytic vector field defiend around 0. Summarizing the discussion above we have:

Lemma 7.1 *Consider a subgroup $\Gamma_0 \subset \text{Diff}^{\omega}(\mathbb{S}^1)$ whose elements fix a point $p \in \mathbb{S}^1$. Suppose that Γ_0 is an Abelian group which is not a finite extension of $\mathbb{Z}$. Then there is an analytic vector field X defined around p and contained in the $C^{\infty}-$closure of Γ_0.*

Proof Suppose first the existence of an element $f \in \Gamma_0$ such that $f'(p) \neq 1$. In an appropriate coordinate around p, the restriction of the elements of Γ_0 is given by hometheties. Hence there is an obvious identification of Γ_0 with a subgroup of $\mathbb{R}^*$ and, since Γ_0 is not a finite extension of $\mathbb{Z}$, this latter subgroup must be dense in $\mathbb{R}^*$. We conclude that the vector field associated to the flow $\Phi^t(x) = e^t x$ is contained in the closure of Γ_0 with respect to the above mentioned neighborhood.

On the other hand, if Γ_0 consists of diffeomorphsims tangent to the identity, then the formal vector field which contains Γ_0 must be convergent since Γ_0 is not a finite extension of $\mathbb{Z}$. It immediately follows that this vector field in contained in the closure of Γ_0. □

Now let us prove that Γ is piecewise ergodic on $\mathbb{S}^1$ unless Γ is a finite extension of $\mathbb{Z}$.

Proposition 7.2 *Let $\Gamma \subset \text{Diff}^{\omega}(\mathbb{S}^1)$ be an infinite finitely generated solvable group. Assume that Γ is not piecewise ergodic on $\mathbb{S}^1$. Then Γ is a finite extension of $\mathbb{Z}$.*

Proof It is well-known that a C^2-diffeomorphism of $\mathbb{S}^1$ whose rotation number is irrational acts ergodically on $\mathbb{S}^1$ (cf. [H-K]). Hence all the elements of Γ must have rational rotation number.

Consider the probability measure ν preserved by Γ. We claim that the support of ν is a finite orbit. Indeed, suppose for a contradiction that the support of ν is the entire $\mathbb{S}^1$. In this case Γ is conjugate to a group of rotations. Besides all these rotations are rational and thus periodic. Since Γ is finitely generated, it follows that Γ is actually finite which is the desired contradiction.

We have then concluded the existence of a finite orbit $\{p_1, \ldots, p_l\}$ to Γ. Let Γ_0 be the subgroup of Γ consisting of diffeomorphisms fixing p_1. Clearly Γ_0 has finite index in Γ. Besides, since Γ_0 preserves orientation, Γ_0 actually fixes all points $p_1, \ldots, p_l$.

Assume that Γ_0 is Abelian. Because Γ_0 cannot be a finite extension of $\mathbb{Z}$ (otherwise γ itself would be a finite extension of $\mathbb{Z}$), Lemma (7.1) ensures the existence of a non-trivial vector field

X_i in the $C^\infty-$closure of Γ_0 defined around p_i ($i = 1, \ldots, p_l$). Therefore the method employed in the proof of Theorem A (the non-solvable case) implies that Γ_0 is "locally ergodic" on the "one-sided" small neighborhhod of p. However the orbit of every point in s intersects a small neighborhood of some p_i, thus Γ_0 and hence Γ is piecewise ergodic.

Suppose now that Γ_0 is solvable but not Abelian. Fix p_i and a local coordinate taking p_i to $0 \in \mathbb{R}$. Because Γ_0 is not Abelian, Γ_0 contains two elements whose "flatness orders" at $0 \simeq p_i$ are different. Using Nakai's method, it is then possible to contruct an analytic vector field X^+ (resp. X^-) defined in $(0, \varepsilon)$ (resp. $(-\varepsilon, 0)$) and contained in the closure of Γ_0 (for some $\varepsilon > 0$). Just as before we can deduce that Γ_0 is piecewise ergodic. The proof of the proposition is over.

□

Now we shall focus on solvable subgroups Γ of $\mathrm{Diff}^\omega(\mathbb{S}^1)$ which are not abelian (nor finite extensions of $\mathbb{Z}$). These groups were previously discussed in [Gh], here those results will be sharpened.

We have seen that such a group Γ possesses a finite orbit $\{p_1, \ldots, p_l\}$. Again we let Γ_0 denote the common stabilizer of the points p_i's. Clearly Γ_0 is not a finite extension of $\mathbb{Z}$. Therefore there exists a non-trivial vector field X_i defined in a neighborhood of p_i ($i = 1, \ldots, l$) which is contained in the closure of Γ_0.

First we shall prove that Γ_0 is not Abelian. In other words, there is no solvable but non-Abelian subgroup of $\mathrm{Diff}^\omega(\mathbb{S}^1)$ which is a finite extension of an Abelian group "greater" than $\mathbb{Z}$.

Lemma 7.3 *Suppose that Γ_0 is an Abelian group which is not a finite extension of $\mathbb{Z}$. Then there is a globally defined $C^\omega-$vector field X on $\mathbb{S}^1$ whose flow contains the elements of Γ_0. Moreover X is unique up to multiplication by a scalar.*

Proof Since Γ_0 has fixed points, it immediately follows that the set of finite orbits of Γ_0 does coincide with the set of its fixed points. Denote by $\{q_1, \ldots, q_s\}$ the set of fixed points of Γ_0. For each q_j ($j = 1, \ldots, s$), there exists a (unique up to parametrization) vector field X_j defined in a neighborhhod of q_j and contained in the closure of Γ_0. Thus the flow of X_j contains the restrictions of elements in Γ_0.

On the other hand, a proposition attributed to G. Hector (cf. [Gh]) ensures that Γ_0 has dense orbits on the connected components of $\mathbb{S}^1 \setminus \{q_1, \ldots, q_s\}$. Thus beginning with X_1 defined around p_1 and using the commutatibility of Γ_0, it is possible to extend the domain of X_1 so as to include the interval $[q_1, q_2)$. Besides, thanks to the uniqueness of the X_i's, X_1, X_2 must agree on the overlaps of their domains. Thus X_1 can in fact be extended to a neighborhhod of q_2. Continuing this procedure, we eventually realize X_1 as a global vector field X defined on $\mathbb{S}^1$. This proves the lemma. □

We are now able to prove that Γ_0 is not Abelian.

Lemma 7.4 Γ_0 *is not Abelian.*

Proof Suppose for a contradiction that Γ_0 is Abelian. The strategy of proof consists of showing that Γ itself is Abelian, hence obtaining a contradiction.

First observe that $\{p_1, \ldots, p_l\}$ (the set of periodic points of Γ) is contained in $\{q_1, \ldots, q_s\}$ which is in turn the set of the singularities of the vector field X.

Next consider $f \in \Gamma \setminus \Gamma_0$. There is $i_0 \neq 1$ such that $f(p_1) = p_{i_0}$ Hence the pull-back of X_{i_0} by f is a constant multiple of X_1 (otherwise there would be another vector field $f^* X_{i_0}$ in the closure of Γ_0 relative to a neighborhhod of p_1). Since all the X_i's glue together into a global vector field X, we conclude that $f^* X$ is a constant multiple of X and hence equal to X.

The discussion above shows that f preserves X for all $f \in \Gamma \setminus \Gamma_0$. In particular the set of singularities of X constitutes a finite orbit for Γ. In other words, the sets $\{p_1, \ldots, p_l\}$ and $\{q_1, \ldots, q_s\}$ do coincide.

Obviously an element $f \in \Gamma$ leaving invariant a connected component of $\mathbb{S}^1 \setminus \{p_1, \ldots, p_l\}$ is contained in the flow of X. Thus let f (resp. h) be an element of $\Gamma \setminus \Gamma_0$ taking p_1 to p_{i_0} (resp. p_{i_1}). We need to prove that f, h do commute. In order to prove this, it is clearly sufficient to show that, in a suitable global coordinate on $\mathbb{S}^1$, the vector field X restricted to an interval $[p_i, p_{i+1}]$ is nothing but the push-forward of X restricted to $[p_1, p_2]$ by a convenient rational rotation. That is, in an appropriate coordinate, the vector field X is invariant under a rational rotation permuting the intervals $[p_i, p_{i+1}]$ $(i = 1, \ldots, l-1)$.

Note that f^l leaves invariant the intervals (p_1, p_{i+1}) (indeed l is the smallest positive integer such that f^l verifies this condition). Let Φ_X denote the flow of X. By the above discussion, there is $t_0 \in \mathbb{R}$ such that $\Phi_X^{-t_0} \circ f^l$ coincides with the identity. Hence, in a suitable coordinate, this element is a rational rotation permuting the intervals $[p_i, p_{i+1}]$. Furthermore this rotation clearly leaves X invariant. This establishes the lemma. □

Going back to the initial groups Γ, Γ_0, it was seen that Γ_0 is a solvable non-Abelian group.

Again $\{p_1, \ldots, p_l\}$ will denote the set of finite orbits of Γ and Γ_0 will be the common stabilizer of the p_i's. We know that Γ_0 is not Abelian. Hence, for every p_i $(i = 1, \ldots, l)$, Γ_0 contains hyperbolic elements at p_i (i.e. there is an element of Γ_0 for which p_i is a hyperbolic fixed point). In particular, there is a neighborhood U_i of p_i in which Γ_0 is $C^\omega-$conjugate to a subgroup of the group $\mathrm{Aff}(k_0)$ given by the elements of the form

$$x \longmapsto \frac{ax}{(1 + bx^{k_0})^{1/k_0}} \quad a \in \mathbb{R}_+^* \, , \; b \in \mathbb{R} \, .$$

Furthermore the "linearizing" coordinate is unique provided that Γ_0 is not discrete.

Note that the elements of Γ_0 which are tangent to the identity (at some p_i) form an Abelian group. This group actually contains the first derived group $D^1\Gamma_0$ of Γ_0 so that it is not reduced to $\{id\}$.

On the other hand, using the local coordinate mentioned above (or even Nakai's method), it is easy to check the existence of a vector field X_i defined on a neighborhood of p_i and contained in the closure of Γ_0. Furthermore X_i commute with (i.e. is preserved by) hyperbolic diffeomorphisms of Γ_0. However X_i is not preserved by the diffeomorphisms of Γ_0 which are tangent to the identity. This implies that pseudo-Lie algebra of vector fields contained in the closure of Γ_0 has dimension ≥ 2. Nonetheless, by virtue of Lie's theorem, this dimension must be precisely 2.

Lemma 7.5 *The vector field X_1 can be extended to a $C^\omega-$vector field X globally defined on $\mathbb{S}^1$.*

Proof Recall that the orbits of Γ_0 are dense in the connected components of $\mathbb{S}^1 \setminus \{p_1, \ldots, p_l\}$. As mentioned, hyperbolic diffeomorphisms of Γ_0 act on X_1 multiplying it by a constant. Thus it becomes clear that X_1 can be extended to $[p_1, p_2)$. However X_1 is uniquely determined up to parametrization since it commutes with an element of Γ_0. Therefore X_1 can be extended to a neighborhhod of p_2. Continuing this procedure, we eventually prove the lemma. □

The proposition below is the last ingredient needed to establish the Theorem A.

Proposition 7.6 *A solvable non-Abelian subgroup Γ of $\mathrm{Diff}^\omega(\mathbb{S}^1)$ is $C^\omega-$conjugate to a finite covering of a subgroup of $\mathrm{Aff}(\mathbb{R})$.*

Proof Let $\{p_1, \ldots, p_l\}$ and Γ_0 be as before. Note that a non-Abelian subgroup of $\mathrm{Aff}(\mathbb{R})$ possesses exactly one fixed point. Thus we shall consider the covering $\mathrm{Aff}_{k_0}(\mathbb{R})$ of $\mathrm{Aff}(\mathbb{R})$ having degree $k_0 \in \mathbb{N}$. Modulo a rotation, p_1 can be supposed fixed for both Γ_0 and $\mathrm{Aff}_{k_0}(\mathbb{R})$. Therefore, in a neighborhhod of p_1, we can identify Γ_0 with a subgroup of $\mathrm{Aff}_{k_0}(\mathbb{R})$ denoted by Γ_{aff}.

Denote by X_{Γ_0} and X_{Aff} the corresponding vector fields associated to these group as in Lemma (7.5) and let ϕ be a local conjugacy between Γ_0 and Γ_{aff}. Next observe that the vector field above are *synchronized* with respect to ϕ, that is the equation

$$\phi \circ \Phi^t_{X_{\Gamma_0}} = \Phi^t_{X_{\mathrm{Aff}}} \circ \phi$$

holds whenever both member are defined (where $\Phi^t_{X_{\Gamma_0}}, \Phi^t_{X_{\mathrm{Aff}}}$ stand for the corresponding local flows). Thus we immediately obtain an extension of ϕ to $[p_1, p_2)$. However, arguing as before, it is possible to extend ϕ to a neighborhood of p_2. So we shall eventually realize ϕ as a global conjugacy between Γ_0 and Γ_{Aff}.

Finally, considering an element $f \in \Gamma \setminus \Gamma_0$, we see that an appropriate iterated of f belongs to Γ_0 (since f permutes the p_i's). Once again we deduce that f is taken by ϕ to an element of $\mathrm{Aff}_{k_0}(\mathbb{R})$. This concludes the proof of the proposition. □

We are finally ready to complete the proof of Theorem A.

Proof of Theorem A We have already seen that, if a group Γ as in the statement of this theorem is non-solvable, then Γ is piecewise ergodic on the tori of dimensions $1, 2$ and 3. Furthermore, when Γ is not conjugate to a finite covering of $\mathrm{PSL}(2, \mathbb{R})$, then Γ is in fact piecewise ergodic on the torus of dimension $n \in \mathbb{N}$.

So it is enough to discuss the case of solvable groups. Hence we assume that Γ is an infinite solvable group which is not a finite extension of $\mathbb{Z}$. Under this assumption, Proposition (7.2) ensures that Γ is piecewise ergodic on $\mathbb{S}^1$. The case of other Abelian groups does not need to be treated (indeed they are never piecewise ergodic on $\mathbb{T}^2$). Finally we just have to consider a non-Abelian subgroup Γ of $\mathrm{Aff}_{k_0}(\mathbb{R})$.

Let $\Gamma \subset \text{Aff}_{k_0}(\mathbb{R})$ be as above. The existence of a vector field X defined on the whole $\mathbb{S}^1$ and contained in the closure of Γ implies that there is a vector field (X, X) defined on $\mathbb{T}^2$ and contained in the closure of the diagonal action $\widetilde{\Gamma}$ of Γ on $\mathbb{T}^2$. In view of the proof of the theorem in the non-solvable case, all we have to do is to show the existence of another vector field (Y, Y) defined on $\mathbb{T}^2$, contained in the closure of the action of $\widetilde{\Gamma}$ on $\mathbb{T}^2$ and linearly independent with (X, X) at generic points. To verify the existence of this vector field (Y, Y), note that there is a point $(p, p) \in \mathbb{T}^2$ which is fixed by a subgroup $\widetilde{\Gamma}_0$ having finite index in $\widetilde{\Gamma}$. Moreover there is also $(h, h) \in \widetilde{\Gamma}_0$ such that h is tangent to the identity at p. On the other hand, there is $(f, f) \in \widetilde{\Gamma}_0$ with f hyperbolic at p and such that $(f, f)^*(X, X)$ is parallel to (X, X). It is now clear that $(h, h)^*(X, X)$ cannot be parallel to (X, X) at generic points. This accomplishes the proof of Theorem A. □

• Further Comments

Let us finish this article by providing a generalization of our previous results. In particular we shall state a version of Theorem A (Theorem 7.8) which parallels Lie's theorem in a even more faithful way.

The following definition is very natural in view of the methods employe in this work.

Definition 7.7 *Consider $\mathbb{S}^1$ as the unit circle in $\mathbb{C}$. A subgroup $\Gamma \subset \text{Diff}^\omega(\mathbb{S}^1)$ will be called non-discrete if, for some $\tau > 0$, there exists a non-trivial sequence of elements $\{h_i\}$ in Γ verifying the conditions below:*

1. *Each h_i admits a holomorphic extension $h_{i,A}$ to the annulus A_τ;*
2. *These extensions converge uniformly to the identity on A_τ.*

According to the definition above, a non-discrete group is necessarily infinite. Also Theorem (2.1) gives a criterium ensuring the existence of many finitely generated non-discrete groups. Using this terminology, we obtain the following theorem which holds also for infinitely generated groups.

Theorem 7.8 *Let $\Gamma \subset \text{Diff}^\omega(\mathbb{S}^1)$ be a non-discrete subgroup of $\text{Diff}^\omega(\mathbb{S}^1)$. Then Γ is always piecewise ergodic on $\mathbb{S}^1$. Furthermore one has:*

1. *Γ is piecewise ergodic on $\mathbb{T}^2$ unless Γ is Abelian.*
2. *If Γ is not Abelian, then Γ is piecewise ergodic on $\mathbb{T}^3$ unless Γ is conjugate to a finite covering of a subgroup of the affine group $\text{Aff}(\mathbb{R})$.*
3. *If Γ is not conjugate to a finite covering of a subgroup of the affine group $\text{Aff}(\mathbb{R})$, then Γ is piecewise ergodic on $\mathbb{T}^4$ unless Γ is conjugate to a finite covering of a subgroup of $\text{PSL}(2, \mathbb{R})$*
4. *If Γ is not conjugate to a finite covering of a subgroup of $\text{PSL}(2, \mathbb{R})$, then Γ is piecewise ergodic on $\mathbb{T}^n$ for all $n \in \mathbb{N}$.*

Before going into the proof of this slightly more general result, it is interesting to compare it with Lie's theorem. Thinking of a vector field as a 1-parameter group, we can say by a small abuse of language that a Lie algebra *acts* on $\mathbb{S}^1$. Of course a non-trivial Lie algebra is always piecewise ergodic on $\mathbb{S}^1$. On the other hand the three possible finite dimensional Lie algebras are characterized by leaving a foliation invariant on the tori of dimensions respectively $2, 3$ and 4. Infinite-dimensional Lie algebras on $\mathbb{S}^1$ will be piecewise ergodic on $\mathbb{T}^n$ for all $n \in \mathbb{N}$ (this can be seen by applying Theorem (3.8)). Summarizing, from a dynamical point of view in which infinite-dimensional Lie algebras may be compared to piecewise ergodicity on $\mathbb{T}^n$ for all n, Theorem (7.8) associates a notion of Lie algebra to a general non-discrete subgroup of $\text{Diff}^{\omega}(\mathbb{S}^1)$. Furthermore this association is in total agreement with the usual notion of Lie algebra and with Lie's theorem. So, from a dynamical viewpoint, $\text{Diff}^{\omega}(\mathbb{S}^1)$ can be considered as a "rather honest" infinite-dimensional Lie group.

Because of Theorem A and Proposition (7.6), the proof of Theorem (7.8) clearly results from the two lemmas below.

Lemma 7.9 *Assume that* $\Gamma \subset \text{Diff}^{\omega}(\mathbb{S}^1)$ *is a non-discrete group. Then* Γ *is piecewise ergodic on* $\mathbb{S}^1$.

Lemma 7.10 *Assume we are given* $\varepsilon, \tau > 0$ *and a non-discrete subgroup* Γ *of* $\text{Diff}^{\omega}(\mathbb{S}^1)$. *Suppose also that* Γ *is not solvable. Then there exists a finite set* $S \subset \Gamma$ *satisfying the following conditions:*

1. *The group* Γ_S *generated by* S *is not solvable;*

2. *Any element* $h \in S$ *possesses a holomorphic extension* h_A *to the annulus* A_τ. *Furthermore one has*

$$\sup_{z \in A_\tau} \| h_A(z) - z \| < \varepsilon \, .$$

Proof of Lemma (7.9) Let h be an element of Γ and suppose for a contradiction that Γ is not piecewise ergodic on $\mathbb{S}^1$. As mentioned, this implies that the rotation number of h is rational.

On the other hand, Lemma (7.10) allows to suppose that Γ is solvable. Therefore either Γ has a finite orbit or it is Abelian (and indeed conjugate to a group of rotations).

Let us first suppose that Γ has a finite orbit. Denote by Γ_0 a finite index subgroup of Γ fixing a point $p \in \mathbb{S}^1$. Because Γ_0 is not discrete, it follows from our previous discussion that the closure of Γ_0 contains a non-trivial vector field. It then follows that Γ_0 is piecewise ergodic which is a contradiction.

Now suppose that Γ_0 is actually conjugate to a group consisting of rotations. Recall that all elements of Γ have rational rotation number so that these rotations are actually rational. Therefore the elements of Γ becomes the identity after a number of iterations. It is well-known that, in this case, Γ is $C^{\omega}-$conjugate to a group $\text{Rot}_{p/q}$ consisting of rational rotations. However Γ and hence $\text{Rot}_{p/q}$ is non-discrete. Thus there is a sequence of rotations $x \mapsto x + p_i/q_i$ belonging to $\text{Rot}_{p/q}$ such that p_i/q_i converge to *zero*. Now it is clear that $\text{Rot}_{p/q}$ is ergodic on $\mathbb{S}^1$. This proves the lemma. □

The proof of Lemma (7.10) is more complicated. First fix a sequence $\{h_j\}$ converging to the identity on some annulus A_τ. For each $i \in \mathbb{N}$, let Γ_i be the subgroup generated by the h_j's with $j \geq i$. Obviously we can suppose that Γ_i is solvable for every i. Moreover, we can also suppose that either all the Γ_i are Abelian or they are all solvable non-Abelian groups.

Once again Γ_i is either Abelian (conjugate to a group of rotations) or it has fixed points (actually if Γ_{i_0} has a finite orbit then Γ_{i_1} will have fixed points for sufficiently large i_1).

Proof of Lemma (7.10) when Γ_i *is conjugate to a group of rotations* Denote by $\{h_{1,i}, h_{2,i}, \ldots\}$ the generators of Γ_i. There exists f in Γ such that f does not commute with some of these elements. In fact, if $f \in \Gamma$ commutes with all the diffeomorphisms $h_{1,i}, h_{2,i}, \ldots$, then f is a rotation in the appropriate coordinate since it commutes with a non-discrete set of rotations.

It follows that $f \circ h_{j,i} \circ f^{-1}$ cannot coincide with rotations in the above mentioned coordinate. Therefore, for j fixed, the group generated by the set $\{h_{1,i}, h_{2,i}, \ldots, f \circ h_{1,i} \circ f^{-1}, f \circ h_{2,i} \circ f^{-1}, \ldots\}$ is not Abelian. If this group were solvable, then Proposition (7.6) would imply that it has fixed points (modulo passing to a finite index subgroup). This is of course impossible. Hence, if j is large enough, this non-solvable group clearly satisfies our assumptions. The proof of the lemma is therefore over. □

It remains to discuss the case in which Γ_i has fixed points.

Proof of Lemma (7.10) general case Consider a $f \in \Gamma$. Observe that for a very large i, all the diffeomorphisms $h_{1,i}, h_{2,i}, \ldots, f \circ h_{1,i} \circ f^{-1}, f \circ h_{2,i} \circ f^{-1}, \ldots$ are close to the identity.

Recall that all the Γ_i are either Abelian or solvable non-Abelian. Because Γ_i is non-discrete, it results that Γ_i is dense in Γ_1 for every i. In particular the group generated by Γ_1 and f is solvable if and only if the group generated by Γ_i and f is solvable for some $i \in \mathbb{N}$. On the other hand, it is clear that the group generated by the diffeomorphisms $h_{1,i}, h_{2,i}, \ldots, f \circ h_{1,i} \circ f^{-1}, f \circ h_{2,i} \circ f^{-1}, \ldots$ is not solvable as long as the group generated by Γ_i and f is not solvable. In other words, we just have to find $f \in \Gamma$ such that the group generated by Γ_1 and f is not solvable.

Observe that Γ_1 has fixed points $\{p_1, \ldots, p_l\}$. Modulo passing to a finite index subgroup Γ_0 of Γ, we can suppose that Γ_0 fixes all these points. Besides we have seen that Γ_0 is non-solvable since Γ is so. However there is a uniquely defined local coordinate around p_1 in which Γ_1 is given as a subgroup of the group

$$x \longmapsto \frac{ax}{(1 + bx^{k_0})^{1/k_0}} \quad a \in \mathbb{R}_+^* \, , \; b \in \mathbb{R} \, .$$

On the other hand there is an element $f \in \Gamma_0$ which does not have the above form in this coordinate since Γ_0 is not solvable. We then conclude that the group generated by Γ_1 and f is not solvable. This establishes the lemma completing the proof of Theorem (7.8). □

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