EARTHQUAKE CURVES

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Abstract

The first two parts of this paper concern homeomorphisms of the circle, their associated earthquakes, earthquake laminations and shearing measures. We prove a finite version of Thurston's earthquake theorem [9] and show that it implies the existence of an earthquake realizing any homeomorphism. Our approach gives an effective way to compute the lamination. We then show how to recover the earthquake from the measure, and give examples to show that locally finite measures on given laminations do not necessarily yield homeomorphisms. One of them also presents an example of a lamination $\mathcal{L}$ and a measure σ such that the corresponding mapping h_σ is not a homeomorphism of the circle but $h_{2\sigma}$ is.

The third part of the paper concerns the dependence between the norm $||\sigma||_{Th}$ of a measure σ and the norm $||h||_{cr}$ of its corresponding quasisymmetric circle homeomorphism h_σ. We first show that $||\sigma||_{Th}$ is bounded by a constant multiple of $||h||_{cr}$. Conversely, we show for any $C_0 > 0$, there exists a constant $C > 0$ depending on C_0 such that for any σ, if $||\sigma||_{Th} \leq C_0$ then $||h||_{cr} \leq C||\sigma||_{Th}$.

The fourth part of the paper concerns the differentiability of the earthquake curve $h_{t\sigma}, t \geq 0$, on the parameter t. We show that for any locally finite measure σ, $h_{t\sigma}$ satisfies the nonautonomous ordinary differential equation

$$\frac{d}{dt}h_{t\sigma}(x) = V_t(h_{t\sigma}(x)),\ t \geq 0,$$

at any point x on the boundary of a stratum of the lamination corresponding to the measure σ. We also show that if the norm of σ is finite, then the differential equation extends to every point x on the boundary circle, and the solution to the differential equation an initial condition is unique.

The fifth and last part of the paper concerns correspondence of regularity conditions on the measure σ, on its corresponding mapping h_σ, and on the tangent vector

$$V = V_0 = \frac{d}{dt}\Big|_{t=0} h_{t\sigma}.$$

We give equivalent conditions on σ, h_σ and V that correspond to h_σ being in $\mathit{Diff}^{\,1+\alpha}$ classes, where $0 \leq \alpha < 1$.

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Introduction

The earthquake theorem of Thurston gives a way to break down the data that determine an orientation preserving homeomorphism of the circle into three parts, a lamination, a shearing measure on that lamination and a normalization. If we consider two homeomorphisms $A \circ h$ and h where A is a Möbius transformation to be in the same class, then it is unnecessary to mention the normalization; the class of h uniquely determines and is determined by its lamination $\mathcal{L}$ and its left shearing measure σ.

The first step in finding the earthquake for the class of h is to construct its lamination. The second step is to find a Möbius transformation E_T for each stratum T of the lamination which has the following properties:

(1) For each pair of strata T_1 and T_2 of the lamination, $E_{T_1}^{-1} \circ E_{T_2}$ is a hyperbolic Möbius transformation whose axis separates T_1 and T_2 and moves T_2 to the left as viewed from T_1.
(2) For every point p on the circle and on the boundary of a stratum T, $E_T(p) = h(p)$.

The third step is to construct the shearing measure σ from E. In [9] Thurston carries out each of these steps, and shows that the homeomorphism h uniquely determines the measure σ.

In the first two sections of this paper we give an alternative approach to these steps based on the finite earthquake theorem. Then we go on to a fourth step, namely, we describe a procedure by which $h = h_\sigma$ can be recovered from σ. This procedure applies to any locally finite measure defined on a lamination. However, the map h_σ obtained from σ will in general only be a homeomorphism of a dense subset of the circle into the circle. We have not found the exact condition on σ that guarantees its associated map h is continuous everywhere. We do provide an example that shows $h_{2\sigma}$ can be a homeomorphism of the circle while h_σ is not.

In the third section we study the quantitative dependence between the Thurston norm of a measure and a cross-ratio distortion norm of its corresponding homeomorphism. The Thurston norm $||\sigma||_{Th}$ of a measure σ is defined to be the supremum over all hyperbolic geodesic segments β of length one in the hyperbolic plane of the total mass of the lines in the support of σ that intersect β. The cross-ratio distortion norm $||h||_{cr}$ of a homeomorphism h is defined as

$$||h||_{cr} = \sup_{Q} |\log |cr(h(Q))||\,,$$

where the supremum is taken over all quadruples $Q = (a, b, c, d)$ of points arranged in counterclockwise order on the circle such that

$$cr(Q) = \frac{(b-a)(d-c)}{(c-b)(d-a)} = 1.$$

We first show that $||\sigma||_{Th}$ is bounded by a constant multiple of $||h||_{cr}$. In order to show the existence of such a constant, we associate each geodesic segment β with two suitable quadruples such that the cross-ratio distortion on one quadruple controls the total mass of β when $||h||_{cr} < 1$, and the cross-ratio distortion on another quadruple controls the total mass of β when $||h||_{cr} \geq 1$. Conversely, we show that for any constant $C_0 > 0$, there exists a constant $C > 0$ depending on C_0 such that for any σ, if $||\sigma||_{Th} \leq C_0$ then $||h||_{cr} \leq C||\sigma||_{Th}$. The work used to achieve this result involves the approximations of h_σ by finite earthquake maps h_{σ_n} and the study of the tangent vector V_t^n of the curve $h_{t\sigma_n}, 0 \leq t \leq 1$, of finite earthquake maps to the variable t. We show that the Thurston norm of the pushforward $h^*_{t\sigma_n}\sigma_n$ of σ_n by $h_{t\sigma_n}$ is bounded by a constant for $0 \leq t \leq 1$, where the constant is also independent of n but depends on C_0. And therefore the Zygmund norm of V_t^n has a uniform upper bound for $0 \leq t \leq 1$. By integrating the differential equation which $h_{t\sigma_n}$ satisfies, we prove that the cross-ratio distortion norm $||h_{\sigma_n}||_{cr} \leq C_1||\sigma_n||_{Th}$ for a constant C_1 depending on C_0. Since $||\sigma_n||_{Th}$ is also bounded by a constant multiple of $||\sigma||_{Th}$, by passing to the limit we complete the proof.

The main result of the fourth section is to extend the ordinary differential equation of the curve of finite earthquake maps to the curve determined by any locally finite measure σ. Furthermore, we show that if the Thurston norm of σ is finite, then the differential equation extends to any point x on the boundary circle, and the earthquake curve is the unique solution to the differential equation if it is normalized.

In the last part of the paper, we first do further study of regularity of the tangent vectors $V_t(x)$ to an earthquake curve determined by a measure σ which satisfies vanishing conditions near the boundary circle. Then we apply them to characterize different smoothness classes of circle homeomorphisms. At the end, we show that the characterizations are also equivalent to the corresponding vanishing conditions on the initial tangent vector to the earthquake curve.

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1. Homeomorphisms

In this section we begin by proving an earthquake theorem for cyclic order preserving maps of a finite subset of the circle into the circle. We go on to show how this theorem implies the existence of a unique earthquake corresponding to any homeomorphism of the circle. After that we show how the earthquake determines a nonnegative shearing measure supported on the lines of its lamination and how this measure gives back the homeomorphism.

1.1. **Earthquakes on finite sets.** Let S be a finite subset of the unit circle consisting of $n \geq 4$ points and h a cyclic, order-preserving map from S into another finite subset $h(S)$ of the unit circle. In this section we show that data determining h up to post-composition by a Möbius transformation can be broken into two parts. The first part is a finite lamination and the second part is a nonnegative measure supported on the lines of the lamination.

A *finite lamination* $\mathcal{L}$ for S is a collection of $n-3$ hyperbolic lines joining the points of S so that no two of the lines in $\mathcal{L}$ intersect and no line of $\mathcal{L}$ joins *adjacent points* of S. Points $a,\ b$ in S are adjacent if they are the endpoints of an interval on the circle that contains no other points of S. $\mathcal{L}$ determines a decomposition of the disc into $n-2$ triangles all of whose vertices are points of S. Let $I_k, 1 \leq k \leq n$, be the intervals on the unit circle whose endpoints are the points of S, labeled in counterclockwise order. The *triangles of the lamination* are triangles whose sides are the n intervals I_k and $n-3$ non-intersecting hyperbolic lines L_j joining pairs of non-adjacent points of S. We call the sides I_k *boundary sides* and the sides L_j *interior sides*.

The triangles T of this decomposition are of three types. They can have two boundary sides, one boundary side or no boundary sides. In the first case, we call such a triangle a *border triangle* and denote the total number of these by t_2. In the second case, we call such a triangle a *boundary triangle* and denote the total number of these by t_1. Finally, in the third case, we call such a triangle an *interior triangle* and denote the total number of these by t_0.

If the vertices of a border triangle are labeled a, b and c in counterclockwise order, then the line joining a to c is one of its sides and the other two sides are intervals from a to b and from b to c. In this case we call the point b the *midpoint of the border triangle*. Note that the points a, b, and the points b and c are adjacent in S.

We will need the following lemma about finite laminations.

Lemma 1. *Let $\mathcal{L}$ be a finite lamination on a subset S of the unit circle containing n points, where $n \geq 4$. Then the decomposition of the disc by $\mathcal{L}$ into triangles must contain at least two border triangles. Moreover, $t_2 = t_0 + 2$ and there is always a border triangle inside any half-plane bounded by a line in $\mathcal{L}$.*

Proof. The total number of boundary sides is

$$n = 2t_2 + t_1.$$

The total number of triangles in the decomposition of the disc determined by $\mathcal{L}$ is

$$n - 2 = t_0 + t_1 + t_2.$$

Eliminating n from these two equations yields

$$t_2 = t_0 + 2.$$

In particular, there are always at least two border triangles.

If we look at any line L of $\mathcal{L}$ we can consider the finite subset S' of S consisting of the two endpoints a and c of L and all of the points of S lying on the boundary of one of the half-planes H_1 bounded by L and just one of the points b of S lying on the boundary of the other half-plane H_2 bounded by L. Let $\mathcal{L}'$ be the lamination of the disc consisting of the line L and the lines of $\mathcal{L}$ that lie in H_1. The triangle with vertices a, b, and c is one border triangle for the decomposition of the disc induced by the $\mathcal{L}'$. Another border triangle must lie in H_1. □

By drawing a few examples, one can see that there are always $n - 2$ triangles. The number of topological types of laminations for a set S of size $n \geq 4$ is the Catalan number

$$\frac{1}{n-1}\binom{2n-4}{n-2}.$$

In particular, when $n = 4$ there are two types, when $n = 5$, there are 5 types, and when $n = 6$, there are 14 types. The fourteen different types for the case $n = 6$ are illustrated in Figure 1.

Now assume we are given a finite lamination $\mathcal{L}$ for a subset S of the circle with $n \geq 4$ points. Also assume we are given a nonnegative measure σ on $\mathcal{L}$. It is the assignment of a nonnegative number μ_j to each of the lines L_j of the lamination. We can also view σ as an atomic measure on $S^1 \times S^1 \setminus (\textit{the diagonal})$. Each point in the support of σ is a pair of endpoints of a line L in $\mathcal{L}$. The support of σ coincides with the pairs of endpoints of those lines L_j of $\mathcal{L}$ for which the weight μ_j is positive.

By leftward isometric shearing along the lines of $\mathcal{L}$, we now describe how such a measure σ determines a map h_σ of the closed disc whose restriction to S is a cyclic order-preserving map h of S into the unit circle. h_σ is a map of the interior of the disc with the following properties:

(1) inside any one of the triangles h_σ is an isometry and
(2) if L is the common hyperbolic line of two neighboring triangles T_1 and T_2, the points of T_2 move to the left compared to the points of T_1.

Let T_1 be any one of the triangles whose sides are intervals I_k on the boundary of the unit circle or interior lines L_j in $\mathcal{L}$. At least one of the sides of T_1 must be a line L_1 in $\mathcal{L}$. Then L_1 is also the side of a second triangle T_2. Construct the hyperbolic Möbius transformation C_1 that preserves the unit circle, that has fixed points at the two endpoints of L_1 and that moves points in the interior of L_1 a hyperbolic distance μ_1 to the left, relative to T_1. Let a_1 and b_1 be the left and right endpoints of L_1 as viewed from the interior of the triangle T_1. Then C_1 is determined by the equation

$$\frac{C_1(z) - b_1}{C_1(z) - a_1} = \lambda_1 \frac{z - b_1}{z - a_1},$$

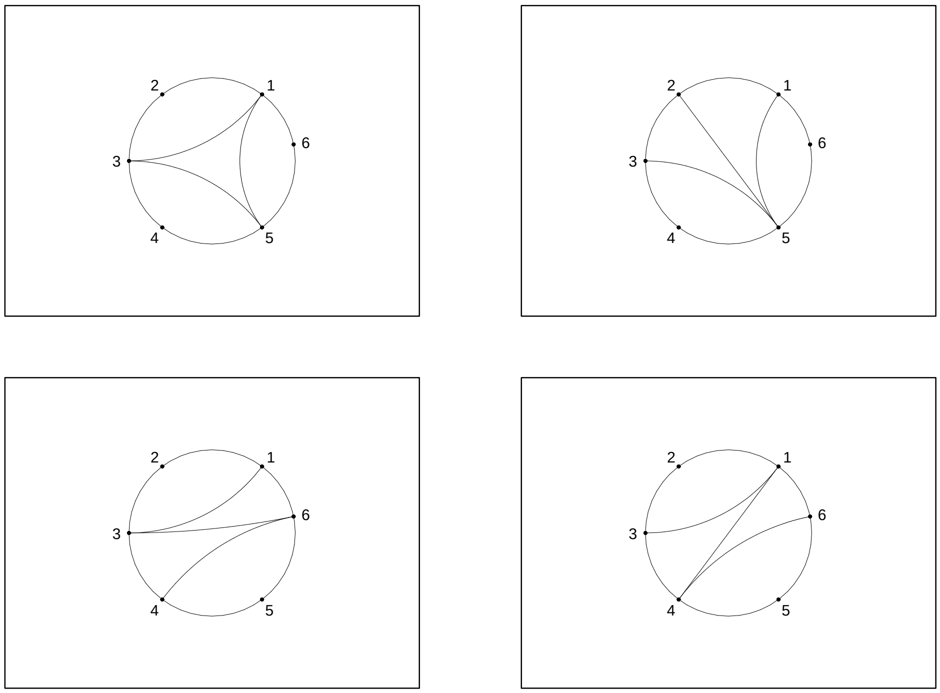

FIGURE 1. Different topological types of laminations on six points. The upper left figure is a star, the upper right a fan, the lower left a left accordion, and the lower right a right accordion. With this labeling, there are two stars, six fans, three left accordions, and three right accordions, making a total of fourteen.

where $\mu_1 = \log \lambda_1$. In the special case that $\mu_1 = 0$, the transformation C_1 is the identity and otherwise $\lambda_1 > 1$. We call C_1 the *comparison map* on the line L_1.

We apply the transformation C_1 to all of the triangles lying on the side of L_1 opposite to the triangle T_1. All of the vertices and all of the edges of these triangles are then moved. If T_2 is a border triangle, the third vertex of T_2 which is not equal to either of the endpoints of L_1 is moved finally by the transformation C_1. In the other case, T_2 must have another side L_2 that belongs to $\mathcal{L}$. It has endpoints a_2 and b_2, one of which is equal to either a_1 or b_1. Now, define C_2 to be the hyperbolic Möbius transformation with fixed points at $C_1(a_2)$ and $C_1(b_2)$, with translation length μ_2 and that moves the line $C_1(L_2)$ to the left relative to the triangle $C_1(T_2)$. We continue in this way down every branch of the *tree of triangles* in the lamination. In the end we obtain a map h_σ of the closed disc whose restriction to S is a cyclic, order-preserving map from S to another finite set in the unit circle. h_σ takes every line of $\mathcal{L}$ to the lines of another lamination $h(\mathcal{L})$, and we call h_σ the *finite left earthquake* for the finite lamination $\mathcal{L}$. The next lemma shows h_σ is determined, up to postcomposition by a Möbius transformation, by $\mathcal{L}$ and the totality of weights assigned to each of the lines in $\mathcal{L}$.

Lemma 2. *Let a finite lamination and measure on this lamination be given, containing triangles T_1 and T_2. Let h_1 and h_2 be the cyclic, order-preserving maps from S into the unit circle constructed as above, starting at the triangles T_1 and T_2,* *respectively. Then there is a Möbius transformation B preserving the unit circle such that $B \circ h_1 = h_2$.*

Proof. In the lamination of the disc by triangles, there is a unique chain of triangles that join T_1 to T_2 with successive comparison maps $C_1, ..., C_k$. The map h_1 fixes T_1 and h_2 fixes T_2 and, thus, h_2 postcomposed by $B = C_k \circ C_{k-1} \cdots C_1$ coincides with h_1 on T_2. By following along other comparison maps that lead to any point of S from the interior of T_2, we find that $B \circ h_2 = h_1$ at every point of S. □

The point of following theorem is that every cyclic, order-preserving map of a finite subset of the unit circle into the unit circle is realized, up to post-composition by a Möbius transformation, by leftward isometric shearing along a finite lamination in amounts corresponding to the nonnegative weights specified by a measure on the lines of this finite lamination. We shall call a map constructed in this way a *finite left earthquake.*

A *stratum of a finite earthquake* is a maximal connected union of triangles in the lamination such that the shearing along neighboring triangles is zero. If shearing weights are positive along each of the $n-3$ lines, then each of the $n-2$ triangles is a stratum.

Of course, a *finite right earthquake* is a mapping created in the same way except that all the shearing is to the right instead of to the left. The measure and lines along which this measure is nonnegative are uniquely determined by the map.

Theorem 1. (Finite earthquake theorem) *Assume h is a cyclic, order preserving map from a finite subset S of the unit circle into the unit circle. Then there exists a finite lamination $\mathcal{L}$ for S and a nonnegative measure σ supported on the pairs of endpoints of lines in $\mathcal{L}$, such that, up to postcomposition by a Möbius transformation, h is the restriction to S of the finite left earthquake h_σ. Moreover, the set S and the map h uniquely determine the measure σ.*

The parallel statement is also true for the realization of a given cyclic, order-preserving map h from a finite subset S of the unit circle into the circle by a finite right earthquake.

Proof. Suppose a, b, c and d are four points of S labelled in counterclockwise order and let A be the Möbius transformation that maps $h(a), h(b), h(c)$ to a, b, c, respectively. If $d' = Ah(d))$, then viewed from the point b, d' lies either to the right or to the left of d. The amount that d' is displaced from d can be measured by the multiplier of the Möbius transformation B that fixes a and c, and maps d to d' and is given by the formula

$$\frac{B(z) - a}{B(z) - c} = \lambda \frac{z - a}{z - c}.$$

If $\lambda \geq 1$, we say d' is moved to the left of d by the amount $\log \lambda$, if $\lambda \leq 1$, we say d' is moved to the right of d by the amount $\log(1/\lambda)$, and if $\lambda = 1$, then $d' = d$.

Lemma 3. *Given the map h and any four points a, b, c and d in S and arranged in counterclockwise order, either h moves d to the left relative to a, b and c, or h moves a to the left relative to b, c and d.*

Proof. Let $Q = \{a, b, c, d\}$ and $cr(Q)$ be the cross-ratio

$$\frac{(d - c)(b - a)}{(c - b)(a - d)}.$$

Then the amount by which h moves d to the left is $\log\lambda$ where $\lambda = \lambda(a,b,c,d)$ is the ratio of cross-ratios,

$$\frac{cr(h(Q))}{cr(Q)}.$$

But, obviously, $\lambda(a,b,c,d) = 1/\lambda(b,c,d,a)$. □

This lemma proves the Theorem when S has 4 points, for either h moves d to the left relative to a,b and c or it moves a to the left relative to b,c and d. In the former case, we let the lamination $\mathcal{L}$ consist of one line L_{ac} joining a to c and the weight assigned to this line is $\log\lambda(a,b,c,d)$. In the latter case, we let $\mathcal{L}$ consist of the line L_{bd} joining b to d and the weight assigned to this line is $\lambda(b,c,d,a)$.

The proof of the existence part of the theorem for $n > 4$ proceeds by induction. One assumes the lamination and the measure exists for every homeomorphism h and every finite subset of the circle with $n-1$ or fewer points. In order to find the finite lamination for a set S of size n we need the following lemma.

Lemma 4. *For every triple $T = \{a,b,c\}$ of points in S select a Möbius transformation A_T such that $A_T \circ h$ fixes a,b and c. There exists a triple T of points in S so that every point of $S \setminus \{a,b,c\}$ is either fixed or moved to the left by $A_T \circ h$.*

Proof. For any point x in S, the set $S \setminus \{x\}$ contains $n-1$ points and by induction we can assume the map h restricted to $S \setminus \{x\}$ is realized by a finite left earthquake on a finite lamination for $S \setminus \{x\}$. By Lemma 1 the lamination for the earthquake on $S \setminus \{x\}$ has at least two border triangles, T_1 and T_2. Assume the points of one of these border triangles are labeled in the counterclockwise direction by a,b and c, and that b is the midpoint. The point x could be on the border side from a to b or on the border side from b to c or in the complement of these two sides. We can assume that it is in the complement, because if it is not in the complement for the triangle T_1, then it is necessarily in the complement for the triangle T_2.

We can also assume h fixes the vertices a,b and c of a border triangle T for the lamination on $S \setminus \{x\}$ and all of the points of $S \setminus \{x\}$ are moved to the left by h, relative to to a, b and c. Also assume the points a,b and c are labelled in counterclockwise order. There are three cases. Either $x' = h(x)$ lies to the left of x, $x' = x$, or x' lies to the right of x. In the first two cases the triangle T satisfies the conditions of the lemma. In the third case, we use the triangle $\tilde{T}$ with vertices x,a and b. Now let A be the Möbius transformation that moves x' back to x and fixes a and b. Then A moves all of the points in the counterclockwise interval from b to a in the counterclockwise direction. Thus, $A \circ h$ and the triangle determined by these three points satisfies the conditions of the lemma. □

This lemma enables one to find inductively the lamination $\mathcal{L}$ for a given finite set S and mapping h. To construct a line of $\mathcal{L}$, one considers all of the triangles T_{abc} with the properties of Lemma 4. For each such triangle one calculates the minimum leftward motion of the points of $S \setminus \{a,b,c\}$. If this minimum is as small as it can be among all such triangles, then the line L_{ac} is a line of $\mathcal{L}$. Subsequent lines of $\mathcal{L}$ are found by continuing the same process for the homeomorphism h restricted to the set $S \setminus \{b\}$. One of the authors [NL] has written a program that finds the lamination for a given finite set S and a given h. We refer to the website *http://comet.lehman.cuny.edu/lakic/html/program.html.*

For the remainder of the proof of the theorem we refer to page 343 of [4]. □

1.2. **Earthquakes on the circle.** In this section we recall from [4] how the finite earthquake theorem can be used to show that any orientation preserving homeomorphism h is determined, up to postcomposition by a Möbius transformation, by a lamination and a totality of leftward comparison maps for this lamination. Conversely, if the lamination and the comparison maps are known to come from a homeomorphism, then, up to postcomposition by a Möbius transformation, the homeomorphism can be reconstructed from the lamination and the comparison maps.

First we need two definitions.

Definition 1. *A geodesic lamination $\mathcal{L}$ is a closed subset of the hyperbolic plane written as a disjoint union of non-intersecting hyperbolic lines. The components of the complement in the hyperbolic plane of the lamination are its gaps and the completions of these gaps in the Poincaré metric are called the complete gaps. The gaps together with the lines in $\mathcal{L}$ are its strata.*

Obviously the union of all of the strata for a lamination is the whole disc, a lamination determines its strata and the totality of its strata determine the lamination. Also, if we are given two laminations, $\mathcal{L}$ and $\mathcal{L}'$, and every stratum of $\mathcal{L}$ is a stratum of $\mathcal{L}'$, then $\mathcal{L} = \mathcal{L}'$.

The intersection of the unit circle with the closure of any stratum T of $\mathcal{L}$ is a closed set of points $bd(T)$ on the unit circle. T is reconstructed from $bd(T)$ by taking the convex hull of the union of all hyperbolic lines joining pairs of points of $bd(T)$. No hyperbolic line in the interior of T is in $\mathcal{L}$ and every hyperbolic line on the boundary of T is in $\mathcal{L}$.

Definition 2. *If $\mathcal{L}$ is a lamination in the hyperbolic plane, a $\mathcal{L}$-left earthquake map is an injective and surjective map E from the hyperbolic plane to the hyperbolic plane that is an isometry on each stratum of $\mathcal{L}$. If T_1 and T_2 are two strata of $\mathcal{L}$, then the comparison map $cmp(T_1, T_2) = (E|T_1)^{-1} \circ (E|T_2)$ must be a hyperbolic transformation with axis that weakly separates T_1 from T_2, and moves points of T_2 to the left relative to T_1.*

A hyperbolic line L weakly separates T_1 from T_2 if any path joining a point in T_1 to a point in T_2 necessarily intersects L. In case a stratum T_1 is a single line on the boundary of a stratum T_2, we allow the comparison map $cmp(T_1, T_2)$ to be the identity, and otherwise we require the translation length of $cmp(T_1, T_2)$ be strictly positive.

Theorem 2. (General earthquake theorem)(Thurston) *Let an orientation preserving homeomorphism h from the unit circle to itself be given. Then there is a left earthquake map E associated to a lamination $\mathcal{L}$, such that h is the restriction of E to the boundary of the unit disc. h uniquely determines the lamination $\mathcal{L}$. Moreover, h determines all of the isometries $E|T$ for every stratum T in $\mathcal{L}$ with the possible exception of the choices for E on any leaf L in $\mathcal{L}$ where E has a discontinuity. If there is a discontinuity, the possible choices for the isometry $E|L$ all have the same image but may differ by translations ranging between the limiting values for E on the two sides of L.*

The parallel statement is also true for the realization of a given homeomorphism h of the unit circle by a right earthquake.

Proof. We use as a model for the hyperbolic plane the upper half-plane $\mathbb{H}$, and take an ascending sequence of finite subsets S_n of $\overline{\mathbb{R}}$ whose union is dense in $\overline{\mathbb{R}}$.

For each set S_n we form the corresponding finite earthquake measure σ_n and the finite earthquake maps $h_n = h_{\sigma_n}$. Since the restriction of h_n to S_n coincides with h restricted to S_n and since both h_n and h are monotone, h_n and h_n^{-1} converge uniformly on the unit circle to h and h^{-1}, respectively.

For each n, let T_n be a triangular part of a stratum associated to a finite lamination of the finite left earthquake map h_n that coincides with h on S_n. Let vertices of T_n be a_n, b_n and c_n. Then the restriction of h_n to T_n is equal to the Möbius transformation M_n for which $M_n^{-1} \circ h_n$ fixes a_n, b_n, and c_n. $M_n^{-1} \circ h_n$ moves every other point of S_n to the left. By passing to a subsequence, we may assume $a_n \to a$, $b_n \to b$, and $c_n \to c$. If the points a, b and c coincide, then we say that the sequence of triangles T_n is degenerating. If all three of the points a, b and c are distinct, then since $M_n(a_n) = h(a_n)$, $M_n(b_n) = h(b_n)$ and $M_n(c_n) = h(c_n)$, the sequence M_n is a normal family, and by passing to another subsequence, we may assume M_n converges to a Möbius transformation M. Taking the limit as $n \to \infty$, we obtain $M(a) = h(a)$, $M(b) = h(b)$, and $M(c) = h(c)$. If T is the triangle with vertices at a, b and c, we say that $M : T \to M(T)$ is the limiting map of the sequence T_n.

Suppose now that $card\{a, b, c\} = 2$. By relabeling we may assume $a \neq b = c$, and by passing to a subsequence again, we may assume $M_n^{-1}(z)$ converges to $F(z)$ for all z. Either F is a Möbius transformation or $F(z)$ equals the same boundary point for all but one value of z. Since we are using $\mathbb{H}$ as the model for the hyperbolic plane, we may take a_n, b_n, and c_n to be real numbers with $a_n < b_n < c_n$ for all n, and so $a < b = c$. If F is not a Möbius transformation, then either $M_n^{-1}(z) \to a$ for all $z \in \overline{\mathbb{H}} \setminus \{h(c)\}$ or $M_n^{-1}(z) \to c$ for all $z \in \overline{\mathbb{H}} \setminus \{a\}$. Let p be a point contained in one of the sets S_{n_0} such that $a < p < c$. Note that p since $S_{n_0} \subset S_n$, p is all of the sets S_n for $n > n_0$. Since $M_n^{-1} \circ h$ moves all points of S_n to the left (with respect to the triangle T_n), we have $M_n^{-1} \circ h(p) > p$ for sufficiently large n. Therefore $M_n^{-1}(h(p))$ cannot converge to a. If q is a point contained in one of the sets S_n such that $q < a$, then $M_n^{-1} \circ h(q) > q$. Therefore $M_n^{-1}(h(q))$ cannot converge to c. This implies that F must be a Möbius transformation, and therefore M_n converges to a Möbius transformation $M = F^{-1}$. If T is the hyperbolic line with endpoints at a and c, we say that $M : T \to M(T)$ is a limiting map of the sequence T_n.

By taking subsequences enough times, we may assume every non-degenerate sequence of triangles T_n has a limit T that is either a triangle or a line and the Möbius transformations M_n converge to a Möbius transformation $M = M(T)$ associated to T. Let $\mathcal{A}$ be the set of all limiting maps $M : T_M \mapsto M(T_M)$ of non-degenerating sequences T_n. The basic properties of the set $\mathcal{A}$ are summarized in the following three lemmas.

Lemma 5. $\mathbb{H} \subset \bigcup_{M \in \mathcal{A}} T_M$.

Proof. Let p be a point in $\mathbb{H}$. Pick a triangle T_n of the finite lamination for the finite earthquake associated to the restriction of h to S_n, such that p is in the closure of T_n. Since all of the triangles T_n contain the point p, the subsequences for which T_n converges cannot degenerate, and the limiting map M contains p in its domain. □

Lemma 6. *For every limiting map M, the composition $M^{-1} \circ h$ fixes the endpoints of T_M and moves every other point to the left.*

Proof. From the finite earthquake theorem, the map $M_n^{-1} \circ h$ fixes the endpoints of T_n and moves every other point of S_n to the left. Since the union of ascending sets S_n is dense in $\mathbb{R}$, the result follows by taking the limit as $n \to \infty$. □

Lemma 7. *Let l_1 and l_2 be distinct geodesic lines such that $l_1 \subset T_{M_1}$ and $l_2 \subset T_{M_2}$ for two limiting maps M_1 and M_2. If $M_1 \neq M_2$, then l_1 does not intersect l_2 in $\mathbb{H}$, and $M_2^{-1} \circ M_1$ is a hyperbolic Möbius transformation whose axis separates l_1 and l_2.*

Proof. The proof is based on the reasoning from the proof of Lemma 4. If a and b are the endpoints of l_1 and c and d are the endpoints of l_2, such that $a < c < b < d$, then the composition $E_2^{-1} \circ E_1$ would have to be a Möbius transformation fixing at least four points. Therefore $M_2^{-1} \circ M_1$ is the identity map, a contradiction. Similar reasoning shows that if $a < b < c < d$, then $M_2^{-1} \circ M_1$ has fixed points in the intervals $[a, b]$ and $[c, d]$. □

Let $\mathcal{L}_0$ be the set of all geodesic lines L such that L is a subset of some T_M. For every L in $\mathcal{L}_0$ choose one of the limiting maps $M = M(L)$ with $L \subset T_M$. Define an equivalence relation on the elements of $\mathcal{L}_0$ be letting L_1 be equivalent to L_2 if $M(L_1) = M(L_2)$. For every equivalence class $\{L\}$ let $\partial(hull(\{L\}))$ be the boundary of the convex hull of all lines L in $\{L\}$. Let $\mathcal{L}$ be the union over all of these equivalence classes of all lines in $\partial(hull(\{L\}))$. Since the sets T_M are limiting sets, clearly, $\mathcal{L}_0$ is closed. Therefore, $\mathcal{L}$ is also closed, and by Lemma 7, $\mathcal{L}$ is a disjoint union of geodesics. We conclude that $\mathcal{L}$ is a lamination.

We define the map E on each stratum of $\mathcal{L}$ by the following method. If a point p belongs to the interior of some triangle T_M, we let $E(p) = M_T(p)$. In the case p belongs to a geodesic line L contained in some T_M, we let $E(p) = M(L)(p)$. By Lemma 7, E is well-defined and E restricted to any stratum s of $\mathcal{L}$ is equal to a single Möbius transformation M_s on that stratum. Lemma 5 implies E is defined for all points p in $\mathbb{H}$. Lemma 6 implies for every stratum s of $\mathcal{L}$, the map $M_s^{-1} \circ h$ fixes the boundary points of s on $\overline{\mathbb{R}}$ and moves all other points of $\overline{\mathbb{R}}$ to the left. By Lemma 7, E is an $\mathcal{L}$-left earthquake. Let q be a point in $\mathbb{R}$. If q is on the boundary of some T_E, then by Lemma 6, $E(q) = h(q)$. If q is not on the boundary of any T_E, by Lemma 5 select a line or triangle T_{M_k} that contains the point $q + i/k$. Then select a point q_k on the boundary of T_{M_k} that is closest to the point $q + i/k$. Since $q_k \to q$ and $E(q_k) = h(q_k)$, by taking the limit, we conclude that $E(q) = h(q)$. Therefore h is the restriction to the real axis of the earthquake E.

It remains to show uniqueness. Suppose E and E' are $\mathcal{L}$-left and $\mathcal{L}'$-left earthquakes yielding the same homeomorphism h. Let p be a point on a line L of $\mathcal{L}$. Since the union of the lines and gaps for $\mathcal{L}'$ fill the hyperbolic plane, either p is in a line L' of $\mathcal{L}'$ or p is in an open gap of $\mathcal{L}'$. In the first case, L' must equal L. If L' were not equal to L, then by considering their four endpoints a, b, c, and d, we would find that h would have to move d both to the left and to the right relative to a, b, and c. In the second case p is in the open gap G', and the line L either crosses a hyperbolic line bounding this gap or it lies in the interior of G' and joins two points in $bd(G')$. In the first case, we again find a line L' in $\mathcal{L}$ that intersects L, and we have just shown this leads to a contradiction. In the second case, $L = L_{ab}$ divides G' into two parts G'_1 and G'_2. Either there are lines of $\mathcal{L}$ that intersect boundary lines of G' or there are two gaps of $\mathcal{L}$, G_1 and G_2 lying on the same sides of L as G'_1 and G'_2, respectively. The gaps and lines of $\mathcal{L}$ cover G'_1 and we can assume none of the lines of $\mathcal{L}$ intersect the boundary lines of G'_1, because no line of $\mathcal{L}$ can intersect a line of $\mathcal{L}'$. Thus either another line L_1 of $\mathcal{L}$ is contained in G'_1 or a gap G of $\mathcal{L}$ has a boundary point c in common with a boundary point of G'_1 that is not equal to either of the endpoints of L. Then E would move all the points on the other side of L to the left relative to a, b and c, whereas E' holds the boundary points of G' fixed relative to a, b, and c. We conclude that $\mathcal{L} = \mathcal{L}'$, and E and E' must agree on every stratum unless it happens to be a line of $\mathcal{L}$ where E and E' are discontinuous. □

1.3. **Construction of shearing measures.** In this section we assume a homeomorphism h of the circle and its associated left earthquake E with lamination $\mathcal{L}$ are given. From this data we construct a nonnegative shearing measure σ for E. Let $\mathcal{M}$ be $S^1 \times S^1 - \{\textit{the diagonal}\}$ factored by the equivalence relation $(a, b) \sim (b, a)$. The support of σ is the closed subset of $\mathcal{M}$ consisting of pairs of endpoints of lines in $\mathcal{L}$.

To define σ it suffices to define the mass $\sigma(I \times J)$, where I and J are disjoint, closed intervals in a sufficiently large class. If there are no lines of $\mathcal{L}$ joining I to J, we let this mass be *zero*. If there is just one line l of $\mathcal{L}$ joining I to J, and if one of the endpoints of that line is interior to either I or J, we let T_1 be the stratum on one side of l, T_2 be the stratum on the other side, and we let $\sigma(I \times J)$ be the translation length of $E(T_1)^{-1} \circ E(T_2)$. If both endpoints of that line meet endpoints of I and J, we do not bother to define $\sigma(I \times J)$; we will still have a sufficiently large class to determine the measure σ.

Otherwise, there are at least two lines in $\mathcal{L}$ joining I to J and let l_1 be the first and l_2 be the last. Label them so that moving around in a counterclockwise direction, the line l_1, a subinterval of I_1 of I, the line l_2, and a subinterval of J_1 of J surround a topological rectangle and all of the lines of $\mathcal{L}$ that join I to J lie inside this rectangle. Let β be the common perpendicular and β' a geodesic segment containing β in its interior such that the length of β'_n equal the length of β plus $\frac{1}{2^n}$. Let $\mathcal{P}_n$ be a sequence of finite subsets of β'_n such that $\mathcal{P}_n \cap \beta'_{n+1}$ is a subset of $\mathcal{P}_{n+1}$, such that the distance between any two adjacent points in β'_n is less than $1/2^n$, and the union of the intervals between adjacent points covers β'_n. Assume $T_1, \ldots, T_m$ are strata of $\mathcal{L}$ arranged in order and such that each T_k contains the k-th point of $\mathcal{P}_n$. Let t_k be the translation length of the comparison map $E(T_k)^{-1} \circ E(T_{k+1})$ and $\sigma_n(\mathcal{P}_n) = \sum_{k=0}^{m} t_k$.

If $\mathcal{Q}_n$ is another sequence of finite subsets of β'_n having the same properties as $\mathcal{P}_n$, we use the following lemma to show that the sequences $\sigma_n(\mathcal{P}_n)$ and $\sigma(\mathcal{Q}_n)$ converge to the same limit.

Lemma 8. *Assume L_S and L_T are nonintersecting axes of hyperbolic Möbius transformations S and T with translation lengths $\tau(S)$ and $\tau(T)$, and assume the axis of $S \circ T$ separates L_S and L_T. Also assume the hyperbolic distance d from L_S to L_T is small. Then*

$$\tau(S) + \tau(T) \le \tau(S \circ T) \le \tau(S) + \tau(T) + O(\min\{\tau(T), \tau(S)\} d^2). \tag{1}$$

Proof. We may assume $S(z) = \lambda_S z$, with $\lambda_S \ge 1$ and fixed points at 0 and ∞. Also we may assume T has fixed points at a and b with $0 < a < b = 1/a$, and with multiplier λ_T. The formula for T is

$$T(z) = \frac{\lambda_T (b-a)(z-a)}{(\lambda_T - 1)(z-a) + (b-a)} + a.$$

Thus,

$$S\circ T(z)=\frac{\lambda_S\lambda_T(b-a)(z-a)}{(\lambda_T-1)(z-a)+(b-a)}+\lambda_S a.$$

Note that the repelling fixed point of $S\circ T$ is a number x between 0 and a, and the multiplier of a hyperbolic Möbius transformation is equal to its derivative evaluated at its repelling fixed point. Thus, from the equation

$$(S\circ T)'(x)=\frac{(b-a)^2\lambda_S\lambda_T}{((\lambda_T-1)(x-a)+(b-a))^2}$$

we obtain

$$\tau(S\circ T)=\tau(S)+\tau(T)-2\log\left(\frac{(b-a)+(\lambda_T-1)(x-a)}{b-a}\right). \tag{2}$$

We claim that

$$0<\left(\frac{(b-a)+(\lambda_T-1)(x-a)}{b-a}\right)<1. \tag{3}$$

The right hand part of (3) follows from $0<x<a<b$ and $\lambda_T>1$. Since x is a fixed point of $S\circ T$,

$$x-\lambda_S a=\frac{\lambda_S\lambda_T(b-a)(x-a)}{(\lambda_T-1)(x-a)+(b-a)}.$$

The left hand side of this equation is negative and the numerator of the fraction on the right hand side is negative. Therefore, $(\lambda_T-1)(x-a)+(b-a)$ is positive and we obtain the left hand part of (3). Therefore equation (2) implies the the left hand part (1).

$$\tau(S\circ T)\le\tau(S)+\tau(T)+O(\tau(T)d^2) \tag{4}$$

follows from (2) and the observations that $|x-a|<a$, $b=1/a$,

$$\left|\frac{x-a}{b-a}\right|\le\frac{a^2}{1-a^2}$$

and when a is small, $a\sim d$. The right hand part of (1) follows from (4) and the same result applied to $T^{-1}\circ S^{-1}$. □

Associated to the finite sets $\mathcal{P}_n$ and $\mathcal{P}_n\cup\mathcal{Q}_n$ there are translation lengths t_j and t_{jk} such that $\sum_k t_{jk}\le t_j$. If d_n is the maximum hyperbolic distance between adjacent points in $\mathcal{P}_n$, the preceding lemma implies $t_j\le\sum_k t_{jk}+t_jd_n^2$. Thus, $\sum_j t_j\le\sum_j\sum_k t_{jk}+d_n^2\sum_j t_j$. Since the last sum is bounded, this implies that the difference between the sum of the translation lengths corresponding to the set $\mathcal{P}_n$ and the sum of the translation lengths corresponding to the set $\mathcal{Q}_n$ approaches zero. We thus obtain a nonnegative number $\sigma(I\times J)$ for every pair of disjoint, closed sets I and J. By taking countable unions and intersections we obtain a sigma-additive, Borel measure σ on $S^1\times S^1\setminus\{diagonal\}$.

To summarize, we have the following proposition concerning approximations of σ, due to Thurston (see [9]).

Proposition 1. *Let β denote a closed hyperbolic geodesic segment which is transversal to the lamination $\mathcal{L}$, $\mathcal{P}=\{p_k\}_{k=0}^n$ a partition of β, and t_k the translation length of the comparison map of the strata containing two adjacent points p_k and p_{k-1}. Then $\sigma(\beta)$ modulo a constant $c(\beta)$ differs from $\sigma(\mathcal{P})=\sum_{k=1}^n t_k$ by an amount* *equal to $O(\sigma(\mathcal{P})d(\mathcal{P})^2)$, where $d(\mathcal{P})$ is the maximal hyperbolic distance between two adjacent points in $\mathcal{P}$, and $c(\beta)$ is independent of $\mathcal{P}$.*

2. Earthquake Measures

In this section we first show how any locally finite measure whose support is a lamination produces a homeomorphism of a dense subset of the unit circle into the unit circle. We go on to show that this construction of an earthquake is compatible with the construction of the measure in the first section. Finally, we show that this homeomorphism may not be extendable to a homeomorphism of the whole unit circle.

2.1. **Constructing earthquakes from measures.** Let σ be a non-negative locally finite measure whose support is a lamination $\mathcal{L}$. To construct a homeomorphism whose earthquake measure equals σ, we use finite approximations. Since the construction is relatively long, we divide it into several steps.

Step I Atomic approximations of σ

Fix a geodesic line l such that l belongs to $\mathcal{L}$. Without loss of generality, we work in the upper half-plane model and assume l joins 0 to ∞. Let $T_1, T_2, T_3, \dots$ be an enumeration of all strata of $\mathcal{L}$ that contain interior points. Since there are at most countably many such strata and the boundary of each such stratum T_i can contain at most countably many sides, there are at most countably many boundary sides of strata with interior points. Let $Q = \{q_1, q_2, q_3, \dots\}$ be the collection of all such sides in $\mathcal{L}$. Choose a sequence of finite laminations $\mathcal{L}_n$, such that

$$l \subset \mathcal{L}_1 \subset \mathcal{L}_2 \subset \mathcal{L}_3 \subset \dots \subset \mathcal{L},$$

in the following way. Divide the hyperbolic disc $D(i, n)$ with center at i and radius n into finitely many pieces, each with diameter less than $\frac{1}{n}$. For every piece P that has non-empty intersection with $\mathcal{L}$, choose one line l_P from $\mathcal{L}$ which intersects that piece, and stipulate that l_P belongs to a lamination $\mathcal{L}'_n$. Also add to $\mathcal{L}'_n$ all lines of $Q_n = \{q_1, q_2, \dots, g_n\} \subset Q$ that intersect $D(i, n)$. More precisely, $\mathcal{L}'_n$ is the union of $\mathcal{L}'_{n-1}$, the lines q from Q_n which intersect $D(i, n)$, and the lines l_P. We say two lines l_1 in l_2 in $\mathcal{L}'_n$ are adjacent if no other line of $\mathcal{L}'_n$ separates l_1 from l_2. Let γ be a geodesic segment that joins l_1 and l_2. Divide γ into finitely many subsegments S_i, each with length less than $\frac{1}{n}$. Let e_i be an endpoint of some subsegment S_i. If there is a line $\tilde{l}$ in $\mathcal{L}$ passing through e_i, we pick such a line $\tilde{l}$ and call it a special line. If there is no such line, then there are two lines $\tilde{l}_1$ and $\tilde{l}_2$ in Q, such that the subsegment of γ joining $\tilde{l}_1$ and $\tilde{l}_2$ contains the point e_1 and does not intersect $\mathcal{L}$. In that case we say that both $\tilde{l}_1$ and $\tilde{l}_2$ are special lines. Let $\mathcal{L}_n$ be the set of all lines in $\mathcal{L}'_n$ plus all special lines obtained using this algorithm for every pair of adjacent lines in $\mathcal{L}'_n$. This refinement $\mathcal{L}_n$ of $\mathcal{L}'_n$ has the property that any two adjacent lines l_1 and l_2 in $\mathcal{L}_n$ are either within distance $\frac{1}{n}$ or there are no lines in $\mathcal{L}$ separating l_1 from l_2.

Note that the following condition holds for every n. If D is a hyperbolic disc with radius $r \geq \frac{1}{n}$ and center $c \in \mathcal{L}$ whose distance to the point i is less than n, then there is a line l_n in $\mathcal{L}_n$ which intersects D.

To assign the atomic measures σ_n to the laminations $\mathcal{L}_n$, we start with the line l, and we let $\sigma_n(l) = \sigma(l)$. Let l_0 be any other line of the lamination $\mathcal{L}_n$. If a and c are two endpoints of l_0 with $a < c$, then either both a and c are non-positive or both a and c are non-negative. Suppose that both a and c are non-negative. The geodesic lines l_0 and l are separated by the lines $l_0, l_1, l_2, \ldots, l_k = l$ in $\mathcal{L}_n$. The endpoints a_i and c_i of the lines l_i satisfy $c \leq c_1 \leq c_2 \leq \ldots \leq c_k = \infty$ and $a \geq a_1 \geq a_2 \geq \ldots \geq a_k = 0$. We let

$$\sigma_n(l_0) = \sigma([a_1, a] \times [c, c_1]) - \sigma(l_1),$$

where $\sigma([a_1, a] \times [c, c_1])$ is the $\sigma-$measure of the set of lines in $\mathcal{L}$ which have one endpoint in $[a_1, a]$ and another in $[c, c_1]$. In the special case when $l_1 = l$, $\sigma_n(l_0) = \sigma([0, a] \times [c, \infty]) - \sigma(l)$. We define σ similarly on geodesic lines of $\mathcal{L}_n$ whose endpoints are nonpositive.

The lamination $\mathcal{L}$ and the measure σ_n determine the finite earthquake map E_n. We normalize E_n by letting E_n be equal to the identity mapping on the stratum that belongs to the second quadrant and has l as a boundary side.

Step II Convergence of finite earthquakes

We first investigate the convergence of the sequence $\{h_1, h_2, h_3, \ldots\}$ of earthquake mappings on each line in $\mathcal{L}_n$. Let l_0 be any line of $\mathcal{L}_n$. Suppose the endpoints of l_0 are non-negative. Again observe that the geodesic lines l_0 and l are separated by the lines $l_0, l_1, l_2, \ldots, l_k = l$ in $\mathcal{L}_n$, and the endpoints a_i and c_i of the lines l_i satisfy $c \leq c_1 \leq c_2 \leq \ldots \leq c_k = \infty$ and $a \geq a_1 \geq a_2 \geq \ldots \geq a_k = 0$.

To fix the values of h_n on l_0 we let

$$h_n|_{l_0} = L_k \circ L_{k-1} \circ \ldots \circ L_1 \circ L_0,$$

where L_i is a hyperbolic Möbius transformation with translation length $\sigma_n(l_i)$, attracting fixed point at c_i, and repelling fixed point at a_i. Similarly, we prescribe the values of h_n on the lines of $\mathcal{L}_n$ with non-positive endpoints, again requesting that h_n takes the maximal possible weight along each such line. By Lemma 1, the translation length τ of $h_n|_{l_0}$ is greater than or equal to $\sum_{i=0}^{k} \sigma_n(L_i) = \sigma([0, a] \times [c, \infty])$. Hence, $\sigma([0, a] \times [c, \infty])$ is a lower bound for τ. The following theorem implies that there is also an upper bound for τ.

Theorem 3. *For any positive integer n_0 and any line l_0 in $\mathcal{L}_{n_0}$, the sequence $(h_n|_{l_0}, n = n_0, n_0 + 1, n_0 + 2, n_0 + 3, \ldots)$ is a normal family of Möbius transformations.*

Before we prove this theorem, we summarize some techniques in the following lemmas and corollaries, which will also be used in Sections 3.1 and 5.5. In these lemmas we use the following notation. Let $\{a, b, c, d\}$ be a quadruple of points on the real axis with $a < b < c < d$, and L, M, R and T stand for the left, middle, right and total intervals, i.e., $L = [a, b], M = [b, c], R = [c, d]$ and $T = [a, d]$. We also identify L, M, R, and T with the corresponding lengths $b - a, c - b, d - c$ and $d - a$.

Lemma 9. *Assume $a < b < c < x < y < d$ and h_x and h_y are simple left earthquake maps, each with multiplier $\lambda > 1$, and supported on the lines that join x to ∞ and y to ∞, respectively. Let L_x, M_x, R_x, T_x and L_y, M_y, R_y, T_y be the images of the intervals L, M, R and T under the mappings h_x and h_y. Then*

$$\frac{L_x R_x}{M_x T_x} > \frac{L_y R_y}{M_y T_y}. \tag{5}$$

Moreover, if $a < y < x < b < c < d$, then

$$\frac{L_x R_x}{M_x T_x} < \frac{L_y R_y}{M_y T_y}. \tag{6}$$

Proof. Both h_x and h_y are equal to the identity on L and M, so $L = L_x = L_y$ and $M = M_x = M_y$. Also,

$$\frac{R_x}{T_x} = \frac{\lambda(d-x)+x-c}{\lambda(d-x)+x-a},$$

and

$$\frac{R_y}{T_y} = \frac{\lambda(d-y)+y-c}{\lambda(d-y)+y-a}.$$

Thus,

$$\frac{R_x}{T_x} > \frac{R_y}{T_y},$$

and inequality (5) follows. The proof of (6) is similar. □

Lemma 10. *With the same notation as in the previous lemma, suppose $a < b < x < y < c < d$. Then*

$$\frac{L_y R_y}{M_y T_y} > \frac{L_x R_x}{M_x T_x}. \tag{7}$$

Proof. $h_x(z)$ equals z for $z < x$ and $\lambda(z-x)+x$ for $z \geq x$, and similarly, $h_y(z)$ equals z for $z < y$ and $\lambda(z-y)+y$ for $z \geq y$. We have $h_x(L) = h_y(L) = L, h_x(R) = R_x = \lambda R = h_y(R) = R_y = \lambda R$. And since $x < y$ and $\lambda > 1$, $h_x(M) = M_x > h_y(M) = M_y, h_x(T) = T_x > h_y(T) = T_y$, which implies the lemma. □

Given a quadruple $Q = \{a, b, c, d\}$ with $a < b < c < d$, denote by $cr(Q)$ the cross ratio $\frac{LR}{MT} = \frac{(b-a)(d-c)}{(c-b)(d-a)}$. For a homeomorphism h defined on the real line, let $cr(h(Q))$ denote the cross-ratio of the quadruple $h(Q)$. We have the following corollaries of the previous lemmas respectively.

Corollary 1. *Let $Q = \{a, b, c, d\}$ be a quadruple on the real line with $a < b < c < d$, and $c \leq s \leq d$ and $d < t$. Suppose that $A_{(s,t)}$ is the hyperbolic Möbius transformation with the repelling fixed point at s and the attracting fixed point at t and its derivative at the repelling fixed point equal to $\lambda > 1$, and $f_{(s,t)} : \mathbb{R} \to \mathbb{R}$ is defined to be equal to $A_{(s,t)}$ on the interval $[s, t]$ and equal to the identity on the complement of $[s, t]$. Let a, b, c, d and λ be temporarily fixed. Then the cross-ratio of the image quadruple $f_{(s,t)}(Q)$ considered as a function of two variables $s \in [y, z]$ and $t \in (z, +\infty)$ decreases in s for each fixed t and increases in t for each fixed s.*

Corollary 2. *With the same notations as in the previous corollary, suppose $b \leq s \leq c$ and $d \leq t$. Then the cross-ratio of the image quadruple $f_{(s,t)}(Q)$ is increasing in s for each fixed t and also increasing in t for each fixed s.*

Now we begin the proof of Theorem 3.

Proof. The Möbius transformation $T_n = E_n|_{l_0}$ is a hyperbolic Möbius transformation with the attracting fixed point in the interval $[c, \infty]$ and the repelling fixed point in the interval $[0, a]$. Let $\tilde{a} = \frac{a+c}{2}$. To prove the theorem, we find an upper bound on $T_n(\tilde{a})$ and a positive lower bound on the cross ratio $|cr_n|$ of the four points $\infty, 0, T_n(\tilde{a}), T_n(c)$. Here $|cr_n| = \frac{T_n(c)-T_n(\tilde{a})}{T_n(\tilde{a})} = \frac{T_n(c)}{T(_n\tilde{a})} - 1$.

If λ_n is the multiplier of T_n and $c_n = T_n(c)$ and both λ_n and c_n approach infinity, then $T_n(\tilde{a})$ approaches infinity. Thus an upper bound on $T_n(\tilde{a})$ implies that if $c_n \to \infty$ then a subsequence of λ_n converges. On the other hand, if c_n is bounded then a positive lower bound on $|cr_n|$ also implies a subsequence of λ_n converges. In either case, $\{T_n\}_{n=1}^{\infty}$ is a normal family.

Let us first show that there exists a positive lower bound for $|cr_n|$. By Corollary 2,

$$\begin{aligned} &cr(\infty, 0, h_n(\tilde{a}), h_n(c)) \\ &= cr(\infty, 0, L_k(L_{k-1} \circ \ldots \circ L_1 \circ L_0(\tilde{a})), L_k(L_{k-1} \circ \ldots \circ L_1 \circ L_0(c))) \\ &\geq cr(\infty, 0, A_k(L_{k-1} \circ \ldots \circ L_1 \circ L_0(\tilde{a})), A_k(L_{k-1} \circ \ldots \circ L_1 \circ L_0(c))), \end{aligned}$$

where A_k is the hyperbolic transformation with translation length $\sigma_n(L_k)$, a repelling fixed point at 0 and an attracting fixed point at c_{k-1}. The Möbius transformation $\tilde{L}_{k-1} = A_k \circ L_{k-1}$ is a hyperbolic transformation with an attracting fixed point at c_{k-1}, a repelling fixed point in $[0, a_{k-1}]$, and translation length $\sigma_n(L_k) + \sigma_n(L_{k-1})$. Move the axis of $\tilde{L}_{k-1}$ to the geodesic connecting 0 to c_{k-2} to have a hyperbolic Möbius transformation A_{k-1}, again applying the Corollary 2, we have

$$\begin{aligned} &cr(\infty, 0, \tilde{L}_{k-1} \circ \ldots \circ L_1 \circ L_0(\tilde{a}), \tilde{L}_{k-1} \circ \ldots \circ L_1 \circ L_0(c)) \\ &\geq cr(\infty, 0, A_{k-1}(L_{k-2} \circ \ldots \circ L_1 \circ L_0(\tilde{a})), A_{k-1}(L_{k-2} \circ \ldots \circ L_1 \circ L_0(c))) \end{aligned}$$

Let $\tilde{L}_{k-2} = A_{k-1} \circ L_{k-2}$. It is a hyperbolic Möbius tranformation with an attracting point at c_{k-2}, a repelling fixed point in $[0, a_{k-2}]$, and translation length $\sigma_n(L_k) + \sigma_n(L_{k-1}) + \sigma_n(L_{k-2})$. Repeating the above process of moving the axis of $\tilde{L}_{k-2}$ finitely many times, we obtain

$$|cr(\infty, 0, T_n(\tilde{a}), T_n(c))| \geq |cr(\infty, 0, A(\tilde{a}), A(c))|,$$

where A is a hyperbolic Möbius transformation with attracting fixed point at c, repelling fixed point at 0, and with translation length equal to $\sum_{i=0}^{k} \sigma_n(L_i) = \sigma([0, a] \times [c, \infty]) = \sigma < \infty$. thus, if $\lambda = e^{\sigma}$ then

$$A(x) = \frac{\lambda c x}{(\lambda - 1)x + c},$$

and $|cr_n|$ is bounded below by $\frac{c-a}{c+a} e^{-\sigma}$.

To obtain an upper bound on $T_n(\tilde{a})$ we apply Corollary 1 to the four points $-\infty, -1, 0$ and $\tilde{a}$. Similar reasonings yield $T(\tilde{a}) \leq \tilde{a} e^{\sigma} = \frac{a+c}{2} e^{\sigma}$. □

Step III The earthquake mapping restricted to the laminations $\mathcal{L}_n$.

Now we define the earthquake E whose earthquake measure equals σ. We first define E on the laminations $\mathcal{L}_n$. Let s be a point in $\bigcup_{n=1}^{\infty} \mathcal{L}_n$. Then there exists n_0 and a line l_0 in $\mathcal{L}_{n_0}$ such that $s \in l_0$ and so $s \in l_0 \subset \mathcal{L}_n$ for all $n \geq n_0$. By Theorem 3, the sequence $E_n|_{l_0}$ has a convergent sequence. There are countably many lines in S, and by diagonalization, we may pass to a subsequence so that the restriction of E_n to any line $\tilde{l}$ in S converges to a Möbius transformation $A = A(\tilde{l})$. We let $E(z) = A(\tilde{l})(z)$ for every z on $\tilde{l}$, and every $\tilde{l}$ in $\bigcup_{n=1}^{\infty} \mathcal{L}_n$.

For any two lines l_0 and l_1 in $\bigcup_{n=1}^{\infty} \mathcal{L}_n$, there exists n_0 such that both l_0 and l_1 belong to $\mathcal{L}_{n_0}$. By the leftward property of the earthquake mapping h_n, the composition

$$(E_n|l_0)^{-1} \circ (E_n|l_1)$$

is a hyperbolic transformation whose axis weakly separates l_1 and l_0, and moves to the left, when viewed from l_0 (for all $n \geq n_0$). By passing to the limit as $n \to \infty$, we see that the same properties hold for the transformation

$$(A(l_0))^{-1} \circ (A(l_1)).$$

Therefore, if l_0 is any line in $\bigcup_{n=1}^{\infty} \mathcal{L}_n$, then the restriction $A(l_0)$ of E to l_0 satisfies the following property: the composition $A(l_0)^{-1} \circ E$ fixes the points on the line l_0 and moves every other point in $\bigcup_{n=1}^{\infty} \mathcal{L}_n$ to the left, when viewed from l_0. We say that E satisfies the leftward property on S.

Step IV Extending the earthquake map to $\mathcal{L}$

Let l_0 be any line in the lamination $\mathcal{L}$. Then there exist a sequence of lines l_n converging to the line l_0, such that $l_n \subset \mathcal{L}_n$ for each n. To see this, take any point s on l_0 and take the hyperbolic disk D_n with center at s and radius $\frac{1}{n}$. For sufficiently large n, there exists a line l_n in $\mathcal{L}_n$ such that l_n intersects D_n. Let A_n be the restriction of h_n to the line l_n. By a generalization of Theorem 3, the sequence A_n is a normal family. Therefore, there exists a subsequence A_{n_k} of A_n which converges to a Möbius transformation $A = A(l_0)$. We let $h(z) = A(l_0)(z)$ for all z on l_0. If l_1 is any line in $\bigcup_{n=1}^{\infty} \mathcal{L}_n$, then the transformation $(E_{n_k}|l_1)^{-1} \circ (E_{n_k}|l_{n_k})$ is a hyperbolic transformation whose axis weakly separates l_1 and l_{n_k}, and moves to the left, when viewed from l_1. By passing to the limit, we conclude that the mapping $A(l_1)^{-1} \circ A(l_0)$ is a hyperbolic transformation whose axis weakly separates l_1 and l_0, and moves to the left, when viewed from l_1.

Suppose now that $\tilde{l}_1$ and $\tilde{l}_2$ are any two lines in $\mathcal{L} \backslash \bigcup_{n=1}^{\infty} \mathcal{L}_n$. Then there exists a line $\tilde{l}$ in $\mathcal{L}$ which separates $\tilde{l}_1$ and $\tilde{l}_2$. Therefore we can find a positive integer n_0 and a line l_0 in $\mathcal{L}_{n_0}$ such that l_0 separates $\tilde{l}_1$ and $\tilde{l}_2$. Using the comparison isometries for h between l_0 and $\tilde{l}_1$, and between l_0 and $\tilde{l}_2$, and applying the same arguments from the previous paragraph, we conclude that the restriction of h to $\mathcal{L}$ satisfies the leftward property.

Step V Extending the earthquake map to the upper half-plane

Suppose a point z in the upper half-plane belongs to the complement of $\mathcal{L}$. Then z is an interior point of a stratum T for $\mathcal{L}$. Let γ be a geodesic ray which starts at the point z and ends at a point on the geodesic line l joining 0 and ∞, such that γ is perpendicular to l. Let a be the first intersection of γ with $\mathcal{L}$. Then a belongs to a line l_0 in $\mathcal{L}$. We let $h(z) = A(l_0)(z)$. Observe that this definition yields $h(w) = A(l_0)(w)$ for all w in T. Therefore, to show that h restricted to the upper half-plane satisfies the leftward property, it is enough to prove it for the comparison mapping A between a stratum T with an interior point, and its boundary side $\tilde{l}$ in $\bigcup_{n=1}^{\infty} \mathcal{L}_n$. Note that A is the comparison map between two boundary sides l_0 and $\tilde{l}$ of T. There exists a positive integer n_0 such that both l_0 and $\tilde{l}$ belong to $\mathcal{L}_{n_0}$. Since there is no line in $\mathcal{L}_n$ with $n \geq n_0$ which separates l_0 and $\tilde{l}$, we conclude that $(E_n|_{l_0})^{-1} \circ (E_n|_{\tilde{l}})$ is a hyperbolic Möbius transformation with axis l_0 and translation length $\sigma(\tilde{l})$. By passing to the limit as $n \to \infty$, we obtain the leftward property for E on $\mathcal{L}$.

Step VI Extending the earthquake map to the real line

The mapping E extends trivially to the Euclidean boundary of any stratum T of $\mathcal{L}$ and we let we let $h(x) = E_T(x)$ for every point x on the boundary of T. The leftward property trivially extends to the set consisting of all boundary points of strata of $\mathcal{L}$.

Let P be the set of all points which belong to the boundary of a stratum of $\mathcal{L}$. Obviously, P contains the points at 0 and ∞ and P is dense in the extended real line. The points of P are called the boundary points of $\mathcal{L}$. If x is any boundary point of $\mathcal{L}$, by the leftward property, $h(x) = A(x)$, where A is a hyperbolic Möbius transformation whose axis separates x from the line l that joins 0 to ∞. Furthermore, A moves its axis to the right when viewed from x. Therefore, $xh(x) \geq 0$ and $h(x) \geq x$ for all points x in P.

Let a and b be any two boundary points of $\mathcal{L}$ with $a < b$. Suppose first that $a < 0 < b$. Then, since $ah(a) \geq 0$ and $bh(b) \geq 0$, we conclude $h(a) \leq 0 \leq h(b)$. Suppose now that both a and b are positive. Since all finite approximations h_n fix 0 and ∞, we conclude $h(a) \leq h(b)$ for all a and b in the boundary of S such that $a < b$. Since any two lines in $\mathcal{L} \backslash \bigcup_{n=1}^{\infty} \mathcal{L}_n$ are separated by a line in $\bigcup_{n=1}^{\infty} \mathcal{L}_n$, $h(a) \leq h(b)$ for all a and b in P with $a < b$. By the leftward property, $h(a) = A(a)$ and $h(b) = B(b)$ where A and B are Möbius transformations such that the comparison map $L = A^{-1} \circ B$ is a hyperbolic transformation whose repelling fixed point s belongs to the interval $[a, b]$ and the attracting fixed point t is outside the interval (a, b). Therefore, $A^{-1} \circ B(b) \neq a$ and hence $h(b) \neq h(a)$. Similar reasoning shows that $h(a) < h(b)$ in the case when a and b are both negative. Therefore, h is strictly increasing on the set P.

Now suppose x is not in P. Then there exists a sequence of lines l_n in $\mathcal{L}$ with endpoints at a_n and b_n, such that $a_n < x < b_n$ and both a_n and b_n converge to x. The sequence $h(a_n)$ is an increasing sequence, while $h(b_n)$ is a decreasing sequence. Let $h(x) = \lim_{n\to\infty} h(a_n)$.

Step VII $\quad$ **h is strictly increasing**

The restriction of h to the set P is a strictly increasing function. Let x and y be any two points on the real line such that $x < y$. By the definition, there exist two sequences x_n and y_n in P, such that $x_n \to x, y_n \to y, h(x_n) \to h(x)$ and $h(y_n) \to h(y)$. (If x or y is in P, we simply take a constant sequence.) Since $h(x_n) < h(y_n)$ whenever $x_n < y_n$ we conclude that $h(x) \leq h(y)$. Furthermore, since P is dense in the real line, there are two points z and w in P such that $x < z < w < y$. Hence $h(x) \leq h(z) < h(w) \leq h(y)$. Therefore, h is strictly increasing on the whole real line.

Step VIII $\quad$ **h is continuous**

The restriction of h to the set P is a continuous function. Let $x \in P$, and let x_n be a sequence in P, such that $x_n \to x$ as $n \to \infty$. If x belongs to a line l_0 in $\mathcal{L}$, then $h(x) = A(l_0)(x)$. Similarly, $h(x_n) = A(l_n)(x_n)$, where l_n is a line in $\mathcal{L}$ containing the point x_n. Furthermore, the mapping $L_n = A(l_0)^{-1} \circ A(l_n)$ is a hyperbolic Möbius transformation whose axis separates points x_n and x, and the attracting fixed point of L_n belongs to the interval I_n from x_n to x. Therefore $|h(x_n) - h(x)| = |A(l_0)(L_n(x_n)) - A(l_0)(x)| \leq diameter(A(l_0)(I_n)) \to 0$ as $n \to \infty$.

Step IX $\quad$ **Pointwise convergence of finite approximations**

Since clearly $h_n(x) \to h(x)$ for all x in the boundary of S, the continuity and the monotonicity of both h_n and h at boundary points yield the following corollary.

Corollary 3. *$h_n(x) \to h(x)$ as $n \to \infty$ for all boundary points x.*

In general, for an arbitrarily locally finite measure $(\sigma, \mathcal{L})$, the map h is not necessarily continuously extendable to the points on the circle which are not in P (called the *accumulation points* with respect to the lamination $\mathcal{L}$). See some examples in Section 2.4. However, if σ is Thurston bounded, then h is continuously extendable to the accumulation points, and therefore becomes a homeomorphism of the circle. In the remaining part of this section, we assume that σ is Thurston bounded.

Lemma 11. *Let l_1 and l_2 be any two lines in $\mathcal{L}$ and let d be the hyperbolic distance from l_1 to l_2. If $d \geq 1$, then the hyperbolic distance from $h(l_1)$ to $h(l_2)$ is greater then C, where C is a positive constant only depending on the total measure of all intermediate lines.*

Proof. Conjugating by a Möbius transformation, we may assume that the endpoints of l_1 are at 0 and ∞, while the endpoints of l_2 are at the points 1 and c with $c > 1$. To prove the lemma it is enough to find an upper bound on the cross ratio $|cr| = |cr(h(\infty), h(0), h(1), h(c)|$. Since the finite earthquake approximations h_n converge to h at the four points a, b, c, d, it is enough to find a uniform bound on $|cr_n| = |cr(h_n(\infty), h_n(0), h_n(1), h_n(c)|$. Postcomposing by another Möbius transformation if necessary, we may assume that h_n fixes all points on the line l_1. Using the method of proving Theorem 3, one can show that the cross ratio $|cr_n|$ is maximized when all lines of $\mathcal{L}$ separating l_1 and l_2 are the same and equal to the line that joins 1 and ∞. Therefore,

$$|cr_n| \leq |cr(\infty, 0, 1, L(c)|,$$

where L is a hyperbolic transformation with the attracting fixed point at 1, repelling fixed point at ∞, and the translation length $\sigma([0,1] \times [c, \infty])$. Therefore

$$|cr| \leq \sigma([0,1] \times [c, \infty])|cr(0, \infty, 1, c)|.$$

Since the function $d^2|cr(0, \infty, 1, c)|$ is bounded away from 0 and from ∞ for any c in a compact interval, the lemma follows. □

Proposition 2. *If σ is Thurston bounded, then h is a homeomorphism on the circle.*

Proof. We only need to show that h is continuous at the accumulation points. Let x be such a point, and $\{l_n\}_{n=1}^{\infty}$ a sequence of lines l_n in $\mathcal{L}$ with endpoints at a_n and b_n such that $a_n < x < b_n$ and both a_n and b_n converge to x. We want to show that h is continuous at x. Since h is strictly increasing, it is enough to show that $h(b_n)$ converges to $h(x)$. We choose the lines in the sequence $l_1, l_2, l_3, \ldots$ so that the distance between any two consecutive lines is minimal possible but yet not less than 1. This forces the measure of all intermediate lines for any consecutive pair to be at most $||\sigma||_{Th}$. Let d_i be the distance from l_i to l_{i+1}, and let $\tilde{d}_i$ be the distance from $h(l_i)$ to $h(l_{i+1})$. By Lemma 11, $\tilde{d}_i \geq C(||\sigma||_{Th})$ for all i. Since the distance from $h(l_i)$ to $h(l_{i+k})$ is greater than or equal to $\sum_{j=i}^{j=i+k} \tilde{d}_j$, we conclude that $\lim_{n\to\infty} h(a_n) = \lim_{n\to\infty} h(b_n)$, and the continuity of h at x follows. □

The monotonicities of h_n's and the continuity of h yield the following.

Theorem 4. *(1) $h_n(x) \to h(x)$ as $n \to \infty$ at any point x on the real line.*

(2) If one considers the unit disk as the hyperbolic plane, then h_n converges to h uniformly on the unit circle.

In the next, we show that the atomic approximations σ_n have uniformly bounded Thurston norm. Let l_0 be any line in the lamination $\mathcal{L}_n$. By Step I there exists a line l_1 in $\mathcal{L}_n$ such that l_0 and l_1 are adjecent lines in $\mathcal{L}_n$ and $\sigma_n(l_0)$ is equal to the $\sigma-$ measure of the set S_0 of all lines in $\mathcal{L}$ which separate l_0 from l_1, including l_0 and excluding l_1. If the distance from l_0 to l_1 is more that $\frac{1}{n}$, then by the construction of $\mathcal{L}_n$ in Step 1, $\sigma(S_0) = \sigma(l_1)$. Therefore, in both cases we conclude that

$$\sigma_n(l_0) \leq ||\sigma||_{Th}. \tag{8}$$

for all lines l_0 in $\mathcal{L}_n$.

Proposition 3.

$$||\sigma_n||_{Th} \leq 3||\sigma||_{Th} \text{ for all } n.$$

Proof. Suppose that l_0 and l_1 are two lines from the lamination $\mathcal{L}_n$ such that the distance from l_0 to l_1 is at most 1. Let $l_1, l_2, l_3, \dots, l_k = l_0$ be the set of all lines in $\mathcal{L}_n$ which weakly separate l_0 and l_1. The lines l_i have the endpoints at the intervals I and J determined by l_0 and l_1. By the definition from Step I, $\sigma_n(l_i) = \sigma(S_i)$ where S_i are corresponding sets, obtained using the link from l to l_i. Therefore, all but at most two of the sets S_i are disjoint subsets of $I \times J$. The inequality (8) yields $\sigma_n(l_i) \leq ||\sigma||_{Th}$ for each i, and the theorem follows. □

2.2. **Recovering measures.** Let σ be a locally finite measure whose support is a lamination $\mathcal{L}$. The construction in 2.1 produces a homeomorphism $h = H(\sigma)$ of a dense subset $P = P(\sigma)$ of the unit circle into the unit circle. Furthermore, there is an earthquake map $E = E_h$ whose lamination is the support $\mathcal{L}$ of σ and whose restriction to P is h and h is continuous at every point of P. The construction in 1.3 produces a locally finite measure $\sigma_1 = F(h)$ using the values of the earthquake extension of h to the unit disk. The next theorem shows that this construction is the inverse of the construction given in 2.1, that is, $\sigma_1 = \sigma$.

Theorem 5. *If σ is a locally finite measure whose support is a lamination, then*

$$F(H(\sigma)) = \sigma.$$

Proof. Let I and J be any two open intervals on the extended real line such that the closures of I and J are disjoint. We have to show that $\sigma(I \times J) = \sigma_1(I \times J)$. $\sigma(I \times J)$ is the σ-measure of the set of all lines in $\mathcal{L}$ which separate two geodesic lines l_0 and l_1. We can assume the set of the endpoints of I and J is the same as the set of the endpoints of the two lines l_0 and l_1. By the additive property of the two measures, we may also assume that the line l_0 separates the line l_1 from the line l joining 0 to ∞. Suppose first that both lines l_0 and l_1 belong to $\mathcal{L}_{n_0}$ for some n_0. Let N be a large positive integer. Choose a sufficiently large n such that $n \geq n_0$ and the set $\{l_1, l_2, l_3, \dots, l_{k(n)} = l_0\}$ of all lines in $\mathcal{L}_n$ which separate l_1 from l_0 satisfies the following property. Any two consecutive lines l_i and l_{i+1} are either within distance less than $\frac{1}{N}$, or there are no lines in $\mathcal{L}$ separating l_i and l_{i+1}. If A_i is the restriction of E_h to l_i, then, by the left hand inequality in Lemma 8, the sum of the translation lengths $\tau((A_i)^{-1} \circ A_{i+1})$ of all comparison mappings is greater than or equal to $\sigma_1(I \times J)$. By the right hand inequality in Lemma 8, $\tau((A_i)^{-1} \circ A_{i+1}) \leq \tau((A_{i,n})^{-1} \circ A_{i+1,n}) + \frac{C}{N^2}$ where $A_{i,n}$ is the restriction of E_{h_n} to l_i. Therefore, the sum of $\tau((A_{i,n})^{-1} \circ A_{i+1,n})$ is greater than or equal to $\sigma_1(I \times J) - \frac{C}{N^2}$. On the other hand, the sum of $\tau((A_{i,n})^{-1} \circ A_{i+1,n})$ is equal to $\sigma(\overline{I} \times \overline{J}) - \sigma(l_0)$. Suppose now that I and J are any two open intervals with disjoint closures. Approximating $I \times J$ by a sequence of the products of subintervals with end points in $S = \bigcup_n \mathcal{L}_n$ we obtain

$$\sigma(I \times J) \geq \sigma_1(I \times J).$$

To prove the converse, let $\epsilon > 0$. Suppose first that both lines l_0 and l_1 belong to $\mathcal{L}_{n_0}$ for some n_0. There exists a positive integer $n \geq n_0$ such that the set $\{l_1, l_2, l_3, \ldots, l_{k(n)} = l_0\}$ of all lines in $\mathcal{L}_n$ satisfies the following property. If A_i is the restriction of E_h to l_i, then the sum of the translation lengths $\tau((A_i)^{-1} \circ A_{i+1})$ of all comparison mappings is less than $\sigma_1(\overline{I} \times \overline{J}) - \sigma_1(l_0) + \epsilon$. Since $\tau((A_i)^{-1} \circ A_{i+1}) = \lim_{k\to\infty} \tau((A_{i,k})^{-1} \circ A_{i+1,k})$ where $A_{i,k}$ is the restriction of E_{h_k} to l_i, we conclude that the sum of $\tau((A_{i,k})^{-1} \circ A_{i+1,k})$ is less than $\sigma_1(\overline{I} \times \overline{J}) - \sigma_1(l_0) + \epsilon$. On the other hand, by the left hand side of (1) in Lemma 8, the sum of $\tau((A_{i,k})^{-1} \circ A_{i+1,k})$ is greater than or equal to $\sigma(\overline{I} \times \overline{J}) - \sigma(l_0)$. Approximating a general $I \times J$ again by a sequence of the products of subintervals with end points in $S = \bigcup_n \mathcal{L}_n$ and letting $\epsilon \to \infty$, we obtain

$$\sigma(I \times J) \leq \sigma_1(I \times J).$$

Therefore σ_1 coincides with σ. □

Corollary 4. *The mapping H is one-to-one on the space of locally finite measures with lamination-support.*

Proof. Let σ and σ_0 be two locally finite measures with lamination-support. If $H(\sigma) = H(\sigma_0)$, then $\sigma = F(H(\sigma)) = F(H(\sigma_0)) = \sigma_0$. □

2.3. **Recovering homeomorphisms.** If a measure σ and its support $\mathcal{L}$ naturally come from an earthquake representation (see Theorem 2) of a homeomorphism h, then the earthquake map E constructed in Section 1.2 is extendable to a circle homeomorphism h_1, and in fact h_1 and h are equal up to postcomposition by a Möbius transformation. This statement is a consequence of the following result, which applies to earthquakes sharing the same lamination and shearing measure, but which do not necessarily extend to homeomorphisms of the circle.

Proposition 4. *Let E_1 and E_2 be two earthquake maps with the same source lamination $\mathcal{L}$. If the shearing measures introduced in Section 1.3 for E_1 and E_2 are the same, then there is a Möbius transformation A such that E_2 is equal to $A \circ E_1$ on the boundary of the unit circle.*

The proof we give is due to Thurston [9], who stated it for the case that the restriction of E_1 to the unit circle is a homeomorphism. Since we are claiming the result in a slightly more general context, we give the complete proof here. We begin with the following lemma which applies to two Möbius transformations with nearly equal translation length and nearby, but possibly intersecting translation axes.

Lemma 12. *Assume that two goedesic lines l_1 and l_2 in the upper half plane are so close that they are contained in a strip S bounded by two semicircles which share a center in the real line and have hyperbolic distance apart equal to d, where*

$0 \leq d \leq \epsilon$. Denote the endpoints of l_1 by a and c with $a < c$. Suppose that A_1 and A_2 are two hyperbolic transformations with axes l_1 and l_2, and translation lengths $\log\lambda_1$ and $\log\lambda_2$ respectively. If $1 \leq \lambda_1, \lambda_2 \leq M$ and $|\log\lambda_1 - \log\lambda_2| \leq Cd^2\max\{\log\lambda_1, \log\lambda_2\}$ for two constants $C, M > 0$, then for sufficient small ϵ, $A_2 \circ A_1^{-1}(x)$ differs from x by at most $O(d(c-a)\max\{\log\lambda_1, \log\lambda_2\})$ for any x in the interval $[a, c]$, where the constant in O only depends on ϵ, C and M.

Proof. Let us first assume l_2 shares the endpoint c with l_1. Denote the other endpoint l_2 by b. Let $B(x) = \frac{x-a}{c-x}$, then $B(l_1)$ is the half imaginary axis and $B(l_2)$ is a geodesic connecting b' to infinity, where $|b'| \leq C_1 d$ and the constant C_1 depends on ϵ. Clearly

$$B \circ A_1 \circ B^{-1}(y) = \lambda_1 y \text{ and } B \circ A_2 \circ B^{-1}(y) = \lambda_2(y - b') + b'.$$

Then

$$B \circ A_2 \circ A_1^{-1} \circ B^{-1}(y) = \lambda_2(\frac{y}{\lambda_1} - b') + b' = \frac{\lambda_2}{\lambda_1}y + b'(1-\lambda_2),$$

and

$$A_2 \circ A_1^{-1}(x) = B^{-1}[\frac{\lambda_2}{\lambda_1}B(x) + b'(1-\lambda_2)].$$

Therefore

$$A_2 \circ A_1^{-1}(x) - x = \frac{(\frac{\lambda_2}{\lambda_1} - 1)(x-a) + b'(1-\lambda_2)(c-x)}{\frac{\lambda_2}{\lambda_1}(x-a) + [1 + b'(1-\lambda_2)](c-x)} \cdot (c-x). \tag{9}$$

For any $a \leq x \leq c$, divide the numerator and the denominator of the above quotient by $x-a$ (or $c-x$) if $x - a \geq c - x$ (resp. $c - x > x - a$). Since $|\log\lambda_1 - \log\lambda_2| \leq Cd^2\max\{\log\lambda_1, \log\lambda_2\}$, $|b'| \leq C_1 d$, $0 \leq d \leq \epsilon$, and $1 \leq \lambda_2 \leq M$,

$$|\frac{\lambda_2}{\lambda_1} - 1| \leq C_2 d^2 \max\{\log\lambda_1, \log\lambda_2\} \text{ and } |b'(1-\lambda_2)| \leq C_3 d\log\lambda_2$$

for two constants $C_2, C_3 > 0$. Then the numerator in (9) is equal to

$$O(d^2\max\{\log\lambda_1, \log\lambda_2\} + d\log\lambda_2) = O(d\max\{\log\lambda_1, \log\lambda_2\}).$$

When ϵ is small enough, both summands in the denominator are positive and at least one of them is bounded away from zero. Therefore

$$A_2 \circ A_1^{-1}(x) - x = O(d|a-c|\max\{\log\lambda_1, \log\lambda_2\}),$$

where the constant O depends on ϵ, C and M.

Similarly, we have the same estimate if l_2 shares the endpoint a with l_1.

If the geodesic l_2 does not share any endpoint with l_1, then we select a geodesic l that joins the endpoint a of l_1 to the endpoint of l_2 which is near c, and let A be the hyperbolic Möbius transformation with the axis l, translation length λ_1, and translating in the same direction as A_1 and A_2. We rewrite

$$A_2 \circ A_1^{-1}(x) - x = [(A_2 \circ A^{-1}) \circ (A \circ A_1^{-1})(x) - (A \circ A_1^{-1})(x)] + [(A \circ A_1^{-1})(x) - x].$$

Then the proof of the lemma follows by the two special cases already proved applied to the Möbius transformations A and A_1 and to the Möbius transformations A and A_2. □

We will use the follwoing straightforward corollary of the previous lemma to derive the ordinary differential equation of general earthquake curves in Section 4.

Corollary 5. *With the same assumptions in the previous lemma, for any $x \in [a, c]$,*

$$|A_2(x) - A_1(x)| = O(d(c-a)\max\{\log\lambda_1, \log\lambda_2\}).$$

Now we begin the proof of Proposition 4.

Proof. Let $E = E_2 \circ E_1^{-1}$. We do not know if the map E is an earthquake map defined on the target lamination of E_1, but we can still consider the comparison maps for E. For any two lines l and l' in the source lamination of E_1 with hyperbolic distance < 1, we show that the comparison map $(E|_{E_1(l)})^{-1}(E|_{E_1(l')})$ is the identity, and therefore E is a single Möbius transformation B, that is, $E_2 = B \circ E_1$.

We seperate the proof into two cases according whether the hyperbolic distance between l and l' is zero or not. We first treat the case in which it is not zero. By conjugating by a Möbius transformation, we may assume l and l' are two semicircles in the upper half-plane centered at the origin. Let r be the segment on the imaginary axis joining l to l'. We may also assume l' joins -1 to 1 and l joins $-a$ to a, with $1 \leq a \leq e$. We also assume there are infinitely many lines in the lamination intersecting r and the intersections are dense in r. Otherwise, we consider subintervals of r in which the lines of the lamination are dense and the gaps between them. Now, let $P_n = \{r_i\}$ be a sequence of partitions of r into short segments, each with hyperbolic length $\leq d(n)$, where $d(n) \to 0$ as $n \to \infty$. Assume lines of the lamination are labeled in order with $l_0 = l, l_1, l_2, \cdots, l_n = l'$, and assume the restriction of E_1 on l_0 is the identity, and $E_1|_{l_i} = A_1 \circ \cdots \circ A_i$ for $1 \leq i \leq n$, and $E_2|_{l_0} = Id$, $E_2|_{l_i} = B_1 \circ \cdots \circ B_i$ for $1 \leq i \leq n$. Then for each $1 \leq i \leq n$, A_i and B_i are hyperbolic transformations which translate in the same direction and with axes are sufficiently close such that they are contained in a strip S_i bounded by two semicircles centered at a point in $[-1, 1]$ with hyperbolic distance $< d$, where $d = O(d(n))$. Let $\log\lambda_i$ denote the translation of A_i and $\log\nu_i$ the translation length of B_i, $1 \leq i \leq n$. Let $\sigma(r) = \sigma_1(r) = \sigma_2(r)$ and $\sigma_i = \sigma_1(r_i) = \sigma_2(r_i)$, $1 \leq i \leq n$. By Proposition 1, both $\log\lambda_i$ and $\log\nu_i$ differ from the measure σ_i by $O(\log\lambda_i d(n)^2) = O(\log\nu_i d(n)^2) = O(\sigma_i d(n)^2)$, and therefore $|\log\lambda_i - \log\nu_i| = O(\sigma_i d(n)^2)$.

The comparison map $A_{l'l} = (E|_{E_1(l)})^{-1}(E|_{E_1(l')})$ is equal to

$$B_n \circ B_{n-1} \circ \cdots \circ B_2 \circ B_1 \circ A_1^{-1} \circ A_2^{-1} \circ \cdots \circ A_{n-1}^{-1} \circ A_n^{-1}. \tag{10}$$

Let x be a point near the origin. We will show that $A_{l'l}$ is the identity by showing that the the long composition (10) for $A_{l'l}$ applied to x is arbitrarily near x.

Let $x_{-i} = A_i^{-1} \circ A_{(i-1)}^{-1} \circ \cdots \circ A_1^{-1}(x)$, $1 \leq i \leq n-1$. By Lemma 12, we have

$$B_n \circ A_n^{-1}(x_{-(n-1)}) = x_{-(n-1)} + O(d(n)\sigma_n),$$

and by the mean value theorem and Lemma 12 again,

$$B_{n-1}(x_{-(n-1)} + O(d(n)\sigma_n)) = B_{n-1}(x_{-(n-1)}) + O(e^{\nu_{n-1}} d(n)\sigma_n)$$
$$= B_{n-1} \circ A_{n-1}^{-1}(x_{-(n-2)}) + O(d(n)e^{\sigma_{n-1}}\sigma_n).$$

Inductively using the mean value theorem and Lemma 12, we have

$$A_{l'l}(x) = x + O(d(n)\sigma_1) + O(d(n)e^{\sigma_1}\sigma_2) + O(d(n)e^{\sigma_1+\sigma_2}\sigma_3) + \cdots$$
$$+ O(d(n)e^{\sigma_1+\sigma_2+\cdots+\sigma_{n-1}}\sigma_n) = x + O(d(n)e^{\sigma(r)}\sigma(r)).$$

Since $\sigma(r)$ is finite and $d(n)$ converges to 0 as n goes to ∞, $A_{l'l}(x) = x$. Thus $A_{l'l}$ is the identity map.

If l_1 and l_2 share an endpoint, after conjugation by a Möbius transformation, we may assume that l_1 is the semicircle connecting -1 to 1 and l_2 is a semicircle connecting -1 to a for some $a > 1$. These two semicircles cut the imaginary axis to the segment r. Using the partitions of r and the same arguement in the previous case, we can again show the comparison map $(E|_{E_1(l)})^{-1}(E|_{E_1(l')})$ is the identity map. ☐

Using Theorem 5 and the previous proposition, we have the following theorem.

Theorem 6. *Let h be a homeomorphism of the unit circle and σ the shearing measure corresponding to the earthquake representation $(E, \mathcal{L})$ of h. If E_1 is the earthquake map constructed in Section 2.1 for $(\sigma, \mathcal{L})$, then E_1 extends to a homeomorphism h_1 on the boundary circle, and $h_1 = A \circ h$ for some Möbius transformation A.*

Proof. Let σ_1 denote the shearing measure induced by E_1. By Theorem 5, $\sigma_1 = \sigma$ with support $\mathcal{L}$. Now by the previous proposition, there exists a Möbius transformation A such that $E_1 = A \circ E$. Since E extends to h on the boundary circle, E_1 extends to $A \circ h$ on the boundary circle. ☐

2.4. **Measures that do not yield homeomorphisms.** In this section we provide examples of locally finite measures σ such that the maps h_σ cannot be extended to homeomorphisms of the whole unit circle. We prefer to work with the upper half plane.

The first example is due to Thurston [8]. Let l_n be the geodesic line connecting $-n$ to infinity, where $n = 0, 1, 2, \cdots, n, \cdots$, and $\mathcal{L}$ the collection of all l_n's. Suppose that the weight on each line is $\ln 2$, and the left earthquake map E restricted on the right upper half plane is the identity map. Then $E|_{l_{n+1}}^{-1} \circ E|_{l_n}(x) = \frac{1}{2}(x + n) - n$, and therefore for each $n > 0$,

$$E|_{l_n}(-n) = -(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}).$$

Clearly $E|_{l_n}(-n)$ converges to -1 as n goes to infinity, hence the map E maps the whole upper half plane into the quarter on the right of l_1. This implies that E does not extend to a homeomorphism on the boundary $\mathbb{R}$.

The second example shows that there is a locally finite measure σ supported on a lamination $\mathcal{L}$ such that E_σ does not extend to a homeomorphism on $\bar{\mathbb{R}}$ but $E_{2\sigma}$ does.

Pick any quadruple $\{a_1, b_1, e_1, f_1\}$ on the real line such that $a_1 < e_1 < f_1 < b_1$. Let L_1 be the geodesic line joining a_1 to b_1. For every $\lambda > 0$, let A_1^λ be the hyperbolic isometry with translation length $\log \lambda$, attracting fixed point at b_1 and repelling fixed point at a_1. Note that $A_1^\lambda(e_1) \to b_1$ and $(A_1^\lambda)^{-1}(f_1) \to a_1$ as $\lambda \to \infty$. Therefore, we may choose sufficiently large λ_1 so that the transformations $A_1 = A_1^{\lambda_1}$ and A_1^{-1} map the points e_1 and f_1 to c_1 and d_1, and e_2 and f_2, respectively, such that

$$a_1 < e_2 < f_2 < e_1 < f_1 < c_1 < d_1 < b_1,$$

$$d_1 - c_1 < \frac{f_1 - e_1}{2}, \text{ and}$$

$$f_2 - e_2 < \frac{f_1 - e_1}{2}.$$

Since $A_1^2(e_2) = c_1$ and $A_1^2(f_2) = d_1$, we can choose two points a_2 and b_2 sufficiently close to e_2 and f_2 respectively, so that

$$a_1 < a_2 < e_2 < f_2 < b_2 < e_1,$$
$$b_2 - a_2 < \frac{f_1 - e_1}{2}, \text{ and}$$
$$A_1^2(b_2) - A_1^2(a_2) < \frac{f_1 - e_1}{2}.$$

Now let L_2 be the geodesic line joining a_2 to b_2. Again let A_2^λ be the hyperbolic isometry with translation length $\log \lambda$, attracting fixed point at b_2 and repelling fixed point at a_2. Note that $(A_1^2 \circ A_2^\lambda)(e_2) \to A_1^2(b_2)$ and $(A_2^\lambda)^{-1}(f_2) \to a_2$ as $\lambda \to \infty$. Therefore, we may choose sufficiently large λ_2 so that the transformations $A_2 = A_2^{\lambda_2}$ satisfies

$$A_2^{-1}(e_2) = e_3, \ A_2^{-1}(e_3) = f_3, \ A_1^2 \circ A_2(e_2) = c_2, \ A_1^2 \circ A_2(f_2) = d_2,$$
$$a_2 < e_3 < f_3 < e_2 < d_1 < c_2 < d_2 < A_1^2(b_2),$$
$$d_2 - c_2 < \frac{f_2 - e_2}{2}, \text{ and}$$
$$f_3 - e_3 < \frac{f_2 - e_2}{2}.$$

Since $A_1^2 \circ A_2^2(e_3) = c_2$ and $A_1^2 \circ A_2^2(f_3) = d_2$, we can choose two points a_3 and b_3 sufficiently close to e_3 and f_3 respectively, so that

$$a_2 < a_3 < e_3 < f_3 < b_3 < e_2,$$
$$b_3 - a_3 < \frac{f_2 - e_2}{2}, \text{ and}$$
$$A_1^2 \circ A_2^2(b_3) - A_1^2 \circ A_2^2(a_3) < \frac{f_2 - e_2}{2} < \frac{f_1 - e_1}{4}.$$

If we continue this construction, we are going to get a sequence of lines L_i with endpoints at a_i and b_i such that both a_i and b_i converge to the same point a on the real line. Let $\mathcal{L}$ be the lamination consisting of lines $L_1, L_2, L_3, \ldots$, and let σ be the measure defined on $\mathcal{L}$ such that the weight of σ on each L_i is equal to $\log \lambda_i$. If we normalize $h = h_\sigma$ to be equal the identity mapping on the segment $[-\infty, a_1]$, then the restriction of h to the line L_i is equal to $A_1 \circ A_2 \circ A_3 \circ \ldots \circ A_i$. The Möbius transformation A_i has axis L_i and weight $\log \lambda_i$. Furthermore, the points $A_1^2 \circ A_2^2 \circ A_3^2 \circ \ldots A_i^2(a_{i+1})$ and $A_1^2 \circ A_2^2 \circ A_3^2 \circ \ldots A_i^2(b_{i+1})$ converge to the same point c as $n \to \infty$.

Theorem 7. *The mapping h_σ is well-defined at each point on the extended real line except at the point a. Furthermore, h_σ is strictly increasing. However, h_σ cannot be continuously extended to the homeomorphism of $\overline{\mathbb{R}}$. On the other hand, the mapping $h_{2\sigma}$ does extend to a homeomorphism of $\overline{\mathbb{R}}$.*

Proof. The first two statements follow from Section 1.4. To verify the third, observe that no point in the interval (e_1, f_1) is in the image of h_σ. Finally, the map $h_{2\sigma}$ extends continuously with $h(a) = c$. □

Corollary 6. *There exists a locally finite measure σ for which $H(\sigma)$ is not a homeomorphism of the circle.*

Corollary 7. *There exist earthquake curves $t \mapsto h_{t\sigma}$ that do not preserve homeomorphisms.*

Proof. For σ as constructed in the previous theorem, h_σ is not a homeomorphism but $h_{2\sigma}$ is. □

One of the consequences of the results of the next section is that if h is quasisymmetric, then so is $h_{t\sigma}$ for all $t \geq 0$.

3. Quasisymmetric homeomorphisms

Let I and J be two contiguous intervals on the unit circle of equal Euclidean length and each of length less than $\pi/4$, and denote the length of I by $|I|$. A circle homeomorphism h is *quasisymmetric* if there is a constant M, such that

$$1/M \leq \frac{|h(J)|}{|h(I)|} \leq M$$

for all such I and J.

An equivalent condition for quasisymmetry can be given in terms of the distortion by h of cross-ratios of quadruples Q on the unit circle. Assume Q consists of four points a, b, c and d on the circle arranged in counterclockwise order and let the cross ratio of Q be

$$cr(Q) = \frac{(d-c)(b-a)}{(c-b)(a-d)}.$$

Define the norm of h to be

$$||h||_{cr} = \sup \left| \log \frac{cr(h(Q))}{cr(Q)} \right|, \tag{11}$$

where the supremum is taken over all Q for which $cr(Q) = -1$. It turns out that h is quasisymmetric if, and only if, $||h||_{cr}$ is finite. $||h||_{cr}$ has the convenient property that $||A \circ h||_{cr} = ||h||_{cr} = ||h \circ B||_{cr}$ for any Möbius transformations A and B.

We also need a norm on the earthquake measure σ for h. Let β be an arbitrary hyperbolic, closed geodesic segment in the unit disc of length 1. Since $\mathcal{L}$ is closed the intersection of the lines of $\mathcal{L}$ is a closed set in β, and if this closed set is non-empty we can select lines l_1 and l_2 in $\mathcal{L}$ that meet β at points on β which are the maximum possible distance apart. The lines l_1 and l_2 together with two closed intervals I and J on the unit circle surround a topological rectangle. We define $mass_\sigma(\beta)$ to be $\sigma(I \times J)$ and

$$||\sigma||_{Th} = \sup mass_\sigma(\beta), \tag{12}$$

where the supremum is taken over all hyperbolic, closed geodesic segments of length 1. We call $||\sigma||_{Th}$ the Thurston norm of σ. The primary purpose of this section is to show there is a positive constant C such that

$$||\sigma||_{Th} \leq C||h||_{cr}$$

and, given any positive constant C_0, there exists a positive constant C_1 such that if $||\sigma|| \leq C_0$, then

$$||h||_{cr} \leq C_1||\sigma||_{Th}.$$

3.1. **Quasisymmetry implies a Thurston bound.** Consider a hyperbolic geodesic segment β of length 1. We will show there is a constant C such that

$$mass_\sigma(\beta) \leq C||h||_{cr}.$$

To prove this we first show there exists a positive constant C_1 such that for any h, if $||h||_{cr} < 1$, then

$$mass_\sigma(\beta) \leq C_1||h||_{cr}. \tag{13}$$

Then we show there exists another positive constant C_2 such that

$$mass_\sigma(\beta) \leq ||h||_{cr} + C_2 \tag{14}$$

for any h. Thus if $||h||_{cr} \geq 1$, then

$$mass_\sigma(\beta) \leq (1 + C_2)||h||_{cr}.$$

Consequently, for any h, we have

$$mass_\sigma(\beta) \leq \max\{C_1, (1 + C_2)\}||h||_{cr}.$$

To prove inequalities (13) and (14) we use Lemmas 9 and Lemma 10.

Let l_1 and l_2 denote the lines in $\mathcal{L}$ that meet β at points the maximum possible distance apart. There are three situations of relative positions of l_1 and l_2 we need to consider: l_1 and l_2 share no endpoints, or one endpoint or two points.

Let us first assume that l_1 and l_2 do not have a common endpoint. In this case we replace β by the geodesic segment perpendicular to both l_1 and l_2, which we continue to denote by β. By a normalization we may assume β is an arc on the imaginary axis between i and ib with $1 \leq b \leq e$. Clearly, $mass_\sigma(\beta) = \sigma([-b, -1] \times [1, b])$ and the length of β is $\log b$. We also assume that l_1 is the geodesic connecting -1 to 1 and l_2 is the one connecting $-b$ to b.

By postcomposition by a Möbius transformation, we can assume that h fixes the geodesic l_2. Consider the sublamination $\mathcal{L}'$ consisting only of those lines of $\mathcal{L}$ that intersect β and the measure σ' which is the measure σ restricted to the closed subset $\mathcal{L}'$. Then σ' induces a homeomorphism h', we assume that h' also fixes the geodesic l_2. To prove the inequalities (13) and (14), we will compare the cross ratio distortions by h and h' on certain quadruples Q and give bounds for the cross ratio distortions by h' on the quadruples. To prove (13), we take $Q = \{b, -\infty, -b, 0\}$, and to prove (14), we take $Q = \{-\infty, -b, \frac{1+(-b)}{2}, 1\}$. Note that in each case $cr(Q) = 1$.

In the situation that l_1 and l_2 share two endpoints, i.e., $l_1 = l_2$, we substitute the above b by 1 and $-b$ by -1 everywhere. In the other situation that l_1 and l_2 share one endpoint, we replace β by a geodesic segment of small hyperbolic length which is perpendicular to l_1 and transversal to l_2 (still call it β); through a conjugation by a Möbius transformation, we normalize β to be an arc on the imaginary axis between i and ib' with $b' > 1$, and assume that l_1 is the geodesic connecting -1 to 1 and l_2 is the geodesic connecting -1 to b with $b > 1$; we can further assume that $b \leq e$ by taking a replacement of β with small enough hyperbolic length; finally we substitute the above $-b$ by -1 everywhere.

The rest of the proofs for the three situations are the same as follows.

Let us begin with the proof of (13). Note first that the cross ratio distortion by h on Q is bounded below by the cross ratio distortion by h' on Q. To estimate the amount that h' distorts the cross ratio of Q, we approximate h' by compositions of finitely many hyperbolic isometries. For any positive integer n, by the definition of $mass_\sigma(\beta)$, there exist finitely many hyperbolic isometries $A_1, A_2, \ldots, A_k$ whose axes L_i are non-intersecting lines with one endpoint on $[-b,-1]$ and the other on $[1,b]$, such that the composition

$$h'_{\sigma_n} = A_1 \circ A_2 \circ \cdots \circ A_k$$

coincides with h' on the intervals $(-\infty,-b]$ and $[b,\infty)$,

$$mass_\sigma(\beta) \le mass_{\sigma_n}(\beta) = \sum_{i=1}^{k} \tau(A_i) < mass_\sigma(\beta) + \frac{1}{n},$$

and

$$|cr(h'_{\sigma_n}(Q)) - cr(h'(Q))| < \frac{1}{n}.$$

For successive lines L_i and L_{i+1} draw a hyperbolic line L' that connects the endpoint of L_i on $[-b,-1]$ to the endpoint of L_{i+1} on $[1,e]$. By inductively applying Lemma 9 to all of the lines L_i, one sees that the amount of the cross ratio by which h'_σ distorts on Q is least if h'_σ is equal to the identity above the line l_1 joining -1 to 1, and below l_1, it is equal to the hyperbolic Möbius transformation with axis l_1 and multiplier λ_n, where $\log\lambda_n = mass_{\sigma_n}(\beta)$. In that case, h'_{σ_n} fixes $-\infty, -1$ and 1; denote $h'_{\sigma_n}(z)$ by w for any z below l_1, then

$$\frac{w+1}{w-1} = \lambda_n \frac{z+1}{z-1},$$

and therefore $h'_{\sigma_n}(0) = \frac{\lambda_n - 1}{\lambda_n + 1}$; the ratio in (11) works out to be $\frac{b+a_n}{b-a_n}$, where $a_n = \frac{\lambda_n - 1}{\lambda_n + 1}$. Now we have

$$e^{||h||_{cr}} \ge cr(h(Q)) \ge cr(h'(Q)) > cr(h'_{\sigma_n}(Q)) - \frac{1}{n} \ge \frac{b+a_n}{b-a_n} - \frac{1}{n}.$$

Let n go to infinity,

$$e^{||h||_{cr}} \ge \frac{b+a}{b-a},$$

where $a = \frac{\lambda-1}{\lambda+1}$ and $\log\lambda = mass_\sigma(\beta)$. Since $1 \le b \le e$ and $0 \le a < 1$,

$$e^{||h||_{cr}} \ge \frac{b+a}{b-a} \ge \frac{e+a}{e-a}.$$

Let

$$v(\lambda) = \frac{e+a}{e-a} = \frac{(e+1)\lambda + (e-1)}{(e-1)\lambda + (e+1)}.$$

Elementary analysis of $v(\lambda)$ yields that $v(\lambda) \le e^{||h||_{cr}}$ and $e^{||h||_{cr}} < \frac{e+1}{e-1}$ imply that $\lambda \le v^{-1}(e^{||h||_{cr}})$. But $v^{-1}(e) = \frac{e^2+1}{2e+1-e^2} < 5$, so by the mean value theorem applied to v^{-1},

$$v^{-1}(e^{||h||_{cr}}) = v^{-1}(1) + (v^{-1})'(u)(e^{||h||_{cr}} - 1) \le 1 + \frac{1}{v'(5)}(e^{||h||_{cr}} - 1),$$

where $1 \le u \le v^{-1}(e^{||h||_{cr}}) < v^{-1}(e) < 5$ since $||h||_{cr} < 1$. Therefore

$$\log\lambda \le \lambda - 1 \le \frac{1}{v'(5)}(e^{||h||_{cr}} - 1) \le \frac{e}{v'(5)}||h||_{cr}$$

when $||h||_{cr} < 1$. Let $C_1 = \frac{e}{v'(5)} = \frac{(6e-4)^2}{4} = (3e-2)^2$. Then

$$mass_\sigma(\beta) = \log\lambda \le C_1||h||_{cr}$$

for any h with $||h||_{cr} \leq 1$. This proves inequality (13). Also note that when the cross-ratio norm $||h||_{cr}$ decreases to 0, the constant C_1 decreases to 1.

In the remainder of this section, we prove inequality (14) by considering the cross-ratio distortion of h on the quadruple $Q = \{-\infty, -b, \frac{1-b}{2}, 1\}$. Just as in the previous case, we first notice that the cross-ratio distortion of h on Q is less than or equal to the cross-ratio distortion of h' on Q. By Lemma 10, the amount by which h'_{σ_n} distorts the cross-ratio of Q is the greatest if we assume all of the mass of σ_n in β is concentrated on one geodesic line joining -1 to b and the map h'_{σ_n} is the identity outside the interval $[-1, b]$ and inside this interval it is given by the Möbius transformation $z \mapsto w = h'_{\sigma_n}(z)$, where

$$\frac{w+1}{w-b} = \lambda_n \frac{z+1}{z-b}.$$

Taking the limit when n goes to infinity, we see that the amount by which h'_σ distorts the cross-ratio of Q is the greatest if we assume all of the mass of σ in β is concentrated on one geodesic line joining -1 to b and the map h'_σ is the identity outside the interval $[-1, b]$ and inside this interval it is given by the Möbius transformation $z \mapsto w = h'_\sigma(z) = A(z)$, where

$$\frac{w+1}{w-b} = \lambda \frac{z+1}{z-b}.$$

In that case, h_σ maps $-\infty$ to $-\infty$, $-b$ to $-b$, $\frac{1-b}{2}$ to $A(\frac{1-b}{2}) = w_1$, and 1 to $A(1) = w_2$. If $B(s) = \frac{1}{b+1}(s+1)$, then $B \circ A \circ B^{-1}$ has multiplier λ, fixed points at 0 and 1, and

$$B \circ A \circ B^{-1}(t) = \frac{\lambda t}{(\lambda - 1)t + 1}. \tag{15}$$

Let $x = B(-b) = \frac{1-b}{1+b}, y = B(\frac{1-b}{2}) = \frac{3-b}{2(b+1)}, z = B(1) = \frac{2}{b+1}$. We must calculate the distortion of the map $\tilde{h}$ that fixes $-\infty$ and $-b$ and maps y to $B \circ A \circ B^{-1}(y)$ and z to $B \circ A \circ B^{-1}(z)$. From (15), the distortion is

$$cr(\tilde{h}(Q)) = \frac{\frac{\lambda z}{(\lambda-1)z+1} - \frac{\lambda y}{(\lambda-1)y+1}}{\frac{\lambda y}{(\lambda-1)y+1} - x}$$

$$= \frac{\lambda(z-y)}{\lambda^2(y-yx) + \lambda\left((y-yx)(1-z) + (yx-x)z\right) + (yx-x)(1-z)}.$$

Thus, the reciprocal of this distortion is

$$\lambda \frac{z(y-yx)}{z-y} + \frac{yz(2z-1) + y - x(y+z)}{z-y} + \frac{x(y-1)(1-z)}{\lambda(z-y)}. \tag{16}$$

Since the last two terms in (16) are positive, we obtain

$$\log \frac{1}{cr(\tilde{h}(Q))} \geq \log \lambda + \log \frac{zy(1-x)}{z-y}.$$

Since $cr(h(Q)) \leq cr(h'_\sigma(Q)) \leq cr(\tilde{h}(Q)) \leq 1$,

$$||h||_{cr} \geq |\log cr(h(Q)| \geq |\log cr(\tilde{h}(Q))| = \log \frac{1}{cr(\tilde{h}(Q))},$$

and hence

$$||h||_{cr} \geq \log \lambda + \log \frac{zy(1-x)}{z-y}.$$

By substituting in the values of x, y and z, the term $\log \frac{zy(1-x)}{z-y}$ is a function of b for $1 \le b \le e$ which has a lower bound $-\log \frac{(e+1)^3}{4e(3-e)}$. Therefore,

$$\log \lambda \le ||h||_{cr} + \log \frac{(e+1)^3}{4e(3-e)}.$$

We have completed a proof of the inequality (14).

Notice that by careful study of the function of λ defined by the formula (16) when λ is near 1, one can also prove inequality (13). The method used in the proof of the inequality (13) has more applications in Section 5.4.

In summary, we have proved the following theorem.

Theorem 8. *There are positive constants C_1 and C_2 such that if h is a quasisymmetric homeomorphism of the circle and σ its left earthquake measure, then*

$$||\sigma||_{Th} \le C_1 ||h||_{cr},$$

and

$$||\sigma||_{Th} \le ||h||_{cr} + C_2, \tag{17}$$

where $C_1 = \max\{(3e-2)^2, 1 + C_2\}$ and $C_2 = \log \frac{(e+1)^3}{4e(3-e)}$.

3.2. **Finite ordinary differential equations.** For any finite earthquake measure σ with lamination $\mathcal{L}$, $t\sigma$ has same lamination and all of its weights have been multiplied by t. By the finite earthquake theorem, if $t \ge 0$, $t\sigma$ determines, up to postcomposition by a Möbius transformation, a homeomorphism $h_t = H(t\sigma)$. If we assume both h and h_t are normalized at three points, then $h_1 = h$, and h_t is a curve of homeomorphisms which satisfies an ordinary differential equation, [3]. The equation is non-autonomous and can be written in the following way:

$$h_{t+s} \circ (h_t)^{-1}(x) = x + sV_t(x) + o(s), \tag{18}$$

where V_t is a vector field determined by the measure σ transported by h_t to the lamination $h_t(\mathcal{L})$.

We now describe a formula for the vector field V_t. In order to do this, we take the upper half-plane as a model for the hyperbolic plane and h and h_t are homeomorphisms of $\overline{\mathbb{R}}$. Define a vector field $E_{ab}(x)$ by the following formula:

$$E_{ab}(x) = \begin{cases} \frac{(x-a)(x-b)}{a-b} & \text{for } a \le x \le b, \\ 0 & \text{otherwise,} \end{cases} \tag{19}$$

For each line L_j in $\mathcal{L}$ with endpoints a_j, b_j, let $p_j = \sigma(\{(a_j, b_j)\})$ be the weight of σ on L_j. Let $L_j(t) = h_t(L_j)$ have endpoints $h_t(a_j) = a_j(t)$ and $h_t(b_j) = b_j(t)$. Then

$$V_t(x) = \sum_j p_j E_{a_j(t) b_j(t)}(x), \tag{20}$$

where the sum is over all endpoints $a_j(t), b_j(t)$ of lines $h_t(L_j)$ in the target lamination $h_t(\mathcal{L})$.

Formula (18) where V_t is given by (20) is verified in two steps. One first shows that it is correct when $t = 0$. At a general point t, one uses the fact that, for $s > 0$, $E_{(t+s)\sigma} \circ (E_{t\sigma})^{-1}$ is a left earthquake whose lamination is $h_t(\mathcal{L})$ and whose measure is $sh_t^*(\sigma)$.

Given a finite earthquake measure σ, there is a linear map E from σ to V given by

$$E(\sigma)(x) = V(x) = \sum_j \sigma(\{(a_j, b_j)\}) E_{a_j b_j}(x),$$

where the sum is over all pairs of endpoints (a_j, b_j) of lines L_j in $\mathcal{L}$. We introduce the following invariant norm on the space of vector fields, V :

$$||V||_{cr} = \sup_Q |cr(Q)\rho(cr(Q))V[Q]|\,, \tag{21}$$

where

$$V[Q] = \frac{V(b) - V(a)}{b - a} - \frac{V(c) - V(b)}{c - b} + \frac{V(d) - V(c)}{d - c} - \frac{V(d) - V(a)}{d - a} \tag{22}$$

and ρ is the infinitesimal form of the Poincaré metric for the sphere punctured at $0, 1$ and ∞. In [4] it is shown that E is a bounded operator in the sense that

$$||E(\sigma)|| \leq C||\sigma||_{Th}. \tag{23}$$

3.3. **Thurston bound implies quasisymmetry.** In this section, we control the norm $||h||_{cr}$ in terms of the norm $||\sigma||_{Th}$. More precisely, we show that for any $C_0 >$, there exists a constant $C > 0$ depending on C_0 such that for any σ, if $||\sigma||_{Th} \leq C_0$ then $||h||_{cr} \leq C||\sigma||_{Th}$.

Let σ be a Thurston bounded measure with lamination $\mathcal{L}$, and h is determined by σ up to postcomposition by a Möbius transformation. Assume two lines L_1 and L_2 in the source lamination have a common perpendicular β with length greater than or equal to 1. In Lemma 11, we have shown the hyperbolic distance between $h(L_1)$ and $h(L_2)$ is greater than or equal to a constant C, where C depends only on the total mass of the lines L in $\mathcal{L}$ which cross the commom perpendicular segment β.

Lemma 13. *Let $C_0 > 0$ and let σ_t be the pushforward by h_t of the measure σ on $\mathcal{L}$. Then for $0 \leq t \leq 1$ and $||\sigma||_{Th} \leq C_0$, we have $||\sigma_t||_{Th} \leq C||\sigma||_{Th}$. The constant C depends only on C_0.*

Proof. Let $\mathcal{L}$ be the support lamination for the measure σ. Let $h_t(l_1)$ and $h_t(l_2)$ be any two lines in $h_t(\mathcal{L})$ whose distance is at most 1. Then the total σ_t-measure of all lines separating $h_t(l_1)$ from $h_t(l_2)$ is $\sigma_t(h_t(I) \times h_t(J))$ where $I = [a, b]$ and $J = [c, d]$ are two disjoint intervals whose endpoints are the endpoints of l_1 and l_2. Choose a sequence of lines $\tilde{l}_1 = l_1, \tilde{l}_2, \tilde{l}_3, \ldots \tilde{l}_k = l_2$ in the following way. Start with $i = 1$ and let $\tilde{l}_{i+1}$ be the line in $\mathcal{L}$ which weakly separates $\tilde{l}_i$ from l_2 and the common geodesic to $\tilde{l}_i$ and $\tilde{l}_{i+1}$ has length $1 + \epsilon$ where ϵ is the smallest possible nonnegative number. Continue this process for $i = 2, 3, 4, \ldots$. Eventually, the distance from $\tilde{l}_i$ to l_2 will be less than 1. In that case, we let $\tilde{l}_{i+1} = l_2$ and stop. This forces the $\sigma-$measure of all intermediate lines for any consecutive pair $\tilde{l}_i$ and $\tilde{l}_{i+1}$ be at most $||\sigma||_{Th}$. Let $\tilde{d}_i$ be the distance from $h_t(\tilde{l}_i)$ to $h_t(\tilde{l}_{i+1})$. By Lemma 11, $\tilde{d}_i \geq C(||\sigma||_{Th})$ for all $i \leq k - 2$. The distance from $h_t(l_1)$ to $h_t(l_2)$ is greater than or equal to $\sum_{j=1}^{j=k-2} \tilde{d}_j$, which is greater than or equal to $(k-2)C(||\sigma||_{Th})$. On the other hand, $\sigma(I \times J) \leq (k - 1)||\sigma||_{Th}$. Therefore,

$$\sigma_t(h_t(I) \times h_t(J)) = \sigma(I \times J) \leq \frac{2||\sigma||_{Th}}{C}, \tag{24}$$

where $C = C(C_0)$ is the constant from Lemma 11. □

Theorem 9. *Let $C_0 > 0$. Then there is a positive constant C, such that if σ is a locally finite measure whose support is a lamination $\mathcal{L}$, and $h = H(\sigma)$, then*

$$||h||_{cr} \leq C||\sigma||_{Th}$$

for all σ with $||\sigma||_{Th} \leq C_0$. The constant C depends only on C_0. In particular, h is a homeomorphism of the unit circle.

Proof. Select a quadruple $Q = \{a, b, c, d\}$, where a, b, c and d are in counterclockwise order and

$$cr(Q) = \frac{(b-a)(d-c)}{(c-b)(d-a)} = 1.$$

Let $I = [a, b]$, $J = [c, d]$ and L_{ad} and L_{bc} be the hyperbolic geodesics joining a to d and b to c, respectively. Finally, let β be the closed segment which is the common perpendicular to the to lines L_{ad} and L_{bc}. Now, select the finite earthquakes with measures σ_n chosen in Section 2.1. Recall that the homeomorphisms h_{σ_n} converge pointwise to h_σ on the unit circle, and

$$||\sigma_n||_{Th} \leq 3||\sigma||_{Th}.$$

Therefore, the theorem follows if we can show

$$|\log cr(h_{\sigma_n}(Q))| \leq C||\sigma_n||_{Th},$$

where the constant C is independent of n. Therefore, it suffices to prove the theorem for finite earthquakes and to treat this case we can use the ordinary differential equation (18).

Assume h is a finite earthquake map with finite earthquake measure σ, and consider the curve of quadruples $Q_t = \{h_t(a), h_t(b), h_t(c), h_t(d)\} = \{a_t, b_t, c_t, d_t\}$. We wish to estimate $|\log cr(Q_1)|$. The curve $g(t) = cr(Q_t)$ takes values on the negative real axis and $g(0) = 1$ and

$$g'(t) = g(t)\left(\log g(t)\right)'(t) = cr(Q_t)V_t[Q],$$

where

$$V_t[Q] = \frac{V_t(b_t) - V_t(a_t)}{b_t - a_t} - \frac{V_t(c_t) - V_t(b_t)}{c_t - b_t} + \frac{V_t(d_t) - V_t(c_t)}{d_t - c_t} - \frac{V_t(d_t) - V_t(a_t)}{d_t - a_t}.$$

Thus, by Lemma 13 and the linear bound of inequality (23), we obtain a bound on

$$\int_0^1 \rho(g(t))|g'(t)|dt.$$

This bound implies $g(1)$ is bounded away from $0, -1$ and ∞ and yields a bound on $|\log|cr(h(Q_1)||$ depending only on $||\sigma||_{Th}$. □

Therefore, we have proved the following corollaries.

Corollary 8. *Let σ be a locally finite measure whose support is a lamination $\mathcal{L}$. Then $h = H(\sigma)$ is a quasisymmetric homeomorphism of the unit circle, if, and only if, σ is Thurston bounded.*

Corollary 9. *If h is quasisymmetric, then there is a unique shearing measure on the lamination $\mathcal{L}$ for h, $h = h_\sigma$ and $h_{t\sigma}$ is quasisymmetric for all $t \geq 0$.*

4. Ordinary Differential Equations

Let σ be an earthquake measure supported on a lamination $\mathcal{L}$ which has finitely many leaves. For any $t \geq 0$, $(t\sigma, \mathcal{L})$ determines up to postcomposition by a Möbius transformation a circle homeomorphism $h_t = h_{t\sigma}$. We assume that for every $t \geq 0$, h_t fixes the same three points. Such a curve is called an earthquake curve determined by $(t\sigma, \mathcal{L})$. Then h_0 is the identity map and for any point x on the circle $h_t(x)$ is differentiable with respect to t and the derivative $\frac{d}{dt}h_t$ satisfies the following nonautonomous ordinary differential equation

$$\frac{d}{dt}h_t(x) = V_t(h_t(x)), \tag{25}$$

where V_t is the vector field given by

$$V_t(x) = \int\int E_{h_t(a)h_t(b)}(x)d\sigma(a,b) + a\ quadratic\ polynomial,$$

where $E_{ab}(x) = \frac{(x-a)(x-b)}{a-b}$ if $x \in [a,b]$, and otherwise $E_{ab}(x) = 0$.

In Section 4.1, we show that any earthquake curve corresponding to a locally finite earthquake measure σ supported on lamination $\mathcal{L}$ safisfies the above ordinary differential equation for any point x contained in the boundary of a stratum. Notice that the union of the boundary points of all strata does not necessarily cover the whole circle, and the earthquake map E_t, determined by $(t\sigma, \mathcal{L})$, is not necessarily extendable continuously. In Section 4.2 we show that if $||\sigma||_{Th} < \infty$, then the ordinary differential equation extends to any point on the boundary circle and the earthquake curve is the unique solution of the equation.

Recently Dragomir Šarić [11] has shown that if σ is Thurston bounded, then for each x on the boundary circle, the earthquake curve $h_t(x)$ extends to a complex analytic function of t in a disk $D(t)$ centered at t and whose radius depends on t and $||\sigma||_{Th}$. One can use this result to give an alternative proof that the earthquake curve satisfies the ordinary differential equation in the quasisymmetric case. However our result is more general and does not depend on Šarić's result.

4.1. **Differentiation of earthquake curves.** Let σ be a locally finite earthquake measure supported on a lamination $\mathcal{L}$, and l_0 be a line in $\mathcal{L}$. Let $E_t = E_{t\sigma}$ be the earthquake map determined by $(t\sigma, \mathcal{L})$ with the normalization of fixing the line l_0 for $t \geq 0$. Let S denote the union of the boundaries of all strata. Then E_t extends to a continuous injective map $h_t = h_{t\sigma}$ from S into the circle. In this section we show that for each $x \in S$, $h_t(x)$ is differentiable in t and it satisfies the ordinary differential equation (25). We first show $h_t(x)$ is differentiable at $t = 0$.

Lemma 14. *For each $x \in S$,*

$$\lim_{t\to 0^+} \frac{h_t(x) - x}{t} = \int\int E_{ab}(x)d\sigma(a,b),$$

where $E_{ab}(x) = \frac{(x-a)(x-b)}{a-b}$ if $x \in [a,b]$, and otherwise $E_{ab}(x) = 0$.

Proof. Let T denote the stratum whose boundary on the circle contains the point x, l the geodesic boundary line of T which is the closest one to l_0 in the hyperbolic metric, and r the geodesic arc perpendicular to both l_0 and l. (In the case that l_0 and l_1 share one endpoint, we may choose any geodesic arc which is transversal to both of them.) Denote by $\mathcal{L}'$ the set of all leaves of $\mathcal{L}$ in the strip bounded by l_0

and l which are parallel to l_0 and l. Let σ' denote the restriction of σ to $\mathcal{L}.'$ and $E'_t = E_{t\sigma'}$ be the earthquake map determined by $(\mathcal{L}', t\sigma')$ normalized to fix the line l_0. Then E'_t extends to a homeomorphism h'_t on the circle $\mathbb{S}^1$. Clearly $h_t(x) = h'_t(x)$ for any $t \geq 0$. Denote by $V(x) = \int\int E_{ab}(x) d\sigma(a,b)$, then

$$V(x) = \int\int E_{ab}(x) d\sigma'(a,b).$$

Let $P = \{z_i\}_{i=0}^n$ be a partition of the tranversal arc r, and d the maximum of the hyperbolic lengths of the small arcs in the partition. Assume that z_0 belongs to l_0 and z_n belongs to l. Denote by T_i the stratum that contains z_i, $0 \leq i \leq n$. Let r_i denote the arc $(z_{i-1}, z_i]$ on r, and l_i a line in $\mathcal{L}'$ intersecting r_i, $1 \leq i \leq n$. Denote in order by T_i the strata containing the endpoints of r_i's, $0 \leq i \leq n$. Let $\sigma'_1 = \sigma'([z_0, z_1])$, and $\sigma'_i = \sigma'(r_i), 2 \leq i \leq n$. And let $\mathcal{L}_n$ denote the collection of $l_i, 1 \leq i \leq n$, ν_n the measure supported on $\mathcal{L}_n$ with the weight σ'_i on l_i, $1 \leq i \leq n$. Let $F_t^n = F_1^t \circ F_2^t \circ \cdots \circ F_n^t$ denote the curve of the finite earthquake maps which fix the line l_0, have lamination $\mathcal{L}_n$ and measure $t\nu_n$, where $t \geq 0$. Let $E_i^t = E'_t|_{T_i}$. Then

$$h_t(x) = h'_t(x) = E'_t(x) = E_1^t \circ E_2^t \circ \cdots \circ E_n^t(x).$$

Clearly each F_i^t is a hyperbolic Möbius transformation with axis l_i and translation length σ'_i. Each E_i^t is also a hyperbolic Möbius transformation with axis intersecting the arc $[z_{i-1}, z_i]$, and by Proposition 1 its translation length differs from σ'_i by an amount $O(\sigma'_i l(r_i))$, where $l(r_i))$ denotes the hyperbolic length of the arc r_i.

Sublemma:

$$|h'_t(x) - F_t^n(x)| = O(td(n)\sigma'(r)e^{t\sigma'(r)}),$$

where $d(n)$ denote the maximum of the hyperbolic lengths of r_i's.

Let us first use this sublemma to prove the differentiability of $h_t(x)$ at $t = 0$.

Define

$$V_n(x) = \int\int E_{ab}(x) d\nu_n(a,b).$$

One can easily show that $V_n(x)$ approaches $V(x)$ as n approaches infinity.

Now we write

$$\frac{h_t(x) - x}{t} - V(x) = \frac{h'_t(x) - x}{t} - V(x)$$

$$= \frac{E'_t(x) - F_t^n(x)}{t} + [\frac{F_t^n(x) - x}{t} - V_n(x)] + [V_n(x) - V(x)].$$

Let ϵ denote an arbitrarily small positive. Because of the sublemma, for every $\epsilon > 0$, if n is sufficiently large, $|\frac{E'_t(x) - F_t^n(x)}{t}| < \frac{\epsilon}{3}$. and $|V_n(x) - V(x)| < \frac{\epsilon}{3}$. For a fixed large value of n, if t is small enough, then $|\frac{F_t^n(x) - x}{t} - V_n(x)| < \frac{\epsilon}{3}$. Therefore there exists $\delta > 0$, such that for any $0 \leq t < \delta$,

$$|\frac{h_t(x) - x}{t} - V(x)| < \epsilon.$$

This means that $\frac{d}{dt} h_t(x)|_{t=0} = V(x)$.

Now we prove the sublemma by a method similar to the one used to prove Proposition 4. Let $x_i^t = E_{n-i+1}^t \circ E_{n-i+2}^t \circ \cdots \circ E_n^t(x)$ and $y_i^t = F_{n-i+1}^t \circ F_{n-i+2}^t \circ \cdots \circ F_n^t(x)$, $1 \leq i \leq n$. By Corollary 5, $x_1^t = E_n^t(x)$ differs from $y_1^t = F_n^t(x)$ by at most $O(l(r_n) t\sigma'(r_n)) = O(d(n) t\sigma'_n)$. By Corollary 5 again,

$$x_2^t = E_{n-1}^t(x_1^t) = F_{n-1}^t(x_1^t) + O(d(n) t\sigma'_{n-1}).$$

By the mean value theorem,

$$|F^t_{n-1}(x^t_1) - F^t_{n-1}(y^t_1)| \leq e^{t{\sigma'}_{n-1}}|x^t_1 - y^t_1)|,$$

and therefore

$$x^t_2 = y^t_2 + O(d(n)t\sigma'_{n-1}) + O(d(n)e^{t{\sigma'}_{n-1}}t\sigma'_n).$$

Inductively using Corollary 5 and the mean value theorem, we have

$$x^t_n = y^t_n + O(d(n)t\sigma'_1) + O(d(n)e^{t\sigma'_1}t\sigma'_2) + O(d(n)e^{t\sigma'_1+t\sigma'_2}t\sigma'_3) + \cdots$$
$$+O(d(n)e^{t\sigma'_1+t\sigma'_2+\cdots+t\sigma'_{n-1}}t\sigma'_n) = x^t_n + O(d(n)e^{t\sigma'(r)}t\sigma'(r)),$$

which proves the sublemma. □

Suppose that t and s are positive. Let h_t denote the curve of earthquake maps that fix l_0, have source lamination $\mathcal{L}$ and measure $t\sigma$. Let $\mathcal{L}_t$ denote the target lamination of h_t and $h^*_t(\sigma)$ be the pushforward of the measure $(\sigma, \mathcal{L})$ under h_t, whose support is $\mathcal{L}_t$. Let $\tilde{h}_s$ be the curve of earthquake maps that fix the line l_0 and have lamination $\mathcal{L}_t$ and measure $sh^*_t(\sigma)$, $s \geq 0$.

Lemma 15. *For each $x \in S$.*

$$h_{t+s}(x) = \tilde{h}_s(h_t(x)).$$

Proof. We use the notations in the proof of the previous lemma. Let $\tilde{E}'_s$, $s \geq 0$, be the earthquake map determined by $(E'_t(\mathcal{L}'), s(h'_t)^*(\sigma'))$ normalized to fix the line $l_0 = h'_t(l_0)$, and $\tilde{h}'_s$ be the extension of $\tilde{E}'_s$ to the boundary circle. We only need to show

$$h'_{t+s}(x) = \tilde{h}'_s(h'_t(x)).$$

Let h^n_t denote the extension of the finite earthquake map F^n_t to the boundary circle. The method used to show the sublemma in the proof of Lemma 14 implies that h^n_t converges to h'_t on the boundary circle. For each finite earthquake curve F^n_t, $F^n_{t+s} \circ (F^n_t)^{-1}$ is also a left earthquake with the source lamination $F^n_t(\mathcal{L}_n)$ and measure $s(h^n_t)^*(\nu_n)$. Let $\tilde{E}^n_s$ denote the finite earthquake curve determined by $(F^n_t(\mathcal{L}_n), s(h^n_t)^*(\nu_n))$ normalized to fix l_0, and $\tilde{h}^n_s$ the extension of $\tilde{E}^n_s$ to $\mathbb{S}^1$. Then for each $y \in \mathbb{S}^1$,

$$h^n_{t+s}(y) = \tilde{h}^n_s(h^n_t(y)).$$

In particular,

$$h^n_{t+s}(x) = \tilde{h}^n_s(h^n_t(x)).$$

As n approaches infinity, $h^n_{t+s}(x)$ approaches to $h'_{t+s}(x)$. In fact, for any $t \geq 0$, h^n_t converges to h'_t unformly on $\mathbb{S}^1$. It remains to show that the right-hand side of the above equation converges to $\tilde{h}'_s(h'_t(x))$. We first show that $\tilde{h}^n_s$ converges to $\tilde{h}'_s$ in C^0 topology. Since h^n_t converges to h'_t uniformly on $\mathbb{S}^1$, the images of the leaves in $\mathcal{L}_n$ under F^n_t are uniformly close to their images under E'_t. Now we write the map $\tilde{h}'_s$ as a long composition of the comparison maps with respect to the finitely many lines in the lamination $E'_t(\mathcal{L}_n)$. Because we require $\tilde{E}'_s$ to fix the line l_0, the comparison maps are hyperbolic Möbius transformations. And of course the finite earthquake map $\tilde{E}^n_s$ is also a long composition of hyperbolic Möbius transformations. Consider respectively the Möbius transformations in these two long compositions. They have nearby axes and nearly equal translation lengths. Again using the method to show the sublemma in the proof of the previous lemma, we conclude that $\max_{y\in\mathbb{S}^1} |\tilde{h}^n_s(y) - \tilde{h}'_s(y)|$ converges to zero as n approaches infinity.

Now we write

$$\tilde{h}_s^n(h_t^n(x)) - \tilde{h'}_s(h'_t(x))$$
$$= [\tilde{h}_s^n(h_t^n(x)) - \tilde{h'}_s(h_t^n(x))] + [\tilde{h'}_s(h_t^n(x)) - \tilde{h'}_s(h'_t(x))],$$

and deduce the convergence of $\tilde{h}_s^n(h_t^n(x))$ to $\tilde{h'}_s(h'_t(x))$. □

Theorem 10. *For each $x \in S$, $h_t(x)$ is differentiable for any $t \geq 0$, and*

$$\frac{d}{dt}h_t(x) = V_t(h_t(x)),$$

where $V_t(y) = \int\int E_{h_t(a)h_t(b)}(y)d\sigma(a,b)$.

Proof. By definition,

$$\frac{d}{dt}h_t(x) = \lim_{s\to 0}\frac{h_{t+s}(x) - h_t(x)}{s}.$$

When $s > 0$, by Lemma 15,

$$\frac{h_{t+s}(x) - h_t(x)}{s} = \frac{\tilde{h}_s(h_t(x)) - h_t(x)}{s} = \frac{\tilde{h}_s(y) - y}{s},$$

where $y = h_t(x)$. By Lemma 14,

$$\lim_{s\to 0^+}\frac{\tilde{h}_s(y) - y}{s} = \tilde{V}(y),$$

where $\tilde{V}(y) = \int\int E_{ab}(y)dh_t^*\sigma(a,b)$. And clearly

$$\tilde{V}(y) = \int\int E_{ab}(y)dh_t^*\sigma(a,b) = \int\int E_{h_t(a)h_t(b)}(y)d\sigma(a,b) = V_t(y).$$

Therefore

$$\lim_{s\to 0^+}\frac{h_{t+s}(x) - h_t(x)}{s} = V_t(h_t(x)).$$

Now we consider the limit $\lim_{s\to 0^+}\frac{h_{t-s}(x)-h_t(x)}{-s}$. Following arguments similar to the arguments in Lemma 15, one can show that $h_{t-s}(x) = \bar{h}_s(h_t(x))$, where $\bar{h}_s$, $s \geq 0$, is the right earthquake map determined by $(h_t(\mathcal{L}), sh_t^*(\sigma))$ normalized to fix l_0. Just as in Lemma 14, one can show that

$$\lim_{s\to 0^+}\frac{\bar{h}_s(y) - y}{s} = \int\int E_{ba}(y)dh_t^*\sigma(b,a) = -\int\int E_{ab}(y)dh_t^*\sigma(a,b) = -V_t(y).$$

Therefore

$$\lim_{s\to 0^+}\frac{h_{t-s}(x) - h_t(x)}{-s} = V_t(h_t(x)).$$

This completes the proof. □

4.2. **Uniqueness of solution.** If the measure σ has bounded Thurston norm then each earthquake map $E_{t\sigma}$, $t \geq 0$, extends to a homeomorphism on the boundary circle (see Proposition 2 in Section 2.1). In this section, we will first show that for a Thurston bounded measure σ, $h_t(x)$ is differentiable on $t \geq 0$ for any point x on the circle. We will also prove that under that condition, the normalized earthquake curve is the unique solution to the ordinary differential equation.

Given a lamination $\mathcal{L}$, a point x on the boundary circle is called a *accumulation point* with respect to $\mathcal{L}$ if there exists an infinite sequence $\{l_n\}_{n=0}^{\infty}$ of distinct, nonintersecting lines in $\mathcal{L}$ such that both endpoints of l_n converge to x in the Euclidean metric. Let S^a denote the set of all accumulation points with respect to $\mathcal{L}$. Let $h_t(x)$ be the curve of earthquake maps determined by $(\mathcal{L}, t\sigma)$ normalized to fix l_0 in $\mathcal{L}$. In order to conclude that the normalized curve $h_t(x)$ is differentiable on $t \geq 0$ for any x on the circle, by Theorem 10 we only need to prove that $h_t(x)$ is differentiable on t for any any accumulation point x of $\mathcal{L}$.

Lemma 16. *For any $x \in S^c$,*

$$\lim_{t\to 0^+} \frac{h_t(x) - x}{t} = \int\int E_{ab}(x) d\sigma(a, b),$$

where $E_{ab}(x) = \frac{(x-a)(x-b)}{a-b}$ if $x \in [a, b]$, and otherwise $E_{ab}(x) = 0$.

Proof. Let $l_0^\perp$ denote the geodesic that passes through x and is perpendicular to l_0. Conjugating by a Möbius transformation, we may assume that $l_0^\perp$ is the imaginary axis of the upper half plane, x is at the origin, and l_0 is the geodesic connecting -1 to 1. Let $\mathcal{L}'$ denote the collection of those lines in $\mathcal{L}$ that connect a point in $[-1, 0)$ to a point in $(0, 1]$, and σ' the restriction of σ to $\mathcal{L}'$. Let $a_n = -e^{-n}$ and $b_n = e^{-n}$, where $n \geq 0$. Let l_n be a line in $\mathcal{L}'$ which connects a point a'_n in $[a_{n-1}, a_n)$ to a point b'_n in $(b_n, b_{n-1}]$. If there is no such a line, then we add such a line with zero weight to the lamination $\mathcal{L}'$. Clearly for each $n \in \mathbb{N}$ the hyperbolic distance between l_{n-1} and l_n is less than or equal to 2, and $|a'_n - b'_n| \leq |a_{n-1} - b_{n-1}| = 2e^{-(n-1)}$.

For each $n \in \mathbb{N}$, let $(\mathcal{L}^{(n)}, \sigma^{(n)})$ denote the restriction of $(\mathcal{L}', \sigma')$ on $[a_0, a'_n) \times (b'_n, b_0]$, and $E_t^{(n)}$ be the curve of the earthquake maps determined by $(\mathcal{L}^{(n)}, t\sigma^{(n)})$ with the normalization of fixing l_0.

Sublemma 1: *If t is small enough such that $2t||\sigma||_{Th} < 1$, then*

$$\max_{y\in\mathbb{S}^1} |E_t^{(n+1)}(y) - E_t^{(n)}(y)| = O(t||\sigma||e^{-n(1-2t||\sigma||)}).$$

Proof. We first notice that the measure $\sigma'([a'_{n-1}, a'_n) \times (b'_{n-1}, b'_n]) = \sigma([a'_{n-1}, a'_n) \times (b'_{n-1}, b'_n])$ is bounded above by $2||\sigma||$ for each $n \in \mathbb{N}$. Clearly

$$0 \leq (E_t^{(n)})^{-1} \circ E_t^{(n+1)}(y) - a'_n \leq e^{2t||\sigma||}(y - a'_n)$$

for each $y \in [a'_n, b'_n]$, and therefore for each $y \in \mathbb{S}^1$,

$$|(E_t^{(n)})^{-1} \circ E_t^{(n+1)}(y) - y| \leq (e^{2t||\sigma||} - 1)|b'_n - a'_n|.$$

And then

$$\max_{y\in\mathbb{S}^1} |E_t^{(n+1)}(y) - E_t^{(n)}(y)| \leq (e^{2t||\sigma||})^n (e^{2t||\sigma||} - 1) 2e^{-(n-1)}.$$

Hence

$$\max_{y\in\mathbb{S}^1} |E_t^{(n+1)}(y) - E_t^{(n)}(y)| = O(t||\sigma||e^{-n(1-2t||\sigma||)}).$$

□

From this sublemma we see that if $t < \frac{1}{2||\sigma||_{Th}}$, then $E_t^{(n)}$ converges uniformly to E'_t on the circle. In the remaining part of the proof, we assume $x = 0$. Then

$$E_t(x) - E_t^{(n)}(x) = E'_t(x) - E_t^{(n)}(x) = O(t||\sigma||e^{-n(1-2t||\sigma||)}).$$

If $V_n(x) = \int\int E_{ab}(x) d\sigma^{(n)}(a, b)$, and $V(x) = \int\int E_{ab}(x) d\sigma(a, b)$ then clearly

$$V(x) = \int\int E_{ab}(x) d\sigma'(a, b).$$

Sublemma 2: $\lim_{n\to\infty} V_n(x) = V(x)$.

Proof. Since

$$V(x) - V_n(x) = \int\int_{[a'_n,0)\times(0,b'_n]} E_{ab}(x)d\sigma'(a,b)$$

$$= O(\sum_{k=n}^{\infty} |a'_k - b'_k| ||\sigma'||) = O(e^{-(n-1)}||\sigma||),$$

$V_n(x)$ converges to $V(x)$ as n goes to infinity. □

We now write

$$\frac{h_t(x) - x}{t} - V(x) = \frac{E'_t(x) - E^{(n)}_t(x)}{t} + \left[\frac{E^{(n)}_t(x) - x}{t} - V_n(x)\right] + [V_n(x) - V(x)].$$

Given $\epsilon > 0$, because of the above sublemmas, there exists a big n, such that $|\frac{E'_t(x)-E^{(n)}_t(x)}{t}| < \frac{\epsilon}{3}$ and $|V_n(x) - V(x)| < \frac{\epsilon}{3}$. Since x is a boundary point with respect to the lamination $\mathcal{L}^{(n)}$, by Theorem 10, for large enough n, there exists $\delta > 0$ such that for any $0 \leq t < \delta$,

$$\left|\frac{E^{(n)}_t(x) - x}{t} - V_n(x)\right| < \frac{\epsilon}{3}.$$

Therefore

$$\lim_{t\to 0^+} \frac{h_t(x) - x}{t} = V(x).$$

□

We need the following lemma, which is analogous to Lemma 15.

Lemma 17. *For each $x \in S^c$ and $t, s \geq 0$,*

$$h_{t+s}(x) = \tilde{h}_s(h_t(x)).$$

Proof. We follow the notations of lemma 16. It suffices to show that $h'_{t+s}(x) = \tilde{h'}_s(h'_t(x))$. Let $h^{(n)}_t$ denote the extension of $E^{(n)}_t$ to the boundary circle, and $\tilde{E}^n_s$ be the curve of earthquake maps determined by $(E^{(n)}_t(\mathcal{L}^{(n)}), s(h^{(n)}_t)^*(\sigma^{(n)}))$ with the normalization of fixing l_0 and $\tilde{h}^{(n)}_s$ the extension of $\tilde{E}^n_s$ to the boundary circle. Since x is a boundary point of some stratum of $\mathcal{L}^{(n)}$, by Lemma 15,

$$h^{(n)}_{t+s}(x) = \tilde{h}^{(n)}_s(h^{(n)}_t(x)).$$

We need to do pass to the limit in this equation. By the Proposition 2 in Section 2.1, $h^{(n)}_t(x)$ converges to $h'_t(x)$ as n goes to infinity. By the same argument used in sublemma 1 in the proof of the Lemma 16, if $s \geq 0$ is small enough, then $\tilde{h}^{(n)}_s$ converges to $\tilde{h'}_s$ uniformly on the boundary circle, where the size of s depends on the Thurston norm of $(h'_t)^*(\sigma')$, and hence depends on t and σ. Now we rewrite

$$h^{(n)}_{t+s}(x) = [\tilde{h}^{(n)}_s(h^{(n)}_t(x)) - \tilde{h'}_s(h^{(n)}_t(x))] + \tilde{h'}_s(h^{(n)}_t(x)).$$

Passing the limit, we finish the proof. □

Theorem 10 and Lemmas 16 and 17 imply the following theorem.

Theorem 11. *Let $(\sigma, \mathcal{L})$ be a Thurston bounded measure and $h_t(x)$, $t \geq 0$, be the curve of the earthquake maps determined by $(t\sigma, \mathcal{L})$ normalized to fix l_0 in $\mathcal{L}$. Then*

$$\frac{d}{dt}h_t(x) = V_t(h_t(x)),$$

where $V_t(y) = \int\int E_{h_t(a)h_t(b)}(y)d\sigma(a,b)$.

In the remainder of this section, we show the uniqueness of the solution to the ordinary differential equation.

Theorem 12. *Let $||\sigma||_{Th} < \infty$. Any normalized solution to the ordinary differential equation (25) must coincide with $h_{t\sigma}$.*

Proof. Note that the vector field $V_{t\sigma}$ is a Zygmund bounded function and thus $V_{t\sigma}(x)$ satisfies a $|\epsilon \log \epsilon|$-modulus of continuity that is uniform for $0 \leq t \leq t_0$.

If we put $\omega(s) = s\log(1/s)$ for $0 < s < 1/2$, ω is called the modulus of continuity. Since $\int_0^{1/2} \frac{ds}{\omega(s)} = \infty$, ω satisfies the W. Osgood criterion for the uniqueness of solution to an ordinary differential equation. The proof of the criterion is simple. Let $x(t) = h_{t\sigma}(x)$ be the solution the solution to (25) and suppose $y(t)$ is another solution with $y(0) = x(0)$. We have

$$\dot{x} = W(t,x) \text{ and } \dot{y} = W(t,y),$$

where

$$W(t,z) = V_t(z),$$

Thus the difference $z(t) = y(t) - x(t)$ satisfies

$$\dot{z} = W(t,y) - W(t,x),$$

and

$$|\dot{z}| \leq C\omega(z),$$

where $\omega(z)$ is the modulus of continuity. We may assume there is a value t_0 where $z_0 = z(t_0) = 0$ and another value $t_1 > t_0$ where $z_1 = z(t_1) > 0$ and $z(t) > 0$ for $t_0 < t < t_1$. We have

$$\int_{z_0}^{z_1} \frac{dz}{\omega(z)} = \int_{t_0}^{t_1} \frac{\dot{z}(t)}{\omega(z(t))}dt,$$

and

$$\int_{t_0}^{t_1} \frac{\dot{z}(t)}{\omega(z(t))}dt \leq C\int_{t_0}^{t_1} dt = C(t_1 - t_0).$$

But this is a contradiction because $\int_{z_0}^{z_1} \frac{dz}{\omega(z)} = \infty$. □

5. **Smoothness Classes**

In this last part of the paper, we first investigate the regularities of the tangent vectors $V_t(x)$ to earthquake curves determined by measures σ which satisfy vanishing conditions near the boundary circle. Then we apply them to characterize different smooth classes of circle homeomorphisms. At the end, we show that the characterizations are also equivalent to the corresponding vanishing conditions on the initial tangent vectors $V_0(x)$ to the earthquake curves.

Let $\alpha \geq 0$, D be a disk in the unit disk of diameter 1 in the hyperbolic metric, and $\delta(D)$ denote the Euclidean distance from D to the unit circle. We say that σ is *vanishing of order* α if

$$mass_\sigma(D) \leq C_1\delta(D)^\alpha, \tag{26}$$

for some constant $C_1 > 0$, where $mass_\sigma(D)$ is the total σ-mass of the lines of the lamination for σ that meet D.

Four distinct points a, b, c and d on the unit circle forms a quadruple. We assume that a, b, c and d are labelled in counterclockwise order. Define the scale s of the quadruple $\{a, b, c, d\}$ to be

$$s(\{a, b, c, d\}) = \min\{|a-b|, |b-c|, |c-d|, |d-a|\}. \tag{27}$$

We say that a homeomorphism h of the unit circle is *smooth of order* α if there exists a constant $C_2 > 0$, such that

$$\left|\log\left|\frac{(h(b)-h(a))(h(d)-h(c))}{(h(b)-h(c))(h(d)-h(a))}\right|\right| \leq C_2 s^\alpha, \tag{28}$$

for any four distinct points a, b, c, d on the unit circle in counterclockwise order such that $cr(a, b, c, d) = \frac{(b-a)(d-c)}{(c-b)(d-a)} = 1$.

Remark. In fact, h is smooth of order 0 means that h is quasisymmetric; h is smooth of order α, $0 < \alpha < 1$, is equivalent to saying that h is $C^{1+\alpha}$ ([10]); and h is smooth of order 1 is equivalent to saying that h is $C^{1+Zygmund}$ [7]. Also see [7] and [5] for applications of these smooth conditions in the study of the dynamics of circle diffeomorphisms.

Theorem 13. *Let $0 \leq \alpha < 1$. Then a circle homeomorphism h is smooth of order α if and only if the measure σ determining h is vanishing of order α. Moreover, the constants C_1 and C_2 can be extimated in terms of each other.*

When $\alpha = 0$, the above theorem is the corollary of Theorems 8 and 9. When $0 < \alpha < 1$, the proof of "if" part is divided into sections 5.2, 5.3 and 5.4, and the proof for "only if" part is given in section 5.5.

Let $h_{t\sigma}, t \geq 0$, be a curve of earthquake maps determined by the measure $t\sigma$, and $V(x)$ denote the tangent vector of $h_{t\sigma}(x)$ at $t = 0$, i.e.,

$$V(x) = \left.\frac{d}{dt}\right|_{t=0} h_{t\sigma}(x) \tag{29}$$

$$= \int\int E_{h_t(a)h_t(b)}(x) d\sigma(a, b) + a \text{ quadratic polynomial},$$

where $E_{ab}(x) = \frac{(x-a)(x-b)}{a-b}$ if $x \in [a, b]$, and otherwise $E_{ab}(x) = 0$. Clearly, if σ is vanishing of order α then it has a finite Thurston norm. And then the norm of V is bounded in the sense of the norm defined in (21) in Section 3.2. We say that the vector V is *vanishing of order* α if there exists a constant $C_3 > 0$, such that

$$|V[Q]| \leq C_3 s^\alpha, \tag{30}$$

for any four distinct points a, b, c, d on the unit circle in counterclockwise order such that $cr(a, b, c, d) = \frac{(b-a)(d-c)}{(c-b)(d-a)} = 1$, where $V[Q]$ is defined in (22) in section 3.2.

Theorem 14. *Let $0 \leq \alpha < 1$. Then the initial tangent vector V given in above formula (30) is vanishing of order α if and only if the measure σ is vanishing of same order. Moreover, the constants C_1 and C_3 can be estimated in terms of each other.*

The proof of "if" part comes out the work of Sections 5.4, and the other half proof is given in Section 5.6.

We have the following corollary from Theorems 13 and 14.

Corollary 10. *Let $0 \leq \alpha < 1$. The following three conditions are equivalent.*

(1) *There exists a constant C_1 such that σ is vanishing of order α with constant C_1.*
(2) *There exists a constant C_2 such that h is smooth of order α with constant C_2.*
(3) *There exists a constant C_3 such that V is vanishing of order α with constant C_3.*

Moreover, the constants C_1, C_2 and C_3 can be estimated in terms of each other.

An open question is to study whether the three conditions in the above corollary are equivalent for $\alpha = 1$.

5.1. **Norms of tangent vectors.** For a point ζ in the unit circle, we let z be the stereographic projection of ζ to the real axis. Then ζ is related to z by the formula $\zeta = T_1(z,)$ where

$$\zeta = T_1(z) = i\frac{i-z}{i+z}.$$

A vector field V on the circle transforms by T to a vector field W_1 on the real axis by the formula

$$W_1(z) = V(T_1(z))(T_1'(z))^{-1} = V(T_1(z))\frac{2}{(i+z)^2}.$$

Similarly, if we use the stereographic projection from the south pole,

$$T_2(z) = -T_1(z),$$

we obtain

$$W_2(z) = V(T_2(z))(T_2'(z)^{-1}) = V(T_2(z))(-\frac{2}{(i+z)^2}).$$

If we let I be the interval from -2 to 2, then $T_1(I)$ and $T_2(I)$ form a covering of the circle by two coordinate patches. We introduce a Zygmund norm on V depending on these coordinate patches by defining

$$||V||_Z = \max\{||W_1||_1, ||W_2||_2\},$$

where

$$||W_j||_j = \sup\left|\frac{W_j(x-s) + W_j(x+s) - 2W_j(x)}{s}\right|,$$

where the supremum is taken over all x and s such that $x - s$ and $x + s$ belong to $[-2, 2]$. We leave it to the reader to show the following lemma.

Lemma 18. *Assume $V(z)\frac{\partial}{\partial z}$ is a continuous vector field on the unit circle normalized to vanish at three points. Then $||V||_Z$ is bounded if and only if*

$$\sup_{x,s}\left|\frac{V(e^{i(x-s)}) + V(e^{i(x+s)}) - 2V(e^{ix})}{s}\right|$$

is bounded. Moreover, there is a constant $C > 0$ such that for $0 < s < 1/2$,

$$\frac{1}{C}\left|\frac{V(e^{i(x-s)}) + V(e^{i(x+s)}) - 2V(e^{ix})}{s}\right|$$

$$\le \max_j \sup_{x,s} \left\{ \left| \frac{W_j(x-s) + W_j(x+s) - 2W_j(x)}{s} \right| \right\}$$
$$\le C \left| \frac{V(e^{i(x-s)}) + V(e^{i(x+s)}) - 2V(e^{ix})}{s} \right|.$$

5.2. **Scales of quadruples.** Let l_1 and l_2 be two nonintersecting geodesic lines. Then the strip S bounded by l_1 and l_2 in the unit disk is of the form $[a, b] \times [c, d]$, where a, d and b, c are the endpoints of l_1 and l_2, respectively. Assume that l_1 and l_2 share no endpoints, let β be the geodesic segment in S perpendicular to both l_1 and l_2. Recall that the scale $s(a, b, c, d)$ of a quadruple $\{a, b, c, c\}$ is defined in (27).

Lemma 19. *If the hyperbolic length of β is equal to 1, then there exists a constant C such that*
$$\frac{1}{C}\delta(\beta) \le s(a, b, c, d) \le C\delta(\beta),$$
where $\delta(\beta)$ is the Euclidean distance from β to the unit circle, and the constant C is universal, independent of α, l_1 or l_2.

Proof. Since the hyperbolic length of β is equal to 1, the cross ratio
$$\left| \frac{(a-b)(c-d)}{(d-a)(c-b)} \right|$$
is equal to
$$\frac{(e-1)^2}{4e} = C < 1.$$
Therefore $|a - b||c - d| = C|d - a||b - c|$. Let $s = s(a, b, c, d)$. By relabelling, if necessary, we may assume that $s = |a - b|$ or $s = |b - c|$. Suppose first that $s = |a - b|$. Then
$$C|d - a||b - c| = s|c - d| \le s(s + |d - a| + |b - c|) \le 3s \max\{|d - a|, |b - c|\}.$$
Therefore,
$$\min\{|d - a|, |b - c|\} \le \frac{3}{C}s.$$
Suppose now that $s = |b - c|$. Similar reasoning yields
$$\min\{|d - c|, |b - a|\} \le 3Cs.$$
Therefore, in all cases,
$$\min\{|d - b|, |a - c|\} \le \frac{6}{C}s.$$
Without loss of generality, we may assume $|a-c| \le |a-b|+|b-c| \le \frac{6}{C}s$. Therefore the Euclidean distance from the endpoint of β which lies on the line l_2 to the unit circle is less than $\frac{6}{C}s$. Therefore,
$$s(a, b, c, d) \ge \frac{C}{6}\delta(\beta).$$

Now we prove the second inequality. Since the hyperbolic length of β equals 1, the Euclidean distance from any point on β to the unit circle is at most $e\delta(\beta)$. By relabelling, we may assume $|a - b| \le |c - d|$. Let l_3 and l_4 be geodesics joining a to b, and c to d, respectively. Let d_0 be the Euclidean distance from 0 to l_3 and let D be a closed Euclidean disc with center at 0 and radius d_0. Since $|a - b| \le |c - d|$, D

intersects the line l_4. Therefore, the set $T = D \bigcup l_3 \bigcup l_4$ separates $l_1 \setminus T$ from $l_2 \setminus T$. Hence there is a point p in β such that $p \in D$. Therefore, a simple calculation yields

$$\delta(\beta) \geq \frac{1}{e}(1-|p|) \geq \frac{1}{e}\frac{1}{2\tan\frac{3\pi}{8}}|a-b| \geq \frac{1}{2e\tan\frac{3\pi}{8}}s(a,b,c,d).$$

Therefore Lemma 19 follows with the constant

$$C = \max\{2e\tan\frac{3\pi}{8}, \frac{24e}{(e-1)^2}\} \leq 23.$$

□

5.3. **Bounds on vanishing measures.**

Lemma 20. *Let σ be an earthquake measure vanishing of order $\alpha > 0$ with constant C_1, and assume the lamination $\mathcal{L}$ of σ contains the line l which joins -1 to 1. If l_1 and l_2 are any two lines in the lamination $\mathcal{L}$, then the total σ-measure of all lines in $\mathcal{L}$ that separate l_1 from l_2 is less than a finite constant C. The constant C depends only on α and C_1.*

Proof. Without loss of generality, we may assume $l_2 = l$, and l_1 connects a point in $[-1,1]$ to another point in $[-1,1]$. Let σ_0 be the pull back of σ by a stereographic projection T_1 or T_2 which moves the endpoints of l_1 to points x, y with $-1 \leq x < y = x+s \leq 1$. Let S_k be a set of all lines in $\mathcal{L}$ which have one endpoint in the interval $[x-(2^k-1)s, x-(2^{k-1}-1)s]$ and another endpoint in the interval $[x+s, 1]$. Observe that

$$\sigma_0([-1,x]\times[x+s,1]) = \sum_{i=1}^{n}\sigma(S_i),$$

where n is chosen so that

$$x-(2^n-1)s \leq -1 \leq x-(2^{n-1}-1)s.$$

Thus, by the smoothness property of σ and Lemma 14,

$$\sigma_0([-1,x]\times[x+s,1]) \leq \sum_{i=1}^{n}(Const)(2^{i-1}s)^{\alpha} \leq$$

$$Const.4^{\alpha}\sum_{i=1}^{n}(2^{\alpha})^{i-n} \leq Const.$$

□

5.4. **Smooth homeomorphisms from vanishing measures.** Suppose that σ is vanishing of order α. Let $V_t(x)$ denote the derivative of an earthquake curve $h_{t\sigma}$ on the variable t. Then

$$h_{(t+s)\sigma}\circ h_{t\sigma}^{-1}(x) = x + sV_t(x) + o(s),$$

where the earthquake measure σ_t of V_t is the pushforward of σ under $h_{t\sigma}$, which is supported on the lamination $h_{t\sigma}(\mathcal{L})$. By Lemma 20, there exists a constant $C > 0$ depending only on α and C_1 such that

$$\sigma_t(I\times J) \leq C, \tag{31}$$

for any two disjoint closed subarcs I and J of the unit circle.

We divide the proof of the smoothness of order α of h into several steps. In the different steps we will use the same letter C for a constant that varies from step to step. The important point is that C depends only on $||\sigma||_{Th}$ and on the constant C_1 in the statement of Theorem 13. We first show that the tangent vectors V_t are uniformly Lipschitz continuous.

Step I There exists a constant C such that

$$|V_t(x) - V_t(y)| \leq C|x-y|, \tag{32}$$

for all x and y on the unit circle, and all $0 \leq t \leq 1$.

Proof. Since Lipschitz continuity is a local condition, it is enough to show (32) for the two restrictions of V_t, one to the upper semi-circle and the other to the lower semi-circle. Therefore, if W_t is the pull-back of the tangent vector V_t to the real axis by one of the stereographic projections, T_1 or T_2, it is enough to show that W_t is uniformly Lipschitz continuous on the interval $[-1,1]$. Let σ_t^* be the measure for W_t and $\mathcal{L}_t^*$ its associated lamination in the upper half-plane. We have

$$W_t(x) - W_t(y) = \iint_{\mathcal{L}_t^*} (E_{ab}(x) - E_{ab}(y)) d\sigma_t^*(a,b) =$$

$$\iint_{\mathcal{L}_t^*} (E_{ab}(x) - E_{ab}(y)) d\sigma_1(a,b) + \iint_{\mathcal{L}_t^*} (E_{ab}(x) - E_{ab}(y)) d\sigma_2(a,b),$$

where σ_1 is the restriction of σ_t^* to the set of all lines in $\mathcal{L}_t^*$ which meet the interval between x and y, and $\sigma_2 = \sigma_t^* - \sigma_1$. Let $W_i(x) = \iint_{\mathcal{L}_t^*} E_{ab}(x) d\sigma_i(x)$. We may assume $-1 \leq x < y = x + s < 1$. The inequality (31) yields

$$\sigma_2([-1,x] \times [x+s,1]) \leq C.$$

On the other hand, if a is a point in $[-1,x]$ and b is a point in $[x+s,1]$, then

$$|E_{ab}(x) - E_{ab}(y)| = |\frac{s(a+b-2x) - s^2}{b-a}| \leq s.$$

Therefore,

$$|W_2(x) - W_2(y)| \leq s\sigma([-1,x] \times [x+s,1]) \leq Cs.$$

To show W_1 is Lipschitz continuous, observe that

$$W_1(x) - W_1(y) = \iint_{\mathcal{L}_t^*} (E_{ab}(x) - E_{ab}(y)) d\sigma_1(a,b) =$$

$$\iint_{\mathcal{L}_t^*} (E_{ab}(x) - E_{ab}(y)) d\sigma_3(a,b) + \iint_{\mathcal{L}_t^*} (E_{ab}(x) - E_{ab}(y)) d\sigma_4(a,b),$$

where σ_3 is the restriction of σ_1 to the set of all lines in $\mathcal{L}$ with endpoints a and b such that $a < x \leq b \leq y$, and $\sigma_4 = \sigma_1 - \sigma_3$. Therefore

$$W_3(x) - W_3(y) = \iint_{\mathcal{L}} (E_{ab}(x) - E_{ab}(y)) d\sigma_3(a,b) =$$

$$\iint_{\mathcal{L}_t^*} \frac{(x-a)(b-x)}{b-a} d\sigma_3(a,b) \leq s \iint_{\mathcal{L}_t^*} d\sigma_3(a,b) =$$

$$s \lim_{c \to 0+} \iint_{[-1,x-c] \times [x,y]} d\sigma_3(a,b) \leq Cs,$$

again by inequality (31).

Similar reasoning shows $|W_4(x) - W_4(y)| \leq C|x-y|$. Therefore $|W_t(x) - W_t(y)| \leq 3C|x-y|$. ∎

In the next step we show that the functions $h_t = h_{t\sigma}$ are uniformly bi-Lipschitz continuous for $0 \le t \le 1$.

Step II There exists a constant C such that

$$\frac{1}{C}|x-y| \le |h_t(x) - h_t(y)| \le C|x-y|, \tag{33}$$

for all x and y on the unit circle, and all $0 \le t \le 1$.

Proof. Let $F(t) = \log|h_t(x) - h_t(y)|$, where x and y are any two distinct points on the unit circle. Step I yields

$$|F'(t)| = |\frac{V_t(h_t(x)) - V_t(h_t(y))}{h_t(x) - h_t(y)}| \le C.$$

Therefore

$$C \ge |F(1) - F(0)| = |\log|\frac{h_t(x) - h_t(y)}{x-y}||,$$

and

$$e^{-C} \le |\frac{h_t(x) - h_t(y)}{x-y}| \le e^C.$$

□

Step III The measures σ_t are uniformly vanishing of order α.

Proof. By Step II, the mapping $h_{t\sigma}$ quasipreserves the scales of quadruples, which are by Lemma 19, equivalent to the Euclidean distance from the common perpendicular segment to the unit circle. By Theorem 9, $h_{t\sigma}$ quasipreserves the hyperbolic length of this segment. See [1] for another proof of this statement. Therefore, the measures σ_t are uniformly vanishing of the same order as $t\sigma$. □

Now we start to prove that h is vanishing of order α. Let a, b, c and d be any four distinct points on the unit circle such that $cr(a, b, c, d) = \frac{(d-c)(b-a)}{(b-c)(d-a)} = 1$. We want to show

$$\log|cr(h(a), h(b), h(c), h(d))| \le (Const)s(a, b, c, d)^\alpha, \tag{34}$$

If we let

$$G(t) = \log|cr(h_t(a), h_t(b), h_t(c), h_t(d))|,$$

then by the proof of Theorem 9,

$$G'(t) = |V_t[h_t(a), h_t(b), h_t(c), h_t(d)]|, \tag{35}$$

Here we are using the square bracket notation defined in equation (22). By Step II,

$$\frac{s(a, b, c, d)}{C} \le s(h_t(a), h_t(b), h_t(c), h_t(d)) \le s(a, b, c, d)\ C.$$

Furthermore, Theorem 9 yields

$$\frac{1}{C} \le |cr(h_t(a), h_t(b), h_t(c), h_t(d))| \le C. \tag{36}$$

Let $s = s(a, b, c, d)$. By the proof of Lemma 19, we may assume $|a-b|+|b-c| \le Cs$. By taking s to be sufficiently small, we may assume that the Euclidean distance from all three points a, b, c to the same one point 1 or -1 is at least $\frac{1}{2}$. We first show that there exists a (universal) constant C such that

$$|V_t[h_t(a), h_t(b), h_t(c), h_t(d)]| \leq Cs^{\alpha}. \tag{37}$$

We may assume that

$$\max\{|h_t(a) - 1|, |h_t(b) - 1|, |h_t(c) - 1|\} \geq \frac{1}{2}. \tag{38}$$

The pull-back W of V_t by the stereographic projection $T_3(z) = iT_1(z)$ vanishes at 0 and ∞. Furthermore, proving inequality (37) is equivalent to proving

$$|W[\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}]| \leq Cs^{\alpha}, \tag{39}$$

where $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ are images under the projection T_3 of $h_t(a), h_t(b), h_t(c), h_t(d)$, respectively. We prove this inequality in Step V. Inequality (38) implies

$$-3 \leq \tilde{a}, \tilde{b}, \tilde{c} \leq 3. \tag{40}$$

We first prove a preliminary Step.

Step IV There exists a constant C such that

$$\left|\frac{W(x+s) + W(x-s) - 2W(x)}{s}\right| \leq Cs^{\alpha}, \tag{41}$$

whenever $-3 \leq x - s < x < x + s \leq 3$. The constant C depends only on α and C_1.

Proof. Let σ_0 be the earthquake measure of $W = W_{\sigma_0}$ and let $\mathcal{L}_0$ be its support lamination. Let $\mathcal{L}_0 = \mathcal{L}_1 \bigcup \mathcal{L}_2$, where $\mathcal{L}_1$ is the set of all lines in $\mathcal{L}$ both of whose endpoints belong to the interval $[x-3s, x+3s]$. Let σ_1 be the restriction of σ_0 to $\mathcal{L}_1$, and let $\sigma_2 = \sigma_0 - \sigma_1$. By Step III, σ_0 is uniformly vanishing of order α. Therefore, the inequality (36) yields

$$\left|\frac{W_{\sigma_1}(x+s) + W_{\sigma_1}(x-s) - 2W_{\sigma_1}(x)}{s}\right| = |W_{\sigma_1}[x-s, x, x+s, \infty]|$$
$$\leq C||\sigma_1||_{Th} \leq Cs^{\alpha}.$$

Thus, it is enough to show

$$\left|\frac{W_{\sigma_2}(x+s) + W_{\sigma_2}(x-s) - 2W_{\sigma_2}(x)}{s}\right| \leq Cs^{\alpha}.$$

Let $\mathcal{L}_2 = \mathcal{L}_3 \bigcup \mathcal{L}_4$, where $\mathcal{L}_4$ is the set of all lines in $\mathcal{L}$ which meet the interval $[x-s, x+s]$. Let σ_3 be the restriction of σ_2 to $\mathcal{L}_3$, and let $\sigma_4 = \sigma_2 - \sigma_3$. If a geodesic line l_{ab} is in $\mathcal{L}_4$, then one of its endpoints a, b belongs to the interval $[x-s, x+s]$. Therefore $E_{ab}(y) = \frac{(y-a)(b-y)}{b-a} \leq 2s$ for every point y inside the interval $[x-s, x+s]$. Thus,

$$\left|\frac{W_{\sigma_4}(x+s) + W_{\sigma_4}(x-s) - 2W_{\sigma_4}(x)}{s}\right| \leq 8(\sigma_0([x-s, x+s] \times [x+3s, \infty])+$$
$$\sigma_0([-\infty, x-3s] \times [x-s, x+s]) \leq Cs^{\alpha},$$

by Lemma 19. Finally we show

$$|\frac{W_{\sigma_3}(x+s) + W_{\sigma_3}(x-s) - 2W_{\sigma_3}(x)}{s}| \leq Cs^{\alpha}.$$

Let g_k be the segment on the geodesic line joining x to ∞, such that the endpoints of g_k are at $x + 2^{k-1}si$ and $x + 2^k si$. Let $\sigma_3(g_k)$ be the total σ_3 measure of all lines in $\mathcal{L}_3$ which intersect g_k. By the vanishing property of σ_0 and Lemma 14, $\sigma(g_k) \leq (Const)(2^{k-1}s)^\alpha$. Furthermore if a line l_{ab} in $\mathcal{L}_3$ has endpoints at a and b, then we may assume $a \leq x - s < x + s \leq b$. Therefore

$$E_{ab}(x+s) + E_{ab}(x-s) - 2E_{ab}(x) = \frac{-2s^2}{b-a} \leq \frac{-2s^2}{2^k s}.$$

Thus,

$$\left| \frac{W_{\sigma_3}(x+s) + W_{\sigma_3}(x-s) - 2W_{\sigma_3}(x)}{s} \right| \leq \sum_{k=1}^{\infty} \frac{2s}{2^k s} \sigma(g_k)$$

$$\leq Cs^\alpha \sum_{k=0}^{\infty} (2^{\alpha-1})^k \leq Cs^\alpha.$$

□

Step V There exists a universal constant C such that

$$\left| W[\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}] \right| \leq Cs^\alpha. \tag{42}$$

Proof. Note first that

$$\left| W[\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}] \right| \leq$$

$$\left| \frac{W(\tilde{a}) - W(\tilde{b})}{\tilde{a} - \tilde{b}} - \frac{W(\tilde{c}) - W(\tilde{b})}{\tilde{c} - \tilde{b}} \right| + \left| \frac{W(\tilde{a}) - W(\tilde{d})}{\tilde{a} - \tilde{d}} - \frac{W(\tilde{c}) - W(\tilde{d})}{\tilde{c} - \tilde{d}} \right|.$$

By the proof of Step IV,

$$\left| \frac{W(\tilde{a}) - W(\tilde{b})}{\tilde{a} - \tilde{b}} - \frac{W(\tilde{c}) - W(\tilde{b})}{\tilde{c} - \tilde{b}} \right| \leq Cs^\alpha.$$

Therefore it is enough to show

$$A = \left| \frac{W(\tilde{a}) - W(\tilde{d})}{\tilde{a} - \tilde{d}} - \frac{W(\tilde{c}) - W(\tilde{d})}{\tilde{c} - \tilde{d}} \right| \leq Cs^\alpha.$$

If $\tilde{d} \geq 4$, then by Step I,

$$A \leq \frac{|W(\tilde{c}) - W(\tilde{a})|}{|\tilde{c} - \tilde{a}|} + |\tilde{c} - \tilde{a}| \frac{|W(\tilde{d}) - W(\tilde{a})|}{|\tilde{d} - \tilde{a}||\tilde{d} - \tilde{c}|} \leq C|c - a|.$$

Therefore we may assume

$$-3 \leq \tilde{a} < \tilde{b} < \tilde{c} < \tilde{d} \leq 4.$$

Now choose x and y so that $\tilde{a}$ is the midpoint of x and $\tilde{c}$ and $\tilde{c}$ is the midpoint of $\tilde{a}$ and y. Note that

$$A = \left| \iint \left(\frac{E_{pq}(\tilde{d}) - E_{pq}(\tilde{c})}{\tilde{d} - \tilde{c}} - \frac{E_{pq}(\tilde{d}) - E_{pq}(\tilde{a})}{\tilde{d} - \tilde{a}} \right) d\sigma(p,q) \right| \leq \sum_{i=1}^{7} A_i,$$

where

$$A_i = \left| \iint \left(\frac{E_{pq}(\tilde{d}) - E_{pq}(\tilde{c})}{\tilde{d} - \tilde{c}} - \frac{E_{pq}(\tilde{d}) - E_{pq}(\tilde{a})}{\tilde{d} - \tilde{a}} \right) d\sigma_i(p,q) \right|,$$

where σ_1 is the restriction of σ to the set of geodesic lines with endpoints p and q inside the interval $[x, y]$,
σ_2 is the restriction of $\sigma - \sigma_1$ to the set of geodesic lines having endpoints p and q with $p \leq \tilde{a}$ and $q \geq \tilde{d}$,
σ_3 is the restriction of $\sigma - \sigma_1 - \sigma_2$ to the set of geodesic lines having endpoints p and q with $p \leq \tilde{a}$ and $\tilde{c} \leq q \leq \tilde{d}$,
σ_4 is the restriction of $\sigma - \sigma_1 - \sigma_2 - \sigma_3$ to the set of geodesic lines having endpoints p and q with $\tilde{a} \leq p \leq \tilde{c}$ and $q \geq \tilde{d}$,
σ_5 is the restriction of $\sigma - \sigma_1 - \sigma_2 - \sigma_3 - \sigma_4$ to the set of geodesic lines having endpoints p and q with $\tilde{c} \leq p \leq \tilde{d}$ and $q \geq \tilde{d}$,
σ_6 is the restriction of $\sigma - \sigma_1 - \sigma_2 - \sigma_3 - \sigma_4 - \sigma_5$ to the set of geodesic lines having endpoints p and q with $\tilde{a} \leq p \leq \tilde{c}$ and $y \leq q \leq \tilde{d}$, and
σ_7 is the restriction of $\sigma - \sigma_1 - \sigma_2 - \sigma_3 - \sigma_4 - \sigma_5 - \sigma_6$ to the set of geodesic lines having endpoints p and q with $p \leq x$ and $\tilde{a} \leq q \leq \tilde{c}$.

Since σ is vanishing of order α, we have $||\sigma_1||_{Th} \leq Cs^{\alpha}$. Thus,

$$A_1 = |V_{\sigma_1}(\tilde{c}, \tilde{d}, \tilde{a}, \infty)| \leq C||\sigma_1||_{Th} \leq (Const)s^{\alpha}.$$

A simple calculation shows

$$A_2 = \iint \frac{\tilde{c} - \tilde{a}}{q - p} d\sigma_2(p, q) \leq \sum_k \iint_{2^k \leq \frac{q - \tilde{a}}{\tilde{c} - \tilde{a}} \leq 2^{k+1}} \frac{\tilde{c} - \tilde{a}}{q - p} d\sigma_2(p, q) \leq$$

$$\sum_k \iint_{[-\infty, \tilde{a}] \times [\tilde{a} + 2^k(\tilde{c} - \tilde{a}), \tilde{a} + 2^{k+1}(\tilde{c} - \tilde{a})]} \frac{\tilde{c} - \tilde{a}}{q - p} d\sigma_2(p, q)$$

$$\leq C \sum_k (2^k s)^{\alpha} \frac{s}{2^k s} \leq Cs^{\alpha}.$$

Similarly,

$$A_3 = \left| \iint \frac{(\tilde{c} - \tilde{a})[(\tilde{d} - \tilde{c})(\tilde{d} - \tilde{a}) + (q - \tilde{d})(\tilde{d} - p)]}{(q - p)(\tilde{d} - \tilde{c})(\tilde{d} - \tilde{a})} d\sigma_3(p, q) \right| \leq 3$$

$$\iint \frac{\tilde{c} - \tilde{a}}{q - \tilde{c}} d\sigma_3(p, q) \leq \sum_k \iint_{2^k \leq \frac{q - \tilde{a}}{\tilde{c} - \tilde{a}} \leq 2^{k+1}} \frac{\tilde{c} - \tilde{a}}{q - \tilde{c}} d\sigma_3(p, q) \leq$$

$$\sum_k \iint_{[-\infty, \tilde{a}] \times [\tilde{a} + 2^k(\tilde{c} - \tilde{a}), \tilde{a} + 2^{k+1}(\tilde{c} - \tilde{a})]} \frac{\tilde{c} - \tilde{a}}{q - \tilde{c}} d\sigma_3(p, q) \leq$$

$$C \sum_k (2^k s)^{\alpha} \frac{s}{2^k s} \leq Cs^{\alpha}.$$

Furthermore,

$$A_4 = \left| \iint \left[\frac{q + p - \tilde{d} - \tilde{c}}{q - p} - \frac{(q - \tilde{d})(\tilde{d} - p)}{(\tilde{d} - \tilde{a})(q - p)} \right] d\sigma_4(p, q) \right| \leq$$

$$2 \iint_{[\tilde{a}, \tilde{c}] \times [\tilde{d}, \infty]} d\sigma_4(p, q) \leq 2 \iint_{[\tilde{a}, \tilde{c}] \times [y, \infty]} d\sigma_4(p, q) \leq Cs^{\alpha}.$$

Similarly,

$$A_5 = \iint \frac{\tilde{c} - \tilde{a}}{(\tilde{d} - \tilde{c})(\tilde{d} - \tilde{a})} E_{pq}(\tilde{d}) d\sigma_5(p, q) =$$

$$\iint_{q-\tilde d\ge \tilde d-\tilde c} \frac{\tilde c-\tilde a}{(\tilde d-\tilde c)(\tilde d-\tilde a)} E_{pq}(\tilde d) d\sigma_5(p,q) +$$
$$\sum_k \iint_{2^{-k-1}\le \frac{q-\tilde d}{\tilde d-\tilde c}\le 2^{-k}} \frac{\tilde c-\tilde a}{(\tilde d-\tilde c)(\tilde d-\tilde a)} E_{pq}(\tilde d) d\sigma_5(p,q) \le$$
$$\frac{\tilde c-\tilde a}{\tilde d-\tilde c}\sigma_5([\tilde c,\tilde d]\times[\tilde d+\tilde d-\tilde c,\infty])+$$
$$\left|\sum_k \iint_{[-\infty,\tilde d]\times[\tilde d+(\tilde d-\tilde c)2^{-k-1},\tilde d+(\tilde d-\tilde c)2^{-k}]} \frac{\tilde c-\tilde a}{(\tilde d-\tilde c)} d\sigma_5(p,q)\right| \le$$
$$C\frac{\tilde c-\tilde a}{\tilde d-\tilde c}(\tilde d-\tilde c)^\alpha + C\frac{\tilde c-\tilde a}{\tilde d-\tilde c}\sum_k \left(\frac{\tilde d-\tilde c}{2^k}\right)^\alpha \le$$
$$C(\tilde c-\tilde a)^\alpha.$$

Furthermore,

$$A_6 = \iint \frac{(q-\tilde c)(\tilde c-p)}{(q-p)(\tilde d-\tilde c)} d\sigma_6(p,q) \le \sigma([\tilde a,\tilde c]\times[y,\infty]) \le Cs^\alpha.$$

Finally,

$$A_7 = \iint \frac{(q-\tilde a)(\tilde a-p)}{(\tilde c-\tilde a)(q-p)} d\sigma_7(p,q) \le \sigma([-\infty,x]\times[\tilde a,\tilde c]) \le Cs^\alpha.$$

□

Step VI Inequality (34) holds.

Proof. By Step V, $|G'(t)| \le Cs^\alpha$. Therefore,

$$|G(1)-G(0)| \le Cs^\alpha,$$

and this inequality yields inequality (34). Thus, h is vanishing of order α. □

5.5. **Vanishing measures from smooth homeomorphisms.** In this section we show that the measure σ is vanishing of order α if h is smooth of order α.

Let D be a disk of hyperbolic diameter 1 in the unit disk, l_1 and l_2 be two geodesic lines in the laminatin $\mathcal{L}$ which bound all lines of $\mathcal{L}$ intersecting D. Suppose that a, d and b, c are the endpoints of l_2 and l_1 respectively, and a, b, c, d are arranged on the unit circle $\mathbb{S}^1$ in counter clockwise direction. Notice that the hyperbolic distance between l_1 and l_2 is between 0 and 1 and the scale of the quadruple $\{a, b, c, d\}$ is not necessarily commeasurable to the Euclidean distance of the disk D to the unit circle. Therefore we will first construct a special quadruple Q such that $cr(Q) = 1$, $cr(h(Q)) \ge 1$ and the scale of Q is commeasurable to $\delta(D)$, and then we apply a similar idea of proving (13) to compare $\log cr(h(Q))$ with $mass_\sigma(D)$ and show that $mass_\sigma(D) = O((\log cr(h(Q)))^\alpha)$.

Lemma 21. *Let $0 \le s \le s_0 < 1$ and $t \ge t_0 > 1$, and $A_{(s,t)}$ be the hyperbolic Möbius transformation with the repelling fixed point at s and the attracting fixed point at t and its derivative at the repelling fixed point equal to $\lambda \ge 1$. Suppose that $f_{(s,t)} : \mathbb{R} \to \mathbb{R}$ is defined to be equal to $A_{(s,t)}$ on the interval $[s,t]$ and equal to the identity on the complement of $[s,t]$. Then the logarithm of the cross ratio distortion of $f_{(s,t)}$ on the quadruple $Q = \{\infty, -1, 0, 1\}$ is greater than or equal to $C \log \lambda$ for* *some constant $C > 0$ when $\log cr(f_{(s,t)}(Q)) < \delta$ for some constant $\delta > 0$, where both C and δ only depend on s_0 and t_0.*

Proof. Let Q denote the quadruple $\{\infty, -1, 0, 1\}$. By Corollary 1,

$$cr(f_{(s,t)}(Q)) \geq cr(f_{(s_0,t_0)}(Q)).$$

Clearly,

$$cr(f_{(s_0,t_0)}(Q)) = \frac{\lambda\frac{1-s_0}{t_0-1}t_0 + s_0}{\lambda\frac{1-s_0}{t_0-1} + 1}.$$

Consider $\log cr(f_{(s_0,t_0)}(Q))$ as a function of λ, applying the inverse function theorem and mean value theorem to it in a small neighborhood of $\lambda = 1$, one can complete the proof. □

Without loss of generality, we may assume $|b-a| = \min\{|b-a|, |d-c|\}$. Through post-composition by a Möbius transformation, we may also assume that the earthquake representation E of h is the identity map on l_2. Let β' be the geodesic which is perpendicular to both l_2 and l_1 if l_2 and l_1 don't share any endpoint, otherwise take β' to be a geodesic perpendicular to l_2 such that the hyperbolic length of the segment on β' between l_2 and l_1 is less than or equal to $\frac{1}{2}$. Let x' and y' denote the endpoints of β' such that d, x', a, y' are arranged in the counterclockwise direction on the unit circle. Given two points u and v on the unit circle, we use $\hat{uv}$ to denote the geodesic connecting u to v. Let $\hat{x'y}$ denote the geodesic passing through x' and perpendicular to the geodesic $\hat{ay'}$, and $\hat{xy}$ the geodesic passing through y and perpendicular to the geodesic $\hat{da}$. One can easily check that the points d, x', x, a, b, y, y', c and a are presented on the unit circle in the counterclockwise direction. Let $Q = \{d, x, a, y\}$. Since the geodesic $\hat{xy}$ is perpendicular to $\hat{da}$, $cr(Q) = 1$. Furthermore, one can show that $cr(h(Q)) \geq 1$ and the scale $s(Q)$ is commensurable to $\delta(D)$. Let A be the Möbius transformation which maps x' to infinity, y' to 0, a to -1 and d to 1. Denote $A(b)$ and $A(c)$ by $-s$ and t, respectively, with $0 < s \leq 1$ and $0 < t \leq 1$. When $\beta' = \hat{y'x'}$ is both perpendicular to l_2 and l_1, $\frac{1}{e} \leq s = t \leq 1$; when l_2 and l_1 share one endpoint, $s = 1$ and $\frac{1}{e} \leq t \leq 1$. As we constructed,

$$A(y) = -\frac{1}{2} \text{ and } A(x) = -2.$$

Denote by $s_0 = cr(\{d, x, a, b\})$ and $t_0 = cr(\{d, x, a, c\})$. Then

$$s_0 = cr(\{A(d), A(x), A(a), A(b)\}) = cr(\{1, -2, -1, -s\}) = 3\frac{1-s}{1+s},$$

and

$$t_0 = cr(\{A(d), A(x), A(a), A(c)\}) = cr(\{1, -2, -1, t\}) = 3\frac{1+t}{t-1}.$$

Clearly,

$$0 \leq s_0 \leq 3\frac{e-1}{e+1} < 1 \text{ and } t_0 \geq 3\frac{e+1}{e-1} > 1.$$

Now let B denote the Möbius transformation which maps the hyperbolic plane to the upper half plane, and d to ∞, a to 0 and x to -1. Then A maps y to 1, b to s_0 and c to t_0. Denote by $\tilde{h} = B \circ h \circ B^{-1}$. Suppose that $\tilde{A}$ is the hyperbolic Möbius transformation with the repelling fixed point at s_0 and attracting fixed point at t_0, and translation length $\log \lambda$, and $\tilde{f}$ is equal to $\tilde{A}$ on the interval $[s_0, t_0]$ and the identity map on the complement of $[s_0, t_0]$ in the real line, where $\lambda = e^{mass_\sigma(D)}$. Using the same idea in the proof of (13), one can show that

$$cr(h(Q)) = cr(\tilde{h}(Q)) \geq cr(\tilde{f}(Q)).$$

Applying the previous lemma, if $\log cr(h(Q)) > 0$ is small enough, then

$$\log cr(h(Q)) \geq C \log \lambda = C \cdot mass_\sigma(D),$$

and hence

$$mass_\sigma(D) \leq \frac{1}{C} \log cr(h(Q)) = O(\delta(D)^\alpha)$$

when $\delta(D)$ is small enough.

From Theorem 8, one can easily see that $mass_\sigma(D) = O(\delta(D)^\alpha)$ when $\delta(D)$ is bigger than a positive constant.

Remark. In fact, we prove the same result of this section for any $\alpha > 0$.

5.6. **Vanishing measures from vanishing initial vectors.** The conclusion of Step V in Section 5.4 includes the "if" part of Theorem 14. To complete the proof of Theorem 14 we must show that if σ is an earthquake measure not vanishing of order α, then there is a sequence of points $x_j - s_j, x_j, x_j + s_j$ such that the corresponding normalized vector V given by formula (29) satisfies

$$\left| \frac{V(x_j + s_j) + V(x_j - s_j) - 2V(x_j)}{s_j^{1+\alpha}} \right| \to \infty.$$

The proof is similar to the work in the above Section 5.5. But we prove it by making a contradiction. Suppose that σ is not vanishing of order α. Then there is a sequence of discs D_j of hyperbolic diameter equal to one such that

$$\frac{mass_\sigma(D_j)}{\delta(D_j)^\alpha} \to \infty.$$

After a change of coordinates by Möbius transformations we may assume the earthquake acts on the upper half-plane and the discs D_j approach zero. Follow the idea used to prove (13), we can see that there exists a sequence of symmetrically triples $x_j - s_j, x_j, x_j + s_j$ with x_j approaching 0 and with $1/Cs_j \leq \delta(D_j) \leq Cs_j$ such that the mapping $h_{t\sigma}$ distorts the quadruple $-\infty, x - s_j, x_j, x_j + s_j$ by an amount greater than or equal to $M_j t s_j^\alpha$, where M_j approaches ∞. Taking the derivative of this distortion with respect to t at $t = 0$ and using the definition of V in (29), we obtain a contradiction.

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