CUSPS IN COMPLEX BOUNDARIES OF ONE-DIMENSIONAL TEICHMÜLLER SPACE

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Abstract. This paper gives a proof of the conjectural phenomena on the complex boundary one-dimensional slices: Every rational boundary point is cusp shaped. This paper treats this problem for Bers slices, the Earle slices, and the Maskit slice. In proving this, we also obtain the following result: Every Teichmüller modular transformation acting on a Bers slice can be extended as a quasi-conformal mapping on its ambient space. We will observe some similarity phenomena on the boundary of Bers slices, and discuss on the dictionary between Kleinian groups and Rational maps concerning with these phenomena. We will also give a result related to the theory of L.Keen and C.Series of pleated varieties in quasifuchsian space of once punctured tori.

Contents

1. Introduction

This paper is concerned with the following problem for once punctured torus groups:

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Problem. *How is the shape of quasifuchsian space described as the subset of representation space?*

By virtue of Bers' simultaneous uniformization theorem, quasifuchsian space is a domain in representation space which is biholomorphically equivalent to the product of two Teichmüller spaces (of mutually orientation reversed surfaces). So, in our case, this space is a two-dimensional complex manifold. On the other hand, C.T.McMullen showed in [Mc98] that the closure of quasifuchsian space is NOT a topological manifold with boundary (see also Ito [Ito1]). Thus the boundary of this space has some complexities. So its shape should be investigated more. Our strategy to conquer the problem is to investigate the shape of the boundary of several slices (or cross sections) via holomorphic mappings from a Riemann surface which can not admit bounded holomorphic functions, for example $\mathbb{C}$ or $\mathbb{C}^* := \mathbb{C} - \{0\}$.

In this paper, we focus on three kinds of slices; *Bers slices*, the *Earle slice*, and *the Maskit slice*. A Bers slice and the Earle slice are considered as cross sections of quasifuchsian space via holomorphic mappings from one dimensional Banach space and $\mathbb{C}^*$ into representation space respectively. The Maskit slice is a cross section of the boundary of quasifuchsian space via a holomorphic immersion from a complex plane. Moreover, although redundant, these three slices each represent Teichmüller space and themselves are deeply related to the moduli of Kleinian groups. Therefore, it is important and seems natural to study the shape of the boundaries of these three slices. The boundaries of their images are said to be the *complex boundaries*. We adopt this terminology after Lipman Bers. He had used this to indicate the boundary of Bers slices in [B81a].

Recently, thanks to beautiful work of Y.N.Minsky on the ending lamination conjecture for once punctured torus groups, each of three embeddings can be extended homeomorphically to Thurston's compactification of the Teichmüller space of once punctured tori with the property that, except at most two laminations, every rational lamination in the Thurston boundary corresponds to a geometrically finite group which contains an accidental parabolic associated with its lamination (cf. [Min] and [Ko]). In this paper, such boundary points of slices are called the *rational boundary points*.

The central subject of the paper is to study the shape of the complex boundaries of above three slices near rational boundary points. We will obtain a concrete proof of the folklore conjecture:

Theorem. *Each rational boundary point is cusp shaped.*

D. Mumford and D. Wright draw the computer graphical picture of the Maskit slice, a parameter space of Kleinian groups on $\mathbb{C}$ few decades ago. Recently, Pictures of the Bers slice and the Earle slice were drawn by P. Liepa and Y. Yamashita respectively. Pictures of linear slices, that is, slices defined by fixing some trace function, had been given by C.T.McMullen and D.J.Wright. Pictures of other one-dimensional parameter spaces, though not for once punctured torus groups, were given by R.Riley, F.W.Gehring and G.J.Martin, and D.J.Wright. These pictures also seem to admit cusps in their boundaries like those in our three slices. Results given in this paper support the conjectural phenomena at the rational boundary points of these slices, see [GeMn], [KSWY], [Mc98], [Mn], [Pe], [W], [Wr], and [Ya].

FIGURE 1. A Bers slice : *Courtesy of Yasushi Yamashita*

We will also show that each element of the Teichmüller modular transformation acting on a Bers slice admits a quasiconformal extension on the whole ambient space. This is in accordance the tidiness of the arrangement of cuspidal points and the self-similarity on the complex boundaries of Bers slices (cf. p.178 of [Mc96]).

By studying the cusps, we treat the behavior of the rational pleating varieties at the boundary of quasifuchsian space. We will see that the local pleating theorem due to L.Keen and C.Series can be extended for geometrically finite boundary groups. Namely, for a given representation near a geometrically finite boundary punctured torus group, we show that if the trace functions corresponding to accidental parabolics are real and greater than 4, the image of the given representation is discrete and the boundary of the convex core of its Kleinian manifold is bent along the corresponding geodesics (see [KS99]).

1.1. **Statements of results.** The main purpose of this paper is to prove the folklore conjecture:

Theorem 1. (Rational boundary points are cusps) *Every rational boundary point of three slices is a* $(2,3)$*-cusp.*

Corollary 1. *All Bers slices, the Earle slice, and the Maskit slice are not quasidisks.*

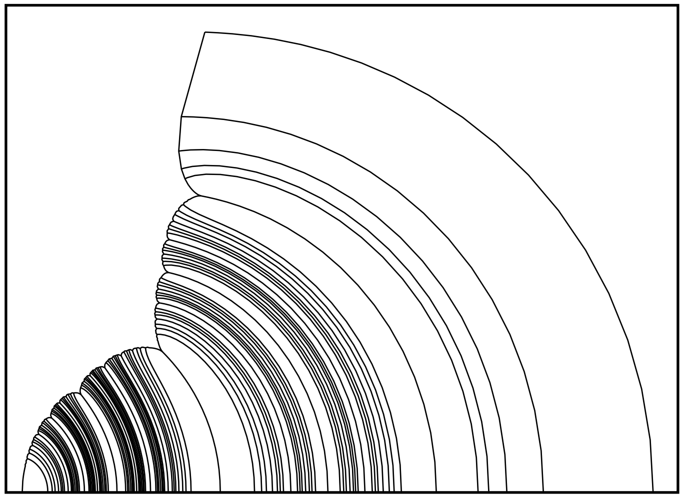

FIGURE 2. The upper half of the Earle slice : *Courtesy of Peter Liepa*

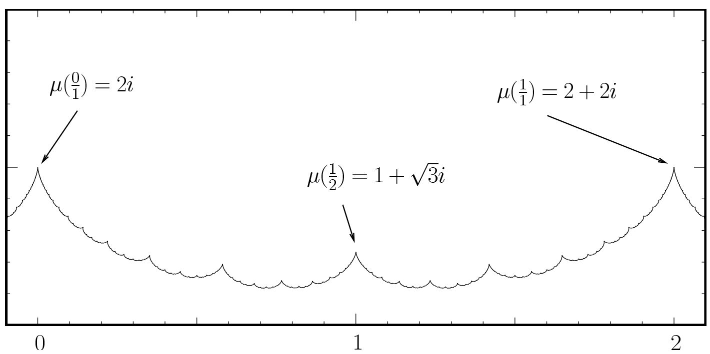

FIGURE 3. The Maskit slice : *Courtesy of David J. Wright*

The definition of a $(2,3)$-cusp is given in §2. Roughly speaking, a $(2,3)$-cusp is a boundary point sandwiched between singularities of two Diocles' cissods. Corollary for the case of the Maskit slice is observed in author's earlier paper [M1].

Theorem 1 is related to the D.J.Wright's conjecture in [W]: In the late 1980's, D.Mumford, C.T.McMullen, and D.J.Wright drew the computer graphical picture of the Maskit embedding and found many cusps on its boundary. In drawing this picture, Wright conjectured that, in the case of the Maskit embedding, every rational boundary point is a simple zero of corresponding trace function, while he found that this conjecture implies that the folklore conjecture in the case of the Maskit embedding is correct. Actually, Theorem 1 is given by proving the following theorem, which contains the affirmative answer to Wright's conjecture:

Theorem 2. (Rational boundary points are simple zeros) *For each of the three slices, every rational boundary point is a simple zero of the trace function corresponding to its accidental parabolic.*

1.2. **Geometry of a Bers slice. Outer radius.** The outer radius of a Bers slice is an important notion from the function theoretic point of view. Furthermore, the extreme point for outer radius often has some extremal properties (cf. Lehto [Leh]).

Corollary 2. (Extreme point for outer radius) *The outer radius of a Bers slice is attained at a boundary point corresponding to a totally degenerate group.*

Indeed, it can be observed from Euclidean geometry that at any $(2,3)$-cusp the outer radius can not be attained.

Self-similarity. In his book [Mc96], C.T.McMullen posed the following problem:

Problem *Is the boundary of a Bers slice self-similar about the fixed points of every hyperbolic modular transformation?*

The following theorem is obtained in part of the proof of the folklore conjecture for Bers slices.

Theorem 3. (Modular transformation has qc-extension) *Every Teichmüller modular transformation acting on a Bers slice has a quasiconformal extension to the whole ambient space. Furthermore, the maximal dilatation of its extension can be dominated by the maximal dilatation of the extremal quasiconformal self-mapping acting on the base surface of its Bers slice, which is homotopic to the homeomorphism corresponding to the given modular transformation.*

Thus, we obtain a version of an answer for his problem.

Corollary 3. *The closure of a Bers slice is weakly quasiconformally self-similar about the fixed points of any hyperbolic modular transformation.*

In this paper, a set $E \subset \hat{\mathbb{C}}$ is called *weakly quasiconformally self-similar* about $e_0 \in E$ if there exist a quasiconformal mapping f of $\hat{\mathbb{C}}$ and at most one point $e_1 \in E \setminus \{e_0\}$ satisfying the following:

(1) $f(E) = E$, $f(e_0) = e_0$, and $\bar{\partial} f \equiv 0$ on the interior of E.
(2) For any neighborhood U of e_0 in $\hat{\mathbb{C}} \setminus \{e_1\}$, $E \setminus \{e_1\} = \cup_{n \in \mathbb{N}} f^n(U \cap E)$ holds.

We observe that the following similarity phenomenon among cusps in the boundary of a Bers slice in §11:

Corollary 4. (All rational boundary points are similar each other) *Every Teichmüller modular transformation acting on a Bers slice is $C^{1+\alpha}$-conformal at all rational boundary points, where $\alpha > 0$ is depend only on the base point of given Bers slice, independent of the choice of elements of modular transformation.*

In Appendix A, we will discuss on similarity phenomena about the boundary of the Mandelbrot set concerning with this corollary.

Hölder continuity at boundary Combining Theorem 3 and the Hölder continuity of quasiconformal mappings (see p.70 of Lehto-Virtanen [LeV]), we also obtain a refinement of a result of Bers in p.51 of [B81a]:

Corollary 5. *Every Teichmüller modular transformation acting on a Bers slice can be extended Hölder-continuously on the closure of its Bers slice.*

1.3. **Extended Local Pleating Theorem.** We extend the local pleating theorem due to L.Keen and C.Series to a neighborhood of geometrically finite boundary groups (cf. [KS99] and §12). The once punctured torus group $[\eta]$ is called a *boundary group* if it is Kleinian but not quasifuchsian[1] If a boundary group is geometrically finite, it has either one or two conjugacy classes of primitive accidental parabolics. In this paper, a geometrically finite boundary group is *of type* n if it admits n conjugacy classes of primitive accidental parabolics.

Theorem 4. (Extended local pleating theorem, The rational case) *Let $[\eta_0]$ be a geometrically finite boundary group. Then there exists a neighborhood $\mathcal{U}$ of $[\eta_0]$ in representation space satisfying the following: Suppose $[\rho_0]$ is of type n and let $\{\mathbf{w}_i\}_{1\leq i\leq n}$ be elements of the fundamental group of a once punctured torus such that $\eta_0(\mathbf{w}_i)$ are primitive accidental parabolics. For $[\eta]\in\mathcal{U}$, if $\mathrm{tr}^2\eta(\mathbf{w}_i)$ are real and greater than 4, then $[\eta]$ is a quasifuchsian representation. Furthermore the convex core of the Kleinian manifold associated with $[\eta]$ is bent along the geodesic corresponding to $\eta(\mathbf{w}_i)$ $(1\leq i\leq n)$.*

Theorem 4 is related to the following result of J.P.Otal in [O]:

Theorem (Otal) *Let η be a Kleinian once punctured torus group. Let $\mathbf{w}$ be an element of the fundamental group of a once punctured torus which represents a simple closed curve. If $\eta(\mathbf{w})$ is hyperbolic and if its translation length is less than some universal constant, then the boundary of convex core of the associated Kleinian manifold is bent along the geodesic corresponding to $\eta(\mathbf{w})$.*

Our theorem is proved without using this result of Otal. The resemblance between ours and Otal's is that both theorems are concerned with when a geodesic corresponding to a hyperbolic element which is sufficiently near parabolic appears as the pleating locus of the convex core. There are two differences between two theses: The first is an advantage of ours. In Otal's case, a given group for which it is to be ascertained whether the associate convex core is bent or not, is assumed to be discrete at the beginning. This is not so in our case. Furthermore our theorem gives a decision as to whether the group is discrete. The second is a disadvantage of ours. In our case, we are looking only near geometrically finite groups. However, Otal's theorem treats any once punctured torus group and points out the universality for the condition of translation length.

1.4. **Outline of the paper.** In §2, we give the notation and definitions used in this paper. The definition of Minsky's end invariants and the statement of his pivot theorem (the special case) are recalled in §3.

§4 treats a relation between the shape of the complex boundary near a rational boundary point and the trace function associated with given rational boundary point. We will show that Theorem 2 implies Theorem 1 in this section (Proposition 4.3). The most crucial reason why rational boundary points are cuspidal is that when groups move along complex boundaries converging to the group of given rational boundary point, the suitable branch of the complex length of corresponding trace function converges to zero along a path running parallel to a circle which is contained in the right half plane and tangent to zero. This phenomenon can be completely analyzed by using Minsky's pivot theorem.

[1]By virtue of Minsky's theorem, boundary groups can be characterized by this property in the case of once punctured torus groups.

In the proof of Theorem 2 the theory of pleating coordinates occupy an important position ([KS99] and [Mc98]). In Sections 5 and 6, we recall notation concerning with geometries and moduli of convex core of Kleinian manifolds, for example, pleating rays, F-peripheral disks and subgroups, and so forth.

In §7, Theorem 10, which is the most important theorem for the proof of Theorem 2, is stated. Theorem 10 tells us that along a pleating ray, the derivative of the complex length of a trace function not associating with the bending locus is dominated by a linear function of the length of bending locus. In §7.3 we define certain quasiconformal deformations of groups on pleating rays which are used to calculate the derivative of trace functions. In §7.4, we state two theorems, Theorem 12 and 13, which imply Theorem 10. We prove Theorem 10 at §7.4. We complete to prove Theorem 12 and 13 at §8 and 9, respectively. In §10, one can find how Theorem 10 contributes to the proof of Theorem 2, independently of proving Theorem 10.

In §11, we will prove Theorem 3. The soul of the proof is to recognize quasifuchsian space and the cotangent bundle over the Teichmüller space as the total spaces of the holomorphic families of Bers slices and their ambient spaces over the Teichmüller space respectively. Applying the global triviality of the cotangent bundle, we can understand that Bers slices move holomorphically in $\mathbb{C}$ when their base surfaces also move complex analytically. After these considerations, the proof is obtained by applying improved lambda-lemma and the Bers'marking trick in [B81a] (that is, diagonal action of the mapping class group on quasifuchsian space). One can read this section independently of the other results.

We prove Theorem 4 in §12. To prove this theorem we will define a local coordinate near a geometrically finite boundary group in representation space using associated trace functions, which is a key tool for proving this theorem.

In Appendix, we will give how our result relates to contribution to increase columns of dictionary between Kleinian groups and Rational maps.

Notation *$A \asymp B$ and $A \lesssim B$ means $A/C < B < CA$ and $A < CB$ for some implicit constant C, respectively. And for $z, w \in \mathbb{H}$ and $c > 0$, $z \approx_c w$ means that a bound c on hyperbolic distance in $\mathbb{H}$ between z and w.*

2. Preliminaries

In this section, we fix our notations and recall some basic definitions and facts.

2.1. $(2,3)$**-cusps.** Let $a, r > 0$. Define

$$C^+(a,r) = \{z = x+iy \in \mathbb{C} \mid |z| < r, y^2 > ax^3, x \geq 0\} \cup \{|z| < r, x < 0\} \text{ and}$$
$$C^-(a,r) = \{z = x+iy \in \mathbb{C} \mid |z| < r, y^2 < ax^3\ x \geq 0\}.$$

Let E be a domain in $\mathbb{C}$ and $\zeta_0 \in \partial E \cap \mathbb{C}$. We say that ζ_0 is a $(2,3)$-*cusp* of E if there exists $a > b > 0$, $r > 0$, and $\theta \in \mathbb{R}$ so that

$$\zeta_0 + e^\theta \cdot C^+(a,r) \subset E, \text{ and } \{\zeta_0 + e^\theta \cdot C^-(b,r)\} \cap \overline{E} = \emptyset.$$

By definition, being $(2,3)$-cusped is a local property. Namely, for a domain E, a boundary point ζ_0 is a $(2,3)$-cusp of E if and only if so is $E \cap U$ for some neighborhood U of ζ_0. A typical example of a $(2,3)$-cusp is the cuspidal point of the cardioid $\{z^2 \in \mathbb{C} \mid |z-1| < 1\}$.

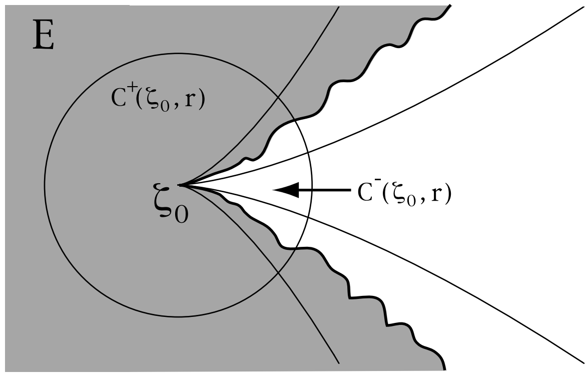

FIGURE 4. A (2,3)-cusp

2.2. **Once punctured torus groups.** Throughout this paper, we denote by Σ an oriented once punctured torus and by π_1 the fundamental group of Σ. For $\mathbf{a}', \mathbf{a}'' \in \pi_1$, the product $\mathbf{a}'\mathbf{a}''$ means the homotopy class of a curve which first passes along a curve in $\mathbf{a}''$ and next passes along a curve in $\mathbf{a}'$.

Enumeration of simple closed curves An ordered pair $(\mathbf{a}, \mathbf{b})$ of generators of π_1 is called *canonical* if the algebraic intersection number of $\mathbf{a}$ and $\mathbf{b}$ with respect to the given orientation of Σ is equal to $+1$. The commutator $[\mathbf{a}, \mathbf{b}] = \mathbf{a}^{-1}\mathbf{b}^{-1}\mathbf{a}\mathbf{b}$ represent a loop around the puncture. Henceforth, we fix a canonical generator pair $(\mathbf{a}, \mathbf{b})$ of π_1.

We review an identification between the homotopy classes of all simple closed curves on Σ and $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{1/0\}$. Henceforth, we will always express rational numbers in the form p/q where p and q are relatively prime integers and $q > 0$.

Set $\mathbf{w}(1/0) = \mathbf{a}^{-1}$ and $\mathbf{w}(n/1) = \mathbf{a}^{-n}\mathbf{b}$ $(n \in \mathbb{Z})$. For $p/q, p'/q' \in \hat{\mathbb{Q}}$ with $pq' - p'q = -1$, we define $\mathbf{w}((p+p')/(q+q')) = \mathbf{w}(p'/q')\mathbf{w}(p/q)$. Then, $\mathbf{w}(p/q) \in \pi_1$ can be defined for all $p/q \in \hat{\mathbb{Q}}$ and the homology class of a simple closed curve in $\mathbf{w}(p/q)$ is equal to that in $\mathbf{a}^{-p}\mathbf{b}^{q}$. See for instance, Keen-Series [KS93], Komori-Series [KoS], Mumford-McMullen-Wright [MMcW], and Wright [W].

Representation space $\mathcal{R}$ A homomorphism from π_1 to $\mathrm{PSL}_2(\mathbb{C})$ is called *admissible* if it sends the commutator $[\mathbf{a}, \mathbf{b}]$ to a parabolic transformation. Denote by $\mathcal{R}$ the $\mathrm{PSL}_2(\mathbb{C})$-*representation space* of π_1, that is, the set of conjugacy classes of admissible homomorphisms. The space $\mathcal{R}$ has a complex structure of two-dimensional complex analytic space so that every irreducible representation corresponds a regular point, and the trace function of $\mathbf{w} \in \pi_1$

$$\mathcal{R} \ni [\eta] \mapsto \mathrm{tr}^2\eta(\mathbf{w}) \in \mathbb{C}$$

becomes holomorphic. Notice that all family of representations given below moves holomorphically on its parameter space.

Once punctured torus groups An admissible homomorphism (or its $\mathrm{PSL}_2(\mathbb{C})$-conjugacy class) is said to be a *once punctured torus group* if it is a conjugacy class of a discrete and faithful representation; by abuse of language, such a representation itself is called by same terminology. Let $\mathcal{D}$ be the set of all once punctured torus groups and $\mathcal{QF}$ the subset of $\mathcal{D}$ which consists of all quasifuchsian representations.

Y.Minsky proved that the closure of $\mathcal{QF}$ coincides with $\mathcal{D}$ (cf.[Min]). $\mathcal{QF}$ is called *quasifuchsian space*.

Let η be a once punctured torus group with invariant component. An invariant component Δ of η is called *distinguished* if η induces an *orientation preserving* homeomorphism from Σ onto $\Delta/\eta(\pi_1)$.

Teichmüller space We denote by $\mathrm{Teich}(\Sigma)$ the *Teichmüller space* of Σ. A point in $\mathrm{Teich}(\Sigma)$ is represented by a pair (f, S) of an orientation preserving homeomorphism f of Σ to an analytically finite Riemann surface S. We say that two pairs (f, S), (f', S') are equivalent if there exists a conformal mapping h from S onto S' which is homotopic to $f' \circ f^{-1}$. The *Teichmüller distance* is defined by

$$d_{\mathrm{Teich}(\Sigma)}((f, S), (f', S')) = \log \inf_h K(h)$$

where h runs over quasiconformal mappings h of S to S' which are homotopic to $f' \circ f^{-1}$ and $K(h)$ is the maximal dilation of h. This infimum is alway attained by the unique so-called *extremal* quasiconformal mapping (cf. Imayoshi-Taniguchi [IT]). We know that the Teichmüller distance coincides with the Kobayashi distance on the Teichmüller space.

Quasifuchsian groups, Bers' uniformization Henceforth, for any 3-manifold M with boundary, the following orientation convention is applied to ∂M: A frame f on ∂M is positive if the frame (f, n) is positive with respect to that of M where n is an inward-pointing vector on ∂M. Notice that if M is a Kleinian 3-manifold, the orientation on ∂M is same as that inherited from the Riemann sphere.

Let $\overline{\Sigma}$ denote Σ with the orientation reversed. Let $x = (f, S) \in \mathrm{Teich}(\Sigma)$ and $y = (g, S) \in \mathrm{Teich}(\overline{\Sigma})$. Bers' simultaneous uniformization tells us that x and y determine a *marked* quasifuchsian group $G(x, y)$ (up to conjugation in $\mathrm{PSL}_2(\mathbb{C})$) such that the quasifuchsian manifold

$$M_y^x := (\mathbb{H}^3 \cup \Omega(x, y))/G(x, y)$$

($\Omega(x, y)$ means the region of discontinuity of $G(x, y)$) is bounded by Riemann surfaces R and S, and such that there exists a homeomorphism between $\Sigma \times [0, 1]$ to M_y^x, compatible with corresponding markings and orientations between their boundaries. Furthermore, let $\eta_{G(x,y)}$ be the representation associated with the marked quasifuchsian group $G(x, y)$. Then the map

$$\mathrm{Teich}(\Sigma) \times \mathrm{Teich}(\overline{\Sigma}) \ni (x, y) \mapsto [\eta_{G(x,y)}] \in \mathcal{QF}$$

becomes biholomorphic.

2.3. **Definitions of Slices. The Bers embedding $\mathcal{B}_G$** In this paper, we treat a Bers slice in somehow extended sense from usual one (cf.e.g. Imayoshi-Taniguchi [IT]). Usually, to define a Bers slice, we consider a fuchsian group acting on the upper half plane and the space of Schwarzian derivatives on the lower half plane, and use univalent functions on the lower half plane to represent deformations of the fuchsian group. In our sense, we use a quasifuchsian group instead of a fuchsian group, and recognize the lower half plane as the invariant component corresponding to the Riemann surface whose orientation induced from the Riemann sphere is *reversing* to that induced from the representation corresponding to given quasifuchsian group.

Let us give more detail definition of Bers slices in our sense. we fix a pair $(x_0, y_0) \in \mathrm{Teich}(\Sigma) \times \mathrm{Teich}(\overline{\Sigma})$. We keep using this point in discussing about Bers slice unless we specify other points. Let $\eta_{G(x_0,y_0)}$ be a representation associated with $G(x_0, y_0)$ as the previous section. We normalize this so that the limit set contains three points 0, 1, and ∞. Denote by $\Omega_+(x_0, y_0)$ the distinguished invariant component of $G(x_0, y_0)$, and by $\Omega_-(x_0, y_0)$ the other component.

Let $(x, y) \in \mathrm{Teich}(\Sigma) \times \mathrm{Teich}(\overline{\Sigma})$, there exists a quasiconformal mapping W_y^x which fixes the three points $\{0, 1, \infty\}$ such that $W_y^x G(x_0, y_0)(W_y^x)^{-1}$ is a quasifuchsian group and

$$[\eta_{G(x,y)}] = [\mathbf{a}' \mapsto W_y^x \circ \eta_{G(x_0,y_0)}(\mathbf{a}') \circ (W_y^x)^{-1}].$$

In the case of $y = y_0$, we may assume that $f_{y_0}^{x,x_0} := W_{y_0}^x \circ (W_{y_0}^{x_0})^{-1}$ is conformal on $\Omega_-(x_0, y_0)$. Let $Q_{-4}(x_0, y_0)$ the complex Banach space of automorphic forms of weight -4 with the hyperbolic L^∞-norm. Then the *Bers embedding* $\beta_{y_0}^{x_0}$ (with respect to $G(x_0, y_0)$) is the mapping from the Teichmüller space $\mathrm{Teich}(\Sigma)$ into a complex vector space $Q_{-4}(x_0, y_0)$ defined by the equation:

$$\beta_{y_0}^{x_0}(x) = S(f_{y_0}^{x,x_0}),$$

where $\mathcal{S}(-)$ means the Schwarzian derivative. By the same argument as in the usual sense, we can see that $\beta_{y_0}^{x_0}$ is embedding and whose image $\mathcal{B}_{G(x_0,y_0)}$ is a bounded domain in $Q_{-4}(x_0, y_0)$. The image is called the *Bers slice* with respect to $G(x_0, y_0)$.

In showing the main theorem, it is convenient to recognize a point in $\mathcal{B}_{G(x_0,y_0)}$ as a point of Teichmüller space via Bers embedding and to specify the distinguished invariant component. Let us introduce the following notation: Henceforth, let $G = G(x_0, y_0)$ for simplicity and let $\varphi \in \mathcal{B}_G$ and $x \in \mathrm{Teich}(\Sigma)$ so that $\varphi = \beta_{y_0}^{x_0}(x)$. Then, we put[2] :

$$\begin{aligned} \rho_\varphi^b(-) &= W_{y_0}^x \circ \eta_{G(x_0,y_0)}(-) \circ (W_{y_0}^x)^{-1} \text{ and} \\ \Omega_\varphi^b &= W_{y_0}^x(\Omega_+(x_0, y_0)). \end{aligned}$$

We know that for $p/q \in \mathbb{Q}$ there exists a unique point $x^b(p/q) \in \partial\mathcal{B}_G$ such that the image of $\rho_{x^b(p/q)}^b$ is a terminal regular b-group with one primitive accidental parabolic transformation $\rho_{x^b(p/q)}^b(\mathbf{w}(p/q))$. A boundary point φ of $\mathcal{B}_G$ corresponds to a geometrically finite group if and only if $\varphi = x^b(p/q)$ for some $p/q \in \mathbb{Q}$. The boundary points $\{x^b(p/q)\}_{p/q \in \hat{\mathbb{Q}}}$ are nothing but the *rational boundary points* of the Bers slice $\mathcal{B}_G$.

As noted in Introduction, the Bers slice $\mathcal{B}_G$ can be recognized as a *slice* of holomorphic mapping from $Q_{-4}(x_0, y_0)$ to $\mathcal{R}$ as follows: Let $\varphi \in Q_{-4}(x_0, y_0)$. In solving the Schwarzian equation $\mathcal{S}(-) = \varphi$, we obtain the monodromy homomorphism $\rho_\varphi^b : \pi_1 \to \mathrm{PSL}_2(\mathbb{C})$ (see e.g. [Hej] and [Sh]). Then we can define the holomorphic mapping

$$Q_{-4}(x_0, y_0) \ni \varphi \mapsto [\rho_\varphi^b] \in \mathcal{R}.$$

Then the Bers slice $\mathcal{B}_G$ is the component of pre-image of quasifuchsian space under this mapping containing the origin (cf. [Sh]).

The Earle embedding $\mathcal{E}$ Most of facts in this subsection can be referred in Komori and Series's paper [KoS].

[2]The superscript "b" comes from the name "Bers slice".

For $d \in \mathbb{C}^*$, we define a homomorphism ρ^e_d from π_1 to $\mathrm{PSL}_2(\mathbb{C})$ by

$$\rho^e_d(\mathbf{a}) = \begin{bmatrix} \frac{d^2+1}{d} & \frac{d^3}{2d^2+1} \\ \frac{2d^2+1}{d} & d \end{bmatrix}, \quad \rho^e_d(\mathbf{b}) = \begin{bmatrix} \frac{d^2+1}{d} & -\frac{d^3}{2d^2+1} \\ -\frac{2d^2+1}{d} & d \end{bmatrix}.$$

The domain $\mathcal{E}$ in $\mathbb{C}$ is defined by

$$\mathcal{E} = \{d \in \mathbb{C} \mid \operatorname{Re} d > 0, \quad [\rho^e_d] \in \mathcal{QF}\}.$$

Denote by Ω^e_d the distinguished invariant component of $\rho^e_d(\pi_1)$. Then as in the case of the Bers embedding, we obtain the isomorphism, called the *Earle embedding*, from $\mathrm{Teich}(\Sigma)$ onto $\mathcal{E}$. We know that $\rho^e_d(\pi_1)$ has an elliptic involution $z \mapsto -z$ which gives the conjugation between $\rho^e_d(\mathbf{a})$ and $\rho^e_d(\mathbf{b})$. Thus, the Earle embedding represents a point of $\mathrm{Teich}(\Sigma)$ by a quasifuchsian group with suitable elliptic involution (cf. Earle [E] and Kra-Maskit [KrM82]). The domain $\mathcal{E}$ is invariant under the complex conjugation $d \mapsto \bar{d}$. One can imagine the shape of the whole $\mathcal{E}$ by taking the reflection of P.Liepa's figure along its bottom break.

The above elliptic involution induces the relation $\mathrm{tr}^2\rho^e_d(\mathbf{w}(p/q)) \equiv \mathrm{tr}^2\rho^e_d(\mathbf{w}(q/p))$ on $\mathbb{C}^*$. It is known that for every $p/q \in \hat{\mathbb{Q}} \setminus \{\pm 1/1\}$, there exists a unique point $x^e(p/q)$ in $\partial\mathcal{E}$ such that the image of $\rho^e_{x^e(p/q)}$ is a maximally parabolic group with parabolic transformations $\rho^e_{x^e(p/q)}(\mathbf{w}(p/q))$, $\rho^e_{x^e(p/q)}(\mathbf{w}(q/p))$, and $\rho^e_{x^e(p/q)}([\mathbf{a},\mathbf{b}])$, and $\operatorname{Im} x^e(p/q) > 0$ if $-1 < p/q < 1$. By a *maximal parabolic group* we mean a Kleinian group with the largest number of the non-conjugate rank 1 maximal parabolic subgroup, see Keen-Maskit-Series [KMsS].

The boundary point d of $\mathcal{E}$ corresponds to a geometrically finite group if and only if $d = x^e(p/q)$ for some $p/q \in \hat{\mathbb{Q}} \setminus \{\pm 1/1\}$. Notice that $x^e(p/q)$ and $x^e(q/p)$ are different. In fact, it holds that $x^e(q/p) = \overline{x^e(p/q)}$ (the complex conjugation). As in the case of the Bers embedding, the *rational boundary points* of $\mathcal{E}$ are $\{x^e(p/q)\}_{p/q \in \hat{\mathbb{Q}} - \{\pm 1/1\}}$.

The Maskit embedding $\mathcal{M}$ In this paper, we deal with the Maskit embedding via so-called the horocyclic coordinate, see Kra [Kr90]. The horocyclic coordinate is defined as follows: For $\mu \in \mathbb{C}$, we define the representation ρ^m_μ of π_1 by

$$\rho^m_\mu(\mathbf{a}) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \rho^m_\mu(\mathbf{b}) = \begin{bmatrix} -i\mu & -i \\ -i & 0 \end{bmatrix}.$$

Then the domain $\mathcal{M}$ is defined as the component of the interior of the set

$$\{\mu \in \mathbb{C} \mid [\rho^m_\mu] \in \mathcal{D}\}$$

which contains $4i$. It is known that for each $\mu \in \mathcal{M}$, $\rho^m_\mu(\pi_1)$ is a terminal regular b-group of type $(1,1)$ (cf. [Kr90]).

Denote by Ω^m_μ, $\mu \in \mathcal{M}$, the invariant component of $\rho^m_\mu(\pi_1)$. Then we can define the biholomorphic mapping, called the *horocyclic coordinate*, of $\mathrm{Teich}(\Sigma)$ onto $\mathcal{M}$ as in subsections above. This mapping is often referred as the Maskit embedding of $\mathrm{Teich}(\Sigma)$ (cf. Kra [Kr88]).

For $\mu \in \mathbb{C}$, we find a unique point $x^m(p/q) \in \partial\mathcal{M} \setminus \{\infty\}$ for $p/q \in \mathbb{Q}$ satisfying that $\rho^m_{x^m(p/q)}(\pi_1)$ is a maximal parabolic group whose parabolic transformations are $\rho^m_{x^m(p/q)}(\mathbf{w}(p/q))$, $\rho^m_{x^m(p/q)}(\mathbf{w}(1/0))$, and $\rho^m_{x^m(p/q)}([\mathbf{a},\mathbf{b}])$. Further, it is known that $\mu \in \partial\mathcal{M}$ corresponds to a geometrically finite group if and only if $\mu = x^m(p/q)$ for some $p/q \in \mathbb{Q}$ (see Keen-Series [KS93]). Therefore the boundary point is a rational boundary point if and only if it equals to $x^m(p/q)$ for some $p/q \in \mathbb{Q}$.

Notation. *To simplify the arguments, we put*

$$\mathbb{Q}_{\mathcal{X}} = \hat{\mathbb{Q}} if\, \mathcal{X} = \mathcal{B}_G, \hat{\mathbb{Q}} - \{\pm 1/1\} if\, \mathcal{X} = \mathcal{E},\ or \mathbb{Q} if\, \mathcal{X} = \mathcal{M}.$$

Let us denote by $(\mathcal{X}_0, \mathcal{X})$ *and* $\{\eta_x\}_{x \in \mathcal{X}_0}$ *the pair and the holomorphic family of admissible representations corresponding to the one of the following three cases:*

(B) $\mathcal{X}_0 = Q_{-4}(G(x_0, y_0))$, $\mathcal{X} = \mathcal{B}_{G(x_0,y_0)}$, *and* $\eta_x = \rho_x^b$.
(E) $\mathcal{X}_0 = \mathbb{C}^*$, $\mathcal{X} = \mathcal{E}$, *and* $\eta_x = \rho_x^e$.
(M) $\mathcal{X}_0 = \mathbb{C}$, $\mathcal{X} = \mathcal{M}$, *and* $\eta_x = \rho_x^m$.

Denote by G_x, $x \in \mathcal{X}$ *the image of* π_1 *under* η_x. *For* $p/q \in \mathbb{Q}_{\mathcal{X}}$, *let* $x(p/q)$ *be the rational boundary point of* $\mathcal{X}$ *corresponding to the geometrically finite group containing* $X_{p/q,x} := \eta_x(\mathbf{w}(p/q))$ *as an accidental parabolic transformation, and in the case of the Earle slice, with the additional condition that* $\operatorname{Im} x(p/q) > 0$ *if* $-1 < p/q < 1$ *and* $x(q/p) = \overline{x(p/q)}$.

3. Complex length and End invariants

According to Minsky's work [Min], every once punctured torus group is determined by its end invariants. In this subsection, after Minsky we recall the definition of end invariants of once punctured torus groups and a relation between end invariants of given group and complex length of loxodromic elements in the group, which follows from Minsky's investigation. Notice that the canonical generators $(\mathbf{a}, \mathbf{b})$ on Σ defined in §2 determine the identification between the Teichmüller space of Σ and the upper-half plane $\mathbb{H}$ via their periods.

Let η be a once punctured torus group. According to Bonahon and Thurston (cf. [Bo86]), η induces an orientation preserving homeomorphism of $\Sigma \times (-1, 1)$ onto the associated manifold $\mathbb{H}^3/\eta(\pi_1)$ and the orientation of Σ agrees with that induced from $\Sigma \times \{1\}$. Let us denote by e_+ the end of $\mathbb{H}^3/\eta(\pi_1)$ corresponding to $\Sigma \times \{1\}$ and by e_- the other end.

The region of discontinuity of $\eta(\pi_1)$ is divided into two disjoint pieces Ω_+ and Ω_- corresponding to e_+ and e_-, respectively. (One (or both) of them may be empty.) There are three possibilities for each of these boundaries, corresponding to three types of end invariants (let s denote either $+$ or $-$):

- Ω_s is a topological disk, and $\Omega_s/\eta(\pi_1)$ is a once punctured torus. η gives the marking on $\Omega_s/\eta(\pi_1)$, and hence determines the point of $\mathrm{Teich}(\Sigma)$ and $\nu_s(\eta) \in \mathbb{H}$.
- Ω_s is an infinite union of disks and $\Omega_s/\eta(\pi_1)$ is a thrice punctured sphere, obtained from the corresponding boundary of $\Sigma \times (0, 1)$ by deleting a simple closed curve $\mathbf{w}(p/q)$ for some $p/q \in \hat{\mathbb{Q}}$. In this case, set $\nu_s(\eta) = p/q$.
- Ω_s is empty. In this case $\nu_s(\eta) \in \mathbb{R} \setminus \mathbb{Q}$. This is defined as the limit of rational numbers corresponding to simple closed curves exiting the end e_s.

We call the pair $(\nu_-(\eta), \nu_+(\eta))$ the *end invariants* of a once punctured torus group η.

Fix $p/q \in \hat{\mathbb{Q}}$. Denote by $\lambda(\eta) = \lambda_{p/q}(\eta)$ be the complex length of $\eta(\mathbf{w}(p/q))$ satisfying that $\mathrm{Re}\lambda(\eta) \geq 0$ and $-\pi < \mathrm{Im}\lambda(\eta) \leq \pi$. This satisfies the equation

$$\mathrm{tr}^2 \eta(\mathbf{a}) = 4\cosh^2(\lambda(\eta)/2).$$

By Minsky's Pivot theorem Theorem 4.1 of [Min], we have the following:

Proposition 3.1. *There exist universal constants $\epsilon_1, c_1 > 0$ so that if* $\mathrm{Re}\lambda(\eta) < \epsilon_1$ *then*
$$\frac{2\pi i}{\lambda(\eta)} \approx_{c_1} T(\nu_+(\eta)) - T(\overline{\nu_-(\eta)}) + i,$$
where $T \in \mathrm{PSL}_2(\mathbb{Z})$ *with* $T(p/q) = \infty$.

Example 5. (F) *Let η be a fuchsian punctured torus group. Then $\nu_+(\eta) = \nu_-(\eta) \in \mathbb{H}$.*

(B) *Let G be a quasifuchsian group of type $(1,1)$. Then $\nu_+(\rho^b_\varphi)$ is the period on $\Omega^b_\varphi/\rho^b_\varphi(\pi_1)$ corresponding to the homology basis $([\mathbf{a}],[\mathbf{b}])$ on Σ by the canonical homeomorphism induced from ρ^b_φ, and $\nu_-(\rho^b_\varphi) \equiv \nu_0$ for some $\nu_0 \in \mathbb{H}$.*

(E) *$\nu_+(\rho^e_d)$ is the period on $\Omega^e_d/\rho^e_d(\pi_1)$ corresponding to the basis $([\mathbf{a}],[\mathbf{b}])$ on Σ. $\nu_-(\rho^e_d)$ satisfies that $\overline{\nu_+(\rho^e_d)}\nu_-(\rho^e_d) = 1$ since $\rho^e_d(\pi_1)$ admits an elliptic involution $z \to -z$ which conjugates $\rho^e_d(\mathbf{a})$ to $\rho^e_d(\mathbf{b})$ (cf.* [Ko]*).*

(M) *$\nu_+(\rho^m_\mu)$ the period on $\Omega^m_\mu/\rho^m_\mu(\pi_1)$ corresponding to the basis $([\mathbf{a}],[\mathbf{b}])$ on Σ, and $\nu_-(\rho^m_\mu) \equiv 1/0$ for all $\mu \in M$.*

4. Trace functions and the geometry of parameter spaces

4.1. **Localization of Holomorphic Families.**

Proposition 4.1. *All Bers slices, the Earle slice, and the Maskit slice are Jordan domains in their ambient spaces. Moreover, Their canonical identifications with the Teichmüller space are extended homeomorphically to the Thurston compactification.*

Proof. See [Min] and [Ko]. □

This asserts local geometric properties near rational boundary points:

Corollary 6. *Let $\mathfrak{q} \in \mathbb{Q}_{\mathcal{X}}$. Then $x(\mathfrak{q})$ admits a system of neighborhoods so that each element U in this system satisfies the following:*

1. *$\mathcal{X} \cap U$ is connected and contractible, and $U \cap \partial\mathcal{X}$ is a Jordan curve whose two end points are contained in ∂U.*
2. *U itself is connected and simply connected.*
3. *Positive end invariant $\nu_+(x) := \nu_+([\eta_x])$ maps $U \cap \overline{\mathcal{X}}$ homeomorphically into $\bar{\mathbb{H}}$ so that $\nu_+(U \cap \mathcal{X}) \subset \mathbb{H}$ and $\nu_+(U \cap \partial\mathcal{X})$ is an open interval in $\hat{\mathbb{R}} = \partial\mathbb{H}$ containing $\mathfrak{q}$.*

4.2. **Geometry of parameter spaces.**
This subsection treats the geometry of the parameter space of holomorphic family of once punctured torus groups defined as in previous subsection.

4.3. $2\pi/n$-**corners.**
We define the notion of $2\pi/n$-corners and give some properties of them.

Definition 6. ($2\pi/n$-corner) *Let E be a proper subdomain in $\mathbb{C}$ and let $e_0 \in \partial E \cap \mathbb{C}$. We say that e_0 is a $2\pi/n$-corner of E ($n \in \mathbb{N}$) if one of the following conditions is satisfied (cf. Figure 5):*

- *If $n = 1$, there exists a disk D in $\mathbb{C}$ such that $0 \in \partial D$ and $e_0 + t^2 \in E$ for any $t \in D$.*
- *If $n = 2$, there exist two disjoint disk D_1 and D_2 such that $\overline{D_1} \cap \overline{D_2} = \{e_0\}$, $D_1 \subset E$, and $D_2 \cap E = \emptyset$.*

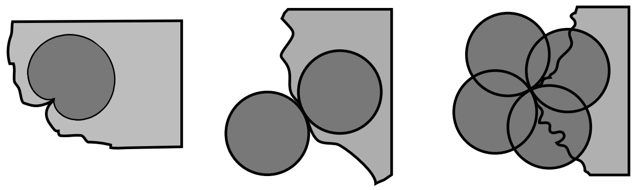

FIGURE 5. $2\pi/n$-corners

- *If $n \geq 3$, there are four disks $\{D_j, D'_j\}_{j=1,2}$ such that D_j and D'_j are tangent at e_0 ($j = 1, 2$), ∂D_1 intersects ∂D_2 at e_0 in angle $2\pi/n$, $(D_1 \cup D_2) \cap E = \emptyset$, and $D'_1 \cap D'_2 \subset E$.*

A 2π-corner is also called an inward-pointing cusp of E.

A $2\pi/n$-corner of E is a corner of opening $2\pi/n$ for E and is that of opening $2\pi(1 - 1/n)$ for the complementally domain $\mathrm{Int}(\complement E)$ of E. The similar notion is given in Pommerenke's book [Po].

Remark 1. *By definition, the notion of $2\pi/n$-corners is a local property. Namely, let E be a proper subdomain of $\mathbb{C}$ and $e_0 \in \partial E \cap \mathbb{C}$. Then e_0 is a $2\pi/n$-corner of E if and only if e_0 is that of $E \cap U$ for some neighborhood U of e_0.*

We note here two basic properties of $2\pi/n$-corners. The first is used in the next subsection and the second at §10.

Lemma 4.1. *Let E be a proper subdomain in $\mathbb{C}$ and $e_0 \in \partial E \cap \mathbb{C}$. Let U be a neighborhood of e_0 and λ a holomorphic function on U. Put n the order of zeros of the function $\lambda - \lambda(e_0)$ at e_0. Then $\lambda(e_0)$ is $2\pi/m$-corner of $\lambda(U_0 \cap E)$ for some small neighborhood U_0 of e_0 if and only if e_0 is a $2\pi/(nm)$-corner of E.*

Proof. We can easy to see this from the local properties of holomorphic functions (cf. Ahlfors [A]). □

Proposition 4.2. *Let $\mathbb{B}$ be a unit disk in $\mathbb{C}$. There is no quasiconformal mapping f such that $f(\hat{\mathbb{C}} - \overline{\mathbb{B}})$ admits an inward-pointing cusp. In particular, no quasiconformal mapping maps a $2\pi/n$-corner ($n \geq 2$) to an inward-pointing cusp.*

Proof. Assume there is such quasiconformal mapping f. We may assume that $f(1) = 0$ and the origin is the inward pointing cusp. Let $\zeta_1, \zeta_2 \in \partial f(\hat{\mathbb{C}} - \overline{\mathbb{B}})$ so that $|\zeta_1| = |\zeta_2|$. Since $f(\hat{\mathbb{C}} - \overline{\mathbb{B}})$ is a quasidisk, These point satisfies $|\zeta_1| < d|\zeta_1 - \zeta_2|$ for some $d > 0$ by Ahlfors's three points property (see Gehring [Ge]). On the other hand, since the origin is an inward pointing cusp, $|\zeta_1 - \zeta_2| = O(|\zeta_1|^2)$. This contradicts the previous inequality. The last claim is proved by using a quasiconformal mapping which maps the sector of angle $2\pi/n$ to the sector of angle π. □

Corollary 7. *No quasi-disk admits inward-pointing cusps.*

4.4. **Inward-pointing cusps versus $(2, 3)$-cusps.** By definition, each $(2, 3)$-cusp is an inward-pointing cusp. However, the converse does not hold.

Example 7. *Define the simple curve by*

$$\mathbb{R} \ni x \mapsto x + ix^{3/2}(1 + \sin(1/x)) \mathit{if}\ x \geq 0 - x - i|x|^{3/2}(1 + \cos(1/x)) \mathit{if}\ x < 0.$$

Then the origin is not a $(2,3)$-cusp of the component of the compliment of the curve containing the negative real axis but an inward pointing-cusp.

The following lemma is used in the proof of Proposition 4.3.

Lemma 4.2. *Let E be a domain with the following property: There exist a neighborhood U of zero and two positive numbers $r_1 > r_2$ so that*

$$\{\zeta \in \mathbb{C} \mid |\zeta - r_2| < r_2\} \subset E \cap U \subset \{\zeta \in \mathbb{C} \mid |\zeta - r_1| < r_1\}.$$

Let $\xi = h(\zeta)$ be an odd univalent function on U. Then the origin is a $(2,3)$-cusp of the domain $E' := \{h(\zeta)^2 \mid \zeta \in E \cap U\}$.

Proof. By considering the case of $E \cap U$ instead of that of E, we may assume that $E \subset U$. By Lemma 4.1, the origin is an inward-pointing cusp of E'. Hence there exists $a, r > 0$ and $\theta \in \mathbb{R}$ so that $e^{\theta} \cdot C^{+}(a,r) \subset E'$. Therefore, to complete this lemma, after taking r so small if necessary we should find a positive number b with a property that $e^{\theta} \cdot C^{-}(b,r) \cap \overline{E'} = \emptyset$.

Especially, Since E' contains a cardioid whose cuspidal point is the origin, we may show the following weaker condition: there exist $\theta' \in \mathbb{R}$ and $b > 0$ so that $e^{\theta'} \cdot C^{-}(b,r) \cap \overline{E'} = \emptyset$. Indeed, if such θ' exists, θ' is automatically equal to θ modulo 2π.

After the above remark, we may assume that $h'(0) = 1$. Since h is an odd univalent function, we have

$$h(-\zeta) = -h(\zeta) \text{ and } |h(\zeta) - \zeta| = O(|\zeta|^3). \tag{1}$$

By definition, $E \cap (-E) = \emptyset$ and $E_0 := E \cup (-E)$ is contained in the union of two circles $\{\zeta | |\zeta \pm r_1| < r_1\}$. Therefore, the equations (1) imply that $h(E) \cap (-h(E)) = \emptyset$, $h(E_0) = h(E) \cup (-h(E))$ and there exist $r_1' > 0$ and a neighborhood V of $w = 0$ so that

$$h(E_0) \cap V \subset \{\zeta \in \mathbb{C} \,|\, |\zeta + r_1'| < r_1'\} \cup \{\zeta \in \mathbb{C} \,|\, |\zeta - r_1'| < r_1'\}.$$

Hence $h(E_0) \cap V$ does not intersect a union of two cones $C' = \{\zeta | |\zeta \pm r_1'| > r_1',\ |\zeta| < r_2'\}$ some small $r_2' > 0$. We can see from the Euclidean geometry that the set $\{\zeta^2 | \zeta \in C'\}$ contains $(-1) \cdot C^{-}(b, (r_2')^2)$ for suitable $b > 0$. This implies that $E \cap (-1) \cdot C^{-}(b,r) = \emptyset$ for some $r > 0$. □

4.5. **Cusps in Parameter space.** The following is the main result of this section.

Proposition 4.3. *Let $\mathfrak{q} \in \mathbb{Q}_{\mathcal{X}}$. If $x(\mathfrak{q})$ is a zero of the function $\mathrm{tr}^2 \eta_x(\mathbf{w}(\mathfrak{q})) - 4$ of order n, then $x(\mathfrak{q})$ is a $2\pi/n$-corner of $\mathcal{X}$. Furthermore, the derivative of the trace function at $x(\mathfrak{q})$ does not vanish if and only if $x(\mathfrak{q})$ is a $(2,3)$-cusp of $\mathcal{X}$.*

Remark 2. *This proposition gives a refinement of the main result of the author's previous paper* [M1]. *The idea of the proof of this proposition is essentially same as that of the result in* [M1].

Proof of Proposition 4.3 Let $\Pi(t) = x(\mathfrak{q}) + t^2$ and Let $\tilde{\mathcal{X}}$ be a lift of $\mathcal{X}$ via the covering mapping Π. To prove the first part of this proposition, it suffices to show from Lemma 4.1 that the origin $t = 0$ is a π/n-corner of $\tilde{\mathcal{X}}$.

Let $\lambda_{\mathfrak{q}}$ denote the complex length of $\eta_{\Pi(t)}(\mathbf{w}(\mathfrak{q}))$ for $t \in \tilde{\mathcal{X}}$. By definition,

$$\mathrm{tr}^2 \eta_{\Pi(t)}(\mathbf{w}(\mathfrak{q})) - 4 = 4 \sinh^2 \frac{\lambda_{\mathfrak{q}}(t)}{2}. \tag{2}$$

Put $\tilde{\mathcal{X}}_0 = \Pi^{-1}(\mathcal{X}_0) \subset \mathbb{C}$. We now have two branched covering spaces $\Pi : \tilde{\mathcal{X}}_0 \to \mathcal{X}_0$ and $\sinh^2(-) : \mathbb{C} \to \mathbb{C}$. Note that these two Π and $\sinh^2(-)$ are branched of order 2 at $x(\mathfrak{q})$ and the origin respectively. Therefore, taking the lift of the map $\mathcal{X}_0 \ni x \mapsto \mathrm{tr}^2\eta_x - 4 \in \mathbb{C}$, we may assume that $\lambda_{\mathfrak{q}}$ is holomorphic on a neighborhood V of the origin $t = 0$ and satisfies that $\mathrm{Re}\,\lambda_{\mathfrak{q}} > 0$ in $\tilde{\mathcal{X}}$ and $\lambda_{\mathfrak{q}}(0) = 0$. Since the holomorphic mapping $\lambda_{\mathfrak{q}}$ is a lift of two covering spaces as above, the mapping commutes with their covering transformations $t \mapsto -t$ of Π and $z \to -z$ of $\sinh^2(-)$. This means that $\lambda_{\mathfrak{q}}$ is an odd function on V. By assumption, $x(\mathfrak{q})$ is a zero of the left hand side of (2) of order n. Therefore the origin is a zero of $\lambda_{\mathfrak{q}}$ of order n. Shrinking V if necessary, we may suppose that the origin is the unique zero of $\lambda_{\mathfrak{q}}$ in V and

$$|\mathrm{Re}\lambda_{\mathfrak{q}}(t)| < \epsilon_1, \quad |\mathrm{Im}\lambda_{\mathfrak{q}}(t)| < \pi,\ t \in V, \tag{3}$$

where ϵ_1 is the positive constant arising in Proposition 3.1 (Minsky's pivot theorem). We may also assume V to be invariant under the covering transformation of $\Pi : \tilde{\mathcal{X}}_0 \to \mathcal{X}_0$ and such that $U := \Pi(V)$ becomes a neighborhood of $x(\mathfrak{q})$ satisfying the conditions (1), (2), and (3) in Corollary 6.

Put $\nu_\pm(x) = \nu_\pm([\eta_x])$ for $x \in \mathcal{X}$. Then by (3), we can apply Proposition 3.1 for ends invariants of $\eta_{\Pi(t)}$ for all $t \in \overline{\tilde{\mathcal{X}}} \cap V$, and hence we obtain

$$\frac{2\pi i}{\lambda_{\mathfrak{q}}(t)} \approx_{c_1} T(\nu_+(\Pi(t))) - \overline{T(\nu_-(\Pi(t)))} + i, \quad t \in \overline{\tilde{\mathcal{X}}} \cap V, \tag{4}$$

where $T \in \mathrm{PSL}_2(\mathbb{Z})$ with $T(\mathfrak{q}) = \infty$.

In the cases of a Bers slice and the Maskit slice, the negative end invariant is a suitable fixed point in $\overline{\mathbb{H}}$. In the Earle slice case $\mathcal{X} = \mathcal{E}$, the negative end invariant of $[\eta_{x(\mathfrak{q})}]$ is $\mathfrak{q}^{-1}$. Since $\overline{\nu_+(x)}\nu_-(x) = 1$ for $x \in \mathcal{E}$ and ν_+ is the Riemann mapping of $\mathcal{E}$, ν_- is continuous on the closure of $\mathcal{E}$. Hence, shrinking V again if necessary, we may suppose that

$$|\mathrm{Re}T(\nu_-(x)) - T(\mathfrak{q}^{-1})| < 1 \text{ and } 0 < \mathrm{Im}T(\nu_-(x)) < 1, \quad x \in U \cap \mathcal{E}.$$

Let I_0 be an interval in $\mathbb{R}$ corresponding to the curve $\nu_+(U) \cap \overline{\mathcal{X}}$ via ν_+. Then on $\Theta \in I_0$, the mapping $\Theta \mapsto x(\Theta) \in U$ is well defined and satisfies that $\nu_+(x(\Theta)) = \Theta$. Take a lift $\tilde{x}(-)$ from I to $\partial\tilde{\mathcal{X}} \cap V$ so that $\Pi \circ \tilde{x}(\Theta) = x(\Theta)$. Then by (4) and from the observation on the negative end invariant, there exist a positive constant c_2 and $\nu_0 \in \overline{\mathbb{H}}$ so that

$$\frac{2\pi i}{\lambda_{\mathfrak{q}}((\tilde{x}(\Theta))} \approx_{c_2} T(\Theta) - \overline{\nu_0} + i, \quad \Theta \in I_0.$$

This implies that

$$\{\lambda \in \mathbb{C} \mid |\lambda - r_1| < r_1\} \subset \lambda_{\mathfrak{q}}(\tilde{\mathcal{X}} \cap V) \subset \{\lambda \in \mathbb{C} \mid |\lambda - r_2| < r_2\} \tag{5}$$

for suitable $r_1, r_2 > 0$. Thus $\lambda = 0$ is a π-corner of a domain $\lambda_{\mathfrak{q}}(V \cap \tilde{\mathcal{X}})$ and hence, the origin $t = 0$ is π/n-corner of $\tilde{\mathcal{X}}$ by Corollary 6.

Next we show the rest of this proposition. If $x(\mathfrak{q})$ is $(2,3)$-cusp of $\mathcal{X}$, the derivative of the trace function at $x(\mathfrak{q})$ does not vanish by above observation. The converse follows from (5) and Lemma 4.2 $\square$

5. The Pleating rays

Details of the contents of this subsection can be found in Epstein-Marden [EpMa], Keen-Series [KS93], Komori-Series [KoS], and McMullen [Mc98]. Notice that for any point $x \in \mathcal{X}$, a Kleinian group G_x admits a distinguished invariant component Ω_x. This is of one of the three kinds; Ω^b_φ, Ω^e_d, or Ω^m_μ.

Let $p/q \in \mathbb{Q}_{\mathcal{X}}$. The *$p/q$-pleating ray* $\mathcal{P}^{\mathcal{X}}_{p/q}$ is the set of points $x \in \mathcal{X}$ so that the boundary component of the convex core of $\mathbb{H}^3/G_x$ facing Ω_x/G_x is bent along the closed geodesic corresponding to $X_{p/q,x}$. In all cases, the p/q-pleating ray is the embedded real analytic curve on $\mathcal{X}$ landing at $x(p/q)$ on which $X_{p/q,x}$ is a hyperbolic transformation.

Remark 3. *This paper treats the pleating rays only in the rational case, that is, the rays corresponding to rational laminations.*

Proposition 5.1. *The translation length of $X_{p/q,x}$ on $\mathcal{P}^{\mathcal{X}}_{p/q}$ gives a real analytic diffeomorphism from $\mathcal{P}^{\mathcal{X}}_{p/q}$ onto its image. Further, from the inside of $\mathcal{P}^{\mathcal{X}}_{p/q}$, x tends to $x(p/q)$ if and only if the translation length tends to zero.*

Here, for a hyperbolic transformation W, the *translation length* of W is the positive real number l satisfying that $\mathrm{tr}^2 W = 4\cosh^2(l/2)$. To prove this proposition, it suffices to show that the derivative of trace function $\mathrm{tr}^2 X_{p/q,x}$ does not vanish on $\mathcal{P}^{\mathcal{X}}_{p/q}$. Indeed, in the cases $\mathcal{X} = \mathcal{B}_G$ or $\mathcal{E}$, this can be proved by combining the Local Pleating Theorem (cf. §12) and the fact that pleating rays are embedded arcs. See Theorem 5.1 of [KoS] and Theorem 7.4 of [Mc98]. The case of $\mathcal{X} = \mathcal{M}$ had already been proved directly in Lemma 5.5 of [KS93].

6. Geometry of F-peripheral disks

In this and the next sections, we treat general once punctured torus groups with distinguished invariant component.

This section deals with basic properties of some F-peripheral subgroups and peripheral disks. Many results of this section seem to be well-known from L.Keen and C.Series' beautiful works. For the convenience of readers, we give a brief proof of them here. Let η be a once punctured torus group with distinguished invariant component Ω. To simplify the argument, we assume that $G := \eta(\pi_1)$ is not a fuchsian group throughout this section.

6.1. **The expanding mapping.** Denote by $\mathcal{C}_\eta$ the convex hull of the limit set of G and by $\partial\mathcal{C}_\eta$ the component of $\partial\mathcal{C}_\eta$ facing Ω. Then $\partial\mathcal{C}_\eta$ is simply connected and has a hyperbolic structure inherited from $\mathbb{H}^3$. Therefore, there exist a hyperbolic structure on Σ and an isometry from the universal covering of Σ to $\partial\mathcal{C}_\eta$. By the *expanding mapping*[3], we here mean this isometry. The expanding mapping induces an isomorphism between π_1 to G compatible with η.

6.2. **F-peripheral, Peripheral subgroups and disks.**

Definition 8. (F-peripheral subgroup) *Let G be a Kleinian group. A fuchsian subgroup H of G is called F-peripheral if the convex hull of the limit set of H is contained in the boundary of convex hull of the limit set of G. Let Ω be a component*

[3]This is not familiar notation.However the author is no good idea to define the name of this mapping.

of G. If the convex hull of $\Lambda(H)$ is contained in the boundary of convex hull of $\Lambda(G)$ facing Ω, the subgroup H is also said to be F-peripheral with respect to Ω.

Lemma 6.1. *Let η, G, Ω, and $\partial\mathcal{C}_\eta$ as in previous subsection. Let $p/q \in \hat{\mathbb{Q}}$. Suppose $\partial\mathcal{C}_\eta$ is bent along the geodesic corresponding to $\eta(\mathbf{w}(p/q))$. Let $X \in G$ be an element conjugate to $\eta(\mathbf{w}(p/q))$ in G and $V \in G$ so that $G = \langle X, V\rangle$. Then a subgroup $H = \langle X, VXV^{-1}\rangle$ is F-peripheral with respect to Ω*

To show Lemma 6.1, we shall prove the following proposition.

Proposition 6.1. *Let H be as in Lemma 6.1. Then, there exists a unique disk $\Delta(H)$ so that $\Delta(H) \subset \Omega$ and $\Lambda(H) = \overline{\Delta(H)} \cap \Lambda(G)$.*

Proof. According to Proposition A.1. of [KS93], H is a fuchsian group with invariant circle C so that the quotient space of a component of $\hat{\mathbb{C}} - C$ by H is a once punctured cylinder whose boundary curves correspond to X and VXV^{-1}. Especially, $\Lambda(H) \subset C$. Since H is represented by a conjugate of the fundamental group of $\Sigma - \mathbf{w}(p/q)$ and the axis of $\eta(\mathbf{w}(p/q))$ is a bending line, by passing through the expanding mapping of $\partial\mathcal{C}_\eta$, we can see that H stabilizes some flat piece N_0 of $\partial\mathcal{C}_\eta$. This implies that $\Lambda(H) \subset N_0 \cap \hat{\mathbb{C}}$. Since $\Lambda(H)$ contains at least 3 points, the hyperbolic plane in $\mathbb{H}^3$ containing N_0 coincides with that whose contour is C. Since $\partial\mathcal{C}_\eta$ is a component of the boundary of the convex set $\mathcal{C}_\eta$, there exists a unique component $\Delta(H)$ of $\hat{\mathbb{C}} - C$ so that the hyperbolic plane whose boundary is C becomes a support plane which separates $\Delta(H)$ and $\mathcal{C}_\eta$. Because $\partial\mathcal{C}_\eta$ faces Ω, the disk $\Delta(H)$ is contained in Ω.

Next, we shall show that $\Lambda(H) = \overline{\Delta(H)} \cap \Lambda(G)$. Immediately, $\Lambda(H) \subset \overline{\Delta(H)} \cap \Lambda(G)$. Let $H' = \langle X, V^{-1}XV\rangle$. Then, H' is also an F-peripheral subgroup containing X and satisfies $\Delta(H') \neq \Delta(H)$. Indeed, assume that $\Delta(H') = \Delta(H)$. Since $VH'V^{-1} = H$, $V(\Delta(H)) = V(\Delta(H')) = \Delta(H)$. Thus $G = \langle H, V\rangle$ stabilizes a disk $\Delta(H)$. This is a contradiction because that means G is fuchsian.

Let σ be the component of $\partial\Delta(H) - \Lambda(H)$ which joins the fixed points of X. Since both $\partial\Delta(H)$ and $\partial\Delta(H')$ contain the fixed points of X, $\partial\Delta(H) \cap \partial\Delta(H')$ consists of the fixed points of X. Then we have $\sigma \subset \Delta(H') \subset \Omega$ because $\partial\Delta(H) - \overline{\sigma}$ contains a limit point of H. By the same reason, the component σ' of $\partial\Delta(H) - \Lambda(H)$ connecting the fixed points of VXV^{-1} is included in Ω.

By applying Theorem A.1 of [KS93] again, we have $\partial\Delta(H) - \Lambda(H) = \cup_{h\in H} h(\sigma) \cup h(\sigma')$. Thus we conclude the assertion. □

Proof of Lemma 6.1 Since the disk $\Delta(H)$ is contained in the invariant component Ω, The hemisphere circled by $\Delta(H)$ intersects the convex hull of $\Lambda(G)$ at most at $\partial\mathcal{C}_\eta$. Since $\Delta(H) \cap \Lambda(G) = \Lambda(H)$, the convex hull of $\Lambda(H)$ is contained in $\partial\mathcal{C}_\eta$. □

Almost all F-peripheral subgroup which is treated in this paper forms as in Lemma 6.1. For sake of simplicity of notation, we define

Definition 9. (Peripheral subgroup, disk) *Let G be a once punctured torus group with invariant component. Assume that a component of the boundary of convex core of the manifold $\mathbb{H}^3/G$ is bent along the simple closed geodesic. The subgroup H of G is called a peripheral subgroup if there exist a primitive hyperbolic element X in G and $V \in G$ such that the axis of X corresponds to a bending locus, $G = \langle X, V\rangle$, and $H = \langle X, V^{-1}XV\rangle$. The disk $\Delta(H)$ as in Proposition 6.1 is called the peripheral disk of H.*

Proposition 6.2. (1) *Let H be a peripheral subgroup of G. Let A be a Möbius transformation. Then AHA^{-1} is a peripheral subgroup of AGA^{-1} with a peripheral disk $A(\Delta(H))$.*

(2) *The stabilizer subgroup of a peripheral disk is its peripheral subgroup.*

(3) *Let X be an element in G conjugate to $\eta(\mathbf{w}(p/q))$. Then there are exactly two peripheral subgroups containing X: $H_1 = \langle X, V^{-1}XV\rangle$ and $H_2 = \langle X, VXV^{-1}\rangle$, where $V \in G$ satisfies $G = \langle X, V\rangle$.*

Proof. (1) By definition.

(2) This follows from L.Keen and C.Series' argument in Lemma 3.4 of [KS94].

(3) ¿From the proof of Proposition 6.1, $\Delta(H_1) \neq \Delta(H_2)$. So $\partial\Delta(H_1)\cap\partial\Delta(H_2)$ consists of the fixed points of X. Let H be an F-peripheral subgroup containing X. Then $\partial\Delta(H)$ passes through the fixed points of X. Since $\Lambda(H) \subset \partial\Delta(H)$ and $\Delta(H) \subset \Omega$, $\Delta(H)$ must coincide one of $\{\Delta(H_i)\}_{i=1,2}$. By applying (2), H agrees with one of those peripheral subgroups. □

Proposition 6.3. *Let H be a peripheral subgroup of G. Let $W \in G$. Then the set of fixed points* $\mathrm{Fix}(W)$ *of W satisfies either* $\mathrm{Fix}(W) \subset \Lambda(H)$ *or* $\mathrm{Fix}(W)\cap\overline{\Delta(H)} = \emptyset$. *Furthermore,* $\mathrm{Fix}(W) \subset \Lambda(H)$ *if and only if $W \in H$*

To prove this proposition, we use the following theorem of Susskind (Theorem 2 of [Suss]).

Theorem. (Susskind) *Let G be a Kleinian group containing no rank 2 parabolic subgroups. Let H_1 and H_2 be a geometrically finite subgroup of G. Then $\Lambda(H_1 \cap H_2) = \Lambda(H_1) \cap \Lambda(H_2)$.*

Proof of Proposition 6.3. Assume that $\langle W\rangle \cap H = \emptyset$. Since G contains no rank 2 parabolic subgroup and H and $\langle W\rangle$ are geometrically finite subgroups of G, we have

$$\Lambda(H) \cap \mathrm{Fix}(W) = \Lambda(H) \cap \Lambda(\langle W\rangle) = \Lambda(H \cap \langle W\rangle) = \emptyset.$$

This proves first statement.

Now, assume that $\mathrm{Fix}(W) \subset \Lambda(H)$ $(\subset \partial\Delta(H))$. By (2) of Proposition 6.2, we have only to show $W(\Delta(H)) = \Delta(H)$.

If W is parabolic, one of disks $\{W(\Delta(H)), \Delta(H)\}$ is contained the other. Since a part of $\Lambda(H)$ lies on their disks, both of disks must coincide. Hence $W \in H$.

Assume W is loxodromic. If $W^2(\Delta(H)) = \Delta(H)$ but $W(\Delta(H)) \neq \Delta(H)$, then $W(\Delta(H)) \cap \Delta(H) = \emptyset$ and $\partial W(\Delta(H)) = \partial\Delta(H)$. Since $\Delta(H)$ is contained in the invariant component Ω, $\Lambda(G) \subset \partial\Delta(H)$. This means G is Fuchsian and contradicts to our assumption. Finally, suppose that that $W(\Delta(H))$, $W^2(\Delta(H)) \neq \Delta(H)$. In this case, we can see that $\cup_{n\in\mathbb{Z}} W^n(\Delta(H)) = \hat{\mathbb{C}} - \mathrm{Fix}(W)$. This implies that $\Lambda(G) \subset \mathrm{Fix}(W)$ because $\Delta(H) \subset \Omega$. Since G is non-elementary, this case can not occur. □

Remark 4. *The author learned Susskind's result from Professor Caroline Series. He would like to thank for her accurate advice.*

Proposition 6.4. *Let H_1 and H_2 be peripheral subgroups of G with respect to the distinguished invariant component Ω. If there is an element $X \in G$ so that X is conjugate to $\eta(\mathbf{w}(p/q))$ and $\partial\Delta(H_1) \cap \partial\Delta(H_2) = \mathrm{Fix}(X)$, then $H_1 \cap H_2 = \langle X\rangle$.*

Proof. Clearly $H_1 \neq H_2$ because $\Delta(H_1) \neq \Delta(H_2)$. Since $\mathrm{Fix}(X) \subset \partial\Delta(H_i)$ ($i = 1, 2$) and $\mathrm{Fix}(X) \subset \Lambda(G)$, $\mathrm{Fix}(X) \subset \Lambda(H)$. Hence X is contained in H_i ($i = 1, 2$) from Proposition 6.3. Since each H_i corresponds to the fundamental group of $\Sigma - \mathbf{w}(p/q)$ and X is conjugate to $\eta(\mathbf{w}(p/q))$, $H_1 \cap H_2$ must agree with $\langle X \rangle$. □

6.3. **Intersections among peripheral disks.** Let η, X, and V be as above. ¿From observations in the previous subsection, the axes of X and $V^{-1}XV$ both lie on $\partial\mathcal{C}_\eta$ and they bound a piece of hyperbolic plane P. The *bending angle* is the angle in the interval $[0, \pi)$ between P and $V(P)$, measured so that it is zero when P and $V(P)$ are coplanar and is near π when P and $V(P)$ are almost parallel, corresponding to the case in which X is nearly parabolic.

Proposition 6.5. *Let η be a once punctured torus group with distinguished invariant component Ω. Then for any ϵ_2 with $0 < \epsilon_2 < \pi$, there exists a constant $\kappa_0 = \kappa_0(\epsilon_2) > 0$ which satisfies the following: Let $p/q \in \hat{\mathbb{Q}}$. Suppose that*

1. *the component $\partial\mathcal{C}_\eta$ of the boundary of the convex hull of the limit set of $G := \eta(\pi_1)$ facing Ω is bent along the axis of $\eta(\mathbf{w}(p/q))$,*
2. *its bending angle is more than ϵ_2, and*
3. *the translation length of $\eta(\mathbf{w}(p/q))$ is less than κ_0.*

Let H be a peripheral subgroup of G and $W \in G - H$. Then $W(\Delta(H)) \cap \Delta(H) \neq \emptyset$ if and only if $\partial W(\Delta(H)) \cap \partial\Delta(H)$ consists of the fixed points of an element of G conjugate to $\eta(\mathbf{w}(p/q))$.

It is easy to see the "if"-part of this proposition. To show the "only if" part, we begin with the following lemma.

Lemma 6.2. *Under the hypothesis of Proposition 6.5, for any ϵ_2 with $0 < \epsilon_2 < \pi$, there exists a constant $\kappa_0 = \kappa_0(\epsilon_2) > 0$ satisfying the following: Let H and H' be peripheral subgroups of G with the property that $H \cap H' = \langle X \rangle$ where X is conjugate to $\eta(\mathbf{w}(p/q))$. Suppose that the bending angle θ' is more than ϵ_0. Then, if the translation length l of $\eta(\mathbf{w}(p/q))$ is less than κ_0, $\Delta(H) \cap \Delta(H')$ is contained in a pre-image of the standard collar of the geodesic in Ω/G which is homotopic to $\mathbf{w}(p/q)$.*

Proof. By taking conjugation if necessary, we may suppose that $X(\zeta) = e^l\zeta$ for some $l > 0$ and $\Delta(H) \cap \Delta(H')$ is the sector $\{\zeta \in \mathbb{C} \mid |\arg\zeta| < (\pi - \theta')/2\}$. Then, we can see that the invariant component Ω contains the sector

$$\Delta(H) \cup \Delta(H') = \{\zeta \in \mathbb{C} \mid |\arg\zeta| < (\pi + \theta')/2\}.$$

Hence by the straight-forward calculation, we have that

$$d_\Omega(\zeta, X(\zeta)) \leq d_{\Delta(H)\cup\Delta(H')}(\zeta, X(\zeta)) \leq \frac{2l}{(\pi + \theta')\cos\{\pi(\pi - \theta')/2(\pi + \theta')\}} \lesssim l.$$

for all $\zeta \in \Delta(H) \cap \Delta(H')$, where the implicit constant in the inequality is depend only on ϵ_2. Therefore there exists a constant $\kappa_0 = \kappa_0(\epsilon_2) > 0$ so that if $l < \kappa_0$, the distance $d_\Omega(\zeta, X(\zeta))$ is less than the Margulis constant. This implies that the intersection $\Delta(H) \cap \Delta(H')$ is contained in a pre-image of the standard collar with respect to $\mathbf{w}(p/q)$ in Ω/G. □

Proof of proposition 6.5. Let κ_0 be as in previous lemma. Assume that the translation length of $\eta(\mathbf{w}(p/q))$ is less than κ_0. Let $W \in G - H$. We suppose that $W(\Delta(H)) \cap \Delta(H) \neq \emptyset$. Notice that $W(\Delta(H))$ is the peripheral disk of $H' =$

WHW^{-1} and $W(\Delta(H)) \neq \Delta(H)$ by (2) of Proposition 6.2. We take $X, X' \in G$ with the properties that both are conjugate to $\eta(\mathbf{w}(p/q))$ and $\partial\Delta(H) \cap \Delta(H')$ (resp. $\partial\Delta(H') \cap \Delta(H)$) is contained in a free side of H (resp. H') whose end points consists of $\mathrm{Fix}(X)$ (resp. $\mathrm{Fix}(X')$). Take peripheral subgroups H_1 and H_1' so that $H \cap H_1 = \langle X \rangle$ and $H' \cap H_1' = \langle X' \rangle$. Then both $\Delta(H) \cap \Delta(H_1)$ and $\Delta(H') \cap \Delta(H_1')$ contains $\Delta(H) \cap \Delta(H')$, and hence both sets do intersect. Since each of both sets is contained in a pre-image of the standard collar with respect to $\mathbf{w}(p/q)$ in Ω/G by the previous lemma, applying Keen-Halpern's collar lemma (cf. e.g Buser [Bu]), we have $\langle X \rangle = \langle X' \rangle$. This concludes the assertion. □

Proposition 6.6. *Under the conditions in Proposition 6.5, if $W \in G - H$ satisfies $W^m(\Delta(H)) \cap \Delta(H) \neq \emptyset$, then $|m| \leq 1$.*

Proof. By Propositions 6.4 and 6.5, there exists $X \in G$ conjugate to $\eta(\mathbf{w}(p/q))$ such that $H \cap W^m H W^{-m} = \langle X \rangle$. Then note that $W^{-m} X W^m \in H$, and $\Lambda(H) \cap W^m(\Lambda(H))$ is the axis of X. Let ϕ be an expanding mapping from $\mathbb{H}^2$ to $\partial\mathcal{C}_\eta$, the boundary of convex full of $\Lambda(G)$ facing Ω. Put $F = \phi^{-1} G \phi$ and let $F_H = \phi^{-1} H \phi$, $\gamma_1 = \phi^{-1} X \phi$, $\gamma_W = \phi^{-1} W \phi$, and $\gamma_2 = \gamma_W^{-m} \gamma_1 \gamma_W^m$. Then these satisfy that $F_H \cap \gamma_W^m F_H \gamma_W^{-m} = \langle \gamma_1 \rangle$ and $\gamma_2 \in F_H$.

Let us assume first that γ_W is hyperbolic. Let P_H be the convex hull of $\Lambda(F_H)$ in $\mathbb{H}^2$. Since the axes $\mathrm{Ax}(\gamma_1)$ and $\mathrm{Ax}(\gamma_2)$ of γ_1 and γ_2 correspond to the geodesic homotopic to $\mathbf{w}(p/q)$, each of both axes is a component of the boundary of P_H.

Here, we claim that the axis of γ_W intersects both the axes $\mathrm{Ax}(\gamma_1)$ and $\mathrm{Ax}(\gamma_2)$. Since $\phi(P_H) = \Lambda(H)$ and $\Lambda(H) \cap W^m(\Lambda(H))$ is the axis of X, $P_H \cap \gamma_W^m(P_H)$ (resp.$P_H \cap \gamma_W^{-m}(P_H)$) is the axis $\mathrm{Ax}(\gamma_1)$ (resp. $\mathrm{Ax}(\gamma_2)$). Hence the fixed points of γ_W lies on the free sides with respect to γ_1 and γ_2. This concludes the claim.

Let us prove this proposition. Assume that $m > 1$. Denote by x_i the intersection point of axes of γ_W and γ_i $(i = 1, 2)$. Since $\gamma_W^m(\mathrm{Ax}(\gamma_1)) = \mathrm{Ax}(\gamma_2)$, $\gamma_W^m(x_1) = x_2$. Since P_H is convex, the part of the axis of γ_W between x_1 and x_2 is contained in the interior of P_H. Thus $\gamma_W(x_1)$ is contained in the interior of P_H and hence $\gamma_W(P_H) \cap P_H$ contains an open set. This means that the convex hulls of $\Lambda(H)$ and $\Lambda(WHW^{-1})$ are contained in same totally geodesic hemisphere, hence $W \in H$ by (2) of Proposition 6.2. This contradicts to the assumption that $W \in G - H$.

If $m < -1$, we may consider W^{-1} instead of W in the argument above.

Finally we shall show that γ_W is not parabolic by contradiction. Assume that γ_W is parabolic. Since $W \notin H$ and $H \cap W^m H W^{-m} = \langle X \rangle$, by the argument as above, we $\gamma_W^m(P_H)$ and P_H intersect at the axis of γ_1. Here, by taking conjugation, we may assume γ_W is a translation. Then by the sublemma below, the axis of γ_1 is vertical line, hence this contradicts to the discreteness of F_H. □

Sublemma. *Let D be a closed hyperbolically convex domain in $\mathbb{H}^2$ with non-empty interior. Let $T(\zeta) = \zeta + 1$ be a translation. If $T(D) \cap D$ is a complete geodesic (that is, a geodesic having infinite length in both directions), it is a vertical line.*

Proof. Since $l := T(D) \cap D$ is complete, l divides the plane $\mathbb{H}^2$ into 2-parts, say E_1 and E_2. Since $T(D) \cap D = l$, both D and $T(D)$ have l as a component of their boundaries. Indeed, if some $\zeta \in l$ lie on the interior $\mathrm{Int}(D)$ of D, there exists a small ball D' in D with center ζ. By assumption, we have that $D' \cap T(D) \subset l \cap D'$. Hence, by the convexity of $T(D)$ and $l \subset T(D)$, a geodesic connecting $\zeta \in l$ and a point of $T(D)$ must lie on $l \cap D'$ near ζ. This means that $T(D) = l$, this contradicts to the assumption that D has non-empty interior. Thus, we may assume that $\mathrm{Int}(D) \subset E_1$ and $T(\mathrm{Int}(D)) \subset E_2$.

Assume that l is not vertical line. Let $a, b \in \mathbb{R}$ be end points of l. Assume first that $|a-b| > 1$. Then $T(l) \cap l \neq \emptyset$. In this case, it is easy to see that $T(D) \cap D$ has interior point, and this is contradiction. hence $|a-b| \leq 1$. By definition of E_i, we can see that D lies on the bottom of l. In this case, D and $T(D)$ can not intersect at l, this contradicts to the assumption. □

At the end of this section, we shall give the following proposition which is used in §10 to prove Theorem 13 stated in §7.4.

Proposition 6.7. (Radius of peripheral disks) *Let η be a once punctured torus group with distinguished invariant component Ω. Assume that η satisfies the conditions (1), (2), and (3) in Proposition 6.5 for some ϵ_2 and $\kappa_0 > 0$. Assume further, by conjugation, that there exists a loxodromic element $W \in G$ which forms $W(\zeta) = e^{\lambda}\zeta$ with $\mathrm{Re}\lambda > 0$ and does not belong to any peripheral subgroup of G. Then for any $r > 0$, there exists δ_0 depend only on $\mathrm{Re}\lambda$ and r such that the radius of a peripheral disk which intersects a disk $\{|\zeta| < r\}$ is less than δ_0.*

To prove this proposition, we begin with the following two observations.

Lemma 6.3. *There exists a universal constant $\mathfrak{a}_0 > 0$ satisfying the following: For a real number b in $[-1, 1]$, there are integers a and c with $2 \leq a \leq \mathfrak{a}_0$ such that $|ab - c| < 1/3$.*

Proof. For $c/a \in \mathbb{Q}$, we consider the following neighborhood

$$\mathcal{N}_{c/a} = \{b \in \mathbb{R} \mid |b - (c/a)| < 1/(6a)\}.$$

Then the collection $\{\mathcal{N}_{c/a}\}_{c/a \in \mathbb{Q}}$ forms a covering of $\mathbb{R}$. Indeed, it is clear that all rational number is contained in the union of the covering. By using the best approximations for irrational numbers (see e.g. Hardy-Wright [HW]), for an irrational number b, we can find $c/a \in \mathbb{Q}$, such that $a \geq 6$ and $|b - (c/a)| < 1/a^2 \leq 1/(6a)$. This means that $b \in \mathcal{N}_{c/a}$ for some $c/a \in \mathbb{Q}$.

Since the interval $[-1, 1]$ is compact, we can find a finite subcollection $\{\mathcal{N}_{c_i/a_i}\}_i$ which covers the interval. Put $\mathfrak{a}_0 = 2 \times \max_i\{a_i\}$. Then for $b \in [-1, 1]$, b satisfies the inequality in the assertion for $a = 2a_i$ and $c = 2c_i$ with $b \in \mathcal{N}_{c_i/a_i}$. □

Lemma 6.4. *Let $\Delta = \{|\zeta - \zeta_0\} < \delta\}$ be a disk in $\mathbb{C}$. Assume that $\Delta \cap \{|\zeta| < r\} \neq \emptyset$ for some $r > 0$. Then for a complex number λ with $\mathrm{Re}\lambda > 0$ and $|\mathrm{Im}\lambda| < \pi/3$, there exists δ_0' depend only on r and $\mathrm{Re}\lambda$ such that $\delta \geq \delta_0'$ implies $e^{\lambda} \cdot \Delta \cap \Delta \neq \emptyset$.*

Proof. The constant δ_0' defined directly to be

$$\delta_0' := \frac{r(1 + e^{\mathrm{Re}\,\lambda} + e^{2\mathrm{Re}\,\lambda})^{1/2}}{1 + e^{\mathrm{Re}\,\lambda} - (1 + e^{\mathrm{Re}\,\lambda} + e^{2\mathrm{Re}\,\lambda})^{1/2}}$$

satisfies the assertion. Indeed if $\delta \geq \delta_0'$, we have $(1 + e^{\mathrm{Re}\,\lambda})\delta \geq (1 + e^{\mathrm{Re}\,\lambda} + e^{2\mathrm{Re}\,\lambda})^{1/2}(r + \delta)$. By definition, the center and the radius of $e^{\lambda} \cdot \Delta$ are $e^{\lambda}\zeta_0$ and $e^{\mathrm{Re}\,\lambda}\delta$, respectively. Moreover $|\zeta_0| < r + \delta$ since $\Delta \cap \{|\zeta| < r\} \neq \emptyset$. By assumption, $|\mathrm{Im}\,\lambda| < \pi/3$, and hence

$$\begin{aligned} |e^{\lambda}\zeta_0 - \zeta_0| &\leq ((1 + e^{\mathrm{Re}\,\lambda} + e^{2\mathrm{Re}\,\lambda})^{1/2}|\zeta_0| \\ &< ((1 + e^{\mathrm{Re}\,\lambda} + e^{2\mathrm{Re}\,\lambda})^{1/2}(r + \delta) \\ &\leq (1 + e^{\mathrm{Re}\,\lambda})\delta, \end{aligned}$$

which means that $e^{\lambda}\cdot\Delta\cap\Delta\neq\emptyset$. □

Proof of Proposition 6.7. Let $\mathfrak{a}_0$ be a universal constant given in Lemma 6.3. Then the constant

$$\delta_0=\delta_0(\mathrm{Re}\lambda,r)=\max_{2\leq a\leq\mathfrak{a}_0}\left\{\frac{r(1+e^{a\mathrm{Re}\lambda}+e^{2a\mathrm{Re}\lambda})^{1/2}}{1+e^{a\mathrm{Re}\lambda}-(1+e^{a\mathrm{Re}\lambda}+e^{2a\mathrm{Re}\lambda})^{1/2}}\right\}$$

is desired one.

Indeed, let Δ be a peripheral disk which intersects a disk $\{|\zeta|<r\}$. Since we consider the expansion $W(\zeta)=e^{\lambda}\zeta$, we may suppose that $|\mathrm{Im}\lambda|\leq\pi$. Hence, by Lemma 6.3, there exist integers a and c such that $2\leq a\leq\mathfrak{a}_0$ and that $|a\,(\mathrm{Im}\lambda/\pi)-c|<1/3$. Since $a\geq 2$, $W^a(\Delta)$ can not intersect Δ by Proposition 6.6. Hence by Lemma 6.4, the radius of Δ is less than δ_0. □

7. Derivative is of linear order

In this section, we will state three theorems, Theorems 10, 12, and 13. The first theorem implies our main theorem here, Theorem 2. The next two theorems are used to show the first one.

7.1. **Statement of Theorem "Derivative is of linear order".**

Let us continue to use the notation defined in previous sections. Let $p/q\in\mathbb{Q}_{\mathcal{X}}$. Denote by $\lambda_{p/q}(x)$ the complex length of $X_{p/q,x}$, that is, $\lambda_{p/q}$ satisfies the equation

$$\mathrm{tr}^2X_{p/q,x}=4\cosh^2(\lambda_{p/q}(x)/2).$$

We take $\lambda_{p/q}$ being holomorphic on $\mathcal{X}$ and satisfying that $\mathrm{Re}\lambda_{p/q}(x)>0$ on $\mathcal{X}$ and $\lambda_{p/q}(x)\in\mathbb{R}$ for a point $x\in\mathcal{P}^{\mathcal{X}}_{p/q}$.

In this and the following sections we fix a rational number $\mathfrak{q}\in\mathbb{Q}_{\mathcal{X}}$. Notice that for $p/q\in\mathbb{Q}_{\mathcal{X}}-\{\mathfrak{q}\}$, $\lambda_{p/q}$ can be extended holomorphically on a neighborhood of $x(\mathfrak{q})$ because $X_{p/q,x}$ is either loxodromic or hyperbolic at $x(\mathfrak{q})$. Denote by $\ell(x)$ the restriction of $\lambda_{\mathfrak{q}}$ to $\mathcal{P}^{\mathcal{X}}_{\mathfrak{q}}$. By Proposition 5.1, $\ell(x)$ is a real analytic diffeomorphism of $\mathcal{P}^{\mathcal{X}}_{\mathfrak{q}}$ onto its image.

The following theorem occupies an important position of the proof of Theorem 2 (Rational boundary points are simple zeros):

Theorem 10. (Derivative is of linear order) *Suppose that $\mathcal{X}=\mathcal{M}$ or $\mathcal{B}_G$ for some quasifuchsian group G of type $(1,1)$. Then for $p/q\in\mathbb{Q}_{\mathcal{X}}-\{\mathfrak{q}\}$,*

$$\left|\frac{d}{dl}(\lambda_{p/q}\circ\ell^{-1})(l)\right|\lesssim l,\ \ as\ \ l\to 0,$$

where the implicit constant depends only on the translation length of $X_{p/q,x(\mathfrak{q})}$.

The proof of Theorem 10 is given after stating Theorems 12 and 13 by assuming these two theorems in §7.4. Their two theorems are proved in §8. To state the two theorems, we begin with the following formula.

7.2. **Parker and Series' Bending Formula.** The contents of this section is due to Parker-Series [PS].

Let η be a once punctured torus group with distinguished invariant component Δ. Assume that $\partial\mathcal{C}_\eta$ is bent along the geodesic corresponding to $\mathbf{w}(p/q)$ for some p/q. Put $X=\eta(\mathbf{w}(p/q))$ and $G=\eta(\pi_1)$. Take $V\in G$ so that $G=\langle X,V\rangle$.

Let us recall the definition of the complex shear. Let ξ be the oriented common perpendicular from $\mathrm{Ax}X$ to $\mathrm{Ax}V^{-1}XV$. Then the *complex shear* is defined to be $\pm d$ where d is the complex distance from ξ to $V(\xi)$. The sign is determined by the condition $\mathrm{Im}\,(\pm d)\in[0,\pi]$. We do not give here the concrete definition of the complex length, but we should know the following property of the complex length that its imaginary part coincides with the bending angle.

Denote by $\lambda(W)$ the complex length of W for a loxodromic element W. Then the relation between the complex shear and $\lambda(X)$ and $\lambda(V)$ is given as follows:

Theorem 11. (Bending formula) *Denote by τ the complex shear. Then*

$$\cosh^2(\tau/2)=\cosh^2(\lambda(V)/2)\tanh^2(\lambda(X)/2).$$

7.3. **Quasiconformal Deformation. Relation between Length and Bending angle.** For $x\in\mathcal{P}_{\mathfrak{q}}^{\mathcal{X}}$, define $\theta(x)\in\mathbb{R}$ so that $\pi-\theta(x)$ is the bending angle, and put $\ell(x)=\lambda_{\mathfrak{q}}(x)$ as in previous section.

Lemma 7.1. $\theta(x)\lesssim\ell(x)$ *as* $x\to x(\mathfrak{q})$.

Proof. Fix $p'/q'\in\mathbb{Q}_{\mathcal{X}}$ so that $\pi_1=\langle\mathbf{w}(\mathfrak{q}),\mathbf{w}(p'/q')\rangle$. Let $\tau(x)$ be the complex shear defined for $X=X_{\mathfrak{q},x}$ and $V=X_{p'/q',x}$. Then the Bending formula gives that

$$\begin{aligned}\sin^2(\theta(x)/2)&=\cos^2(\mathrm{Im}\,\tau(x)/2)\leq|\cosh^2(\tau(x)/2)|\\&=|\cosh^2(\lambda_{p'/q'}(x)/2)|\cdot|\tanh^2(\ell(x)/2)|\\&=4^{-1}|\mathrm{tr}^2X_{p'/q',x}|\tanh^2(\ell(x)/2).\end{aligned}$$

Since the trace function $\mathrm{tr}^2X_{p'/q',x}$ is holomorphic at $x=x(\mathfrak{q})$ and $\sin\theta\asymp\theta$ and $\tanh l\asymp l$ as $\theta,l\to0$ hold, the assertion is established. □

Remark 5. *By the proof of Lemma 7.1, we can take the implicit constant in the inequality to be dependent only on the absolute value of* $\mathrm{tr}^2X_{p'/q',x}$ *at* $x(\mathfrak{q})$.

Quasiconformal deformations. We define a quasiconformal deformation of the group on a pleating ray. This is the central tool for proving Theorem 10 (Derivative is of linear order).

Set ℓ and θ be as in the previous sections. Let $x\in\mathcal{P}_{\mathfrak{q}}^{\mathcal{X}}$. Let H_1 and H_2 be the peripheral subgroups containing $X_{\mathfrak{q},x}$. Since H_i acts on $\Delta(H_i)$, we can consider the axis σ_i of $X_{\mathfrak{q},x}$ in $\Delta(H_i)$ as in 2-dimensional hyperbolic geometry. Each σ_i is a circular arc orthogonal to $\partial\Delta(H_i)$ at the fixed points of $X_{\mathfrak{q},x}$, and σ_1 and σ_2 bound the sector F contained in $\Delta(H_1)\cup\Delta(H_2)$. this F is uniquely determined, see Figure 6.

Set $F_{[B]}=B^{-1}(F)$ for $[B]\in\langle X_{\mathfrak{q},x}\rangle\backslash G_x$. (Notice that F is invariant under the action of $X_{\mathfrak{q},x}$.) Then, for $[B_1],[B_2]\in\langle X_{\mathfrak{q},x}\rangle\backslash G_x$, on has $F_{[B_1]}\cap F_{[B_2]}=\emptyset$ if $[B_1]\neq[B_2]$.

Fix a Möbius transformation A sending the fixed points of $X_{\mathfrak{q},x}$ to $\{0,\infty\}$. Then $A(F)$ becomes a sector with center at origin whose central angle is $\pi-\theta(x)$. Set

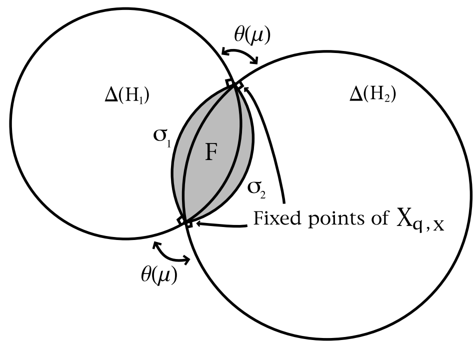

FIGURE 6. The set F

$\hat{\tau}(\zeta) = A(\zeta)\overline{A'(\zeta)}/\overline{A(\zeta)}A'(\zeta)$ on F and $\hat{\tau}(\zeta) = 0$ otherwise. Define the Beltrami differential τ_x, $x \in \mathcal{P}_{\mathfrak{q}}^{\mathcal{X}}$ by

$$\tau_x(\zeta) = \frac{1}{\ell(x)} \sum_{[B]\in\langle X_{\mathfrak{q},x}\rangle\backslash G_x} \hat{\tau}(B(\zeta))\overline{\frac{\partial B}{\partial \zeta}(\zeta)}\left(\frac{\partial B}{\partial \zeta}(\zeta)\right)^{-1}.$$

The differential τ_x is compatible with G_x, that is, $\tau_x \circ g \cdot \bar{g'} = \tau_c \cdot g'$ for all $g \in G_x$. Furthermore, $\|\tau_x\|_\infty = 1/\ell(x)$ and the support of τ_x is $\cup_{[B]\in\langle X_{\mathfrak{q},x}\rangle\backslash G_x} F_{[B]}$.

For $t \in \mathbb{C}$ with $|t| < \ell(x)$, let w^t be a solution on $\hat{\mathbb{C}}$ of the Beltrami equation $\overline{\partial} w^t = t\tau_x \partial w^t$. Then we can define a holomorphic mapping Φ_x from a disk $\{|t| < \ell(x)\}$ to $\mathcal{X}$ such that $\Phi_x(0) = x$ and $G_{\Phi_x(t)}$ is conjugate to $w^t G_x (w^t)^{-1}$.

7.4. **Proof of Theorem "Derivative is of linear order".**

Theorem 12. *Define τ_x as in previous section. Then, there exists $l_1 > 0$ so that*

$$|d\,\lambda_{\mathfrak{q}}[\tau_x]| > 1/2 \textit{ for } \ 0 < l(x) < l_1.$$

Theorem 13. (Remaining curves are not so deformed) *Let τ_x be the Beltrami differential defined as above. Then for $p/q \in \mathbb{Q}_{\mathcal{X}} - \{\mathfrak{q}\}$,*

$$|d\,\lambda_{p/q}[\tau_x]| \lesssim \ell(x) \textit{ as } \ x \to x(\mathfrak{q}).$$

Furthermore, the implicit constant is depend only on the translation length of $X_{p/q,x(\mathfrak{q})}$.

Theorems 12 and 13 are proved in §8 and 10 respectively. For the rest of this subsection, we shall show that these two imply Theorem 10.

Indeed, take $l > 0$ sufficiently small and let $x \in \mathcal{P}_{\mathfrak{q}}^{\mathcal{X}}$ with $\ell(x) = l$. Let Φ_x be the holomorphic mapping of the disk $\{|t| < \ell(x)\}$ to $\mathcal{X}$ defined as in §7.3. ¿From Theorem 12, we may suppose that Φ_x and $\lambda_{\mathfrak{q}} \circ \Phi_x$ are both conformal at $t = 0$. Since the restriction of $\lambda_{\mathfrak{q}}$ on $\mathcal{P}_{\mathfrak{q}}^{\mathcal{X}}$ agrees with $\ell(x)$ and $\Phi_x(0) = x$, we obtain

$$\begin{aligned}\left|\frac{d}{dl}(\lambda_{p/q} \circ \ell^{-1})(l)\right| &= \left|\frac{d}{dt}(\lambda_{p/q} \circ \Phi_x)\Big|_{t=0}\right| \cdot \left|\frac{d}{dl}(\Phi_x^{-1} \circ \ell^{-1})\Big|_l\right| \\ &= |d\,\lambda_{p/q}[\tau_x]| \cdot |d\,\lambda_{\mathfrak{q}}[\tau_x]|^{-1} \lesssim l.\end{aligned}$$

The implicit constant is dependent only on the translation length of $X_{p/q,x(\mathfrak{q})}$ by Theorem 12 and the later part of Theorem 13. $\square$

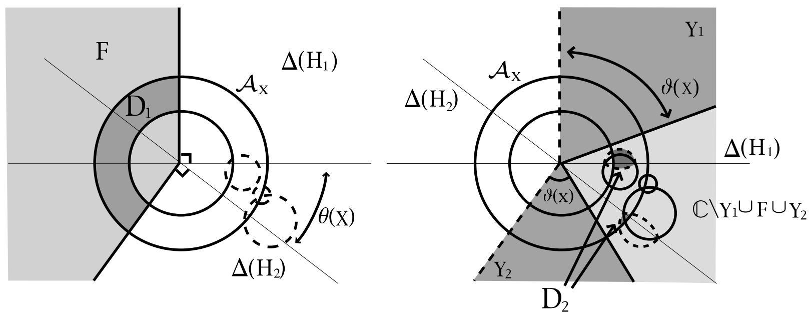

FIGURE 7. D_1 and D_2

8. Differential on Pinched curve under qc-deformation

8.1. **Gardiner's formula.** First we recall the differential formula for the complex length of loxodromic elements under quasiconformal deformations.

Let $g(\zeta) = e^{\lambda}\zeta$ with $\mathrm{Re}\lambda > 0$. Let ν be a Beltrami differential on $\mathbb{C}$ compatible with g. We denote by f^t a solution on $\hat{\mathbb{C}}$ of the equation $\overline{\partial} f^t = t\nu\partial f^t$ for $|t| < 1/\|\nu\|_\infty$. Define the holomorphic function $\lambda(t)$ on $\{|t| < 1/\|\nu\|\}$ by $\mathrm{tr}^2 f^t g (f^t)^{-1} = 4\cosh^2(\lambda(t)/2)$ and $\lambda(0) = \lambda$.

Proposition (F.Gardiner [Ga]**)** *Under the notation above,*

$$\left.\frac{d\lambda}{dt}\right|_{t=0} = \frac{1}{\pi}\int_{\{1<|\zeta|<e^{\mathrm{Re}\lambda}\}} \nu(\zeta)\frac{dudv}{\zeta^2}, \quad \zeta = u + iv.$$

Gardiner proved this formula for fuchsian groups. However, his proof can be applied to the cases of general Kleinian groups.

8.2. **Proof of Theorem 12.** By Lemma 7.1 there exists $l_0 > 0$ so that for $x \in \mathcal{P}^{\mathcal{X}}_{\mathfrak{q}}$ with $\ell(x) < l_0$, the inequality $\theta(x) < \pi/2$ holds. Take such $x \in \mathcal{P}^{\mathcal{X}}_{\mathfrak{q}}$. We conjugate G_x by a Möbius transformation sending the fixed points of $X_{\mathfrak{q},x}$ to $\{0,\infty\}$. To avoid confusion, we use the same symbols to represent G_x, $X_{\mathfrak{q},x}$, etc. Here, we may suppose that $X_{\mathfrak{q},x}(\zeta) = e^{\ell(x)}\zeta$.

Let H_1 and H_2 be the peripheral subgroups of G_x containing $X_{\mathfrak{q},x}$. We may assume that $\Delta(H_1) = \mathbb{H}$ and that the intersection $\Delta(H_1)\cap\Delta(H_2)$ is contained in the left half-plane. Since $\mathrm{tr}^2 X_{\mathfrak{q},\Phi_x(t)} = \mathrm{tr}^2\, w^t X_{\mathfrak{q},x}(w^t)^{-1}$ and $\lambda_{\mathfrak{q}}\circ\Phi_x(0) = \ell(x)$, it follows from Gardiner's formula that

$$d\,\lambda_{\mathfrak{q}}[\tau_x] = \left.\frac{d}{dt}(\lambda_{\mathfrak{q}}\circ\Phi_x)\right|_{t=0} = \frac{1}{\pi}\int_{\mathcal{A}_x}\tau_x(\zeta)\frac{dudv}{\zeta^2}, \tag{6}$$

where $\zeta = u + iv$ and $\mathcal{A}_x = \{\zeta\in\mathbb{C} \mid 1 < |\zeta| < e^{\ell(x)}\}$. By definition, $F = \{\pi/2 < \arg\zeta < 3\pi/2 - \theta(x)\}$ and $\tau_x = \zeta/\,\ell(x)\bar{\zeta}$ on F. Let $D_1 = F\cap\mathcal{A}_x$ and $D_2 = (\mathrm{Supp}(\tau_x)\cap\mathcal{A}_x)\setminus D_1$, see Figure 7. We denote by J_k, $k = 1,2$ the integral over D_k. Then the integral in (6) is equal to $J_1 + J_2$. A simple calculation shows that $J_1 = (\pi - \theta(x))/\pi$.

Let us consider the integral J_2. We take $\vartheta(x) > 0$ to be defined by $\tan\vartheta(x) = 1/\sinh(\ell(x)/2)$. Keen-Halpern's Collar lemma asserts that the sector $Y_1 = \{\pi/2 - \vartheta(x) < \arg\zeta \le \pi/2\}$ is disjoint from lifts of boundary geodesics of $\Delta(H_1)/H_1$ with exception of the axis of $X_{\mathfrak{q},x}$ in $\Delta(H_1)$, the positive imaginary axis. This implies that $Y_1 \cap D_2 = \emptyset$. One shows in the same way that the sector $Y_2 = \{3\pi/2 - \theta(x) \leq \arg\zeta < 3\pi/2 - \theta(x) + \vartheta(x)\}$ is disjoint from D_2. Hence D_2 is contained in the sector $\mathbb{C} \setminus (Y_1 \cup F \cup Y_2)$ and the annulus $\mathcal{A}_x$. This implies $|J_2| < (\pi - 2\vartheta(x) + \theta(x))/\pi$ (cf. Figure 7). Thus we have

$$|d\,\lambda_{\mathfrak{q}}[\tau_x]| \geq |J_1| - |J_2| > 2(\vartheta(x) - \theta(x))/\pi \to 1 \text{ as } \ x \to x(\mathfrak{q}).$$

by the definition of $\vartheta(x)$ and Lemma 7.1.

9. Differential on Remaining curves under qc-deformation

Let $p/q \in \mathbb{Q}_{\mathcal{X}} - \{\mathfrak{q}\}$. In this section, we conjugate G_x, $x \in \mathcal{P}_{\mathfrak{q}}^{\mathcal{X}}$ so that the fixed points of $X_{p/q,x}$ map to $\{0, \infty\}$. As in the proof of Theorem 12, we continue to use by the same words to represent G_x and $X_{p/q,x}$, etc. Here, we suppose $X_{p/q,x}(\zeta) = e^{\lambda_{p/q}(x)}\zeta$.

9.1. **Proof of Theorem 13 (Remaining curves are not so deformed).** Take $l_2 > 0$ so that $\theta(x) < \pi/4$ and $\mathrm{Re}\,\lambda_{p/q}(x) < 2\mathrm{Re}\,\lambda_{p/q}(x(\mathfrak{q}))$ if $\ell(x) < l_2$. We may suppose in addition that l_2 is less than the $\kappa_0 > 0$ defined in Proposition 6.5 for $\epsilon_2 = 3\pi/4$. We now fix such x.

Let $[B] \in \langle X_{\mathfrak{q},x}\rangle\backslash G_x$. We suppose that $B^{-1}X_{\mathfrak{q},x}B(\zeta) = (a_B\zeta + b_B)/(c_B\zeta + d_B)$ with $a_Bd_B - b_Bc_B = 1$. Then it follows from $\theta(x) < \pi/2$ that $F_{[B]}$ is contained in a disk with center $(a_B - d_B)/2c_B$ and of radius $\sinh(\ell(x)/2)/|c_B|$. This implies

$$\mathrm{Area}(F_{[B]}) \leq \pi\sinh^2(\ell(x)/2)/|c_B|^2 \lesssim \ell(x)^2|c_B|^{-2},$$

where Area$(-)$ means the 2-dimensional Lebesgue measure.

Recall that the support of τ_x is the disjoint union of the sets $\{F_{[B]}\}$, $[B] \in \langle X_{\mathfrak{q},x}\rangle\backslash G_x$ (cf. §7.3). Applying Gardiner's formula again, we have

$$\begin{aligned}
|d\,\lambda_{p/q}[\tau_x]| &= \left|\frac{d}{dt}\lambda_{p/q}\circ\Phi_x\Big|_{t=0}\right| \\
&= \left|\frac{1}{\pi}\int_{\{1<|\zeta|<e^{\mathrm{Re}\lambda_{p/q}(x)}\}} \tau_x(\zeta)\frac{dudv}{\zeta^2}\right| \\
&\leq \frac{1}{\pi}\int_{\{|\zeta|\leq 2\mathrm{Re}\,\lambda_{p/q}(x(\mathfrak{q}))\}} |\tau_x(\zeta)|dudv \\
&\leq \frac{1}{\pi\ell(x)}{\sum}'_{[B]}\mathrm{Area}(F_{[B]}) \lesssim \ell(x){\sum}'_{[B]}|c_B|^{-2},
\end{aligned}$$

where ${\sum}'_{[B]}$ means the summation over all $[B] \in \langle X_{\mathfrak{q},x}\rangle\backslash G_x$ with $F_{[B]} \cap \{|\zeta| \leq 2\mathrm{Re}\,\lambda_{p/q}(x(\mathfrak{q}))\} \neq \emptyset$. Thus, Theorem 13 follows from Proposition 9.1 which is proved below.

Proposition 9.1. *With notation as above, we have*

$${\sum}'_{[B]}|c_B|^{-2} = O(1)$$

as $\ell(x) \to 0$, where the right-hand side is dependent only on the translation length of $X_{p/q,x(\mathfrak{q})}$.

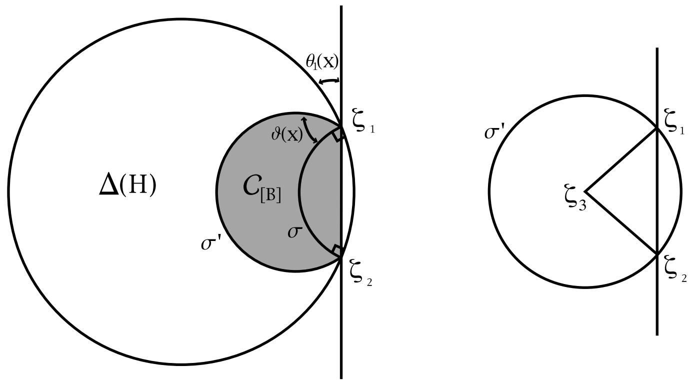

FIGURE 8. The domain $\mathcal{C}_{[B]}$

Proof. In this proof, we use the symbols defined in the above proof of Theorem 13 frequently.

Let $[B]$ be an element in $\langle X_{\mathfrak{q},x}\rangle\backslash G_x$ with $F_{[B]} \cap \{|\zeta| \leq 2\mathrm{Re}\,\lambda_{p/q}(x(\mathfrak{q}))\} \neq \emptyset$. Fix a peripheral subgroup H of G_x containing $B^{-1}X_{\mathfrak{q},x}B$. Let σ be the axis of $B^{-1}X_{\mathfrak{q},x}B$ in $\Delta(H)$ considered as the 2-dimensional hyperbolic disk. Let $\{\zeta_k\}_{k=1,2}$ be the fixed point of $B^{-1}X_{\mathfrak{q},x}B$. We denote by N_H the Nielsen region of H in $\Delta(H)$. Then

$$\sigma' := \{z \in N_H \mid \sinh(d_{\Delta(H)}(z,\sigma))\sinh(\ell(x)/2) = 1\}$$

is a circular arc in $\Delta(H)$ whose end points are $\{\zeta_k\}_{k=1,2}$, where $d_{\Delta(H)}$ means hyperbolic distance on $\Delta(H)$. Therefore, as in Figure 8, the curve consisting of σ' and the line segment joining ζ_1 and ζ_2 bounds a domain $\mathcal{C}_{[B]}$ in $\Delta(H)$. Keen-Halpern's collar lemma asserts that the sets $\mathcal{C}_{[B]}$ are pairwise disjoint for $[B] \in \langle X_{\mathfrak{q},x}\rangle\backslash G_x$.

Let us define a continuous function $\vartheta(x)$ on $\mathcal{P}^{\mathcal{X}}_{\mathfrak{q}} \cup \{x(\mathfrak{q})\}$ by the equation

$$\tan\vartheta(x) = 1/\sinh(\ell(x)/2), \quad \vartheta(x(\mathfrak{q})) = \pi/2.$$

Denote by $\theta_1(x)$ $(< \pi/2)$ the outer angle between the line through two points ζ_1 and ζ_2 and the circle $\partial\Delta(H)$ at ζ_1 (cf. the left picture of Figure 8). By definition, $\theta_1(x) < \theta(x)$ and hence there exists $l_3 > 0$ such that $\theta_1(x) < \vartheta(x)$ whenever $\ell(x) < l_3$. Notice that $|\zeta_1 - \zeta_2| = 2\sinh(\ell(x)/2)/|c_B|$ and the angle between σ' and σ at ζ_1 is equal to $\vartheta(x)$. Hence the angle between σ' and the line segment between ζ_1 and ζ_2 at ζ_1 is equal to $\pi/2 + \vartheta(x) - \theta_1(x)$. Therefore, we obtain that

$$\begin{aligned}\mathrm{Area}(\mathcal{C}_{[B]}) &= \frac{\sinh^2(\ell(x)/2)}{2|c_B|^2}\left\{\frac{\pi + 2(\vartheta(x)-\theta_1(x))}{\cos^2(\vartheta(x)-\theta_1(x))} + 2\tan(\vartheta(x)-\theta_1(x))\right\} \\ &\gtrsim |c_B|^{-2}\left\{\frac{\ell(x)}{\cos(\vartheta(x)-\theta_1(x))}\right\}^2, \end{aligned} \tag{7}$$

whenever $\ell(x) < l_3$, since $\vartheta(x) - \theta_1(x) > 0$. Indeed, let ζ_3 be the center of the circle containing σ'. Then the set $\mathcal{C}_{[B]}$ is divided into two part (cf. the right picture of Figure 8): the sector enclosed by σ' and two segments between ζ_3 and ζ_i for $i = 1, 2$, and the triangle with vertices ζ_1, ζ_2, and ζ_3. The first part in the bracket of the inequality above corresponds to the area of the sector, the second is the area of the triangle.

A simple calculation shows

$$\begin{aligned}\ell(x)^{-1}\cos(\vartheta(x)-\theta_1(x)) &= \ell(x)^{-1}(\cos\vartheta(x)\cos\theta_1(x)+\sin\vartheta(x)\sin\theta_1(x)) \\ &< \ell(x)^{-1}\cos\vartheta(x)+\ell(x)^{-1}\sin\theta(x) \\ &\lesssim \ell(x)^{-1}\tanh(\ell(x)/2)+\ell(x)^{-1}\sin\ell(x)=O(1).\end{aligned}$$

by Lemma 7.1. Together with (7) we can find $l_4>0$, which are independent of the choice of the classes $[B]\in\langle X_{\mathfrak{q},x}\rangle\backslash G_x$, such that if $\ell(x)<l_4$,

$$|c_B|^{-2}\lesssim \mathrm{Area}(\mathcal{C}_{[B]}).$$

By applying Proposition 6.7 (Radius of peripheral disks), the assumption $F_{[B]}\cap\{|\zeta|\leq 2\mathrm{Re}\,\lambda_{p/q}(x(\mathfrak{q}))\}\neq\emptyset$ implies that $\Delta(H)$ is contained in the disk of radius $2\mathrm{Re}\,\lambda_{p/q}(x(\mathfrak{q}))+\delta_0$ with center 0, where δ_0 depends only on $\mathrm{Re}\,\lambda_{p/q}(x(\mathfrak{q}))$. Therefore we conclude that

$$\begin{aligned}{\sum}'_{[B]}|c_B|^{-2} &\lesssim {\sum}'_{[B]}\mathrm{Area}(\mathcal{C}_{[B]})\leq \mathrm{Area}(\{|\zeta|\leq 2\mathrm{Re}\,\lambda_{p/q}(x(\mathfrak{q}))+\delta_0\}) \\ &= \pi(2\mathrm{Re}\,\lambda_{p/q}(x(\mathfrak{q}))+\delta_0)^2\end{aligned}$$

as $\ell(x)\to 0$, and the last constant depends only on $\mathrm{Re}\,\lambda_{p/q}(x(\mathfrak{q}))$ that is, the translation length of $X_{p/q,x(\mathfrak{q})}$. □

10. Non-vanishing theorem for Derivative of trace functions

10.1. Non-vanishing theorem. We shall prove Theorem 2:

Theorem 2 (Rational boundary points are simple zeros) *For all $\mathfrak{q}\in\mathbb{Q}_{\mathcal{X}}$, it holds that*

$$\left.\frac{d}{dx}\mathrm{tr}^2 X_{\mathfrak{q},x}\right|_{x=x(\mathfrak{q})}\neq 0.$$

In particular, for all $\mathfrak{q}\in\mathbb{Q}_{\mathcal{X}}$, $x(\mathfrak{q})$ is a simple zero of the holomorphic function $\mathrm{tr}^2X_{\mathfrak{q},x}$.

Remark 6. *In the case of $\mathcal{X}=\mathcal{M}$, this theorem gives an affirmative answer of a conjecture of D.J.Wright* [W].

10.2. Key Lemma. We show the following:

Lemma 10.1. *Suppose that $\mathcal{X}=\mathcal{M}$ or $\mathcal{B}_G$ for some quasifuchsian group G of type $(1,1)$. Fix $\mathfrak{q}\in\mathbb{Q}_{\mathcal{X}}$. For $p/q\in\mathbb{Q}_{\mathcal{X}}$,*

we have

$$\left|\left.\frac{d}{dx}\mathrm{tr}^2X_{p/q,x}\right|_{x=x(\mathfrak{q})}\right|\lesssim\left|\left.\frac{d}{dx}\mathrm{tr}^2X_{\mathfrak{q},x}\right|_{x=x(\mathfrak{q})}\right|,$$

where the implicit constant of the inequality is dependent only on the translation length of $X_{p/q,x(\mathfrak{q})}$. Hence, if the derivative of $\mathrm{tr}^2X_{p/q,x}$ *at $x(\mathfrak{q})$ does not vanish for some p/q, then that of* $\mathrm{tr}^2X_{\mathfrak{q},x}$ *at $x(\mathfrak{q})$ is also not zero.*

Proof. Let $p/q\in\mathbb{Q}_{\mathcal{X}}$ as in the assumption. If $X_{p/q,x(\mathfrak{q})}$ is parabolic, $\mathcal{X}$ is the Maskit slice and $X_{p/q,x}$ is parabolic for all $x\in\mathcal{X}$ unless $p/q=\mathfrak{q}$. Therefore the derivative of $\mathrm{tr}^2X_{p/q,x}$ at $x=x(\mathfrak{q})$ vanish, and hence the inequality above holds.

Assume that $X_{p/q,x(\mathfrak{q})}$ is loxodromic. We may assume that $p/q \neq \mathfrak{q}$. By virtue of Theorem 10 (Derivative is of linear order), we have

$$\left|\frac{d}{dl}(\lambda_{p/q}\circ \ell^{-1})(l)\right| \lesssim l \text{ as } \ l \to 0. \tag{8}$$

Let $x \in \mathcal{P}^{\mathcal{X}}_{\mathfrak{q}}$ with appropriately small $\ell(x)$. Integrating (8) from $l=0$ to $\ell(x)$, we obtain

$$|\lambda_{p/q}(x) - \lambda_{p/q}(x(\mathfrak{q}))| \lesssim \ell(x)^2,$$

Since $X_{\mathfrak{q},x(\mathfrak{q})}$ is parabolic, the trace function of $X_{\mathfrak{q},x}$ satisfies

$$|\mathrm{tr}^2 X_{\mathfrak{q},x} - 4| \asymp \ell(x)^2$$

for $x \in \mathcal{P}^{\mathcal{X}}_{\mathfrak{q}}$ near $x(\mathfrak{q})$. Hence we have

$$|\lambda_{p/q}(x) - \lambda_{p/q}(x(\mathfrak{q}))| \lesssim |\mathrm{tr}^2 X_{\mathfrak{q},x} - 4| \tag{9}$$

for $x \in \mathcal{P}^{\mathcal{X}}_{\mathfrak{q}}$ with $\ell(x) \to 0$. Since $\mathrm{tr}^2 X_{p/q,x} = 4\cosh^2(\lambda_{p/q}(x)/2)$, dividing the inequality (9) by $|x - x(\mathfrak{q})|$ and letting $x \to x(\mathfrak{q})$ imply the inequality in the assertion. By Theorem 10, we can see that the implicit constant of the inequality is dependent only on the translation length of $X_{p/q,x(\mathfrak{q})}$. □

10.3. **Proof of Theorem 2.** We shall show Theorem 2 in all slices.

(a) the case $\mathcal{X} = \mathcal{M}$.

In this case, $\mathcal{X} = \mathcal{M}$, $\mathcal{X}_0 = \mathbb{C}$, $x(\mathfrak{q}) = x^m(\mathfrak{q})$, and $\eta_x = \rho^m_x$. We shall show the following:

Lemma 10.2. *There exists a universal constant $C_M > 0$ such that*

$$\left|\frac{d}{d\mu}\mathrm{tr}^2\rho^m_\mu(\mathbf{w}(\mathfrak{q}))\Big|_{\mu = x^m(\mathfrak{q})}\right| \geq C_M.$$

for all $\mathfrak{q} \in \mathbb{Q}$.

Proof. A simple calculation shows that $\mathrm{tr}^2\rho^m_\mu(\mathbf{w}(n/1)) = -(\mu - 2n)^2$, and hence

$$\frac{d}{d\mu}\mathrm{tr}^2\rho^m_\mu(\mathbf{w}(n/1)) = -2(\mu - 2n).$$

Let us consider annular domains

$$\mathcal{A}_n := \{\mathrm{Re}\mu - 2n| < 3,\ |\mathrm{Im}\mu - 2| < 3\} - \{\mathrm{Re}\mu - 2n| < 1/3,\ |\mathrm{Im}\mu - 2| < 1/3\}$$

for all $n \in \mathbb{Z}$. By definition, for $\mu \in \mathcal{A}_n$, the translation length of $\mathrm{tr}^2\rho^m_\mu(\mathbf{w}(n/1))$ lies in between $2\sinh^{-1}(1/6)$ and $2\sinh^{-1}(3/2)$, and the absolute value of the derivative of this trace function is greater than 2/3. Furthermore, a collection $\{\mathcal{A}_n\}_n$ covers the strip $\{-1 < \mathrm{Im}\mu < 5\}$ and hence this collection covers the boundary $\partial\mathcal{M}$. Indeed, according to Proposition 2.6 of Keen-Series [KS93] (see also Kra [Kr90]), the boundary $\partial\mathcal{M}$ is contained in the strip $\{0 < \mathrm{Im}\mu < 2\}$. Hence, by Lemma 10.1, the absolute value of the derivative of $\mathrm{tr}^2\rho^m_\mu(\mathbf{w}(\mathfrak{q}))$ at $x = x^m(\mathfrak{q})$ is greater than a universal constant, say $C_M > 0$. □

(b) the case $\mathcal{X} = \mathcal{E}$.

In this case, $\mathcal{X} = \mathcal{E}$, $\mathcal{X}_0 = \mathbb{C}^*$, $x(\mathfrak{q}) = x^e(\mathfrak{q})$, and $\eta_x = \rho^e_x$. First, we show

Proposition 10.1. *Let $p/q, p'/q' \in \hat{\mathbb{Q}}$ with $p/q \neq p'/q'$. Define a holomorphic function on representation space $\mathcal{R}$ by*

$$\Psi([\eta]) = (\mathrm{tr}^2\eta(\mathbf{w}(p/q)), \mathrm{tr}^2\eta(\mathbf{w}(p'/q'))).$$

Denote by $[\rho_0]$ the representation with the condition that its image $\rho_0(\pi_1)$ is a maximal parabolic group with parabolic transformations $\rho_0(\mathbf{w}(p/q))$, $\rho_0(\mathbf{w}(p'/q'))$, and $\rho_0([\mathbf{a}, \mathbf{b}])$. Then Ψ has maximal rank at $[\rho_0]$.

Proof. Since ρ_0 is irreducible, $[\rho_0]$ is a regular point of $\mathcal{R}$. We may assume that $\nu_+([\rho_0]) = p/q$ and $\nu_-([\rho_0]) = p'/q'$.

Take ς_1 be an isomorphism of π_1 so that $\varsigma_1(\mathbf{a}) = \mathbf{w}(p'/q')$. Define $x_1 \in \mathbb{Q}$ by $\mathbf{w}(x_1)$ is conjugate to $\varsigma_1^{-1}(\mathbf{w}(p/q))$. By the injectivity of end invariants, $[\rho^m_{x^m(x_1)} \circ \varsigma_1^{-1}] = [\rho_0]$ (see also [KMsS]).

Let ς_2 be an isomorphism of π_1 so that $\varsigma_2(\mathbf{a}) = \mathbf{w}(p/q)$, and let $x_2 \in \mathbb{Q}$ such that $\mathbf{w}(x_2)$ is conjugate to $\varsigma_2^{-1}(\mathbf{w}(p'/q'))$. Put $\mathcal{I}(z) = \bar{z}$. Then it holds that

$$[\mathcal{I}\overline{\rho^m_{x^m(x_2)}} \circ \varsigma_2^{-1}\mathcal{I}] := [\pi_1 \ni \mathbf{a}' \mapsto \mathcal{I} \circ \overline{\rho^m_{x^m(x_2)}}(\varsigma_2^{-1}(\mathbf{a}')) \circ \mathcal{I}] = [\rho_0].$$

Let us consider the holomorphic mappings ψ_1 and ψ_2 of a sufficiently small disk $\{|t| < \epsilon\}$ to $\mathcal{R}$ defined by

$$\psi_1(t) = [\rho^m_{x^m(x_1)+t} \circ \varsigma_1^{-1}],$$
$$\psi_2(t) = [\mathcal{I}\overline{\rho^m_{x^m(x_2)+\bar{t}}} \circ \varsigma_2^{-1}\mathcal{I}].$$

By definition, their satisfy that $\psi_k(0) = [\rho_0]$ for $k = 1, 2$ and

$$\Psi \circ \psi_1(t) = (4,\ \mathrm{tr}^2\rho^m_{(x^m(x_1)+t)}(\mathbf{w}(x_1))),$$
$$\Psi \circ \psi_2(t) = (\overline{\mathrm{tr}^2\rho^m_{x^m(x_2)+\bar{t}}(\mathbf{w}(x_2))},\ 4) = (\mathrm{tr}^2\rho^m_{x^m(x_2)+t}(\mathbf{w}(x_2)),\ 4),$$

Since $\mathrm{tr}^2\rho^m_\mu(\mathbf{w}(x_2))$ is a polynomial of real coefficients on μ-plane. Together with Theorem 2 for $\mathcal{X} = \mathcal{M}$, we complete the proof of Proposition 10.1. □

Proof of Theorem 2 Define the mapping ψ from $\mathbb{C}$ to $\mathcal{R}$ by $\psi(d) = [\rho^e_d]$. Then

$$\frac{d}{dc}\mathrm{tr}^2\psi(c)(\mathbf{ab}^{-1})\Big|_{c=x^e(\mathfrak{q})} = \frac{d}{dc}\mathrm{tr}^2\rho^e_c(\mathbf{ab}^{-1})\Big|_{c=x^e(\mathfrak{q})} = 8x^e(\mathfrak{q})(4x^e(\mathfrak{q})^2 + 2).$$

Since $\rho^e_{x^e(\mathfrak{q})}$ is faithful and $\mathrm{tr}^2\psi(x^e(\mathfrak{q}))(\mathbf{ab}^{-1}) = (4x^e(\mathfrak{q})^2 + 2)^2$, $4x^e(\mathfrak{q})^2 + 2 \neq 0$. At $c = 0$, the representation ρ^e_c is divergent. Hence it is impossible that $x^e(\mathfrak{q}) = 0$. Therefore, the mapping ψ has the maximal rank at $d = x^e(\mathfrak{q})$. Define a function by

$$\Psi([\eta]) = (\mathrm{tr}^2\eta(\mathbf{w}(\mathfrak{q})), \mathrm{tr}^2\eta(\mathbf{w}(\mathfrak{q}^{-1}))).$$

The condition $\mathfrak{q} \neq \pm 1/1$ implies $\mathfrak{q} \neq \mathfrak{q}^{-1}$. Hence from Proposition 10.1 Ψ has the maximal rank at $[\rho^e_{x^e(\mathfrak{q})}]$. Since $\mathrm{tr}^2\rho^e_d(\mathbf{w}(\mathfrak{q})) = \mathrm{tr}^2\rho^e_d(\mathbf{w}(\mathfrak{q}^{-1}))$ for $d \in \mathbb{C}^*$, and

$$\Psi \circ \psi(d) = (\mathrm{tr}^2\rho^e_d(\mathbf{w}(\mathfrak{q})), \mathrm{tr}^2\rho^e_d(\mathbf{w}(\mathfrak{q}^{-1}))),$$

we conclude the desired statement. □

(c) the case $\mathcal{X} = \mathcal{B}_G$

In this case, there exists a quasifuchsian group G of type $(1, 1)$ such that $\mathcal{X} = \mathcal{B}_G$, $\mathcal{X}_0 = Q_{-4}(G)$, $x(\mathfrak{q}) = x^b(\mathfrak{q})$, and $\eta_x = \rho^b_x$. To prove Theorem 2 in this case, we will show

Proposition 10.2. *There exists a constant $C_G > 0$ dependent only on the group G such that*
$$\left| \frac{d}{d\varphi} \mathrm{tr}^2 \rho^b_\varphi(\mathbf{w}(\mathfrak{q})) \Big|_{\varphi = x^b(\mathfrak{q})} \right| \geq C_G.$$
for all $\mathfrak{q} \in \hat{\mathbb{Q}}$.

We first state the following lemma.

Proposition 11.2. *For any $\varphi_0 \in \partial\mathcal{B}_G$, there exists $\mathfrak{p} \in \hat{\mathbb{Q}}$ such that $\rho^b_{\varphi_0}(\mathbf{w}(\mathfrak{p}))$ is loxodromic and that the derivative of the trace function $\mathrm{tr}^2\rho^b_\varphi(\mathbf{w}(\mathfrak{p}))$ at $\varphi = \varphi_0$ does not vanish.*

The proof of Proposition 11.2 needs some notation and definitions. So it may induce some confusion to readers. Hence we postpone proving this theorem until the next section. Let us prove Proposition 10.2 with assuming Proposition 11.2.

Proof of Proposition 10.2. For any $\varphi \in \partial\mathcal{B}_G$, we can find $\mathfrak{p} = \mathfrak{p}_\varphi \in \hat{\mathbb{Q}}$ with the conditions in Proposition 11.2. Hence there exists a neighborhood $\mathcal{U}'_\varphi$ of φ such that for $\varphi' \in \mathcal{U}'_\varphi$, $\rho^b_{\varphi'}(\mathbf{w}(\mathfrak{p}))$ is loxodromic and the derivative of $\mathrm{tr}^2\rho^b_\varphi(\mathbf{w}(\mathfrak{p}))$ at φ' does not vanish. Let $\mathcal{U}_\varphi$ be a neighborhood of φ whose closure is compact in $\mathcal{U}'_\varphi$. Then the collection $\{\mathcal{U}_\varphi\}$, $\varphi \in \partial\mathcal{B}_G$ forms a covering of the compact set $\partial\mathcal{B}_G$, and hence there exists a finite collection $\{\mathcal{U}_{\varphi_i}\}_i$ which covers $\partial\mathcal{B}_G$.

Let $\mathfrak{q} \in \hat{\mathbb{Q}}$. Take i so that $x^b(\mathfrak{q}) \in \mathcal{U}_{\varphi_i}$. Then by Lemma 10.1, the absolute value of the derivative of $\mathrm{tr}^2\rho^b_\varphi(\mathbf{w}(\mathfrak{q}))$ at $\varphi = x^b(\mathfrak{q})$ is greater than $C_{G,i} > 0$, which depends only on the translation length of $\rho^b_{\varphi_i}(\mathbf{w}(\mathfrak{p}_{\varphi_i}))$ and the absolute value of the derivative of $\mathrm{tr}^2\rho^b_\varphi(\mathbf{w}(\mathfrak{p}_{\varphi_i}))$ on $\varphi \in \mathcal{U}_{\varphi_i} \cap \partial\mathcal{B}_G$. Thus $C_G := \min_i\{C_{G,i}\}$ is a desired constant. □

11. Teichmüller modular group acting on Bers slices

In this section, we shall prove the quasiconformal extendability of Teichmüller modular transformation acting on one dimensional Teichmüller space. We use frequently the notion and symbols for Bers slices defined in §2.

Remark 7. *L.Bers* [B81a] *had proved that every Teichmüller modular transformation acting on $\mathcal{B}_G$ is extended continuously on its closure.* [4]

11.1. **The improved λ-lemma.** For more details, the reader is referred to [MSS], [BR], [EKK], [SuTh], [Sg92], or [Sl].

Definition 14. (Labeled holomorphic motions ([Mc87])) *A labeled holomorphic motion of a set A (in $\hat{\mathbb{C}}$) over a complex manifold M with distinguished point $m \in M$ is a map $f : M \times A \to \hat{\mathbb{C}}$ such that:*

(i) *For any fixed $a \in A$, $f(\lambda, a)$ is a holomorphic function of $\lambda \in M$;*
(ii) *For any fixed $\lambda \in M$, $f(\lambda, a)$ is an injective function of $a \in A$; and*
(iii) *$f(m, a) = a$ for all $a \in A$.*

Theorem 15. (The improved λ-lemma) *Suppose that M is conformally equivalent to the unit disk in $\mathbb{C}$. Then every holomorphic motion f of a set A over M with distinguished point $m \in M$ can be extended as that of $\hat{\mathbb{C}}$ over M with the following property: For all $\lambda \in M$, $f(\lambda, \cdot)$ is a quasiconformal mapping of*

[4] Here we consider the one-dimensional case. Compare with [KeTh].

the Riemann sphere onto itself with dilatation $\exp(d_M(m,\lambda))$*, where* d_M *is the Kobayashi-hyperbolic distance on* M *with curvature* -1*.*

11.2. **Quasiconformal extension of Modular transformations.** To give a proof of Theorem 3 (Modular transformation has qc-extension), we fix notation and definitions used in the proof.

Teichmüller modular group Let $\mathrm{Mod}(\Sigma)$ be the mapping class group of Σ. An element $[\omega] \in \mathrm{Mod}(\Sigma)$ induces the isomorphism $[\omega]_*$ of $\mathrm{Teich}(\Sigma)$ by the equation

$$[\omega]_*((f,R)) = (f \circ \omega^{-1}, R).$$

$[\omega] \in \mathrm{Mod}(\Sigma)$ determines the element $[\bar{\omega}] \in \mathrm{Mod}(\overline{\Sigma})$ by conjugating by the canonical orientation reversing involution. This satisfies that

$$[\bar{\omega}]_*(y) = \overline{[\omega]_*(\bar{y})} \quad \text{for all } y \in \mathrm{Teich}(\overline{\Sigma}).$$

Two propositions We give two propositions to show the quasiconformal extendability of Teichmüller modular transformation.

Definition 16. (Affine mapping $L_y^{x,z}$) *Let* $x, z \in \mathrm{Teich}(\Sigma)$ *and* $y \in \mathrm{Teich}(\overline{\Sigma})$*. The affine mapping* $L_y^{x,z}$ *of* $Q_{-4}(z,y)$ *to* $Q_{-4}(x,y)$ *is defined by the equation*

$$L_y^{x,z}(\varphi) = \varphi \circ (f_y^{z,x}) \times (f_y^{z,x})'^2 + S(f_y^{z,x}).$$

Henceforth, we fix two points $x_0 = (f_0, R_0) \in \mathrm{Teich}(\Sigma)$ and $y_0 = (g_0, S_0) \in \mathrm{Teich}(\overline{\Sigma})$ as in §2. Recall that for $(x,y) \in \mathrm{Teich}(\Sigma) \times \mathrm{Teich}(\overline{\Sigma})$, W_y^x is a quasiconformal mapping on $\hat{\mathbb{C}}$ such that W_y^x fixes 0, 1, and ∞, and that a representation

$$\pi_1 \ni \mathbf{a}' \mapsto W_y^x \circ \eta_{G(x_0,y_0)}(\mathbf{a}') \circ (W_y^x)^{-1} \in \mathrm{PSL}_2(\mathbb{C})$$

is conjugate to $\eta_{G(x,y)}$.

Proposition 11.1. *The following three hold:*

1. $L_{y_0}^{x_0,z} \circ \beta_{y_0}^{z} = \beta_{y_0}^{x_0}$ *for all* $z \in \mathrm{Teich}(\Sigma)$*.*
2. *For* $[\omega] \in \mathrm{Mod}(\Sigma)$*, there is a Möbius transformation* P *such that*
$$G(x_0, [\bar{\omega}]_*^{-1}(y_0)) = PG([\omega]_*(x_0), y_0)P^{-1}$$
as Kleinian groups (i.e. both sides coincide without marking) and
$$\Omega_-(x_0, [\bar{\omega}]_*^{-1}(y_0)) = P(\Omega_-([\omega]_*(x_0), y_0)).$$
3. *Let* P *be as above. Define a linear mapping* $\mathcal{L}_P$ *from* $Q_{-4}(x_0, [\bar{\omega}]_*^{-1}(y_0))$ *to* $Q_{-4}([\omega]_*(x_0), y_0)$ *by* $\mathcal{L}_P(\varphi) = \varphi \circ P(P')^2$*. Then*
$$\beta_{y_0}^{[\omega]_*(x_0)} \circ [\omega]_*(z) = \mathcal{L}_P \circ \beta_{[\bar{\omega}]_*^{-1}(y_0)}^{x_0}(z), \text{ for } z \in \mathrm{Teich}(\Sigma).$$

Proof. The first equation follows from the well-known transform (cf. [Leh])

$$S(h_1 \circ h_2) = S(h_1) \circ h_2 \cdot (h_2')^2 + S(h_2).$$

The second and third are obtained by the normalization of of W_y^x and by applying the marking trick due to Bers [B81a] (cf. Ito [Ito2]) to see the diagonal action of the modular group.

Indeed, denote by q_w a quasiconformal mapping on $\hat{\mathbb{C}}$ so that $q_w(\Omega_-(x_0,y_0)) = \Omega_-(x_0,y_0)$ and q_w corresponds to the lift of $f_0 \circ \omega^{-1} \circ f_0^{-1}$ on $\Omega_+(x_0,y_0)$ and the lift of $g_0 \circ \bar{\omega}^{-1} \circ g_0^{-1}$ on $\Omega_-(x_0,y_0)$. Then $q_\omega G(x_0,y_0)q_\omega^{-1} = G(x_0,y_0)$ as Kleinian groups.

Denote by P the Möbius transformation sending three points 0, 1, and ∞ to $W^{x_0}_{[\bar{\omega}]_*^{-1}(y_0)}(q_\omega(0))$, $W^{x_0}_{[\bar{\omega}]_*^{-1}(y_0)}(q_\omega(1))$, and $W^{x_0}_{[\bar{\omega}]_*^{-1}(y_0)}(q_\omega(\infty))$, respectively. Notice that two quasiconformal mappings $P^{-1}\circ W^{x_0}_{[\bar{\omega}]_*^{-1}(y_0)}$ and $W^{[\omega]_*(x_0)}_{y_0}\circ q_\omega^{-1}$ uniformize two markings $f_0 : \Sigma \to R_0$ and $g_0\circ\omega : \overline{\Sigma}\to S_0$, and take same values on three points $\{q_\omega(0), q_\omega(1), q_\omega(\infty)\}$. Hence we have

$$P^{-1}\circ W^{x_0}_{[\bar{\omega}]_*^{-1}(y_0)} = W^{[\omega]_*(x_0)}_{y_0}\circ q_\omega^{-1},$$

on the limit set of $G(x_0, y_0)$, because the value of a such quasiconformal mapping on the limit set is determined by the corresponding markings. Thus we obtain that

$$G(x_0, [\bar{\omega}]_*^{-1}(y_0)) = PG([\omega]_*(x_0), y_0)P^{-1}$$

as Kleinian groups and $\Omega_-(x_0, [\bar{\omega}]_*^{-1}(y_0)) = P(\Omega_-([\omega]_*(x_0), y_0))$.

Suppose $z \in \mathrm{Teich}(\Sigma)$. Then by the same argument as above, there exists a Möbius transformation P_z so that $P_z^{-1}\circ W^{z}_{[\bar{\omega}]_*^{-1}(y_0)} = W^{[\omega]_*(z)}_{y_0}\circ q_\omega^{-1}$. Thus we conclude that

$$\begin{aligned}
\beta^{[\omega]_*(x_0)}_{y_0}\circ[\omega]_*(z) &= S\left(f^{[\omega]_*(z),[\omega]_*(x_0)}_{y_0}\right)\\
&= S\left(W^{[\omega]_*(z)}_{y_0}\circ q_\omega^{-1}\circ(W^{[\omega]_*(x_0)}_{y_0}\circ q_\omega^{-1})^{-1}\right)\\
&= S\left(P_z^{-1}\circ W^{z}_{[\bar{\omega}]_*^{-1}(y_0)}\circ(P^{-1}\circ W^{x_0}_{[\bar{\omega}]_*^{-1}(y_0)})^{-1}\right)\\
&= S\left(W^{z}_{[\bar{\omega}]_*^{-1}(y_0)}\circ(W^{x_0}_{[\bar{\omega}]_*^{-1}(y_0)})^{-1}\circ P\right)\\
&= \mathcal{L}_P\left(S\left(f^{z,x_0}_{[\bar{\omega}]_*^{-1}(y_0)}\right)\right) = \mathcal{L}_P\circ\beta^{x_0}_{[\bar{\omega}]_*^{-1}(y_0)}(z).
\end{aligned}$$

□

¿From the normalization of W^x_y, the set

$$\mathcal{T}^* := \{(y,\varphi) \mid y\in\mathrm{Teich}(\overline{\Sigma}), \varphi\in Q(x_0,y)\}$$

is recognized as the cotangent space on $\mathrm{Teich}(\overline{\Sigma})$ (cf. Bers [B81b]). We should note that, by virtue of a theorem of Hejhal ([Hej]), the mapping

$$\mathrm{Teich}(\overline{\Sigma})\ni y\mapsto (y, \beta^{x_0}_y(z))\in\mathcal{T}^*$$

is holomorphic for all $z\in\mathrm{Teich}(\Sigma)$. Since the Teichmüller space is biholomorphic to a bounded domain of $\mathbb{C}$, the cotangent space $\mathcal{T}^*$ is holomorphically trivial by a theorem of Grauert (cf. e.g. §9 (D) of Bers [B81b]).

Definition 17. (linear mapping ℓ_y) *Define a linear mapping from $Q_{-4}(x_0,y)$ to $\mathbb{C}$ by a trivialization of $\mathcal{T}^*$:*

$$\mathcal{T}^*\ni(y,\varphi)\mapsto(y,\ell_y(\varphi))\in\mathrm{Teich}(\overline{\Sigma})\times\mathbb{C}$$

11.3. **Quasiconformal extension.** First we prove

Theorem 18. *Any two Bers slices are quasiconformally equivalent. More precisely, for any $(x,y)\in\mathrm{Teich}(\Sigma)\times\mathrm{Teich}(\overline{\Sigma})$, there exists a quasiconformal mapping h^x_y of $Q_{-4}(x_0,y_0)$ to $Q_{-4}(x,y)$ such that*

$$h^x_y\circ\beta^{x_0}_{y_0}(z) = \beta^x_y(z)$$

for all $z\in\mathrm{Teich}(\Sigma)$. Furthermore, the dilatation of h^x_y is $\exp\{d_{\mathrm{Teich}(\overline{\Sigma})}(y,y_0)\}$.

Proof. Let $z \in \mathrm{Teich}(\Sigma)$. Put $c_z = \ell_{y_0} \circ \beta_{y_0}^{x_0}(z) \in \mathbb{C}$ and $A = \{c_z \mid z \in \mathrm{Teich}(\Sigma)\} \cup \{\infty\}$. Then we define the mapping H from $\mathrm{Teich}(\overline{\Sigma}) \times A$ to $\hat{\mathbb{C}}$ by

$$H(y, c) = \ell_y \circ \beta_y^{x_0}(z) \text{if } c = c_z,\ z \in \mathrm{Teich}(\overline{\Sigma}), \infty \text{if } c = \infty.$$

We check here that H is a labeled holomorphic motion of A over $\mathrm{Teich}(\overline{\Sigma})$ with distinguished point y_0. Indeed, by definition, $H(y_0, c_z) = \ell_{y_0} \circ \beta_{y_0}^{x_0}(z) = c_z$. Fix $y \in \mathrm{Teich}(\overline{\Sigma})$, Since $\beta_y^{x_0}$ is injective and $\beta_y^{x_0}(z) \neq \infty$, for $z \in \mathrm{Teich}(\Sigma)$, $H(y, c)$ is also injective. ¿From a theorem of Hejhal,

$$H(y, c_z) = \ell_y \circ \beta_y^{x_0}(z)$$

is holomorphic for $y \in \mathrm{Teich}(\overline{\Sigma})$ and fixed $z \in \mathrm{Teich}(\Sigma)$. Finally, we note that $H(y, \infty) \equiv \infty$ is also holomorphic on $y \in \mathrm{Teich}(\overline{\Sigma})$.

Therefore, the improved λ-lemma tells us that H can be extend as a labeled holomorphic motion $\hat{H}$ of $\hat{\mathbb{C}}$ over $\mathrm{Teich}(\overline{\Sigma})$ with distinguished point y_0. Now, we define a quasiconformal mapping h_y^x of $Q_{-4}(x_0, y_0)$ to $Q_{-4}(x, y)$ by

$$h_y^x(\varphi) = \mathcal{L}_y^{x,x_0} \circ (\ell_y)^{-1} \circ \hat{H}(y, \ell_{y_0}(\varphi)).$$

Then h_y^x satisfies the equation in the statement of this theorem. Indeed,

$$\begin{aligned} h_y^x \circ \beta_{y_0}^{x_0}(z) &= \mathcal{L}_y^{x,x_0} \circ (\ell_y)^{-1} \circ \hat{H}(y, \ell_{y_0} \circ \beta_{y_0}^{x_0}(z)) \\ &= \mathcal{L}_y^{x,x_0} \circ (\ell_y)^{-1} \circ \hat{H}(y, c_z) \\ &= \mathcal{L}_y^{x,x_0} \circ (\ell_y)^{-1} \circ \ell_y \circ \beta_y^{x_0}(z) \\ &= \mathcal{L}_y^{x,x_0} \circ \beta_y^{x_0}(z) = \beta_y^x(z). \end{aligned}$$

By the improved lambda lemma, the dilatation of h_y^x is $\exp\{d_{\mathrm{Teich}(\overline{\Sigma})}(y, y_0)\}$, since all $\mathcal{L}_y^{x,x_0}$, ℓ_y, and ℓ_{y_0} are affine mappings. □

Now, we restate and prove Theorem 3:

Theorem 3. *For $[\omega] \in \mathrm{Mod}(\Sigma)$, there is a $K([\omega], y_0)$-quasiconformal mapping h_ω of $Q_{-4}(x_0, y_0)$ onto itself such that*

$$h_\omega \circ \beta_{y_0}^{x_0}(z) = \beta_{y_0}^{x_0} \circ [\omega]_*(z),\ \textit{for } z \in \mathrm{Teich}(\Sigma),$$

where $K([\omega], y_0) = \exp\{d_{\mathrm{Teich}(\overline{\Sigma})}(y_0, [\bar{\omega}]_*(y_0))\}$.

Proof. This is proved by the argument parallel to that of the proof of Theorem 18.

The later part of the proof is conclusion of the improved lambda lemma. Let us prove the first part. Define a quasiconformal mapping h_ω by

$$h_\omega(\varphi) = L_{y_0}^{x_0,[\omega]_*(x_0)} \circ \mathcal{L}_P \circ \left(\ell_{[\bar{\omega}]_*^{-1}(y_0)}\right)^{-1} \circ \hat{H}\left([\bar{\omega}]_*^{-1}(y_0), \ell_{y_0}(\varphi)\right)$$

for $\varphi \in Q_{-4}(x_0, y_0)$, where P and $\mathcal{L}_P$ are taken and defined as in (3) of Proposition 11.1. Then h_ω satisfies the condition in the assertion. Indeed, since $\hat{H}([\bar{\omega}]_*^{-1}(y_0), \cdot)$ is $K([\omega], y_0)$-quasiconformal and all $L_{y_0}^{x_0,[\omega]_*(x_0)}$, $\ell_{[\bar{\omega}]_*^{-1}(y_0)}$, $\mathcal{L}_P$, and ℓ_{y_0} are affine, h_ω is also $K([\omega], y_0)$-quasiconformal. In addition, for $z \in \mathrm{Teich}(\Sigma)$,

$$\begin{aligned} h_\omega \circ \beta_{y_0}^{x_0}(z) &= L_{y_0}^{x_0,[\omega]_*(x_0)} \circ \mathcal{L}_P \circ \left(\ell_{[\bar{\omega}]_*^{-1}(y_0)}\right)^{-1} \circ H\left([\bar{\omega}]_*^{-1}(y_0), c_z\right) \\ &= L_{y_0}^{x_0,[\omega]_*(x_0)} \circ \mathcal{L}_P \circ \beta^{x_0}_{[\bar{\omega}]_*^{-1}(y_0)}(z) \\ &= L_{y_0}^{x_0,[\omega]_*(x_0)} \circ \beta_{y_0}^{[\omega]_*(x_0)} \circ [\omega]_*(z) = \beta_{y_0}^{x_0} \circ [\omega]_*(z). \quad \square \end{aligned}$$

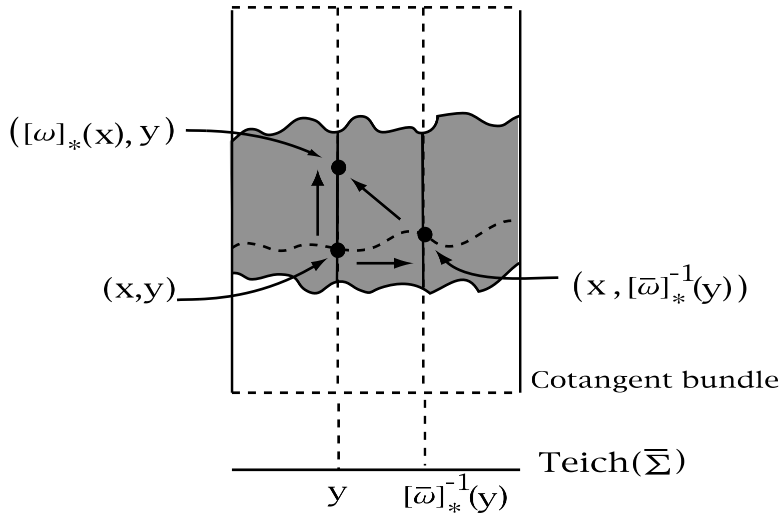

FIGURE 9. Process of qc extension: The vertical direction means the action on a slice via $[\omega]_*$, which is (essentially) factorized into two parts. One is the horizontal part represented by a qc map arising by application of the lambda lemma (Theorem 18). The other is a diagonal part, which is the linear map $\mathcal{L}_P$.

Remark 8. *By definition, the constant $K([\omega], y_0)$ coincides with the maximal dilatation of the extremal quasiconformal self-mapping of S_0 which is homotopic to $g_0 \circ \bar{\omega} \circ g_0^{-1}$. Equivalently, let φ_0 be the point in its Bers slice corresponding to the Fuchsian group. Then $\log K([\omega], y_0)$ equals the hyperbolic distance between φ_0 and its image under $[\omega]_*$.*

Since the boundary of Bers slices moves holomorphically on the base surfaces, we have (cf. [Ast])

Corollary 8. *The Hausdorff dimension of the complex boundary of Bers slices varies continuously with the complex structures of the base surfaces.*

11.4. **Similarity at cusps.** Recall that a point ζ in a measurable set $\Omega \subset \mathbb{C}$ is said to be a *δ-measurable deep point*, $\delta > 0$ if

$$\mathrm{area}(B(\zeta, r) - \Omega) = O(r^{2+\delta}).$$

where $B(\zeta, r)$ is a disk of center ζ with radius r.

A quasiconformal mapping ϕ of $\mathbb{C}$ is said to be $C^{1+\alpha}$-conformal at $\zeta \in \mathbb{C}$ if $\phi'(\zeta)$ exists and

$$\phi(\zeta + t) = \phi(\zeta) + \phi'(\zeta)t + O(|t|^{1+\alpha}).$$

Notice that any $C^{1+\alpha}$-conformal mapping ϕ at ζ is also $C^{1+\alpha'}$-conformal at ζ for all $0 < \alpha' < \alpha$. Further, we note that for two quasiconformal mapping ϕ_1 and ϕ_2 on $\mathbb{C}$, if ϕ_1 (resp. ϕ_2) is $C^{1+\alpha_1}$ (resp. $C^{1+\alpha_2}$)-conformal at $\zeta \in \mathbb{C}$. (resp. at $\phi_1(\zeta)$), then $\phi_2 \circ \phi_1$ is $C^{1+\alpha}$-conformal at ζ, where $\alpha = \min\{\alpha_1, \alpha_2\}$.

In [Mc96], McMullen showed the following result:

Theorem 19. (Theorem 2.25 of McMullen [Mc96]) *Let E be a measurable set on $\mathbb{C}$, and $x \in \mathbb{C}$ a δ-measurable deep point. Then any quasiconformal mapping with* *vanishing Beltrami differential on E is $C^{1+\alpha}$-conformal at ζ, where α is dependent only on δ and the maximal dilatation of given quasiconformal mapping.*

We restate and prove a corollary of Theorem 3 in Introduction:

Corollary 4. (All rational boundary points are similar each other) *There exists $\alpha > 0$ depend only on the base surface of given Bers slice such that every Teichmüller modular transformation acting on given Bers slice is $C^{1+\alpha}$-conformal at all rational boundary points.*

Proof. Since every boundary point is a $(2,3)$-cusp, a simple calculation shows that any rational boundary point is $1/2$-measurable deep point. Since every Teichmüller modular transformation acts on a Bers slice holomorphically and the set of rational boundary points are countable. Hence we obtain the $C^{1+\alpha}$-conformality by Theorem 19.

Next we can choose $\alpha > 0$ to be dependent only on the base surface. Let $[\omega_1]_*$ and $[\omega_2]_*$ be generators of the Teichmüller modular transformation acting on given Bers slice. Suppose that $[\omega_i]_*$ is $C^{1+\alpha_i}$-conformal at each rational boundary points. Since the deepness of each rational boundary point is a universal constant $1/2$, by Remark 8, the constant α_i depends only on the dilatation of the Teichmüller mapping acting on the base surface of given Bers slice representing $[\omega_i]_*$.

Let α be the supreme of $\min\{\alpha_1, \alpha_2\}/2$. for all generators $\{[\omega_1]_*, [\omega_2]_*\}$. Then $\alpha > 0$ depends only on the base surface by definition, and has the property in the assertion. □

By using the same argument, we have

Corollary. (All rational boundary point are similar) *The quasiconformal map between two Bers slices given in Theorem 18 is $C^{1+\alpha}$-conformal at all rational boundary points for some $\alpha > 0$.*

11.5. **Trace function on the boundary of a Bers slice.** At the end of Section 11, we shall show the following proposition, which completes the proof of Theorem 2.

Proposition 11.2. *Let $G = G(x_0, y_0)$. For any $\varphi_0 \in \partial\mathcal{B}_G$, there exists $\mathfrak{p} \in \hat{\mathbb{Q}}$ such that $\rho^b_{\varphi_0}(\mathbf{w}(\mathfrak{p}))$ is loxodromic and that the derivative of the trace function $\mathrm{tr}^2\rho^b_{\varphi}(\mathbf{w}(\mathfrak{p}))$ at $\varphi = \varphi_0$ does not vanish.*

Two show this proposition, we recall the Jørgensen parameter of once punctured torus groups: Let $a, b, c \in \mathbb{C}$ with $c \neq 0$ and $a^2 + b^2 + c^2 = abc$. Then we construct an admissible homomorphism (cf.§2) $\eta_{(a,b,c)}$ by

$$\eta_{(a,b,c)}(\mathbf{a}) := \begin{bmatrix} a - b/c & a/c^2 \\ a & a/c \end{bmatrix}, \eta_{(a,b,c)}(\mathbf{b}) := \begin{bmatrix} b - a/c & -b/c^2 \\ -b & b/c \end{bmatrix}.$$

Notice that

$$\mathrm{tr}^2\eta_{(a,b,c)}(\mathbf{a}) = a^2, \mathrm{tr}^2\eta_{(a,b,c)}(\mathbf{b}) = b^2, \text{ and } \mathrm{tr}^2\eta_{(a,b,c)}(\mathbf{ab}) = c^2.$$

We also note that a point (a, b, c) in the 2-dimensional affine variety $\mathcal{V} := \{a^2 + b^2 + c^2 = abc\}$ is a regular point if $abc \neq 0$. We first note the following well-known lemma.

Lemma 11.1. *For $[\eta] \in \mathcal{D}$. there exists a regular point $(a, b, c) \in \mathcal{V}$ such that $[\eta] = [\eta_{(a,b,c)}]$.*

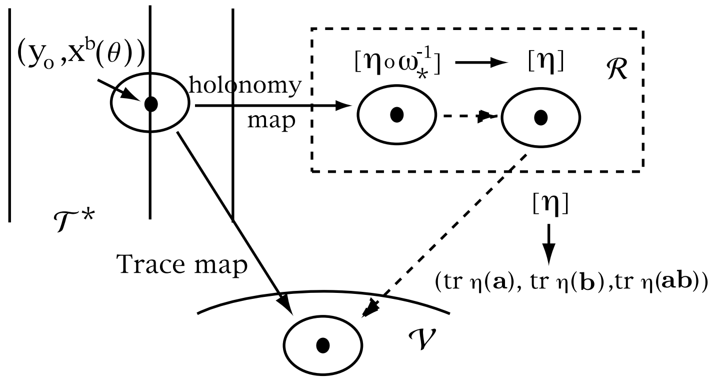

FIGURE 10. Trace map is locally biholomorphic.

The proof of this result is due to T.Jørgensen [Jo]. We here sketch his way to prove.

Proof. Let A and B be lifts $\eta(\mathbf{a})$ and $\eta(\mathbf{b})$ on $\mathrm{SL}_2(\mathbb{C})$, respectively. Since ρ is a discrete representation, we have $\mathrm{tr}ABA^{-1}B^{-1} = -2$ (cf.e.g. Theorem 5.37 of Matsuzaki-Taniguchi [MT]). Hence $a := \mathrm{tr}A$, $b := \mathrm{tr}B$, and $c := \mathrm{tr}AB$ satisfies $a^2 + b^2 + c^2 - abc = 0$, Furthermore, by assumption, $abc \neq 0$. This means that $(a, b, c) \in \mathcal{V}$ and this point is a regular point.

We take a conjugation so that $\eta([\mathbf{a}, \mathbf{b}])(z) = z + 2$, and after this, we take a conjugation again by the translation which takes the pole $(AB)^{-1}(\infty)$ of AB to zero. Then we can observe that $[\eta] = [\eta_{(a,b,c)}]$. Here the pole of a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a/c. □

Hence we have the following proposition.

Proposition 11.3. *Let ω be an orientation preserving homeomorphism on Σ. Then the holomorphic mapping*

$$\mathcal{V} \ni (a, b, c) \mapsto [\eta_{(a,b,c)} \circ (\omega_*)^{-1}] \in \mathcal{R}$$

admits a local inverse mapping at a point (a, b, c) which satisfies $[\eta_{(a,b,c)}] \in \mathcal{D}$.

Remark 9. *By definition, a local inverse mapping forms*

$$[\eta] \mapsto (\mathrm{tr}\,\eta \circ \omega_*(\mathbf{a}), \mathrm{tr}\,\eta \circ \omega_*(\mathbf{b}), \mathrm{tr}\,\eta \circ \omega_*(\mathbf{ab})) \in \mathcal{V}$$

Proof. It is easy to see that the mapping

$$\mathcal{V} \ni (a, b, c) \mapsto [\eta_{(a,b,c)}] \in \mathcal{R}$$

is injective and holomorphic. By Lemma 11.1, a set $\mathcal{D}$ is contained in the image of the mapping above. Since $[\eta] \in \mathcal{D}$ and its corresponding point in $\mathcal{V}$ are both regular point of the ambient manifolds, the mapping above admits local inverse mapping at a neighborhood of $[\eta]$ (cf. Figure 10. The mappings here are represented by broken lines). Since the mapping $\mathcal{R} \ni [\eta] \mapsto [\eta \circ (\omega_*)^{-1}] \in \mathcal{R}$ is a holomorphic automorphism of $\mathcal{R}$ (cf.McMullen [Mc96]), we conclude the assertion. □

Proof of Proposition 11.2. Recall that $G = G(x_0, y_0)$. Let $\varphi_0 \in \partial\mathcal{B}_G$. By definition, we can recognize φ_0 as a point $(y_0, \varphi_0) \in \mathcal{T}^*$. By a theorem of Hejhal [Hej] (see also [Mc98]), the holonomy mapping

$$\mathcal{T}^* \ni (y, \varphi) \mapsto [\rho^b_\varphi] \in \mathcal{R}$$

is an analytic local homeomorphism. We now take a neighborhood $\mathcal{U}$ of $(y_0, \rho^b_{\varphi_0}) \in \mathcal{T}^*$, at which the holonomy mapping is homeomorphic.

Let ω be an orientation preserving homeomorphism of Σ so that all $\rho^b_\varphi \circ \omega_*(\mathbf{a})$, $\rho^b_\varphi \circ \omega_*(\mathbf{b})$, and $\rho^b_\varphi \circ \omega_*(\mathbf{ab})$ are loxodromic at $\varphi = \varphi_0$. Such ω does exist, because $\rho^b_{\varphi_0}$ can admit at most two accidental parabolic transformations.

By composing the local inverse mapping defined in Proposition 11.3 and the homeomorphism defined by the inverse of the holonomy mapping (cf. Figure 10), we conclude that the trace map

$$\mathcal{U} \ni (y, \varphi) \mapsto (\operatorname{tr} \rho^b_\varphi \circ \omega_*(\mathbf{a}), \operatorname{tr} \rho^b_\varphi \circ \omega_*(\mathbf{b}), \operatorname{tr} \rho^b_\varphi \circ \omega_*(\mathbf{ab})) \in \mathcal{V}$$

is well-defined and biholomorphic. This means that, on the differential of the trace map along a fiber of $\mathcal{T}^*$ passing through (y_0, φ_0), one of their coordinates of the right-hand side does not vanish. This implies the assertion. □

12. Extended Local Pleating Theorem

12.1. **Pleating varieties, Local and Limit pleating theorem.** This section treats the definition of the pleating varieties in quasifuchsian groups after Keen and Series [KS99], and gives the statement of the local pleating theorem.

Let $\mathcal{F}$ be fuchsian space in $\mathcal{Q}F$. A component of the boundary of convex core of the manifold corresponding to $[\eta]$ is said to be *positive* if it faces the boundary surface corresponding to the distinguished invariant component of $[\eta]$. The other component is called *negative*. Each component is a *pleated surface* in the sense of Thurston. We denote by $pl^+([\eta])$ (resp. $pl^-([\eta])$) the *bending lamination* of the positive (resp. negative) component.

Denote by $\mathcal{M}L$ and $\mathcal{PM}L$ the space of measured laminations and projective measured laminations on Σ, respectively. For $\nu_1, \nu_2 \in \mathcal{M}L$, we define

$$\mathcal{P}L^\pm_{\nu_1} = \{[\eta] \in \mathcal{Q}F - \mathcal{F} \mid pl^\pm([\eta]) = \nu_1\},$$
$$\mathcal{P}L_{\nu_2,\nu_1} = \mathcal{P}L^+_{\nu_1} \cap \mathcal{P}L^-_{\nu_2}.$$

We call $\mathcal{P}L^\pm_{\nu_1}$ the ν_1-*pleating varieties* and $\mathcal{P}L_{\nu_2,\nu_1}$ the ν_1, ν_2-*pleating plane*. The connection between this notion and ours can be found from the following equation: Let ν_1 be a rational lamination whose support is homotopic to $\mathbf{w}(\mathfrak{q})$. For $\mathcal{X} = \mathcal{B}_G$ or $\mathcal{E}$,

$$\mathcal{P}L^+_{\nu_1} \cap \mathcal{X} = \mathcal{P}^{\mathcal{X}}_{\mathfrak{q}}.$$

In [KS99], L.Keen and C.Series proved the following theorem:

Theorem 20. (Non-singularity of pleating varieties) *Let $p/q, p'/q' \in \hat{\mathbb{Q}}$ with $p/q \neq p'/q'$. Let ν_1 and ν_2 be rational measured laminations whose supports are homotopic to $\mathbf{w}(p/q)$ and $\mathbf{w}(p'/q')$ respectively. Then $\mathcal{P}L_{\nu_1,\nu_2}$ and $\mathcal{P}L_{\nu_2,\nu_1}$ are the components of the pre-image of $\mathbb{R}_{>0} \times \mathbb{R}_{>0}$ under the mapping:*

$$\mathcal{Q}F - \mathcal{F} \ni [\eta] \mapsto (\operatorname{tr}^2\eta(\mathbf{w}(p/q)), \operatorname{tr}^2\eta(\mathbf{w}(p'/q'))).$$

Furthermore, this mapping is non-singular on $\mathcal{P}L_{\nu_1,\nu_2}\cup\mathcal{P}L_{\nu_2,\nu_1}$ *and the boundary of* $\mathcal{P}L_{\nu_1,\nu_2}\cup\mathcal{P}L_{\nu_2,\nu_1}$ *can be computed by solving* $\mathrm{tr}^2\eta(\mathbf{w}(p/q))=4$ *and* $\mathrm{tr}^2\eta(\mathbf{w}(p'/q'))=4$ *on this component.*

Here we note that this theorem does not mention the non-singularity of their trace functions at the boundary of these pleating varieties, which is the one of the main theorems of this section.

To prove this theorem, they showed useful results, called the *Local pleating theorem* and the *Limit pleating theorem* (Theorems 5.1 and 8.1 of [KS99]), which for convenience we explain below. For $\nu_1\in\mathcal{M}L$, We denote by $\lambda_{\nu_1}([\eta])$ the complex length of the lamination ν_1 of the quasifuchsian manifold associated with $[\eta]$. By definition if $\nu_1=t\gamma$ where $t>0$ and γ is a simple closed curve on Σ, this is just t times the complex length defined from the trace function of the corresponding simple closed curve. In the case of irrational laminations, we define the length function by limiting process (cf. [KS99]). The real part of the complex length is nothing but the usual length of lamination ν_1. The complex function λ_{ν_1} takes positive real values on $\mathcal{P}L^+_{\nu_1}\cup\mathcal{F}\cup\mathcal{P}L^-_{\nu_1}$.

Theorem.(Local pleating theorem) Let $\nu_1\in\mathcal{M}L$. Every group $[\eta_0]$ in $\mathcal{P}L^+_{\nu_1}\cup\mathcal{F}\cup\mathcal{P}L^-_{\nu_1}$ has a neighborhood U in $\mathcal{Q}F$ such that for $[\eta]\in U$, $\lambda_{\nu_1}([\eta])\in\mathbb{R}$ implies $[\eta]\in\mathcal{P}L^+_{\nu_1}\cup\mathcal{F}\cup\mathcal{P}L^-_{\nu_1}$.

Theorem.(Limit pleating theorem) Let ν_1 and ν_2 be mutually distinct rational laminations. Let $\{[\eta_n]\}_n$ be a sequence on $\mathcal{P}L_{\nu_1,\nu_2}$. If $\mathrm{Re}\,\lambda_{\nu_i}([\eta_n])\to c_i\geq 0$ for each i, then $\{[\eta_n]\}_n$ contains a subsequence with algebraic limit $[\eta_\infty]$. Further, $[\eta_\infty]$ is quasifuchsian if and only if $c_i>0$ for $i=1,2$.

12.2. **Extended Local pleating theorem.** We now consider how to extend the local pleating theorem and non-singularity theorem of pleating varieties for the geometrically finite boundary groups. By using notation given in the previous section, we can restate Theorem 4 as follows: Recall that a geometrically finite boundary group is said to be of type n ($n=1,2$) if it admits n distinct conjugacy classes of accidental parabolic transformations.

Theorem 4(Extended local pleating theorem) *Let* $[\eta_0]$ *be a geometrically finite boundary group. Then there exists a neighborhood* $\mathcal{U}$ *of* $[\eta_0]$ *in* $\mathcal{R}$ *so that the one of the following holds:*

1. *Suppose that* $[\eta_0]$ *is of type* 1. *Let* $p/q\in\mathbb{Q}$ *be such that* $\mathbf{w}(p/q)$ *corresponds to the accidental parabolic of* $[\eta_0]$. *Let* ν_1 *be a measured lamination whose support is homotopic to* $\mathbf{w}(p/q)$. *Then either* $\mathcal{U}\cap\mathcal{P}L^+_{\nu_1}$ *or* $\mathcal{U}\cap\mathcal{P}L^-_{\nu_1}$ *is the pre-image under the holomorphic mapping*
$$\mathcal{U}\ni[\eta]\mapsto\mathrm{tr}^2\eta(\mathbf{w}(p/q))$$
of $\mathbb{R}_{>4}=\{t\in\mathbb{R}|r>4\}$. *Furthermore, if* $\mathcal{U}\cap\mathcal{P}L^+_{\nu_1}$ *is not empty,* $\mathcal{U}\cap\mathcal{P}L^-_{\nu_1}$ *is empty and vice versa.*
2. *Otherwise, let* $p/q,p'/q'\in\mathbb{Q}$ *correspond to accidental parabolics of* $[\eta_0]$. *Let* ν_1 *and* ν_2 *be measured laminations whose supports are homotopic to* $\mathbf{w}(p/q)$ *and* $\mathbf{w}(p'/q')$ *respectively. Then* $\mathcal{U}\cap\mathcal{P}L_{\nu_1,\nu_2}$ *is the pre-image of the mapping*
$$\mathcal{U}\ni[\eta]\mapsto(\mathrm{tr}^2\eta(\mathbf{w}(p/q)),\mathrm{tr}^2\eta(\mathbf{w}(p'/q')))$$
of $\mathbb{R}_{>4}\times\mathbb{R}_{>4}$.

Furthermore, in all cases, the preimages of these mappings are connected and contractible.

Proof. We first prove the case of the type 2 geometrically finite boundary group. ¿From Proposition 10.1, the mapping

$$\Psi_0([\eta]) = (\mathrm{tr}^2\eta(\mathbf{w}(p/q)), \mathrm{tr}^2\eta(\mathbf{w}(p'/q')))$$

has maximal rank at $[\eta_0]$. Hence there exists a neighborhood $\mathcal{U}$ of $[\eta_0]$ so that Ψ_0 is biholomorphic on $\mathcal{U}$, $\mathcal{U}$ does not intersect the fuchsian slice, and $\Psi_0(\mathcal{U})$ is a poly disk with center $(4,4)$ and radius $r_0 > 0$. Put $N_2 = \{(t,s) \in \mathbb{R} | 4 < t, s < 4 + r_0\}$. By considering the shrinking deformation on the length of the bending lamination on $\mathcal{P}L_{\nu_1,\nu_2}$, we can find a sequence in $\mathcal{P}L_{\nu_1,\nu_2}$ which converges to $[\eta_0]$. This means that $\mathcal{U} \cap \mathcal{P}L_{\nu_1,\nu_2}$ is not empty and contained in $\Psi_0^{-1}(N_2)$. By virtue of the local and limit pleating theorem, $\mathcal{U} \cap \mathcal{P}L_{\nu_1,\nu_2}$ is open and closed in $\Psi_0^{-1}(P)$. Thus

$$\mathcal{U} \cap \mathcal{P}L_{\nu_1,\nu_2} = \{[\eta] \in \mathcal{U} \mid 4 < \mathrm{tr}^2\eta(\mathbf{w}(p/q)), \mathrm{tr}^2\eta(\mathbf{w}(p'/q')) < 4 + r_0\}.$$

Next we assume that $[\eta_0]$ is of type 1. Since representation space $\mathcal{R}$ has anticonformal involution which is induced from the complex conjugation $\zeta \mapsto \bar{\zeta}$ on the phase space $\hat{\mathbb{C}}$ and since the ends of each Kleinian manifold are switched by this involution, we may suppose that the accidental parabolic of $[\eta_0]$ corresponds to the positive end invariant of $[\eta_0]$ that is, $\nu_+([\eta_0]) \in \hat{\mathbb{Q}}$ and $\nu_-([\eta_0]) \in \mathbb{H}$. Therefore $G_0 := \eta_0(\pi_1)$ has the invariant component, which is no longer distinguished. Let $y_0 = (g_0, S_0) \in \mathrm{Teich}(\overline{\Sigma})$ be the conjugate of the point of $\mathrm{Teich}(\Sigma)$ corresponding to $\nu_-([\eta_0]) \in \mathbb{H}$.

Here, we use the notation given in the previous section frequently. Notice that $\mathcal{T}^*$ is recognized as the space of projective structures on Σ (see McMullen [Mc98]). Since η_0 is faithful and of type 1 boundary group, $[\eta_0]$ lies on the image of holonomy map so that $[\eta_0]$ corresponds to the standard projective structure. Hence $[\eta_0]$ is on the image by the holonomy map of the boundary of the Bers slice $\beta^{x_0}_{y_0}(\mathrm{Teich}(\overline{\Sigma}))$. Since the holonomy map is local analytic homeomorphism by a theorem of Hejhal, there exist a neighborhood $\mathcal{U}'$ of $[\eta_0]$ in $\mathcal{R}$ and a holomorphic mapping proj, which is induced from the projection of the cotangent bundle to the Teichmüller space, from $\mathcal{U}'$ to $\mathrm{Teich}(\overline{\Sigma})$ so that $\mathrm{proj}([\eta_0]) = y_0$ and any $[\eta] \in \mathcal{U}'$ is induced from the projective structure whose under complex structure associates with $\mathrm{proj}([\eta])$.

We define the holomorphic mapping Ψ_0 on $\mathcal{U}'$ by

$$\Psi_0 : \mathcal{U}' \ni [\eta] \mapsto (\mathrm{proj}([\eta]), \mathrm{tr}^2\eta(\mathbf{w}(p/q))) \in \mathrm{Teich}(\overline{\Sigma}) \times \mathbb{C}. \tag{10}$$

We claim that Ψ_0 is biholomorphic on a neighborhood of $[\eta_0]$: Let F_0 be a fuchsian group of type $(1,1)$ such that $\Lambda(F_0) = \mathbb{R}$ and satisfies $\mathbb{L}/F_0 = S_0$. We consider the Bers slice whose center F_0. As was said above, $[\rho^b_{x^b(p/q)}] = [\eta_0]$ and $\mathrm{proj}([\rho^b_\varphi]) = y_0$ on some neighborhood of $x^b(p/q)$ in $Q_{-4}(F_0)$. By Theorem 2

$$\frac{d}{d\varphi}\mathrm{tr}^2\rho^b_\varphi(\mathbf{w}(p/q))\bigg|_{\varphi = x^b(p/q)} \neq 0. \tag{11}$$

A quasiconformal deformation of η_0 induces a holomorphic mapping s_0 from a neighborhood N_0 of y_0 in $\mathrm{Teich}(\overline{\Sigma})$ into $\mathcal{U}'$ so that $s_0(y_0) = [\eta_0]$, $\mathrm{proj} \circ s_0(y) = y$ and

$$\mathrm{tr}^2 s_0(y)(\mathbf{w}(p/q)) \equiv 4$$

on N_0. Together with (11), Ψ_0 has maximal rank at $[\eta_0]$.

Shrink $\mathcal{U}_0$ so that on it Ψ_0 is biholomorphic, with image a product of a simply connected domain $\mathrm{proj}(\mathcal{U}_0)$ and a small disk with center 4 and radius r_0, and so that $\mathcal{U}_0 \cap \mathcal{QF}$ does not intersect the fuchsian slice. Then we will show that

$$\mathcal{U}_0 \cap \mathcal{P}L^+_{\nu_1} = \{[\eta] \in \mathcal{U}_0 \mid \mathrm{tr}^2\eta(\mathbf{w}(p/q)) \in \mathbb{R}_{>4}\}.$$

On $\mathcal{P}L^+_{\nu_1}$, the complex length of ν_1 takes real values. So, the left hand side is contained in the right. We show the opposite direction. Let $N_1 = \{(y,t) \in \mathrm{Teich}(\overline{\Sigma})\times\mathbb{R} \mid 0 < t < 4+r_0\}$. Fix $y \in \mathrm{proj}(\mathcal{U}_0)$. Let F_y be a fuchsian group of type $(1,1)$ so that $\Lambda(F_y) = \mathbb{R}$ and $\mathbb{L}/F_y = S$ where $y = (g,S)$. We consider here the Bers slice $\mathcal{B}_{F_y}$ with center F_y. By definition of the mapping s_0, the p/q-pleating ray in $\mathcal{B}_{F_y}$ lands at the rational boundary point $x^b(p/q)$ corresponding to the lamination with support $\mathbf{w}(p/q)$ and $[\rho^b_{x^b(p/q)}] = [s_0(y)]$.

Let $[\eta] \in \Psi_0^{-1}(N_1)$ with $\mathrm{proj}([\eta]) = y$. Let $\varphi \in Q_{-4}(F_y)$ so that $[\eta] = [\rho^b_\varphi]$. We know that the p/q-pleating ray is a connected component of the set

$$\{\varphi \in Q_{-4}(F_y) - \{0\} | \mathrm{tr}^2\rho^b_\varphi(\mathbf{w}(p/q)) > 4\}.$$

By Theorem 2, a set

$$\{\varphi \in Q_{-4}(F_y) \mid \mathrm{tr}^2\rho^b_\varphi(\mathbf{w}(p/q)) \in \mathbb{R}\}$$

is an real analytic curve near $x^b(p/q)$. Hence by taking r_0 sufficiently small, for $\varphi \in Q_{-4}(F_y)$, if $[\rho^b_\varphi] \in \Psi_0^{-1}(N_1)$, φ lies on the p/q-pleating ray in $\mathcal{B}_{F_y}$. Since $\mathcal{U}_0$ does not intersect the fuchsian locus, applying Local pleating theorem and Limit pleating theorem, we have that the set consisting of $[\eta] \in \Psi_0^{-1}(N_1)$ which is also contained in $\mathcal{P}L^+_{\nu_1}$ is open and closed in $\Psi_0^{-1}(N_1)$. This means that $[\eta] \in \mathcal{P}L^+_{\nu_1}$ for $[\eta] \in \mathcal{U}_0$ with $4 < \mathrm{tr}^2\eta(\mathbf{w}(p/q)) < 4 + r_0$. Thus, we may let $\mathcal{U}$ be the resulting neighborhood of $[\eta_0]$. $\square$

Appendix A. Relation to complex dynamics

A similarity phenomenon, Corollary 4, on Bers slices gives columns in the dictionary between rational maps and Kleinian groups. In particular, this corresponds to the similarity of parabolic points of the Mandelbrot set M via tuning. The *tuning* is a homeomorphism from the Mandelbrot set to its copy in itself defined by Renormalization (cf.e.g. §5 of Lyubich [Lyu]).

We will not give the concrete definition of notion in complex dynamics, e.g. tuning and a little Mandelbrot copy, and so forth. However we should know here the following four properties:

(1) There are two kinds of Mandelbrot copies: *primitive* copies and *satellite* copies (cf. e.g. nor []).
(2) If a Mandelbrot copy is primitive, the corresponding tuning is extended quasiconformally to the whole complex plane. If a copy is satellite, the tuning is quasiconformal at $M - \{1/4\}$ (Theorem 5.5 of [Lyu]).
(3) Every parabolic point is a 1/2-deep point for the interior of the Mandelbrot set (cf. e.g. Lemma 6.1 and 6.2 of [Mil]).
(4) A tuning is holomorphic on the interior of the Mandelbrot set (cf. Douady-Hubbard [DH]).

Applying (2), (3), and (4) together with Theorem 19, we have

Theorem 21. *Let M be the Mandelbrot set and M_c a Mandelbrot copy. Then the tuning from M to M_c is $C^{1+\alpha}$-conformal at every parabolic points except for $1/4 \in \partial M$. If M_c is primitive, the tuning is also $C^{1+\alpha}$-conformal at $1/4$.*

Thus we obtain the following table:

Once puncture torus groups	Quadratic polynomials
Bers slice $\mathcal{B}_G$	Mandelbrot set M
Rational boundary point	Parabolic point
Rational boundary points are 1/2-deep points (Theorem 1)	Parabolic points are 1/2-deep points.
Teichmüller modular transformation acting on a Bers slice	Tuning
Teichmüller modular transformation has qc-extension(Theorem 3)	Tuning has qc-extension (locally) (M.Lyubich)
Teichmüller modular transformation is $C^{1+\alpha}$-conformal at rational boundary points (Corollary 4)	Tuning is $C^{1+\alpha}$-conformal at parabolic points (Theorem 21)

About the 4-th column, we note that the root of a small Mandelbrot copy is an inward-pointing cusp (cf.e.g. Lemma 6.2 of [Mil]), which also corresponds to our main theorem, Theorem 1. Many people observed some similarity phenomena in the theory of complex dynamics (cf. e.g. [Lei], [Lyu], [R1], and [R2]).

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