A UNIVERSAL PARABOLIC MAP

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Of all the beautiful results of modern complex dynamics perhaps the most fascinating is the one-dimensional renormalization theory. This theory serves to explain the universality phenomena, such as the celebrated Feigenbaum-Coullett-Tresser universality in transition to chaos in unimodal maps, and at the same time is a rich source of results on the geometry of parameter spaces of analytic maps. There are two branches of the renormalization theory in one-dimensional dynamics, one dealing with unimodal maps, the other with critical homeomorphisms of the circle. Renormalization became a part of complex dynamics mainly due to the work of Sullivan ([Sul1, Sul2, MvS]), and Douady-Hubbard [DH2]. An important first step was made earlier by H. Epstein [E, EE]. We will not attempt to give a survey of the results here, the interested reader is referred to the papers of Lyubich [Lyu3, Lyu4] for an account of the unimodal renormalization theory (of which these papers are the culmination), and to the paper of one of the authors [Ya3] for a description of the theory of critical circle maps. The two theories are closely intertwined, but possess some interesting differences, some of which are discussed in the latter reference.

This paper is concerned with a very special case of renormalization in one-dimensional dynamics, the so-called *parabolic renormalization*. This construction arises as a degenerate case of the usual renormalization, when the periods of the renormalized maps become infinite. The first mention in the literature of the universality phenomenon associated to it was in the work of Lanford [Lan1, Lan2], who incorporated the parabolic renormalization of critical circle maps into the general renormalization framework. Similarly to the usual renormalization, the parabolic renormalization yields various results of geometric nature. In a note of Devaney-Douady [DD] (see also [BDDS] for a published account) it is used to explain the occurence of certain sequences of small copies of the Mandelbrot set $\mathcal{M}$ inside $\mathcal{M}$. And more importantly, it features prominently in the celebrated article of Shishikura [Sh] on the Hausdorff dimension of the boundary of $\mathcal{M}$. The "thick" quadratic Julia sets with Hausdorff dimension almost 2 are created in that paper by applying the second iteration of a parabolic renormalization procedure to a quadratic polynomial with a parabolic periodic orbit.

In the setting of critical circle maps most of the conjectures on the parabolic renormalization were resolved by one of the authors in [Ya2]. The parallel questions in the unimodal setting were handled by Hinkle in [Hin]. However, in [Ya3] a new approach to the renormalization of critical circle mappings was presented to handle the hyperbolicity problem. The classical (i.e. twenty years old) approach is to define the renormalization

Stony Brook IMS Preprint #2001/4
April 2001

in the language of *critical commuting pairs*. This is a forced measure due to a principal difficulty in constructing a well-defined renormalized circle map. This difficulty was finessed in [Ya3]. The resulting definition easily extends to the parabolic case (and has been known before in this setting), and, due to its naturality, radically simplifies the existing theory. In this paper we re-do the relevant results of [Ya2] and extend them to completely settle the Lanford's conjectures for the parabolic renormalization of critical circle maps. This new renormalization theory is notably simplier than the usual one. One of the most technical aspects, the complex *a priori* bounds can be replaced by an elementary argument of C. Petersen. More importantly, the *renormalization towers* introduced by McMullen in [McM2] in this context become bi-infinite extensions of *parabolic towers* developed by one of the authors in his study of geometric limits of rational maps [Ep1]. This allows us to invoke the "soft" results of [Ep1] to replace some of the most challenging parts of the usual renormalization theory.

The title of the paper refers to the attracting fixed point of the parabolic renormalization, whose existence we prove. It is a critical circle map with a parabolic fixed point, which should be thought of as having the rotation number represented by an infinite continued fraction, each entry of which equals to ∞.

Finally, let us comment on the relation of this work to other open questions in holomorphic dynamics. Of great interest is the question of understanding the action of the parabolic renormalization of quadratic maps used by Shishikura, which we have mentioned above. The convergence (or divergence) of this procedure is related to the measure-theoretic properties of quadratic Julia sets, the "holy grail" of the subject. Much of our approach translates to the Shishikura's setting, however, some crucial ingredients are missing. Another interesting question is the following. The new definition of renormalization of circle maps given in [Ya3] was used there to show the hyperbolicity of the periodic orbits of renormalization, but not their existence. The latter is done in the classical context. In the degenerate case studied in this paper, however, we handle both questions simultaneously. Perhaps this can give clues on how to do the same in the general case.

1. Preliminaries

Some notations. We use dist and diam to denote the Euclidean distance and diameter in $\mathbb{C}$. The notation $D_r(z)$ will stand for the Euclidean disk with center $z \in \mathbb{C}$ and radius r. The unit disk $D_1(0)$ will be denoted $\mathbb{D}$. By the circle $\mathbb{T}$ we understand the affine manifold $\mathbb{R}/\mathbb{Z}$, it is naturally identified with the unit circle $S^1 = \partial\mathbb{D}$. The real translation $x \mapsto x+\theta$ projects to the rigid rotation by angle θ, $R_\theta : \mathbb{T} \to \mathbb{T}$. For two points a and b in the circle $\mathbb{T}$ which are not diametrically opposite, $[a, b]$ will denote the shorter of the two arcs connecting them. As usual, $|[a, b]|$ will denote the length of the arc. For two points $a, b \in \mathbb{R}$, $[a, b]$ will denote the closed interval with endpoints a, b without specifying their order. The cylinder in this paper, unless otherwise specified will mean the affine manifold $\mathbb{C}/\mathbb{Z}$. Its equator is the circle $\{\operatorname{Im} z = 0\}/\mathbb{Z} \subset \mathbb{C}/\mathbb{Z}$. We denote π the natural projection $\mathbb{C} \to \mathbb{C}/\mathbb{Z}$. We shall say that $U \subset \mathbb{C}/\mathbb{Z}$ is an *equatorial annulus* if U is a topological annulus with a smooth boundary and $U \supset \mathbb{T}$. Given a hyperbolic Riemann surface X, we shall denote by $d_X(\cdot,\cdot)$ the Poincaré distance on X. We will sometimes use a symbol ∞ as a number with the conventions $1/\infty = 0$, $1 + \infty = \infty$.

For an equatorial annulus U we shall denote $\mathbf{A}(U)$ the space of bounded analytic maps $\phi : U \to \mathbb{C}/\mathbb{Z}$, such that $\phi(\mathbb{T})$ is homotopic to $\mathbb{T}$, equipped with the uniform metric. We shall turn $\mathbf{A}(U)$ into a Banach manifold as follows. Denote $\tilde{U}$ the lift $\pi^{-1}(U) \subset \mathbb{C}$. The space of functions $\tilde{\phi} : \tilde{U} \to \mathbb{C}$ which are analytic, continuous up to the boundary, and 1-periodic, $\tilde{\phi}(z+1) = \tilde{\phi}(z)$, becomes a Banach space when endowed with the sup norm. Denote that space $P(U)$. For a function $\phi : U \to \mathbb{C}/\mathbb{Z}$ denote $\check{\phi}$ an arbitrarily chosen lift $\pi(\check{\phi}(\pi^{-1}(z))) = \phi$. Observe that $\phi \in \mathbf{A}(U)$ if and only if $\tilde{\phi} = \check{\phi} - \mathrm{Id} \in P(U)$. We use the local homeomorphism between $P(U)$ and $\mathbf{A}(U)$ given by

$$\tilde{\phi} \mapsto \pi \circ (\tilde{\phi} + \mathrm{Id}) \circ \pi^{-1}$$

to define the atlas on $\mathbf{A}(U)$. The coordinate change transformations are given by $\tilde{\phi}(z) \mapsto \tilde{\phi}(z+n) + m$ for $n, m \in \mathbb{Z}$, therefore with this atlas $\mathbf{A}(U)$ is a Banach manifold.

We assume that the reader is familiar with the definition and basic properties of quasiconformal mappings. We use the notation σ_0 for the trivial Beltrami differential, representing the standard complex structure in a planar domain.

Critical circle mappings. A critical circle map is an orientation preserving homeomorphism $\mathbb{T} \to \mathbb{T}$ of class C^3 with a single critical point c. A further assumption is made that the critical point is of cubic type. This means that for a lift $\bar{f} : \mathbb{R} \to \mathbb{R}$ of a critical circle map f with critical points at integer translates of $\bar{c}$,

$$\bar{f}(x) - \bar{f}(\bar{c}) = (x - \bar{c})^3(\mathrm{const} + O(x - \bar{c})).$$

We note that all the renormalization results will hold true if in the above definition "3" as the order of smoothness and the order of the critical point is replaced by any other odd number. To fix our ideas, we will always place the critical point of f at $0 \in \mathbb{T}$.

Being a homeomorphism of the circle, a critical circle map f has a well-defined rotation number, denoted $\rho(f) \in \mathbb{T}$. It is not difficult to show (see e.g. [MvS]) that when this number is irrational the map f is topologically semi-conjugate to the rigid rotation $R_{\rho(f)}$. A celebrated result of Yoccoz [Yoc] implies that the semi-conjugacy is, in fact, a true conjugacy. If x_0 is a periodic point of f with period q, and $x_0, x_1, \ldots, x_{q-1}$ is the orbit of x_0 listed in the counterclockwise order, then the combinatorial rotation number of x_0 is p/q where $f(x_0) = x_p$. Recall that $\rho(f)$ is a rational number equal to p/q in the reduced form if and only if f has a periodic point with combinatorial rotation number p/q.

We will find it useful to express the rotation number as a continued fraction

$$\rho(f) = \cfrac{1}{r_0 + \cfrac{1}{r_1 + \cfrac{1}{r_2 + \cdots}}} \tag{1.1}$$

with $r_i \in \mathbb{N} \cup \{\infty\}$ (further on we will abbreviate such an expression as $[r_0, r_1, r_2, \dots]$ for typographical convenience.) The expansion (1.1) is infinite if, and only if, the rotation number is irrational, in which case the terms r_i are uniquely defined. A rational number has more than one expansion. Below we will resolve this ambiguity with the help of the dynamics of f.

Let f be a critical circle map, and denote $\bar{f} : \mathbb{R} \to \mathbb{R}$ the lift of f with $\bar{f}(0) \in [0,1)$. Let us define a possibly finite sequence of positive rationals $\{p_m/q_m\}$ as follows. If there is a natural number m such that $\bar{f}^{m+1}(0) - 1 \in [0, \bar{f}(0))$ then $p_0/q_0 = 1/m$, otherwise, the sequence is empty. In the former case set I_0 to be the projection of the interval $[0, \bar{f}(0)]$ onto the circle. Now inductively define p_m/q_m, and I_m as follows: let q_m be the smallest natural number for which $f^{q_m}(I_{m-1}) \ni 0$, and let $p_m \in \mathbb{N}$ be such that $\bar{f}^{q_m}(\bar{I}_{m-1}) - p_m \ni 0$ (here $\bar{I}_m$ is the lift of I_m containing 0), and set $I_m \subset \mathbb{T}$ to be the closed arc between 0, $f^{q_m}(0)$ which does not contain $f^{q_{m-1}}(0)$. It is easy to see that when $\rho(f)$ is irrational, the numbers p_m/q_m are the finite convergents of the infinite continued fraction (1.1):

$$p_m/q_m = [r_0, r_1, \dots, r_{m-1}] \tag{1.2}$$

On the other hand, if $\rho(f)$ is rational, the sequence $\{p_m/q_m\}$ is finite, and if n is the number of its terms, then the equations (1.2) uniquely define a finite continued fraction

$$\sigma(f) = [r_0, r_1, \dots, r_{n-1}, \infty] \tag{1.3}$$

Of course, $\sigma(f)$ is a continued fraction expansion of $\rho(f)$, but now it is canonically prescribed. Since it reflects the ordering of the critical orbit of f with respect to the standard orientation of the circle, we will refer to $\sigma(f)$ as the combinatorial type of f. As a useful example, note that maps with the combinatorial types $[\infty]$ and $[1, \infty]$ have the same rotation number $0 \in \mathbb{T}$. When the rotation number of f is irrational we will denote $\sigma(f)$ the continued fraction (1.1), and again call it the combinatorial type of f.

Note that we may still define the quantities $\rho(f)$ and $\sigma(f)$ in the case when f leaves invariant a topological circle $S \ni 0$ (rather than the affine circle $\mathbb{T}$), and restricts to a homeomorphism $S \to S$.

An important one-parameter family of examples of critical circle maps, the so-called standard (or Arnold's) maps, is obtained as follows. Given any $\theta \in \mathbb{R}$ the map

$$A_\theta(x) = x + \theta - \frac{1}{2\pi} \sin 2\pi x$$

commutes with the unit translation: $A(x+1) = A(x) + 1$. Therefore it has a well-defined projection to an endomorphism of the circle. We denote this endomorphism f_θ, although in reality it only depends on the class $\theta \bmod \mathbb{Z}$. An elementary computation shows that every A_θ is strictly monotone, and has critical points at integer values of x, all of cubic type. Therefore, each f_θ is a critical circle mapping. The considerations of monotone dependence on parameter imply that

$$\theta \to \rho(\theta) \equiv \rho(f_\theta)$$

is a continuous monotone map of $\mathbb{T}$ onto itself. Whenever $t \in \mathbb{T}$ is irrational, $\rho^{-1}(t)$ is a single point. For $t = p/q$ the set $\rho^{-1}(t)$ is a closed interval, for every parameter value in this interval the homeomorphism f_θ has a periodic orbit with combinatorial rotation number p/q. This orbit has eigenvalue one at the two endpoints of the interval.
The significance of these examples in the Renormalization Theory is due in part to the following rigidity property of Arnold's maps:

Proposition 1.1 (cf. [dF1, dF2]). *Every real-analytic self-map of $\mathbb{C}/\mathbb{Z}$ which is topologically conjugate to a member of the family $\{f_\theta\}$ belongs to this family itself.*

In combination with the properties of the dependence $\rho(\theta)$ quoted above, this yields a corollary:

Proposition 1.2. *Let g be a real-analytic self-map of the cylinder $\mathbb{C}/\mathbb{Z}$ topologically conjugate to a map f_θ. If $\rho(\theta)$ is irrational, or f_θ has a periodic orbit with eigenvalue one in $\mathbb{T}$, then $g \equiv f_\theta$.*

2. Parabolic renormalization

A review of parabolic bifurcation. We present here a review of the theory of parabolic bifurcations, as applied in particular to an analytic critical circle mapping. A detailed account of the theory can be found in a recent paper of Shishikura [Sh], we only give a brief summary of the relevant facts.

Let f be an analytic map defined in some region $\mathrm{Dom}(f) \subset \mathbb{C}$. Assume that f has a parabolic periodic point p of period n with unit eigenvalue: $(f^n)'(p) = 1$. A topological disk U^A is called an *attracting petal* of f at p if $\bar{U}^A \ni \{p\}$ and $f^n(\bar{U}^A) \subset U^A \cup \{p\}$. Similarly, U^R is a *repelling petal* for f if it is an attracting petal for the local branch of f^{-1} which fixes p. We further say that p is a *simple parabolic point* if any two attracting petals have a non-empty intersection. This condition is equivalent to the existence of a pair of petals U^A, U^R whose union forms a punctured neighborhood of p.

Theorem 2.1 (Fatou Coordinates). *In the above notation,*

$$\bigcap_{k=0}^{\infty} f^{nk}(\mathrm{cl}\, U^A) = \{p\}, \ \text{and} \ \bigcap_{k=0}^{\infty} f^{-nk}(\mathrm{cl}\, U^R) = \{p\}.$$

Moreover, there exist injective analytic maps

$$\phi^A : U^A \to \mathbb{C} \text{ and } \phi^R : U^R \to \mathbb{C},$$

unique up to post-composition by translations, such that

$$\phi^A(f^n(z)) = \phi^A(z) + 1 \text{ and } \phi^R(f^n(z)) = \phi^R(z) - 1.$$

The Riemann surfaces $C^A = U^A/\eta_0$ and $C^R = U^R/\eta_0$ are conformally equivalent to the cylinder $\mathbb{C}/\mathbb{Z}$.

We denote $\pi_A = \pi_{A,f} : U^A \to C^A$ and $\pi_R = \pi_{R,f} : U^R \to C^R$ the natural projections. The quotients C^A and C^R are customarily referred to as *Écalle–Voronin cylinders*; we will find it useful to regard these as Riemann spheres with distinguished points $\pm$ filling in the punctures. Any conformal *transit homeomorphism* $\tau : C^A \to C^R$ fixing the ends $\pm$ is a translation in suitable coordinates. Lifiting it produces a map $\bar{\tau} : U^A \to \mathbb{C}$ satisfying

$$\tau \circ \pi_A = \pi_R \circ \bar{\tau}.$$

We will write $\tau \equiv \tau_\theta$, and $\bar{\tau} = \bar{\tau}_\theta$, where

$$\phi^R \circ \bar{\tau} \circ (\phi^A)^{-1}(z) \equiv z + \theta \bmod(z).$$

Of particular importance to us will be the case when f is an analytic critical circle map, defined by analytic extension in some annular neighborhood $\mathrm{Dom}(f) \subset \mathbb{C}/\mathbb{Z}$ of $\mathbb{T}$. In this case we may select $\mathbb{T}$-symmetric petals U^A and U^R. The intersections $\mathbb{T} \cap U^A$ and $\mathbb{T} \cap U^R$ will project to the *natural equators* of the corresponding Écalle-Voronin cylinders $E^A \subset C^A$ and $E^R \subset C^R$.

Let us make some additional assumptions on the global properties of the map f. A connected closed set $B \subset \mathrm{Dom}(f)$ is called a *proper immediate basin* of the parabolic point p if $\overset{\circ}{B}_f$ is a connected component of the basin of p such that $f^n(B) = B$, and the map $f^n : B \to B$ is proper. The degree of the basin is the degree of f^n as a branch covering of B. The standard considerations imply:

Proposition 2.1. *A proper immediate basin contains a critical value of f^n.*

Under additional assumptions some further conclusions can be drawn:

Proposition 2.2. *Suppose f has a simple parabolic periodic point p which possesses an immediate basin B, and assume that B contains a single critical point c of of f^n of degree d. Then the following holds:*

- *The parabolic point p has a unique immediate basin B.*
- *The domain B is simply-connected.*
- *The attracting Fatou coordinate extends via the equation $\phi^A(f^n(z)) = \phi^A(z) + 1$ to an infinite-degree branched covering*

  $$\phi^A : B \to \mathbb{C}.$$

  It is branched at the points $z \in B$ for which $f^{nl}(z) = c$ for some $l \geq 0$. All critical points of ϕ^A have the same degree d.

We illustrate the situation with the next figure, depicting the immediate basin of a critical circle map with a simple parabolic fixed point. The crescents are two fundamental domains for the dynamics of f in the neighborhood of p, which project to C^A and C^R under the Fatou coordinates.

We now turn our attention to small perturbations of parabolic maps. For an analytic critical circle map g which is sufficiently close to f in the uniform metric on $\mathrm{Dom}(f)$, the parabolic point splits into a complex conjugate pair of repelling fixed points $p_g \in \mathbb{H}/\mathbb{Z}$ and

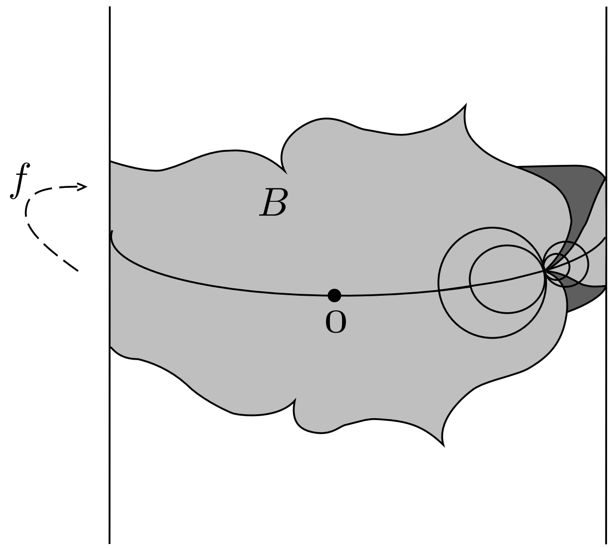


FIGURE 1. The immediate basin of a parabolic fixed point of a critical circle map

$\bar{p}_g$ with multipliers $e^{2\pi i \pm \alpha(g)}$. It is easily seen from the Holomorphic Index Formula that if we select a a neighborhood $\mathcal{W}(f) \subset \mathbf{A}(\mathrm{Dom}(f))$ small enough, then for $g \in \mathcal{W}(f)$ we have $|\arg \alpha(g)| < \pi/4$. In this situation one may still speak of attracting and repelling petals:

Lemma 2.3 (Douady Coordinates). *For every critical circle map $g \in \mathcal{W}(f)$ there exist $\mathbb{T}$-symmetric topological discs U_g^A and U_g^R whose union is a neighborhood of p, and injective analytic maps*

$$\phi_g^A : U^A \to \mathbb{C} \text{ and } \phi_g^R : U_f^R \to \mathbb{C}$$

unique up to post-composition by translations, such that

$$\phi_g^A(g^n(z)) = \phi_g^A(z) + 1 \text{ and } \phi_g^R(g^n(z)) = \phi_g^R(z) - 1.$$

The quotients $C_g^A = U_g^A/g^n$ and $C_g^R = U_g^R/g^n$ are Riemann surfaces conformally equivalent to $\mathbb{C}/\mathbb{Z}$.

Assigning the value at an arbitrarily selected basepoint specifies a Fatou or Douady coordinate uniquely. For further convenience the following choice will be made. Select $k, m \in \mathbb{N}$ large enough so that $g^k(0) \in U_g^A$ and $(g|_{\mathbb{T}})^{-m}(0) \in U_g^R$ throughout $\mathcal{W}(f)$, and set

$$\phi^A(f^k(0)) = \phi_g^A(g^k(0)) = 0, \text{ and } \phi^R((f|_{\mathbb{T}})^{-m}(0)) = \phi_g^R((g|_{\mathbb{T}})^{-m}(0)) = 0.$$

The following fundamental theorem first appeared in [DH]:

Theorem 2.2 (Continuity of Douady Coordinates). *With these normalizations we have*

$$\phi_g^A \to \phi^A \text{ and } \phi_g^R \to \phi^R$$

uniformly on compact subsets of U^A and U^R respectively.

Moreover, select the smallest $n(g) \in \mathbb{N}$ for which $g^{n(g)}(a) \geq r$. Then

$$g^{n(g)}(z) = (\phi_g^R)^{-1} \circ T_{\theta(g)+K} \circ \phi_g^A$$

wherever both sides are defined. In this formula $T_a(z)$ denotes the translation $z \mapsto z + a$, $\theta(g) \in [0, 1)$ is given by

$$\theta(g) = 1/\alpha(g) + \underset{\alpha(g)\to\infty}{o(1)} \mod 1,$$

and the real constant K is determined by the choice of the basepoints a, r. Thus for a sequence $\{g_k\} \subset \mathcal{W}(f)$ converging to f, the iterates $g_k^{n(g_k)}$ converge locally uniformly if and only if there is a convergence $\theta(g_k) \to \theta$, and the limit in this case is a certain lift of the transit homeomorphism τ_θ for the parabolic map f.

Definition of parabolic renormalization. Let f be an analytic critical circle map with a simple parabolic orbit of period n. Denote p the element of the orbit given by $f^n(0) \to p$. Let $z \in C^R$, and take an arbitrary preimage $\zeta = \pi_R^{-1}(z) \in U^R$. Suppose that the forward orbit of ζ intersects the attracting petal U^A, and denote $n(\zeta)$ the first moment that $f^{n(\zeta)}(\zeta) \in U^A$. Define $E_f(z) = \pi_A(f^{n(\zeta)}(\zeta)) \in C^A$. Clearly this expression does not depend on the choice of the preimage ζ, and defines an analytic map from a neighborhood of the equator of C^R to C^A. It is not difficult to see that the restriction $E_f|_\mathbb{T}$ is a critical circle map.

Recall the definition of cylinder renormalization $\mathcal{R}_{\text{cyl}}$ given in [Ya3]. The continuity of Douady coordinates implies:

Proposition 2.4. *Suppose f_k is a sequence of Epstein critical circle maps with $f_k \to f$ and $\mathcal{R}_{\text{cyl}}(f_k) \to g$. Assume also that f has a simple parabolic fixed point. Then*

$$g = \tau \circ E_f$$

where $\tau : C^A \to C^R$ is a transit isomorphism.

Fix a rational number $p/q \in [0, 1)$, $(p, q) = 1$, and let σ be a continued fraction expansion of p/q of the form (1.3). Select the smallest value $\theta = \theta(\sigma) \in \mathbb{T}$ for which the composition $\tau_\theta \circ E_f$ has a parabolic periodic orbit, and the combinatorial type σ. We call $\tau_{\theta(\sigma)} \circ E_f$ the *σ-parabolic renormalization of f* and write

$$\mathcal{P}_\sigma(f) = \tau_{\theta(\sigma)} \circ E_f.$$

In the case when $\sigma = [\infty]$, (which implies that the map has a parabolic fixed point), we drop the index σ. Our apparently arbitrary choice of the particular transit map which produces the desired combinatorial type will not influence our analysis, since in the relevant to us cases it will be uniquely defined (see Proposition 2.7).

For any irrational number $\rho \in (0, 1)$ the value $\theta = \theta(\rho) \in \mathbb{T}$ such that $\rho(\tau_{\theta(\rho)} \circ E_f) = \rho$ is unique, due to the standard considerations of monotonicity. If σ is the continued fraction expansion of ρ (1.1), we define the *σ-parabolic renormalization of f* as

$$\mathcal{P}_\sigma(f) = \tau_{\theta(\rho)} \circ E_f.$$

As a corollary of Proposition 2.2 we have the following:

Proposition 2.5. *Suppose f has a simple parabolic periodic point p as above. Assume that the basin B of p is proper, and that 0 is the only critical point in B. Then the corresponding map E_f has a maximal analytic extension to an annulus $\mathrm{Dom}(E_f)$, and*

$$E_f : \mathrm{Dom}(E_f) \to \mathbb{C}/\mathbb{Z}$$

is an infinite degree branched covering. It is ramified over a single critical value $E_f(0) = 0 \in \mathbb{C}/\mathbb{Z}$.

Corollary 2.6. *Let f be as above, p/q a rational number in $[0, 1)$, and σ a continued fraction expansion of ρ with terms in $\mathbb{N} \cup \{\infty\}$. Then the parabolic orbit of the map $\mathcal{P}_\sigma(f)$ is simple, and with a proper immediate basin of degree 3.*

Proof. The argument is standard. Firstly note that by Proposition 2.5 the renormalization $\mathcal{P}_{p/q}(f)$ is an infinite degree branched covering $\mathrm{Dom}(E_f) \to \mathbb{C}/\mathbb{Z}$ with a single critical value. Set $g = \mathcal{P}_{p/q}(f)$. Let p be a parabolic periodic point of g and let $U^A = U^A_0$ be an attracting petal at p. We inductively define U^A_{-i} for $i > 0$ to be the connected component of the preimage $g^{-q}(U^A_{-(i-1)})$ containing $U^A_{-(i-1)}$. The covering property of g implies that $g^q : U^A_{-i} \to U^A_{-(i-1)}$ is a proper map. Hence, if we denote $B = \cup U^A_{-i}$, the iterate g^q properly maps B onto itself. By definition, B is an immediate basin. Since the branched covering $g : \mathrm{Dom}(E_f) \to \mathbb{C}$ posesses a unique critical value, there are no other immediate basins. This implies that B contains a single critical point. Finally, the existence of two disjoint attracting petals would imply that B is multiply-connected, and hence p is a simple parabolic point. □

Proposition 2.7. *Again, let f be as in Proposition 2.5. Then for every rational or irrational combinatorial type σ there exists a unique $\theta \in \mathbb{T}$ such that*

$$\sigma(\tau_\theta \circ E_f) = \sigma,$$

and if σ is a rational continued fraction, then $\tau_\theta \circ E_f$ has a parabolic fixed point.

Proof. When σ is irrational, the claim follows from the standard considerations of monotone dependence on the parameter. Let us consider the case when σ is rational. A classical argument due to Fatou implies that since $E_f : \mathrm{Dom}(E_f) \to \mathbb{C}/\mathbb{Z}$ is a branched covering map, every non-repelling periodic orbit of $g_\theta \equiv \tau_\theta \circ E_f$ must attract a critical value of g_θ. Hence only one such orbit may exist. Now let g_{θ_0} have a parabolic orbit with combinatorial rotation number p/q, such that $\sigma(g_{\theta_0}) = \sigma$. Denoting $\bar{g}_\theta$ the lift of g_θ with $\bar{g}_\theta(0) \in [0, 1)$, we have $\bar{g}^q_{\theta_0}(x) = x + p$ for some values of $x \in \mathbb{R}$. Uniqueness of the non-repelling orbit of g_{θ_0} implies that the graph of $\bar{g}_{\theta_0}$ must lie either entirely above, or entirely below the line $y = x + p$. Since the dependence $\theta \mapsto \bar{g}_\theta(x)$ is strictly monotone, the values of θ_0 realizing each of these two possibilities is unique, and by our conventions they correspond to different combinatorial types representing the rotation number p/q. □

For ease of reference let us further denote $\mathbf{Par}_\sigma$ the collection of analytic maps f defined in an equatorial annulus $\mathrm{Dom}(f) \subset \mathbb{C}/\mathbb{Z}$ containing 0, and such that: 0 is a cubic critical point of f, the map f leaves invariant an equatorial topological circle $S \ni 0$, the restriction $f : S \to S$ is a homeomorphism of combinatorial type σ, and finally, the map f has a parabolic periodic orbit in S whose immediate basin contains 0, and is proper and of degree 3. As seen from the next proposition, the conditions may be weakened:

Proposition 2.8. *Suppose that f is an analytic map of an equatorial annulus $\mathrm{Dom}(f) \ni 0$ with a cubic critical point at 0, and a parabolic periodic orbit whose immediate basin contains 0, and is proper and of degree 3, and contains in its closure an equatorial curve which passes through the parabolic points. Then there is an equatorial topological circle S which contains the critical point 0 and the parabolic orbit, such that f homeomorphically maps S to S.*

The proof is a simple application of the existence of the Fatou coordinates, and is left to the reader.

To conclude this section let us formulate a useful observation, which follows directly from the uniqueness of the Fatou coordinates:

Proposition 2.9. *Suppose $f, g \in \mathbf{Par}_{\sigma_0}$ and there exists a neighborhood $U_f \supset \mathbb{T}$ containing the immediate parabolic basin of f and a K-quasiconformal homeomorphism $h : U_f \to h(U_f)$ such that $h \circ f \circ h^{-1} = g$. Then the projection $\hat{h} = \pi_{R,g} \circ h \circ (\pi_{R,f})^{-1}$ is a K-quasiconformal homeomorphism of $\mathbb{C}/\mathbb{Z}$ which is also a conjugacy:*

$$\hat{h} \circ \mathcal{P}_\sigma(f) \circ \hat{h}^{-1} = \mathcal{P}_\sigma(g) \text{ for any combinatorial type } \sigma.$$

3. Statements of the results

For a finite continued fraction $\sigma_1 = [r_0, \ldots, r_n]$ with $r_m \in \mathbb{N} \cup \infty$ and a possibly infinite continued fraction $\sigma_2 = [s_0, \ldots, s_m, \ldots]$ with $s_m \in \mathbb{N} \cup \infty$ let us denote $\sigma_1 * \sigma_2$ the continued fraction $[r_0, \ldots, r_n, s_0, \ldots, s_m, \ldots]$. Let us denote Σ^- the collection of all formal continued fractions $[r_i]_{i=-\infty}^0$ where $r_i \in \mathbb{N} \cup \{\infty\}$, and the symbol ∞ appears infinitely often, and let Σ^+ be the set of all formal continued fractions $[r_i]_{i=0}^\infty$ with $r_i \in \mathbb{N} \cup \{\infty\}$. We also let Σ be the set of bi-infinite formal continued fractions $[r_i]_{i=-\infty}^\infty$ with $[r_i]_{i=-\infty}^0 \in \Sigma^-$ and $[r_i]_{i=0}^\infty \in \Sigma^+$. Every continued fraction in Σ^-, can be factored as

$$[r_i]_{i=-\infty}^0 = \cdots * \sigma_{-n} * \sigma_{-(n-1)} * \cdots * \sigma_0 \tag{3.1}$$

where a continued fraction σ_n is a finite fraction whose terms are natural numbers except for the last one, which is ∞ (and represents, therefore, a combinatorial type of a critical circle map with a rational rotation number). We proceed to formulate the principal result of this paper:

Theorem 3.1. (Main Theorem) *There exists a collection $\mathcal{A}$ of analytic critical circle maps and a bijection $\iota : \Sigma^- \to \mathcal{A}$, which is continuous with respect to the weak topology on Σ^- and the uniform topology on $\mathcal{A}$, and has the following properties:*

(a) *the image* $f_{[r_i]_{-\infty}^0} \equiv \iota([r_i]_{-\infty}^0) \in \mathbf{Par}_{\sigma_0}$, *where* σ_n *is as in (3.1) and*

$$\mathcal{P}_{\sigma_1}(f_{[r_i]_{-\infty}^0}) = f_{\cdots*\sigma_{-n}*\sigma_{-(n-1)}*\cdots*\sigma_0*\sigma_1}$$

(b) *for any* $f \in \mathcal{P}_\sigma$ *for some* σ, *any* $g \in \mathcal{A}$, *and an arbitrarily chosen infinite sequence of rational combinatorial types* $\sigma_1, \dots, \sigma_n, \dots$ *we have*

$$\operatorname{dist}(\mathcal{P}_{\sigma_n} \circ \cdots \circ \mathcal{P}_{\sigma_1} f, \mathcal{P}_{\sigma_n} \circ \cdots \circ \mathcal{P}_{\sigma_1} g) \xrightarrow{i\to\infty} 0$$

in the uniform metric on $\mathbb{T}$

(c) *suppose* f *is an analytic critical circle map with* $\rho(f) = [a_0, a_1, \dots] \in \mathbb{R} \setminus \mathbb{Q}$, *and assume that for some sequence* $n_j \to \infty$

$$a_{n_j+i} \to r_i \text{ for all } i \le 0, \text{ where } [r_i]_{-\infty}^0 \in \Sigma^-.$$

Then $\mathcal{R}_{\mathrm{cyl}}^{n_j} f \to \iota([r_i]_{-\infty}^0)$.

(d) *finally, let* $\bar{s} = [r_i]_{i=-\infty}^0 \in \Sigma^-$ *be a periodic sequence with period* k, $r_{i+k} = r_i$, *and let* f *be the corresponding periodic point of the parabolic renormalization in* $\mathcal{A}$. *Suppose the finite formal continued fraction* $[r_{-k+1}, \dots, r_0]$ *factors into a product of combinatorial types* $\sigma_m * \cdots * \sigma_1$, *and denote*

$$P = \mathcal{P}_{\sigma_1} \circ \cdots \circ \mathcal{P}_{\sigma_m} : f \mapsto f.$$

Then there exists a neighborhood $U_f \subset \mathbb{T}$ *in which* f *is defined, and a neighborhood* Y *of* f *in* $\mathbf{A}(U_f)$ *such that* $W = \mathbf{Par}_{\sigma_1} \cap \mathbf{A}(U_f) \cap Y$ *is an analytic submanifold which* P *leaves invariant, the restriction* $P : W \to W$ *is analytic, and the differential* $D_f(P|_W)$ *is a compact operator whose spectral radius is strictly less than* 1. *Moreover, for every* $g \in \mathbf{Par}_{\sigma_1}$ *there exists an iterate* $P^n(g) \in W$, *hence the rate of convergence* $P^k g \to f$ *is eventually geometric.*

Let us make several remarks. Firstly, the renormalization convergence statement in paper [Ya2], formulated for critical commuting pairs, is stronger than the statement (c) in the above theorem. However, the proofs of [Ya2] require different (and quite more involved) analytic techniques.

Also, the proof of continuity of the mapping ι requires the tower rigidity result proved in [Ya2]. Otherwise, our exposition is self-contained. We note that we were unable to show that ι is a homeomorphism, although we expect it to be one. The continuity of ι^{-1} would follow from the negative answer to the following question:

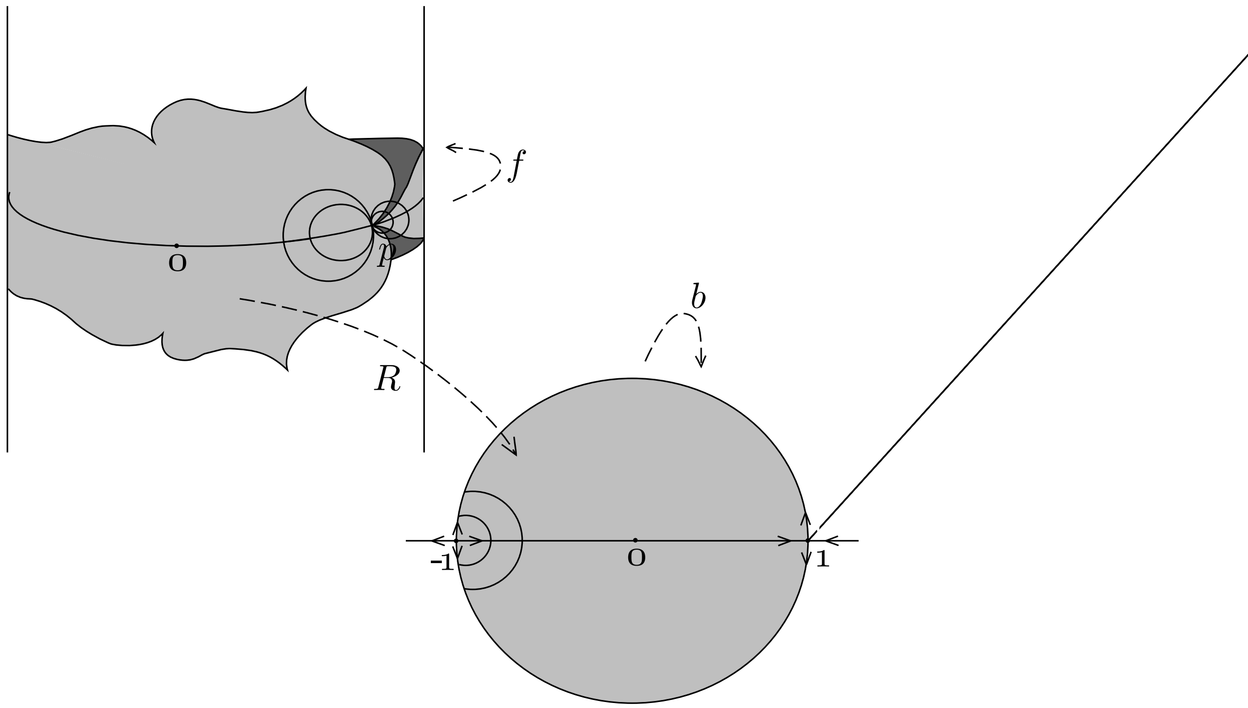
Do there exist rational combinatorial types σ, σ_1, *an analytic cylinder-renormalizable critical circle map* f, *and a critical circle map* $g \in \mathbf{Par}_\sigma$ *such that* $\mathcal{R}_{\mathrm{cyl}} f = \mathcal{P}_{\sigma_1} g$?

The corresponding statement for commuting pairs is quite obvious (see [Ya2]).

A corollary of the Main Theorem is the existence of the universal parabolic map announced in the title of this paper:

Theorem 3.2. (The Universal Parabolic Map) *There exists a critical circle map* f_* *with a parabolic fixed point in* $\mathbb{T}$ *with* $\sigma(f) = [\infty]$ *such that the following holds:*

FIGURE 2. Petersen's argument



(a) $\mathcal{P}f_* = f_*$

(b) *for any combinatorial type* σ *and any* $f \in \mathbf{Par}_\sigma$ *the sequence* $\mathcal{P}^n f \to f_*$

(c) *also for any analytic critical circle map* f *with* $\rho(f) = [a_0, a_1, \dots]$ *with* $a_i \to \infty$*, the renormalizations* $\mathcal{R}^{\circ n}_{\text{cyl}} f \to f_*$

(d) *finally there exists a neighborhood* $U \supset \mathbb{T}$ *in which* f_* *is defined and a neighborhood* Y *of* f_* *in* $\mathbf{A}(U)$ *such that setting* $W = Y \cap \mathbf{Par}_{[\infty]}$ *we have* $f_* : W \to W$ *analytic, and* $D_{f_*}(\mathcal{P}|_Y)$ *is a compact operator whose spectral radius is less than* 1.

4. PETERSEN'S COMPLEX *a priori* BOUNDS

Of crucial importance in what follows will be the following observation of Petersen [Pet] establishing a complex *a priori* bound on a parabolic renormalization of a critical circle map:

Theorem 4.1 (Petersen). *Let* $f \in \mathbf{Par}_\sigma$ *be a critical circle mapping, and let* E_f *as before denote the Écalle mapping associated with one of the parabolic points of* f*. Then the modulus of the annulus* $\mathrm{Dom}(E_f)$ *is a universal positive constant, which does not depend on* f*.*

Proof. The proof is illustrated in Fig. 4 for the case when f has a parabolic fixed point. As an excercise, the reader is invited to determine the combinatorial type of f in the figure.

Let n be the period of the parabolic orbit of f, and let p be the element of the orbit with $f^n(0) \to p$. Denote B_1, B_2 the two immediate parabolic basins of f whose boundaries contain p labelled so that $B_2 \ni 0$. Let $\vartheta : B_1 \to \mathbb{D}$ be a real-analytic Riemann map specified uniquely by the requirement that the critical point of $f^n|_{B_1}$ is mapped to 0. The map

$$b \equiv \vartheta \circ f^n \circ \vartheta^{-1} : \mathbb{D} \to \mathbb{D}$$

is a cubic Blaschke product. The local dynamical picture at p implies that one of the fixed points -1, 1 of b is parabolic, and the other is repelling. To fix the ideas assume that $b'(1) = 1$. Observe that the real Blaschke product of degree 3 satisfying the conditions $b(1) = 1$, $b'(1) = 1$, $b(-1) = -1$, $b'(0) = 0$, $b''(0) = 0$ is unique. In particular, the repelling eigenvalue $b'(-1) > 1$ does not depend on the choice of f. Note, that the annulus $\mathrm{Dom}(E_f)$ is equal to the quotient $(U^R \cap B_1)/f^n$, for an arbitrary choice of the repelling petal U^R at p. Hence, it is isomorphic to $A = (D_r(-1) \cap \mathbb{D})/b$ for a sufficiently small positive r. The symmetry of the construction implies that A is a cylinder whose modulus is equal to half of the modulus of the quotient torus of -1,

$$\mathrm{mod}(\mathrm{Dom}(E_f)) = \mathrm{mod}(A) = \frac{\pi}{\log(b'(-1))}.$$

□

In fact, the following significantly stronger statement will be proved in §6 independently of the above result, under an extra assumption on f:

Theorem 4.2. *Assume in addition that f extends to a branched covering map $\mathrm{Dom}(f) \mapsto \mathbb{C}/\mathbb{Z}$ with a single critical value. Let $\phi : \mathrm{Dom}(E_f) \to A_r$ denote the $\mathbb{T}$-symmetric conformal isomorphism onto a round annulus $A_r = \{|\operatorname{Im} z| < r\}/\mathbb{Z}$, with the normalization $\phi(0) = 0$. Then the composition*

$$\Psi = R_{-E_f(0)} \circ E_f$$

does not depend on the choice of f.

5. Towers of parabolic maps

Let $\mathcal{T} = (f_i)_{-n_1}^{n_2}$, $0 \leq n_i \leq \infty$ be a sequence of analytic critical circle maps. Assume that for each $i < n_2$ the map f_i has a simple parabolic orbit with a proper basin of degree 3. In the case when $n_2 < \infty$, we will require that $\rho(f_{n_2}) \in \mathbb{R} \setminus \mathbb{Q}$. We call $\mathcal{T}$ a *parabolic tower* if for every $i < n_2$

$$\mathcal{P}_{\sigma(f_{i+1})}(f_i) = f_{i+1}.$$

In the case when $n_1 = \infty$ we call $\mathcal{T}$ a *bi-infinite tower*, otherwise we say that $\mathcal{T}$ is a *forward-infinite tower*, and call f_{-n_1} the *base map of* $\mathcal{T}$. We use the word "infinite" even when $n_2 < \infty$, since in this case $\rho(f_{n_2}) \notin \mathbb{Q}$, hence f_{n_2} is infinitely renormalizable, and we may think of the successive renormalizations $\mathcal{R}^n_{\mathrm{cyl}} f_{n_2}$ as the forward part of the tower. For a bi-infinite tower $\mathcal{T} = (f_i)_{-\infty}^{n_2}$ and $n < n_2$ let us denote $\mathcal{T}|_{\geq n}$ the forward-infinite tower

$(f_i)_n^{n_2}$. The approach to renormalization convergence via rigidity of bi-infinite towers was pioneered by McMullen [McM2] in the context of Feigenbaum renormalization. Since his original proof, towers were utilized in similar contexts by others (see for example [dFdM2], and the work of one of the authors [Ya2]). Independently, the special case of forward-infinite parabolic towers considered in this paper was a central subject of the study by the other author [Ep1] of geometric limits of finite-type complex-analytic maps. The present paper combines the use of the rigidity theorem of [Ep1] and some of McMullen's arguments.

Let us define the *combinatorial type* of a tower $\mathcal{T} = (f_i)_{-n_1}^{n_2}$ to be the formal continued fraction $\sigma(\mathcal{T})$ which in the case when $n_2 = \infty$ is the product $\cdots * \sigma(f_{-i}) * \cdots \sigma(f_0) * \cdots \sigma(f_i) * \cdots$, and when $n_2 < \infty$ is the product $\cdots * \sigma(f_{-i}) * \cdots \sigma(f_0) * \cdots * \sigma(f_{n_2-1}) * \sigma(f_{n_2})$. Thus, in the case when $\mathcal{T}$ is a bi-infinite tower, we have $\sigma(\mathcal{T}) \in \Sigma$. A forward-infinite tower of any given combinatorial type may be constructed in an obvious way, by considering successive parabolic renormalizations of a parabolic critical circle map. Let us show how examples of bi-infinite towers can be constructed. Consider a sequence of forward-infinite towers $\mathcal{T}_i = (f_{k,i})_{-m_i}^{\infty}$ with $m_i \to \infty$. By Petersen's complex *a priori* bounds, for every $k \in \mathbb{Z}$ the maps $f_{k,i}$ form a normal family in a neighborhood of the circle. We may therefore find a subsequence i_l for which

$$f_{k,i_l} \to g_k \text{ for all } k \in \mathbb{Z}.$$

Every g_k is a critical circle map. If we additionally assume that for every $k < n_2$ the denominators of the fractions $p_{k,i}/q_{k,i} = \rho(f_{k,i})$ form a bounded sequence, then every g_k has a rational rotation number p_k/q_k, and is necessarily parabolic. The considerations of continuity imply that $g_{k+1} = \mathcal{P}_{\sigma(g_k)} g_k$, and so the sequence $(g_k)_{-\infty}^{n_2}$ forms a parabolic tower. We will call a tower constructed in this way a *limiting bi-infinite tower.* As an excercise, we leave to the reader the proof of the following:

Proposition 5.1. *For any $n_2 \leq \infty$ there exists For any formal continued fraction $\bar{\sigma} \in \Sigma$ there exists a limiting bi-infinite tower $\mathcal{T}$ with $\sigma(\mathcal{T}) = \bar{\sigma}$.*

Consider a forward-infinite tower $\mathcal{T} = (f_i)_{-n_1}^{n_2}$. Since $\mathcal{P}_{\sigma(f_{i+1})}(f_i) = f_{i+1}$, we may naturally identify the domain of definition $\text{Dom}(f_{i+1})$ with a subdomain of the repelling Fatou cylinder $C^R_{f_i}$. Denote $\pi_{A,i} : U^A_{f_i} \to C^A_{f_i}$, $\pi_{R,i} : U^R_{f_i} \to C^R_{f_i}$ the projections to the Fatou cylinders of f_i, and let $(\pi_{A,i})^{-1}$, $(\pi_{R,i})^{-1}$ be their arbitrary inverses. Let τ_{i+1} be the transit isomorphism from

$$\mathcal{P}_{\sigma(f_{i+1})}(f_i) = \tau_{i+1} \circ E_{f_i},$$

and g_{i+1} its lift $(\pi_{R,i})^{-1} \circ \tau_{i+1} \circ \pi_{A,i}$. By Proposition 2.2 the mapping g_{i+1} extends to a branched covering $B_i \to \mathbb{C}/\mathbb{Z}$ where B_i is a component of the immediate basin of f_i. Let $p_i^{\mathcal{T}}$ be the projection $\pi_{R,i-1} \circ \cdots \circ \pi_{R,-n_1+1} \circ \pi_{R,-n_1}$ defined on an appropriate subset of $\text{Dom}(f_{-n_1})$ and set $\hat{f}_i = (p^{\mathcal{T}})_i^{-1} \circ f_i \circ p_i^{\mathcal{T}}$, $\hat{g}_{i+1} = (p_i^{\mathcal{T}})^{-1} \circ g_{i+1} \circ p_i^{\mathcal{T}}$. By the *dynamics of the tower* $\mathcal{T}$ we will understand the abelian semigroup of analytic maps $D(\mathcal{T})$ generated by the collection of maps $\{\hat{f}_i, \hat{g}_{i+1}\}$ for all i between $-n_1$ and n_2. The *orbit* of a point $x \in \mathbb{C}/\mathbb{Z}$ under the tower $\mathcal{T}$ is the set $O_{\mathcal{T}}(x) = \cup h(x)$, where the union is over all $h \in D(\mathcal{T})$ for which $h(x)$ is defined. Clearly the definitions of $D(\mathcal{T})$ and $O_{\mathcal{T}}(x)$ do not depend on the choices we have made. A point $x \in \mathrm{Dom}(f_{-n_1})$ will be called *escaping* if $O_{\mathcal{T}}(x) \setminus \mathrm{Dom}(f_{-n_1}) \neq \emptyset$. The *filled Julia set* $K(\mathcal{T})$ is defined to be the collection of all non-escaping points in $\mathrm{Dom}(f_{-n_1})$, the *Julia set* $J(\mathcal{T})$ is its boundary. By virtue of its definition, $J(\mathcal{T})$ is totally invariant under $D(\mathcal{T})$.

Given a parabolic tower $\mathcal{T} = (f_i)_{-n_1}^{n_2}$, for every $n \geq -n_1$ we will call the collection of maps $D(\mathcal{T}|_{\geq n})$ *the elements of level* n of $\mathcal{T}$.

Consider two towers $\mathcal{T}_1 = (f_i^1)_{-n_1}^{n_2}$. For every $i < n_2$ and $j = 1, 2$ the map f_i^j has a parabolic orbit; denote $\pi_{A,f_i^j} : U^A_{f_i^j} \to \mathbb{C}/\mathbb{Z}$, $\pi_{R,f_i^j} : U^R_{f_i^j} \to \mathbb{C}/\mathbb{Z}$ the corresponding projections. We say that $\mathcal{T}_2 = (f_i^2)_{-n_1}^{n_2}$ are quasiconformally (analytically, etc.) conjugate if there exists a sequence of maps $h_i : \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ of the corresponding regularity such that

$$h_i \circ f_i^1 \circ h_i^{-1} = f_i^2 \text{ and } \pi_{R,f_i^2} \circ h_i \circ (\pi_{R,f_i^1})^{-1} = h_{i+1}.$$

Note that by Proposition 2.9 in the case of forward-infinite towers this is equivalent to the existence of $h : \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ which conjugates the corresponding elements of $D(\mathcal{T}_i)$.
The parts (a)-(c) of our Main Theorem will follow from the following combinatorial rigidity property of bi-infinite towers:

Theorem 5.1 (**Rigidity of a bi-infinite tower).** *Any two bi-infinite parabolic towers of the same combinatorial type are identical.*

A key role in the proof will be played by the analysis of [Ep1], from which we extract the following two properties of forward-infinite towers:

Theorem 5.2 ([Ep1] **Structure of the Julia set of a forward tower**)**.** *Let $\mathcal{T}$ be a forward-infinite parabolic tower whose base map $f = \mathcal{P}_\sigma g$ for some $g \in \mathbf{Par}_{\sigma(g)}$. Then the Julia set of $\mathcal{T}$ is a compact full set with empty interior, $J(\mathcal{T}) = K(\mathcal{T})$, and the repelling periodic orbits of elements of $D(\mathcal{T})$ form a dense subset in $J(\mathcal{T})$. Moreover, for any compact subset $C \subset J(\mathcal{T})$ and any open set U with $U \cap J(\mathcal{T}) \neq \emptyset$, there exists $h \in D(\mathcal{T})$ with $h(U \cap \mathrm{Dom}(h)) \supset C$.*

Theorem 5.3 ([Ep1] **Rigidity of a forward tower**)**.** *Again let $\mathcal{T}$ be a forward-infinite parabolic tower whose base map $f = \mathcal{P}_\sigma g$ for some $g \in \mathbf{Par}_{\sigma(g)}$. Then there exist no line fields on $J(\mathcal{T})$ which are invariant under $D(\mathcal{T})$.*

6. Pinched cubic-like maps

We introduce a modification of the Douady-Hubbard polynomial-like mapping (see [DH2]) suited to our needs. We say that a triple $F = (f, U, V)$ is a *pinched cubic-like map* if U and V are two simply-connected open sets in $\mathbb{C}$, $U \subset V$, $V \setminus U \neq \emptyset$, $\partial U \cap \partial V \neq \emptyset$ and $f : U \to V$ is a branched covering of degree 3. As usual, the filled Julia set of F is defined to be the set of non-escaping points:

$$K(F) = \{z \in U;\ f^n(z) \in U \text{ for } z \in \mathbb{N}\},$$

and the Julia set $J(f) = \partial K(f)$. In accordance with the established terminology, two pinched cubic-like maps $F = (f, U_f, V_f)$ and $G = (g, U_g, V_g)$ are *hybrid equivalent* if there exists a quasiconformal map (*the hybrid equivalence*) $\psi : V_f \to V_g$ such that

$$\psi \circ f \circ \psi^{-1} = g \text{ and } \bar{\partial}\psi|_{K(F)} \equiv 0 \text{ a.e.}$$

The reason for the introduction of pinched cubic-like mappings is the following:

Proposition 6.1. *Let $f \in \mathbf{Par}_{p/q}$, and assume additionally that $f : \mathrm{Dom}(f) \to \mathbb{C}/\mathbb{Z}$ is a proper analytic mapping of an annulus onto the whole cylinder, having a single critical value. Denote B_f the component of the immediate parabolic basin of f containing 0. Then there exists a pinched cubic-like map $F = (f^q, U_f, V_f)$ such that $V_f \supset U_f \supset B_f$, and $K(F) = \overline{B_f}$. The intersection $\partial V_f \cap \partial U_f$ consists of the two parabolic points p_1, p_2 in ∂B_f (see Fig. 6).*

Proof. Denote P_f the postcritical set of f, and note that

$$f : \mathrm{Dom}(f) \setminus f^{-1}(P_f) \longrightarrow (\mathbb{C}/\mathbb{Z}) \setminus P_f$$

is an unbranched analytic covering map, and hence a local isometry between the hyperbolic Riemann surfaces. Since the inclusion $\mathrm{Dom}(f) \setminus f^{-1}(P_f) \hookrightarrow (\mathbb{C}/\mathbb{Z}) \setminus P_f$ is strict, the Schwarz Lemma implies that the branches of f^{-1} contract the hyperbolic metric on $(\mathbb{C}/\mathbb{Z}) \setminus P_f$. For a small $k > 0$ denote γ_k the simple closed curve around B_f consisting of points lying at distance k from ∂B_f in the hyperbolic metric of $(\mathbb{C}/\mathbb{Z}) \setminus P_f$. Denote V the neighborhood of B_f bounded by γ_k, and U the connected component of $f^{-q}(V)$ containing 0. The contracting property of f^{-1} implies that $U \subset V$. By the same consideration, the filled Julia set of the triple $F = (f^q, U, V)$ is $\overline{B_f}$. Finally, if k is chosen small enough, the only points of the parabolic periodic orbit in $\bar{V}$ are p_1, p_2. Since both of these points lie in P_f, and are the only such points in ∂B_f, the intersection $\partial V \cap \partial U = \{p_1, p_2\}$. □

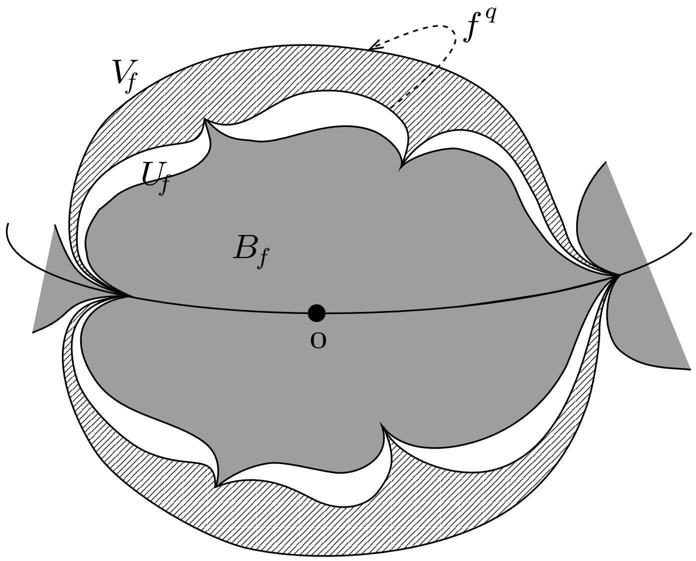
We next show:

Theorem 6.1. *Let $f \in \mathbf{Par}_{p/q}$, $g \in \mathbf{Par}_{s/t}$ both satisfy the conditions of Proposition 6.1. Again denote B_f (respectively B_g) the immediate basin of f (of g) containing 0. Then there exist pinched cubic-like maps $F = (f^q, U_f, V_f)$, $G = (g^t, U_g, V_g)$ such that the following holds:*

- *$V_f \setminus U_f$ is an annulus pinched at the two parabolic points in ∂B_f, and similarly for $V_g \setminus U_g$;*
- *$K(F) = \overline{B_f}$ and $K(G) = \overline{B_g}$;*
- *F and G are quasiconformally equivalent; and moreover, when f and g are critical circle maps, F and G are hybrid equivalent.*

Proof. The general case will clearly follow if we establish the claim in the particular situation when f is an Arnold's map. At the first step of the proof we will present a pair of cubic-like maps F, G as above and a quasiconformal map $\psi_0 : V_f \to V_g$ with the following properties:

FIGURE 3. A pinched cubic-like restriction



- ψ_0 conjugates the action of $f^q|_{U_f}$ and $g^t|_{U_g}$ on their postcritical sets;
- $\psi_0(V_f \setminus U_f) = V_g \setminus U_g$;
- $\psi_0 \circ f^q \circ \psi_0^{-1}(z) = g^t(z)$ for $z \in \partial U_g$.

To do this we will slightly modify the construction of the domain of the cubic-like map given in the above Proposition.

Let us denote p_f, p'_f the elements of the parabolic orbit in ∂B_f. To fix our ideas, let B_f be the basin of attraction of p_f under the iterate f^q. Similarly label p_g, p'_g in the boundary of B_g. We shall begin the construction of the domains of the cubic-like maps and the hybrid equivalence $\psi : V_f \to V_g$ near p_f, p_g.

Let U_f^A, U_g^A be a pair of attracting petals of the parabolic points p_f, p_g respectively, and ϕ_f^A, ϕ_g^A the corresponding attracting Fatou coordinates. Let us select the repelling petals U_f^R, U_g^R so that in the attracting Fatou coordinates their boundaries form horizontal lines $\{\operatorname{Im} z = \pm N\}$ for some large value of N. Denote $\phi_f^R : U_f^R \to \mathbb{C}$ the repelling Fatou coordinate normalized so that $\{\operatorname{Re} z = 0\} \subset \phi_f^R(U_f^R)$, and define ϕ_g^R in a similar fashion. The images $\phi_f^R(\partial B_f)$, $\phi_g^R(\partial B_g)$ are two curves unrolling horizontally towards $-\infty$, invariant under $z \mapsto z+1$. Select $M > 0$ so that the strip $\{|\operatorname{Im} z| < M\}$ does not intersect $\phi_f^R(B_f) \cup \phi_g^R(B_g)$. Let $f(x)$ be a strictly decreasing smooth function with $f(0) > 0$ and $\lim_{x\to-\infty} f(x) = M$. Denote $L^\pm$ the curves

$$L^\pm = \{\operatorname{Im} z = \pm f(\operatorname{Re} z),\ \operatorname{Re} z \le 0\},$$

and set $l_f^\pm = (\phi_f^R)^{-1}(L^\pm)$, $l_g^\pm = (\phi_g^R)^{-1}(L^\pm)$. By construction, l_f^+ and l_f^- are two smooth arcs meeting at the parabolic point p_f, whose preimages by the branch of f^{-q} fixing p_f lie closer to ∂B_f, and similarly for $l_g^\pm$. These curves will become parts of the boundaries of the domains of the two cubic-like maps. Before completing the constructions of these domains, let us explain how the quasiconformal map ψ is constructed in a small neighborhood of p_f. Note that the impression of the postcritical set of f and g on the attracting Fatou cylinders is a single point $\hat{P}_f = \phi^A(P_f \cap U_f^A)/\mathbb{Z}$, and similarly for $\hat{P}_g$. Let us define $\psi_0 : U_f^R \to U_g^R$ as $(\phi_g^R|_{\phi_g^R(U_g)})^{-1} \circ \phi_f^R$. Let us now select an arbitrary quasiconformal map

$$H : \phi_f^A(U_f^A \setminus U_f^R)/\mathbb{Z} \to \phi_g^A(U_g^A \setminus U_g^R)/\mathbb{Z}$$

which maps $\hat{P}_f$ to $\hat{P}_g$ and coincides with $\phi_g^A \circ \psi_0 \circ (\phi_f^A)^{-1} \bmod \mathbb{Z}$ on the boundary of the domain of definition. The map H has a lift $U_f^A \setminus U_f^R \to U_g^A \setminus U_g^R$ which coincides with ψ_0 on $\partial U_f^R \cap U_f^R$. Together the two mappings give us a desired quasiconformal map from a neighborhood of p_f to a neighborhood of p_g which conjugates the dynamics of f^q and g^t on the postcritical sets and maps $l_f^\pm \to l_g^\pm$. Having made clear the idea of the construction, we leave it to the reader to construct the boundary of the domain V_f and the map ψ_0 near p_f'. To complete the construction of V_f outside the neighborhoods of p_f, p_f', let us set it equal to a smooth curve γ_k' lying between γ_k and γ_{k+1} (see the proof of the preceding Proposition), and similarly for V_g. Extend ψ_0 to a quasiconformal homeomorphism $h : V_f \to V_g$ which is a conjugacy on the postcritical sets in an arbitrary fashion. Let U_f and U_g be the connected components of $f^{-q}(V_f)$, $g^{-t}(V_g)$ respectively, containing 0, and define ψ_0 as the lift of h via the commuting diagram:

$$\begin{array}{ccc} U_f & \xrightarrow{\psi_0} & U_g \\ \big\downarrow{\scriptstyle f^q} & & \big\downarrow{\scriptstyle g^t} \\ V_f & \xrightarrow{h} & V_g \end{array}$$

(note that this definition agrees with the definition of ψ_0 on $U_f^R \cup U_f^A$). As an excercise, the reader is invited to verify that the map ψ_0 may be extended to a homeomorphism $V_f \to V_g$ which agrees with h on ∂V_f (this only needs to be checked near the first preimages of p_f and p_f'). We have thus obtained the cubic-like maps $F = (f^q, U_f, V_f)$ and $G = (g^t, U_g, V_g)$ and the quasiconformal map ψ_0 with the desired properties. The Schwarz Lemma again imples that $K(F) = \overline{B_f}$ and $K(G) = \overline{B_g}$.

We may now utilize the standard pull-back argument to obtain a quasiconformal conjugacy between F and G. Inductively define $\psi_n : U_f \to U_g$ as a lift of ψ_{n-1} via

$$\begin{array}{ccc} U_f & \xrightarrow{\psi_n} & U_g \\ \big\downarrow{\scriptstyle f^q} & & \big\downarrow{\scriptstyle g^t} \\ V_f & \xrightarrow{\psi_{n-1}} & V_g \end{array}$$

and extend it by setting $\psi_n \equiv \psi_1$ in $V_f \setminus U_f$. Since f and g are analytic, all the maps ψ_n have the same quasiconformal dilatation bound, and we may select a locally uniformly converging subsequence $\psi_{n_k} \to \psi_\infty$. The limiting quasiconformal map is a conjugacy between the pinched cubic-like maps, however it may happen that $\mu = \psi_\infty^* \sigma_0$ does not vanish at almost every point. When G is real-analytic this can be rectified as follows. We define a Beltrami differential $\nu(z)$ on $\mathbb{C}/\mathbb{Z}$ by setting $\nu(z) = (f^n)^* \mu(z)$ for all z such that $f^n(z) \in B_f$, and $\nu(z) = \sigma_0$ elsewhere. By virtue of the definition, $||\nu||_\infty = ||\mu||_\infty$ and $f^* \nu = \nu$. By the Measurable Riemann Mapping Theorem, there exists a quasiconformal mapping $\phi : \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ such that $\phi^* \sigma_0 = \nu$. The invariance property of ν implies that $\phi \circ f \circ \phi^{-1}$ is an analytic map, which by Proposition 1.2 is equal to f. As the preimages of $\bar{B}_f$ are dense in ∂B_f, $\phi|_{\partial B_f} = \mathrm{Id}$. We define

$$\psi(z) = \begin{cases} \psi_\infty(z) \text{ for } z \notin B_f \\ \psi_\infty \circ \phi^{-1}(z) \text{ for } z \in B_f \end{cases}$$

By the Bers Sewing Lemma, $\psi : V_f \to V_g$ is quasiconformal, and hence a desired hybrid equivalence. □

We are now equipped to give the proof of Theorem 4.2 announced previously.

Proof of Theorem 4.2. Let f and g be two maps satisfying our assumptions, and $\psi : V_f \to V_g$ a hybrid equivalence given by the preceding theorem. Denote $\pi_{A,f} : U_f^A \to \mathbb{C}/\mathbb{Z}$, $\pi_{R,f} : U_f^R \to \mathbb{C}/\mathbb{Z}$, $\pi_{A,g} : U_g^A \to \mathbb{C}/\mathbb{Z}$, and $\pi_{A,g} : U_g^A \to \mathbb{C}/\mathbb{Z}$ the corresponding projections, normalized, as before, so that $E_f(0) = E_g(0) = 0$. Since $\psi(B_f) = B_g$ it induces two $\mathbb{T}$-symmetric analytic isomorphisms

$$\psi_R = \pi_{R_g} \circ \psi \circ \pi_{R_f}^{-1} : \mathrm{Dom}(E_f) \to \mathrm{Dom}(E_g), \text{ and } \psi_A = \pi_{A_g} \circ \psi \circ \pi_{A_f}^{-1} : \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}.$$

The second map fixes 0, hence $\psi_A \equiv \mathrm{Id}$. As $\phi \circ f \circ \phi^{-1} = g$, we have

$$E_g \circ \phi_R \equiv \psi_A \circ E_f \equiv E_f.$$

The claim of the theorem follows from the uniqueness of the Riemann mapping. □

Theorem 6.2. *For every $\mu > 0$ there exists $K = K(\mu) > 1$ such that the following holds. Suppose that $f : \mathrm{Dom}(f) \to \mathbb{C}/\mathbb{Z}$ and $g : \mathrm{Dom}(g) \to \mathbb{C}/\mathbb{Z}$ satisfy the assumptions of Theorem 6.1, the annuli $\mathrm{Dom}(f)$, $\mathrm{Dom}(g)$ are bounded, and $\mu = \min(\mathrm{mod}(\mathrm{Dom}(f)), \mathrm{mod}(\mathrm{Dom}(g)))$. Then for every rational combinatorial type σ, there exists a $\mathbb{T}$-symmetric K-quasiconformal homeomorphism $\phi : \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ such that*

$$\phi \circ \mathcal{P}_\sigma(f) \circ \phi^{-1} = \mathcal{P}_\sigma(g).$$

We will be most interested in the situation when f and g are two parabolic renormalizations of maps with proper basins. In this case the existense of a universal μ is guaranteed by the Petersen's *a priori* bound.

Proof. By the Grötzsch Inequality, there is an equatorial annulus $U = U(\mu) \supset \mathbb{T}$ in which both f and g are defined. By real *a priori* bounds there exists a K_1-quasisymmetric conjugacy $h : \mathbb{T} \to \mathbb{T}$, $h \circ \mathcal{P}_\sigma(f) \circ h^{-1} = \mathcal{P}_\sigma(g)$. Since critical circle maps in $\mathbf{A}(U)$

form a normal family, the constant K_1 may be chosen depending only on μ. Let ψ be the hybrid conjugacy from Theorem 6.1, and as before let an analytic homeomorphism $\psi_R : \mathrm{Dom}(E_f) \to \mathrm{Dom}(E_g)$ fixing 0 be given by $\pi_{R_g} \circ \psi \circ \pi_{R_f}^{-1}$. By its definition, the map ψ_R extends to a quasiconformal map of a larger annulus $\pi_{R,f}(V_f) \to \pi_{R,g}(V_g)$. Denote ϕ_0 a quasiconformal homeomorphism $\mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ such that $\phi_0|_{\mathbb{T}} \equiv h$ and $\phi_0 \equiv \psi$ on an open neighborhood of $\partial\,\mathrm{Dom}(E_f)$. Again by a normal family argument, we may choose ϕ_0 with a dilatation bound depending only on μ; denote this bound $K = K(\mu)$. Now inductively define $\phi_i : \mathrm{Dom}(E_f) \to \mathrm{Dom}(E_g)$ as a lift of ϕ_{i-1} via the commutative diagram

$$\begin{array}{ccc} \mathrm{Dom}(E_f) & \xrightarrow{\phi_i} & \mathrm{Dom}(E_g) \\ \Big\downarrow{\scriptstyle \mathcal{P}_\sigma(f)} & & \Big\downarrow{\scriptstyle \mathcal{P}_\sigma(g)} \\ \mathbb{C}/\mathbb{Z} & \xrightarrow{\phi_{i-1}} & \mathbb{C}/\mathbb{Z} \end{array}$$

and set $\phi_i \equiv \phi_0$ outside $\mathrm{Dom}(E_f)$. Clearly all maps ϕ_i have the same quasiconformal dilatation. Thus we may select a subsequence $\phi_{i_k} \to \phi$ uniformly on compact sets. By construction, ϕ is the desired conjugacy. $\square$

We conclude this section with a step towards proving Theorem 5.1:

Theorem 6.3. *Any two bi-infinite towers of the same combinatorial type are quasiconformally conjugate.*

Proof. Let $\mathcal{T}_1 = (f_i)_{-\infty}^{\infty}$, $\mathcal{T}_2 = (g_i)_{-\infty}^{\infty}$ be two bi-infinite towers with $c(\mathcal{T}_1) = c(\mathcal{T}_2)$. As follows from Theorem 6.2 for every $i \in \mathbb{Z}$ there exists a K-quasiconformal map $h_i : \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ with $h_i \circ f_i \circ h_i^{-1} = g_i$. By Proposition 2.9, for every $j > i$ the map h_i induces a conjugacy $h_{i,j} \circ f_j \circ h_{i,j}^{-1} = g_j$. Now select a subsequence h_{i_k} such that $h_{i_k,j} \to \hat{h}_j$ uniformly on compact subsets of $\mathbb{C}/\mathbb{Z}$ for every $j \in \mathbb{Z}$. By definition, the sequence $\{\hat{h}_j\}$ is the desired conjugacy. $\square$

7. McMullen's hyperbolic expansion argument

In this section we conclude the proof of Theorem 5.1. We utilize the argument developed by McMullen [McM2] in the context of quadratic-like maps to rule out the existence of non-trivial quasiconformal conjugacies between bi-infinite towers:

Theorem 7.1. *Let $\mathcal{T}_1$, $\mathcal{T}_2$ be two bi-infinite towers, and let $h = (h_i)$ be the quasiconformal conjugacy from Theorem 6.3. Then for every i, $(\partial h_i/\partial \bar{z})/(\partial h/\partial z) = 0$ almost everywhere in $\mathbb{C}/\mathbb{Z}$.*

We will require some preliminary definitions. For a differentiable map $f : X \to Y$ of hyperbolic Riemann surfaces, $||f'(x)||_{X,Y}$ will stand for the norm of the derivative with respect to the hyperbolic metrics; we will simply write $||f'(x)||$ if $X = Y = (\mathbb{C}/\mathbb{Z}) \setminus \mathbb{T}$. For the proofs of the next two lemmas see [McM2]:

Lemma 7.1. *There exists a continuous and increasing function $C(s) < 1$ with $C(s) \to 0$ as $s \to 0$ such that for the inclusion $\iota_{X,Y}$ of a hyperbolic Riemann surface X into a hyperbolic Riemann surface Y,*

$$||\iota'_{X,Y}(z)||_{X,Y} < C(s),$$

where $s = d_Y(x, Y \setminus X)$.

Lemma 7.2. *Let $f : X \to Y$ be an analytic map between hyperbolic Riemann surfaces with nowhere vanishing derivative. Suppose, in addition, that $X \subset Y$. Then for $z_1, z_2 \in X$ we have*

$$||f'(z_1)||_{Y,Y}^{1/\alpha} \leq ||f'(z_2)||_{Y,Y} \leq ||f'(z_1)||_{Y,Y}^{\alpha},$$

where $\alpha = exp(K d_Y(z_1, z_2))$ with a universal constant $K > 0$.

A consequence of the above expansion principles is the following:

Proposition 7.3. *Let $\mathcal{T}_{-\infty}^{n_2}$ be a bi-infinite tower, set $\mathcal{T}_n = \mathcal{T}|_{\geq n}$ and let $h \in D(\mathcal{T}_n)$. We have the following:*

(I) *$||h'(z)|| \geq 1$ for any $z \in \mathrm{Dom}(h)$, and $h(z) \notin \mathbb{T}$;*
(II) *there exists a universal constant $C > 1$ such that if $z \in \mathrm{Dom}(f_n)$ and $h(z) \in (\mathbb{C}/\mathbb{Z}) \setminus \mathrm{Dom}(f_n)$ then $||h'(z)|| > C$;*
(III) *moreover, suppose z and $h(z)$ is as in (II), and $k < n$. Then denoting $\zeta = (p_n^{\mathcal{T}|_{\geq k}})^{-1}(z)$, $\hat{h} = (p_n^{\mathcal{T}|_{\geq k}})^{-1} \circ h \circ p_n^{\mathcal{T}|_{\geq k}}$ for some branch of $(p_n^{\mathcal{T}|_{\geq k}})^{-1}$ we have $||\hat{h}'(\zeta)|| > C$.*

Proof. Fix $n \in \mathbb{Z}$. Select a sequence $\theta_k \in \mathbb{T}$ such that if we denote $g_k = R_{\theta_k} \circ f_n$, then g_k is a critical circle mapping with an irrational rotation number $\rho(g_k) = [r_0^k, r_1^k, \ldots, r_i^k, \ldots]$ with $r_i^k \overset{k\to\infty}{\longrightarrow} \infty$. By Theorem 2.2 for every $h \in D(\mathcal{T}_n)$ there exists a sequence of iterates m_k such that $g_k^{m_k} \to h$ uniformly on compact subsets of $\mathrm{Dom}(h)$. By definition, every g_k is a branched covering $\mathrm{Dom}(f_n) \to \mathbb{C}/\mathbb{Z}$ with a single critical value $R_{\theta_k}(f_n(0))$ which is contained in $\mathbb{T}$. The map

$$g_k : X_k \equiv \mathrm{Dom}(f_n) \setminus (g_k^{-1}(\mathbb{T})) \to Y \equiv (\mathbb{C}/\mathbb{Z}) \setminus \mathbb{T}$$

is unbranched, and hence a local isometry between the hyperbolic Riemann surfaces. By the Chain Rule, $||g_k'(z)|| = ||g_k'(z)||_{X_k,Y}/||\iota'_{X_k,Y}(z)||$. By Lemma 7.1, $||g_k'(z)|| > 1$ and the first claim follows.

Now suppose $z \in \mathrm{Dom}(f_n)$ is such that $f_n(z) \notin \mathrm{Dom}(f_{-n})$. By Petersen's complex *a priori* bounds, there exists a universal annulus $U \supset \mathbb{T}$ such that $g_k \in \mathbf{A}(U)$. Since $\mathbb{T}$ is invariant under g_k, normal family considerations imply that there exists another universal annulus $V \Subset U$ around $\mathbb{T}$ such that $g_k(x) \in \mathrm{Dom}(f_n)$ for $x \in V$. On the other hand, the same considerations imply that the preimage $g_k^{-1}(\mathbb{T}) \setminus V$ is a bounded Euclidean distance away from the point z. Hence there exists a universal constant $d > 0$ such that $\mathrm{dist}_Y(z, \partial X_k) \leq d$. The second claim follows from Lemma 7.1 and the Chain Rule.

The third claim follows easily from the second and the Koebe Distortion Principle. $\square$

Theorem 7.2. *Let $\mathcal{T}$ be a bi-infinite tower, let $z = z_i \in \mathrm{Dom}(f_i)$. For $n \le i$ let $z_n \in (p_i^{\mathcal{T}|_{\ge n}})^{-1}(z)$. Then* $\mathrm{dist}(z_n, J(\mathcal{T}|_{\ge n})) \underset{n\to-\infty}{\longrightarrow} 0$.

Proof. Assume that $z_n \notin J(\mathcal{T}|_{\ge n})$. Then for every k between i and $n-1$ there exists an element $h_k \in D(\mathcal{T}|_{\ge k})$ such that $h_k(z_k) \notin \mathrm{Dom}(f_k)$. Denote $\zeta_k^n = (p_k^{\mathcal{T}|_{\ge n}})^{-1}(h_k(z_k)) \in \mathrm{Dom}(f_n)$, and let $\psi_{k,n} \in D(\mathcal{T}|_{\ge n})$ be the element of level n mapping ζ_{k-1}^n to ζ_k^n. By Proposition 7.3 (III) we have $||\psi'_{k,n}(\zeta_{k-1}^n)|| > C$. By a normality argument based on Petersen's *a priori* bounds there exists a universal $s > 0$ such that $\mathrm{dist}_{(\mathbb{C}/\mathbb{Z})\setminus\mathbb{R}}(\zeta_i^n, J(\mathcal{T}|_{\ge n})) < s$. Denote α_n the hyperbolic geodesic of length $l(\alpha_n) < s$ connecting $h_n(z_n)$ to $J(\mathcal{T}|_{\ge n})$. Let α'_n be the connected component of the preimage $\psi_{i,n}^{-1} \circ \psi_{i+1,n}^{-1} \circ \cdots \circ \psi_{n-1,n}^{-1}(\alpha_n)$ containing ζ_i^n. Note that $\mathrm{dist}(\zeta_i^n, J(\mathcal{T}|_{\ge n})) \le l(\alpha'_k)$. As we have seen, $||D\psi_{i,n}^{-1} \circ \psi_{i+1,n}^{-1} \circ \cdots \circ \psi_{n-1,n}^{-1}(h_n(z_n))|| > C^{n-i}$. By Lemma 7.2 this inequality holds along α'_n with C replaced by $C_1(C, s)$, and hence $l(\alpha'_n) \to 0$. □

Proof of Theorem 7.1. Suppose that for some i the Beltrami differential $\mu_i(z) = (\partial h_i/\partial \bar{z})/(\partial h \partial z)$ is supported on a set of positive measure. Let $u_i(z) = \mu_i(z)/|\mu_i(z)|$ denote the corresponding line field. By Theorem 5.3 we may select a point z of almost continuity of $u(z)$ such that $z_n = (p_i^{\mathcal{T}|_{\ge n}})^{-1}(z) \notin J(\mathcal{T}|_{\ge n})$ for every $n \le i$. Let ζ_k^n, $\psi_{k,n}$ be as above. By normality considerations based on Petersen's complex *a priori* bounds we may find a subsequence $n_k \to -\infty$ such that $f_{j+n_k} \to g_j$ uniformly on compact subsets of the domain, $\zeta_{n_k}^{n_k} \to \zeta$ and $u_{j+n_k} \to u'_j$ in w^*-sense. Denote $\mathcal{T}'$ the tower $(g_j)_{-\infty}^{\infty}$.

Let D be a small disk around ζ in $(\mathbb{C}/\mathbb{Z}) \setminus \mathbb{T}$. Denote D_{n_k} its univalent preimage around z_{n_k} by an inverse branch $h_{n_k}^{-1}$ of an element of $D(\mathcal{T}|_{\ge n_k})$ and $D'_{n_k} = p_i^{\mathcal{T}|_{\ge n}}(D_{n_k}) \ni z$. Proposition 7.3 together with Lemma 7.2 implies that the Euclidean diameters of D'_{n_k} shrink to 0, and z is well inside D'_{n_k}. The latter implies that the univalent maps $h_{n_k} \circ p_i^{\mathcal{T}|_{\ge n}}$ form a normal family in a neighborhood of z. As the line field u_i is almost constant in D'_{n_k}, we see that the line field u'_0 is univalent in an open neighborhood D of ζ.

By Theorem 5.2 there exist elements $h_1, h_2 \in D(\mathcal{T}'|_{\le 0})$ mapping D onto 0 and $g_0^2(0)$. By invariance, this means that u' is locally univalent at 0 and $g_0^2(0)$, which implies a contradictory behaviour at $g_0(0)$. □

8. Proof of the Main Theorem

Construction and properties of the mapping ι. Firstly let construct the mapping ι, and the set $\mathcal{A}$ as its image $\mathcal{A} = \iota(\Sigma^-)$. Let $[r_i]_{i=-\infty}^0 \in \Sigma^-$. Continue this formal continued fraction in an arbitrary fashion to a bi-infinite continued fraction $\bar{\sigma} \in \Sigma$. By Proposition 5.1 there exists a bi-infinite tower $\mathcal{T}_{\bar{\sigma}} = (f_i^{\bar{\sigma}})_{i=-\infty}^{n_2}$ with $\sigma(\mathcal{T}_{\bar{\sigma}}) = \bar{\sigma}$. We set

$$\iota([r_i]_{-\infty}^0) = f_0^{\bar{\sigma}}.$$

Let us see that this definition is independent of the choice of the forward part of $\bar{\sigma}$. Indeed, suppose $\bar{\sigma}' = [s_i]_{-\infty}^{\infty}$ with $s_i = r_i$ for $i \le 0$, and $\mathcal{T}_{\bar{\sigma}'} = (f_i^{\bar{\sigma}'})_{i=-\infty}^{n'_2}$ is a bi-infinite tower with $\sigma(\mathcal{T}_{\bar{\sigma}'}) = \bar{\sigma}'$. Let us factor the forward part of $\bar{\sigma}$ into the product of combinatorial types $\sigma_1 * \sigma_2 * \cdots$. Set $h_0 = f_0^{\bar{\sigma}'}$ and inductively define $h_i = \mathcal{P}_{\sigma_i} h_{i-1}$ for $i > 0$. Then the bi-infinite tower $(\ldots, f_{-i}^{c'}, \ldots, f_0^{c'} = h_0, \ldots, h_i, \ldots)$ has the combinatorial type $\bar{\sigma}$ and by Theorem 5.1 it must coincide with $\mathcal{T}_{\bar{\sigma}}$. Hence $f_0^{\bar{\sigma}'} = f_0^{\bar{\sigma}}$. To show the continuity of ι, let us assume the contrary. In this case the considerations of compactness following from complex *a priori* bounds imply that there exist two distinct bi-infinite sequences of critical circle maps $(f_i)_{-\infty}^{\infty}$, $(g_i)_{-\infty}^{\infty}$ such that $\sigma(f_i) = \sigma(g_i)$ for every $i \in \mathbb{Z}$, and $f_{i+1} = \mathcal{R}_{\text{cyl}} f_i$ or $f_{i+1} = \mathcal{P}_{\sigma(f_{i+1})} f_i$, and similarly for g_i. Consider first the case, when there exists an infinite sequence $i_k \to -\infty$ such that $\rho(f_{i_k}) \in \mathbb{Q}$. Then f_{i_k} and g_{i_k} are necessarily parabolic, and the existence of such two sequences is ruled out by the Tower Rigidity Theorem. In the complementary case, the existence of two such sequences contradicts the (non-parabolic) Tower Rigidity Theorem of [Ya2].

Convergence to $\mathcal{A}$.
Firstly, observe that the claim (a) is automatically satisfied.

To deal with the claim (b), assume the contrary. Set $f_i = \mathcal{P}_{\sigma_i} \circ \cdots \circ \mathcal{P}_{\sigma_1}(f)$ and similarly for g_i. Then there exists $\epsilon > 0$ and a sequence $\{n_k\}$ such that $\operatorname{dist}(f_{n_k}, g_{n_k}) > \epsilon$. Passing to a further subsequence, we may ensure that $f_{n_k+j} \to \hat{f}_j$, $g_{n_k+j} \to \hat{g}_j$ for all $j \in \mathbb{Z}$. Then $(\hat{f}_j)$, $(\hat{g}_j)$ are distinct bi-infinite sequences of critical circle maps with the same properties as above, their existence is ruled out in the same way.

To prove the claim (c), note that by real *a priori* bounds every sequence of natural numbers has a subsequence j_k such that $\mathcal{R}_{\text{cyl}}^{n_{j_k}+i} f \to g_i$ for $i \leq 0$. By complex *a priori* bounds, whenever g_i is parabolic, with combinatorial type σ, $g_i \in \mathbf{Par}_\sigma$. Therefore, $g_0 \in \mathcal{A}$, and by Theorem 5.1, it is independent of the subsequence j_k, and is equal to $\iota([r_i]_{-\infty}^0)$.

Periodic orbits of $\mathcal{P}$ are sinks. And finally, let us proceed to proving (d). Let us select an annulus U_f around $\mathbb{T}$ so that f is defined in U_f, and U_f compactly contains the immediate basin B of the parabolic orbit of f. By the Implicit Function Theorem, there exists an open neighborhood $X \subset \mathbf{A}(U_f)$ of the map f such that

$$X_{\sigma_1} = \{g \in X \text{ of combinatorial type } \sigma_1, \text{ having a parabolic periodic orbit}\}$$

is a codimension one analytic submanifold in $\mathbf{A}(U_f)$. Since the boundary of B is a topological repeller (see the proof of Proposition 6.1), there exists a sub-annulus $A \Supset B$ in U_f such that the three-fold preimage $f^{-1}(A) \supset B$ is properly contained in A. Therefore, for all maps in X_{σ_1} sufficiently close to f the immediate basin of the parabolic orbit is compactly contained in U_f and is proper. Clearly, such maps satisfy the conditions of Proposition 2.8 and hence belong to $\mathbf{Par}_{\sigma_1}$. Denote $Y \subset X$ a neighborhood such that $Y_{\sigma_1} \equiv Y \cap X_{\sigma_1} = Y \cap \mathbf{Par}_{\sigma_1}$ and for $g \in Y_{\sigma_1}$ the domain of Pg contains U_f. By Theorem 6.1 every $g \in Y_{\sigma_1}$ has a pinched cubic-like restriction $K(g)$-quasiconformally conjugate to that of f. Let us choose an open (in the induced topology) subset $W \subset Y_{\sigma_1}$ such that for $g \in W$ the dilatation $K(g)$ may be bounded by K_0 with the property that for a normalized K_0-quasiconformal map $q : \mathbb{C}/\mathbb{Z} \to \mathbb{C}/\mathbb{Z}$ the image $q(\operatorname{Dom}(f)) \Supset U_f$. Then, $P : W \to W$.

Since $U_f \Subset \mathrm{Dom}(f)$, the operator $D_f(P|_W)$ is compact by Koebe Theorem. Let us demonstrate that the spectral radius of D_f is less than 1. Observe first that $P^n g \to f$ uniformly for $g \in W$. Indeed, otherwise, there is an $\epsilon > 0$, and sequences $g_k \in W$, $n_k \to \infty$ such that $\mathrm{dist}(P^{n_k} g_k, f) > \epsilon$. Reasoning as in part (b) we find that this contradicts Theorem 5.1. Hence there is an open neighborhood V of f and $N \in \mathbb{N}$ such that $P^N(V) \subset V$. The claim follows by the Schwarz Lemma in a Banach manifold. □

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