# On the scaling structure for period doubling

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**Abstract**

We describe the order on the ratios that define the generic universal smooth period doubling Cantor set. We prove that this set of ratios forms itself a Cantor set, a Conjecture formulated by Coullet and Tresser in 1977. We also show that the two period doubling renormalization operators, acting on the codimension one space of period doubling maps, form an iterated function system whose limit set contains a Cantor set.

## 1 Definitions and Statement of results

A *unimodal map with critical exponent* $\alpha > 1$ is an interval map that can be written in the form $f = \psi \circ q_t \circ \phi$, where $\psi$ and $\phi$ are orientation preserving $C^3$ diffeomorphisms of $[0,1]$, and $q_t : [0,1] \to [0,1]$ with $t \in (0, \frac{1}{2}]$ is the *standard folding map (with critical exponent* $\alpha > 1$) defined by

$$q_t(x) = 1 - \frac{|x-t|^\alpha}{|1-t|^\alpha},$$

that "folds" the interval at its unique critical point $t$. The space of orientation preserving diffeomorphisms of the interval $[0,1]$ with fixed smoothness is denoted by $\mathrm{Diff}^k([0,1])$. The space of unimodal maps with fixed critical exponent $\alpha > 1$ and fixed smoothness can be represented by

$$\mathcal{U} = \mathrm{Diff}^k([0,1]) \times (0, \frac{1}{2}] \times \mathrm{Diff}^k([0,1]).$$

It carries what we call $C^k$*-distances* $d_k$, $k \le 3$, which combines the two $C^k$ distances on each of the two diffeomorphisms $\psi$ and $\phi$ with the distance between the parameters $t$ of the folding parts. Notice that in general, the critical point of $f$ is $c_f = \phi^{-1}(t) \neq t$. Let $p_f$ be the unique fixed point of $f \in W$. A map on the interval is *renormalizable* if it exchanges some number $N_1$ of subintervals. Then the return map on one of these subintervals can again be renormalizable, exchanging this time $N_2$ intervals. If the process continues forever, one says the map is *infinitely renormalizable*. For precise definitions and an account of

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the theory, see for instance [dMvS]. Except otherwise specified when we say renormalizable, we mean renormalizable in the sense of period doubling, *i.e.,* the map exchanges two intervals. Thus, for an infinitely renormalizable map, $N_1 = N_2 = \cdots = 2$.

Fix a critical exponent $\alpha > 1$. We consider the set $W$ of maps $f : [0,1] \to [0,1]$ with $f(c_f) = 1$ and $f(1) = 0$ that are infinitely renormalizable. The critical point defines two invariant intervals

$$U_f = [f^2(c_f), f^4(c_f)] \quad \text{and} \quad V_f = [f^3(c_f), f(c_f)]\,.$$

To these two intervals correspond two *renormalization operators* $R_0 : W \to W$ and $R_1 : W \to W$ defined by:

$$R_0 f = [f^2|V_f]\,, \quad \text{and} \quad R_1 f = [f^2|U_f]\,,$$

where $[\cdot]$ means *affine rescaling to obtain a unimodal map on* $[0,1]$ *that sends its critical point to 1 and 1 to 0.*

Observe, that both operators preserve $W$ and that $R_1$ is the critical point period doubling renormalization operator which has been most studied in the literature (see in particular [La], [Ly], [Mc], [S2], [dMvS] and references therein for the case when $\alpha$ is an even integer, and [E1],[E2] and [Ma2] for arbitrary $\alpha > 1$).

Let $T_n$ be the set of all words of length $n$ over the alphabet $\{0,1\}$. We denote by $T$ the set of all infinite words of the form $w1^\infty$ over the alphabet $\{0,1\}$, and by $\overline{T}$ the set of all infinite words over the alphabet $\{0,1\}$, equipped with the usual metric. Notice that each $T_n$ naturally embeds into $T$. For any word $\tau \in \overline{T}$, we will write $\tau_{\{n\}} \in T_n$ for the initial segment of length $n$ of $\tau$. We are going to consider the iterated function system generated by $R_0$ and $R_1$. To this end, we define:

$$R_{\tau_{\{n\}}} = R_{\tau(1)} \circ \cdots \circ R_{\tau(n)} : W \to W\,,$$

and we will prove:

**Theorem 1.1.** *For any fixed point $f_0$ of $R_0$, there is a Hölder-continuous map $h : \overline{T} \to W$ such that for any $\tau \in \overline{T}$*

$$\lim_{n\to\infty} R_{\tau_{\{n\}}} f_0 = h(\tau).$$

*Moreover, the convergence of the sequence $\{R_{\tau_{\{n\}}} f_0\}$ is exponential in the $C^2$-metric.*

*A similar statement holds for any fixed point $f_1$ of $R_1$.*

**Remark 1.2.** *For any $\alpha > 1$, the existence of a fixed point $f_1$ of $R_1$ is proven in [E1],[E2] and [Ma2]. We show here (see Corollary 2.4) that the existence of a fixed point $f_1$ for $R_1$ is equivalent to the existence of a fixed point $f_0$ for $R_0$. The uniqueness of $f_1$ in the case when $\alpha$ is an even integer was proven in [S2]. In the sequel we will fix $f_0$ and $f_1$ to be fixed points of respectively $R_0$ and $R_1$.*

**Remark 1.3.** *The set $h(\overline{T})$ of limits $\lim_{n\to\infty} R_{\tau_{\{n\}}} f_0$ is denoted by $A \subset W$. Here the notation $A$ represents the fact that we believe, but do not prove, that the set $A$ is indeed the attractor of the iterated function system generated by $R_0$ and $R_1$, and in particular does not depend on the initial point, chosen here to be $f_0$.*

The next two Theorems depend on convexity conditions on $f_0$ and $R_1(f_0)$.

**Convexity Conditions 1.4.** *We assume that:*

- **C1** $f_0|[(f_0)^3(c_{f_0}), 1)$ *is strictly convex,*
- **C2** $R_1(f_0)|[(R_1(f_0))^3(c_{R_1(f_0)}), 1]$ *is strictly convex.*

**Remark 1.5.** *In section 4 we will show that* **C1** *actually holds true in the case when successive $R_1$ renormalizations of a convex function converge to $f_1$: this is known to be the case when $\alpha$ is an even integer. Furthermore, as we will explain, one can check that both* **C1** *and* **C2** *hold true in the most important case of generic (quadratic) critical points, $\alpha = 2$.*

Recall that a Cantor set is a perfect and totally disconnected compact metric space.

**Theorem 1.6.** *If the Convexity Conditions C1 and C2 hold true, then the limit set $A$ of orbits of $f_0$ under the interated function system defined by $R_0$ and $R_1$ is a Cantor set.*

For completeness and to fix notations and definitions, we include some basic discussion of the scaling function, whose origin is rather diffuse: first conjectures about a form of it appeared in [CT], the name and a form of it come from [F], while what was arguably the first theorem about it was in a never circulated work by Feigenbaum and Sullivan cited in [S1]. The literature on scaling functions is extensive and discusses scaling functions beyond the context of dynamics. In particular, in [KSV] a relation with the thermodynamic formalism appeared.

Let $\Lambda$ be the invariant Cantor set of $f_0$. In the sequel we will remind the dynamical construction of covers of $\Lambda$ by finitely many intervals. These covers, called cycles, form a refining nest of covers of this Cantor set. The scaling function contains the infinitesimal geometrical information on how these covers refine. It will be shown that the Cantor set $\Lambda$ is, from a geometrical point of view, very different from the well known middle third Cantor set, in which each refinement is done everywhere in the same manner. In particular, it will be shown in Theorem 1.8 that in the Cantor set $\Lambda$ there are no two places where the refinement is done in the same manner.

Although, the Cantor set $\Lambda$ is the invariant set of a non expanding map, it is also the invariant Cantor set of an expanding interval map, the so-called *presentation function* [R], [S1], a great remark that Rand attributes to Misiurewicz.

As we next recall, this directly follows from $f_0$ being a renormalization fixed point that is expanding to the right of $p_{f_0}$.

Let $U = U_{f_0}$ and $V = V_{f_0} = [1-v, 1]$. Observe, that the affine (scaling) map $s : [0,1] \to [0,1]$ defined by $s : x \mapsto v \cdot (x-1) + 1$ is a homeomorphism from $\Lambda$ to $\Lambda \cap V$. This is a direct consequence of the fact that $s$ conjugates $f_0 = R_0(f_0) = s^{-1} \circ f_0^2 \circ s$ to $f_0^2$. Also the restriction,

$$f_0|V : \Lambda \cap V \to \Lambda \cap U,$$

is a homeomorphism so that the the map $g : [0,1] \to U$ defined by $g = (f_0|V) \circ s$ is a homeomorphism from $\Lambda$ to $\Lambda \cap U$. Let $F : [0,1] \to [0,1]$ be the multivalued function defined by the two branches

$$F_0 = s : [0,1] \to [0,1] \quad \text{and} \quad F_1 = g : [0,1] \to [0,1] \,.$$

The branch $F_0 = s$ is affine, contracting, and orientation preserving while the branch $F_1 = g$ is orientation reversing. Furthermore, the absolute value of the derivative of $F_1$ strictly increases as a consequence of the Convexity Condition C1, so that $F_1$ is also contracting (as $p_{f_0}$ is an expanding fixed point). It follows that the invariant set of the iterated function system $F = \{F_0, F_1\}$ is $\Lambda$, the invariant Cantor set of $f_0$.

The cover $\{U, V\}$ of $\Lambda$ we call *the cycle of the first generation*. The two intervals of this cycle are permuted by the map $f_0$. The Cantor set $\Lambda$ is the intersection of a decreasing sequence of covers we call respectively the *cycles of generation* $n$: the cycle of generation $n$ is the cover of $\Lambda$ consisting of $2^n$ intervals which are permuted by $f_0$. The intervals that form the $n^{th}$ cycle can be described as follows.

The construction of the cycles is using the iterated function system generated by $F_0$ and $F_1$. We will use a different notation for the words describing sequences of compositions of these maps that will be different from the one we used in the definition of the iterated function system generated by $R_0$ and $R_1$. Namely, we write $\Sigma_n$ for the set of words $w = w(1)w(2)\ldots w(n)$ of length $|w| = n$ over the alphabet $\{0,1\}$, and $\Sigma$ for the set of infinite sequences over the alphabet $\{0,1\}$ with the usual metric. Let

$$I_w = F_{w(n)} \circ \cdots \circ F_{w(1)}([0,1]).$$

The $n^{th}$ cycle consists of the intervals $I_w$ with $w$ a word of length $n$.

**Lemma 1.7.** *The way $f_0$ permutes these intervals is described by addition* mod $2^n$ *on the words indexing the intervals. In particular, if $c$ is the critical point of $f_0$ then $c \in I_{1^n}$ and $f_0(c) \in I_{0^n}$. Moreover, $f(I_{1^n}) = I_{0^n}$ and*

$$f_0 : I_w \to I_{w+1},$$

*is a diffeomorphism for each word not equal to $1^n$, $n \geq 1$.*

*Proof.* Let $w$ be a word of length $n-1$. Then

$$I_{w1} = F_1(I_w) = f_0 \circ s(I_w) = f_0(I_{w0}),$$

which proves that $f_0$ permutes the intervals as stated. □

The orientation of an interval $I_w$ is defined to be the number

$$o(w) = (-1)^{\#(w)},$$

where $\#(w)$ is the number of 1's in $w$. The shift of a word $w = w(1)w(2)\ldots w(n)$ is defined as

$$\sigma(w) = w(2)w(3)\ldots w(n).$$

Observe, that

$$I_w \subset I_{\sigma(w)}.$$

In particular, the $n^{th}$ cycle has two intervals in each interval of the $(n-1)^{th}$ cycle:

$$I_{0w}, I_{1w} \subset I_w.$$

The scaling function $q_n : w \mapsto (0,1)$ assigns to each word $w$ of length $n$ the ratio

$$q_n(w) = \frac{|I_w|}{|I_{\sigma(w)}|}.$$

The a priori bounds on the possible values of $q_n$, as presented for example in [Ma1] imply

$$|I_w| \leq \rho^{|w|}$$

for some fixed $\rho < 1$. From this and the smoothness of $f_0$ it follows that the sequence $q_n$ converges to a Hölder function $q : \Sigma = \{0,1\}^{\mathbf{N}} \to (0,1)$. This function $q$ is what we call the *scaling function*, in minor departure from some previous authors.

The next theorem describes properties of the scaling function. To formulate this Theorem we need an order on $\Sigma$: with $w$ standing for the maximal word such that $w_1 = ww^1$ and $w_2 = ww^2$, we say that $w_1$ is strictly smaller than $w_2$ (or $w_1 \prec w_2$)if and only if

$$(-1)^{\#(w)} \cdot w^1(1) < (-1)^{\#(w)} \cdot w^2(1).$$

**Theorem 1.8.** *If the Convexity Conditions hold true then $q$ is strictly monotone.*

*Furthermore, under the same hypothesis, there exists constants $C > 0$ and $r < 1$ such that if $w_1 \prec w_2$ and $w_1(k) = w_2(k)$ whenever $k \leq n$ then*

$$q(w_2) \geq q(w_1) + Cr^n$$

**Remark 1.9.** *If the Convexity Conditions C1 and C2 hold true, the proof of the last Theorem confirms the 1977 Conjecture in [CT] that the limit set of the ratios $q_n(w)$ defining the period doubling Cantor set is a itself a Cantor set.*

In particular, we thus have the following

**Theorem 1.10.** *In the case of quadratic critical point, $\alpha = 2$, we have the following.*

- *The Convexity Conditions holds true.*
- *The universal period doubling scaling function $q$ is strictly monotone and the range forms a Cantor set.*
- *The limit set $A$ of orbits of $f_0$ under the interated function system defined by $R_0$ and $R_1$ is a Cantor set.*

**Acknowledgements 1.** *H.Epstein and O.E. Lanford discovered a relation between the fixed points of $R_0$ and $R_1$. Roughly speaking this relation states that if $f(x) = h(x^2)$ represents the fixed point of $R_1$ then $g(x) = (h(x))^2$ represents the fixed point of $R_0$. This result was not published. However, it was the main inspiration for section 2. In particular, Lemma 2.4 contains this result.*

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# 2 Decompositions and Convergence

The notion of decomposition, introduced in [Ma2], is a tool to describe the combinatorial aspects of universality. In this section, after some background on decompositions, we prove the convergence properties stated in Theorem 2.1.

The set $T_n$ is ordered by the embedding into the natural numbers defined by

$$\tau(1)\tau(2)\dots\tau(n) \mapsto \sum_{i=1}^{n} \tau(i) \cdot 2^{n-i}.$$

Consider also the embedding $j_n : T_n \to T_{n+1}$ defined by

$$j_n : \tau \mapsto \tau 1.$$

This embedding preserves the order. Observe that $T$ inherites an order from the orders on the sets $T_n$, which extends to the order on $\overline{T}$ such that $\tau^1 \leq \tau^2$ iff $\tau^1_{\{n\}} \leq \tau^2_{\{n\}}$ for all $n \geq 1$. The elements of $\overline{T}$ are called *decomposition times*.

For the order $<$, the successor in $T_n$ of $1^n \in T_n$ is $0^n \in T_n$ and the predecessor in $T_n$ of $0^n \in T_n$ is $1^n$. The successor of $\tau \in T$ in $T_n$ is denoted by $\tau^{n+}$ and the predecessor is denoted by $\tau^{n-}$.

The nonlinearity of an orientation preserving diffeomorphism $\phi \in \mathrm{Diff}^2([0,1])$ is

$$\eta_\phi = D \ln D\phi \in C^0([0,1]).$$

A *decomposed unimodal map* is a map

$$\tilde{f} : T \to \mathrm{Diff}^3([0,1]) \cup (0, \frac{1}{2}]$$

with the following properties

- $\tilde{f}(1^\infty)$, the *folding* part of $\tilde{f}$ represents an element $q_t$ of the standard folding family, so we have $\tilde{f}(1^\infty) = t \in (0, \frac{1}{2}]$,
- $\tilde{f}(\tau) \in \mathrm{Diff}^3([0,1])$ for $\tau \neq 1^\infty$, (the *diffeomorphic parts* of $\tilde{f}$).
- $$\sum_{\tau \in T \setminus \{1^\infty\}} |\eta_{\tilde{f}(\tau)}|_0 < \infty.$$
- $$\sum_{\tau \in T \setminus \{1^\infty\}} |D\eta_{\tilde{f}(\tau)}|_0 < \infty.$$

The set $U$ of decomposed unimodal maps carries the metric $d$ defined by

$$d(\tilde{f}, \tilde{g}) = \sum_{\tau \in T \setminus \{1^\infty\}} |\eta_{\tilde{f}(\tau)} - \eta_{\tilde{g}(\tau)}|_1 \; + \; |\tilde{f}(1^\infty) - \tilde{g}(1^\infty)| \, .$$

The two summability conditions for decomposed unimodal maps allow to define what we call *compositions* associated to decomposed unimodal maps. Namely, if one considers a finite set $T_n$ of decomposition times, the composition associated to $\tilde{f}$ and $T_n$ is defined as

$$O(\tilde{f}, n) = \tilde{f}(1^{n-1}0) \circ \cdots \circ \tilde{f}(0^{n-1}1) \circ \tilde{f}(0^n) \circ q_{\tilde{f}(1^n)} \, ,$$

otherwise speaking, the folding part followed by the diffeomorphic parts in the order of the decomposition times (so that the end result of the composition is a unimodal map). In [Ma2] it is shown that this composition, when defined for decomposed unimodal maps over the sets $T_n$, extends to a composition operator

$$O : U \to \mathcal{U},$$

where $\mathcal{U}$ is equipped with the $C^2$ metric, which is a Lipschitz map. This composition operator is based on a choice. Namely, the composition starts with the folding part $q_{\tilde{f}(1^n)}$. We could as well start at any decomposition time $\tau \in T_N$, $N \geq 1$ and consider for each $n \geq N$ the compositions defined by

$$O(\tau, \tilde{f}, n) = \tilde{f}(\tau^{n-}) \circ \cdots \circ \tilde{f}(0^{n-1}1) \circ \tilde{f}(0^n) \circ q_{\tilde{f}(1^n)} \circ \tilde{f}(1^{n-1}0) \circ \cdots \circ \tilde{f}(\tau^{n+}) \circ \tilde{f}(\tau)\,.$$

The same proof which was used in [Ma2] to construct $O(\tilde{f})$ shows the pointwise convergence of the sequence $O(\tau, \tilde{f}, n)$ as $n \to \infty$, thus defining a map

$$O : T \times U \to \mathcal{U}.$$

Observe that $O(1^\infty, \tilde{f})$ is the operator studied in [Ma2].

This construction can be generalized even more. Fix $\tilde{f} \in U$ and choose $\tau_2 > \tau_1$ in $T_N$. For each $n \geq N$ define the diffeomorphism

$$O_{\tau_1}^{\tau_2}(\tilde{f}, n) = \tilde{f}(\tau_2^{n-}) \circ \cdots \circ \tilde{f}(\tau^{n+}) \circ \tilde{f}(\tau) \circ \cdots \circ \tilde{f}(\tau_1^{n+}) \circ \tilde{f}(\tau_1).$$

It follows from [Ma2] that these maps converge, and we set

$$O_{\tau_1}^{\tau_2}(\tilde{f}) = \lim_{n \to \infty} O_{\tau_1}^{\tau_2}(\tilde{f}, n).$$

Moreover, there is a constant $K_{\tilde{f}}$ such that

$$|O_{\tau_1}^{\tau_2}(\tilde{f}) - \mathrm{id}|_2 \leq K_{\tilde{f}} \cdot \sum_{\{\tau \in T | \tau_2 > \tau \geq \tau_1\}} |\eta_{\tilde{f}(\tau)}|_0.$$

**Lemma 2.1.** *The operator $O$ extends continuously to an operator*

$$O : \overline{T} \times U \to \mathcal{U}.$$

*In particular, for each $\tilde{f} \in U$ there exists a constant $K_{\tilde{f}} > 0$ such that for any pair $\tau_2, \tau_1 \in \overline{T}$ with $\tau_2 \geq \tau_1$,*

$$d_2(O(\tau_2, \tilde{f}), O(\tau_1, \tilde{f})) \leq K_{\tilde{f}} \cdot \sum_{\{\tau \in T | \tau_2 > \tau \geq \tau_1\}} |\eta_{\tilde{f}(\tau)}|_0.$$

*Moreover for each $\tau_3 > \tau_2 > \tau_1 \in \overline{T}$ and $\tilde{f} \in U$*

$$O_{\tau_1}^{\tau_3}(\tilde{f}) = O_{\tau_2}^{\tau_3}(\tilde{f}) \circ O_{\tau_1}^{\tau_2}(\tilde{f}).$$

*Proof.* Fix $\tilde{f} \in U$ and choose $\tau_2 > \tau_1$ in $T_N$. Let $h = O_{\tau_1}^{\tau_2}(\tilde{f})$. The construction of $h$ implies directly

$$h \circ O(\tau_1, \tilde{f}) = O(\tau_2, \tilde{f}) \circ h.$$

This construction can be done for every pair of $\tau_1', \tau_2' \in [\tau_2, \tau_1] \cap T$. Hence, there is a constant which only depends on $\tilde{f}$ such that

$$d_2(O(\tau_2', \tilde{f}), O(\tau_1', \tilde{f})) \leq \text{ Const } \cdot \sum_{\{\tau \in T | \tau_2 > \tau \geq \tau_1\}} |\eta_{\tilde{f}(\tau)}|_0.$$

From this we get the continuous extension of $O$ to $\overline{T} \times U$, together with the estimate stated in the Lemma. The composition rule clearly holds for the operators $O_{\tau_1}^{\tau_2}(\tilde{f}, n)$ and hence for the continuous extension of $O$. □

We will also write $O_\tau(\cdot)$ for $O(\tau,\cdot)$. Let $\mathcal{U}_0$ be the set of renormalizable unimodal maps and $U_0 = (O_{1^\infty})^{-1}(\mathcal{U}_0)$. A renormalization operator $R : U_0 \to U$ is constructed in [Ma2] such that

$$O_{1^\infty} \circ R = R_1 \circ O_{1^\infty}.$$

A decomposed unimodal map $\tilde{f} \in U_0$ is said to be $n$ *times renormalizable* iff $f = O(\tilde{f}) \in \mathcal{U}$ is $n$ times renormalizable: we then set $f = \phi \circ q_t$ with $t \in (0, \frac{1}{2}]$. This means there are pairwise disjoint intervals $I_\tau^{f,n}$, $\tau \in T_n$, forming the $n^{th}$ cycle of $f$, such that

- $t \in I_{1^n}^{f,n}$,
- $f : I_\tau^{f,n} \to I_{\tau^{n+}}^{f,n}$ is a diffeomorpishm, whenever $\tau \neq 1^n$,
- $f : I_{1^n}^{f,n} \to I_{0^n}^{f,n}$ is onto.

Let $g : I \to J$ be an endormorphism which has either one or non critical point. Then $[g] : [0,1] \to [0,1]$ is a either a unimodal map or an orientation preserving diffeomorphism obtained by affine scaling of domain and image of $g$.

**Lemma 2.2.** *Let $\tilde{f} \in U$ $n$ times renormalizable and $O(\tilde{f}) = f = \phi \circ q_t \in \mathcal{U}_0$ with $t \in (0, \frac{1}{2}]$. For $n \geq 1$ and $\tau \in T_n \subset T$*

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$$O_\tau^{\tau^{n+}}(R^n \tilde{f}) = [f | I_\tau^{f,n}],$$
-
$$O_\tau^{\tau^{n+}0^\infty}(R^n \tilde{f}) = [q_t | I_\tau^{f,n}],$$
-
$$O_{\tau^{n+}0^\infty}^{\tau^{n+}}(R^n \tilde{f}) = [\phi | q_t(I_\tau^{f,n})].$$

The reader is revered to [Ma2] for the precise definition of the renormalization operator $R : U_0 \to U$, from which the Lemma immediately follows. This lemma indeed captures all the properties of the renormalization operator $R$ that we will need.

**Proposition 2.3.** *For every $\tau \in T_n \subset T$*

$$O_\tau \circ R^n = R_\tau \circ O_{1^\infty}\ ,$$

*and*

$$O_{\tau 0^\infty} \circ R^n = R_\tau \circ O_{0^\infty}\ .$$

*Proof.* Let $\tilde{f} \in U$ be $n \geq 1$ times renormalizable and $O(\tilde{f}) = O_{1^\infty}(\tilde{f}) = f = \phi \circ q_t \in \mathcal{U}_0$ with $t \in (0, \frac{1}{2}]$. As in the proof of Lemma 3.1 we show that for every $n \geq 1$ and $\tau \in T_n \subset T$

$$R_\tau(f) = [f^{2^n} | I_\tau^{f,n}].$$

Let $\tau_1 = \tau$, $\tau_k = \tau_{k-1}^{n+}$, for $k = 2, 3, \ldots, 2^n$. The composition rule for the operators $O_{\tau_1}^{\tau_2}$ and Lemma 2.2 implies

$$\begin{aligned} O_\tau \circ R^n(\tilde{f}) &= O_{\tau_{2^n}}^{\tau_1}(R^n \tilde{f}) \circ \cdots \circ O_{\tau_2}^{\tau_3}(R^n \tilde{f}) \circ O_{\tau_1}^{\tau_2}(R^n \tilde{f}) \\ &= [f | I_{\tau_{2^n}}^{f,n}] \circ \cdots \circ [f | I_{\tau_2}^{f,n}] \circ [f | I_{\tau_1}^{f,n}] \\ &= [f^{2^n} | I_{\tau_1}^{f,n}] \\ &= R_{\tau_1}(f) \\ &= R_\tau \circ O_{1^\infty}(\tilde{f}). \end{aligned}$$

The second equation is proved similarly. □

**Lemma 2.4.** *The operators $R_0$ and $R_1$ have fixed points. Furthermore, for any even integer $\alpha$, both operators $R_0$ and $R_1$ have a unique fixed point.*

*Proof.* It was shown in [Ma2] that the operator $R$ has a fixed point. The previous proposition implies that a fixed point $\tilde{f} \in U_0$ of $R$ produces fixed points of $R_0$ and $R_1$. Namely,

$$R_1(O_{1^\infty}(\tilde{f})) = O_{1^\infty}(\tilde{f})$$

and

$$R_0(O_{0^\infty}(\tilde{f})) = O_{0^\infty}(\tilde{f}).$$

**Claim 2.5.** *For each fixed point $f \in \mathcal{U}$ of $R_1$ (or $R_0$) there exists a unique fixed point of $R$, say $\tilde{f} \in U$ such that $O_{1^\infty}(\tilde{f}) = f$ (or $O_{0^\infty}(\tilde{f}) = f$).*

*Proof.* Let $f = \phi \circ q_t \in \mathcal{U}$ be a fixed point of $R_1$ (the case of a fixed point for $R_0$ can be treated the similarly). Choose $\tilde{f} \in U$ such that

$$O_{1^\infty}(\tilde{f}) = f.$$

For example, consider $\tilde{f} \in U$ defined by

- $\tilde{f}(1^\infty) = q_t$ ,
- $\tilde{f}(01^\infty) = \phi$ ,
- $\tilde{f}(\tau) = \text{id}$ for $\tau \neq 1^\infty, 01^\infty$.

The definition of $\tilde{f}$ and the fact that $O_{1^\infty} \circ R = R_1 \circ O_{1^\infty}$, implies

$$O_{1^\infty}(R^n \tilde{f}) = f \, , n \geq 1.$$

We will show

$$\lim_{n \to \infty} R^n \tilde{f} = \hat{f} \in U,$$

with

$$R\hat{f} = \hat{f} \text{ and } O_{1^\infty}(\hat{f}) = f.$$

Let $n \geq 1$ and $\tau_3 > \tau_2 > \tau_1 \in T_{n+1}$ three consecutive decomposition times in $T_{n+1}$ with $\tau_3, \tau_1 \in T_n$. Observe, that $\tau_3$ and $\tau_1$ are consecutive points in $T_n$. From Lemma 2.2 we get

$$\begin{aligned} O_{\tau_1}^{\tau_3}(R^{n+1}\tilde{f}) &= O_{\tau_2}^{\tau_3}(R^{n+1}\tilde{f}) \circ O_{\tau_1}^{\tau_2}(R^{n+1}\tilde{f}) \\ &= [f|I_{\tau_2}^{f,n+1}] \circ [f|I_{\tau_1}^{f,n+1}] \\ &= [f^2|I_{\tau_1}^{f,n+1}] \\ &= [f|I_{\tau_1}^{f,n}], \end{aligned}$$

where we used that $f$ is a fixed point of $R_1$. Again from Lemma 2.2 we get $[f|I_{\tau_1}^{f,n}] = O_{\tau_1}^{\tau_3}(R^n\tilde{f})$. Hence,

$$O_{\tau_1}^{\tau_3}(R^{n+1}\tilde{f}) = O_{\tau_1}^{\tau_3}(R^n\tilde{f}).$$

This should be interpreted as $R^{n+1}\tilde{f}$ being a refinement of $R^n\tilde{f}$. In [AMM] it has been shown that there is a constant $K > 0$ and $\rho < 1$ such that

$$\sum_{\tau_1 \in T_n} |(O_{\tau_2}^{\tau_3}(R^{n+1}\tilde{f}) - \text{id})|_2 \leq K \cdot \rho^n.$$

This implies that $\lim_{n\to\infty} R^n\tilde{f} = \hat{f} \in U$. In particular, this implies that $\hat{f}$ is a fixed point of $R$ which projects by $O_{1^\infty}$ to $f$. This concludes the existence part of the Claim.

We can use Lemma 2.2 to identify $\tilde{f}(\tau)$, $\tau \in T_N$. Namely,

$$\begin{aligned} \tilde{f}(\tau) &= \lim_{n\to\infty} R^n\tilde{f}(\tau) \\ &= \lim_{n\to\infty} O_\tau^{\tau^{n+}0^\infty}(R^n\tilde{f}) \\ &= \lim_{n\to\infty} [q_t|I_\tau^{f,n}] \\ &= [q_t|I_\tau^{f,N}], \end{aligned}$$

where we used that $f$ is a fixed point of $R_1$ to obtain the last equality. This implies the uniqueness part of the Claim. □

It has been shown in [S2] that the operator $R_1$ has a unique fixed point when $\alpha$ is an even integer. Now the uniqueness part of the Lemma follows by using the Claim. □

**Proof of Theorem 1.1** Let $f_0$ be a fixed point of $R_0$ and $\tilde{f}_0 \in U$ the unique fixed of $R$ with $O_{0^\infty}(\tilde{f}_0) = f_0$. Let $h : \overline{T} \to W$ be defined by

$$h(\tau) = O_\tau(\tilde{f}_0).$$

For any $\tau_1, \tau_2 \in \overline{T}$ let $|\tau_2 - \tau_1|$ be the maximal length for which initial segments of the word $\tau_1$ and $\tau_2$ of that length agree. In [AMM] it has been shown that there is a constant $K > 0$ and $\rho < 1$ such that

$$\sum_{\tau_2 > \tau > \tau_1} |\eta_{\tilde{f}_0(\tau)}|_0 \leq K \cdot \rho^{|\tau_2 - \tau_1|}.$$

Recall that $\tau_{\{n\}}$ is the word consisting of the first $n$ symbols of a word $\tau \in \overline{T}$. From Lemma 2.1 we get

$$d_2(h(\tau_{\{n\}}0^\infty), h(\tau)) \leq K \cdot \rho^n.$$

Theorem 1.1 follows from Proposition 2.3. Namely,

$$\begin{aligned} R_{\tau_{\{n\}}} f_0 &= R_{\tau_{\{n\}}} \circ O_{0^\infty} \tilde{f}_0 \\ &= O_{\tau_{\{n\}}0^\infty} \circ R^n \tilde{f}_0 \\ &= O_{\tau_{\{n\}}0^\infty} \tilde{f}_0 \\ &= h(\tau_{\{n\}}0^\infty) \to h(\tau), \end{aligned}$$

where the convergence is exponential.

# 3 The monotonicity of the scaling function

The monotonicity of the scaling function $q$, as formulated in Theorem 1.8 is based on the following combinatorial Lemmas. First we will concentrate on these Lemmas and proof Theorem 1.8. Secondly, Theorem 1.8 is used to prove Theorem 1.6.

Although decomposition times and the words used to define the intervals $I_w$ are conceptually different, the following Lemma shows that they are strongly related.

**Lemma 3.1.** *For every word $w$ of length $n$*

$$R_w(f_0) = [f_0^{2^n} | I_w].$$

*Proof.* The proof is by induction in $n$. For $n = 1$ the Lemma restates the definition of $R_0$ and $R_1$. Assume the Lemma holds for some $n \geq 1$. Choose a word $w$ of length $n$ and consider the two intervals $I_{0w}$ and $I_{1w}$. These intervals are contained in $I_w$ and each contains a boundary point of $I_w$. Using the induction hypothesis $R_w(f_0) = [f_0^{2^n} | I_w]$ and the fact that $f_0^{2^n} | I_w$ permutes $I_{0w}$ and $I_{1w}$ we get that $R_{0w}(f_0) = R_0(R_w(f_0))$ and $R_{1w}(f_0) = R_1(R_w(f_0))$ correspond to either of $f^{2^{n+1}} | I_{0w}$ or $f^{2^{n+1}} | I_{1w}$.

It is left to identify which of the two intervals correspond to $U_{R_w(f_0)}$ resp. $V_{R_w(f_0)}$. The map $f_0$ permutes the intervals $I_{w'}$ with $|w'| = n + 1$ according to addition mod.$2^n$ on the words indexing the intervals, as described in Lemma 1.7. Observe,

$$1w = 0w + 2^n \cdot 1.$$

This means that $f_0^{2^n}|I_{0w}$ is monotone because $0w+k\cdot 1$, $k<2^n$ never equals the word $1^{n+1}$ and $f_0|I_{1^{n+1}}$ is the only place where monotonicity of $f_0$ fails. Hence,

$$R_{0w}(f_0) = R_0([f_0^{2^n}|I_w]) = [(f_0^{2^n})^2|I_{0w}]$$

and

$$R_{1w}(f_0) = R_1([f_0^{2^n}|I_w]) = [(f_0^{2^n})^2|I_{1w}].$$

□

In the sequel we will identify $R_w(f_0)$ with $f_0^{2^n}|I_w$.

**Lemma 3.2.** *For every pair of words $w$ and $w^0$, the map*

$$R_{w^0}(f_0) : I_{w0w^0} \to I_{w1w^0},$$

*is monotone and onto.*

*Proof.* Let $|w^0| = n$. The action of $f_0$ on the intervals of length $|w|+1+|w^0|$ is described by addition mod.$2^n$ on the words indexing the intervals (see Lemma 1.7). In particular,

$$w1w^0 = w0w^0 + 2^n \cdot 1.$$

Hence

$$f_0^{2^n}(I_{w0w^0}) = I_{w1w^0}.$$

By construction we have

$$I_{w1w^0}, I_{w0w^0} \subset I_{w^0}.$$

Now the Lemma follows from $R_{w^0}(f_0) = [f_0^{2^n}|I_{w^0}]$, which was obtained in the previous Lemma. □

**Proposition 3.3.** *If the Convexity Condition holds then there exist constants $C>0$ and $r\in(0,1)$ with the following property. Let $w$ be a word of with $|w| = n$.*

*If $o(w) = +1$ then*

- *$w0 < w1$ and $w00 < w01 < w11 < w10$*
- *$q_{n+1}(w0) < q_{n+1}(w1)$*
- *$q_{n+2}(w00) < q_{n+2}(w01) < q_{n+2}(w11) < q_{n+2}(w10)$*
- *$q_{n+2}(w11) > q_{n+2}(w01) + Cr^n$*

*If $o(w) = -1$ then*

- *$w1 < w0$ and $w10 < w11 < w01 < w00$*
- *$q_{n+1}(w1) < q_{n+1}(w0)$*
- *$q_{n+2}(w10) < q_{n+2}(w11) < q_{n+2}(w01) < q_{n+2}(w00)$*

- $q_{n+2}(w01) > q_{n+2}(w11) + Cr^n$

*Proof.* The construction of the intervals $I_w$ imply immediately the following. If $o(w) = +1$ then the interval $I_w$ contains the right boundary point of $I_{\sigma(w)}$. And if $o(w) = -1$ then $I_w$ contains the left boundary point of $I_{\sigma(w)}$. Using this, the convexity of $F_1$ and the fact that $F_0$ is affine we get

**Claim 3.4.**

$$o(w) \cdot q_{n+1}(w0) < o(w) \cdot q_{n+1}(w1),$$

*for every word $w$ with $|w| = n$.*

The case with $o(w) = -1$ of the Proposition can be proved similarly as the first case. We will only present the proof in the case $o(w) = +1$. The first statement is merely the definition of the order on the symbol space. The second follows directly from the Claim above. The Claim also implies

$$q_{n+2}(w00) < q_{n+2}(w01),$$

and

$$q_{n+2}(w11) < q_{n+2}(w10).$$

To study the middle inequality observe that

$$I_{\sigma(w)01} \cup I_{\sigma(w)11} \subset I_1.$$

First observe, $o(w01) = -1$ (and $o(w11) = 1$). In particular the negatively oriented interval $I_{w01}$ contains the left boundary point of the interval $I_{\sigma(w)01}$. Moreover,

$$I_{\sigma(w)01} \subset I_{01} \subset [0, f_0^4(c_{f_0})],$$

where $0 \in I_1$ is the left boundary point of $I_1$.

By Lemma 3.2 we have

$$R_1(f_0) : I_{\sigma(w)01} \to I_{\sigma(w)11}.$$

The Convexity Condition states that the absolute value of the derivative of this map decreases strictly on the interval $[0, f_0^6(c_{f_0})]$. Now using

$$I_{w01} \subset I_{\sigma(w)01} \subset [0, f_0^6(c_{f_0})]$$

and that $I_{w01} \subset I_{\sigma(w)01}$ contains the left boundary point of $I_{\sigma(w)01}$ we get

$$q_{n+2}(w01) < q_{n+2}(w11).$$

From the a priori bounds described in for example [Ma1] we know that there are constants $C > 0$ and $r \in (0, 1)$ such that

$$|I_w| \geq C \cdot r^{|w|}$$

for all words $w$. This implies the final estimate of the Proposition. □

Let $w$ be a word with $|w| = k$. Then define the interval

$$J_w = [q_{k+1}(w0), q_{k+1}(w1)].$$

**Proof of Theorem 1.8:** This Theorem is reformulated in

**Claim 3.5.** *Let $w$ be a word with $|w| = k$ and $|wh| = n$. Then*

$$q_n(wh) \in J_w.$$

*In particular,*

$$J_{wh} \subset J_w.$$

*Moreover, if $w^1$ and $w^2$ are distinct words of length $k$ then $J(w^1)$ and $J(w^2)$ are disjoint and the distance between them is larger than $Cr^k$.*

*Proof.* The proof of the first part of the Claim is by induction in $n$. For $n = 2$ the statement follows from the Proposition 3.3. Assume the Claim holds for all words $wh$ with $|wh| \leq n$.

Consider a word $wh = w\hat{h}h^1h^2$. with $|wh| = n+1$ and $|h^1| = |h^2| = 1$. Then the Proposition implies for every pair of symbol $x, y$

$$q_{n+1}(w\hat{h}xy) \in [q_{n+1}(w\hat{h}10), q_{n+1}(w\hat{h}00)].$$

In particular,

$$\begin{aligned} q_{n+1}(wh) &\in [q_{n+1}(w\hat{h}10), q_{n+1}(w\hat{h}00)] \\ &= [q_n(w\hat{h}1), q_n(w\hat{h}0)] \\ &\subset J_w, \end{aligned}$$

The above equality follows from the fact that $q_{n+1}(w\hat{h}10) = q_n(w\hat{h}1)$ because the interval $I_{w\hat{h}10}$ is obtained from $I_{w\hat{h}1}$ by applying the affine branch $F_0$. The other boundary is treated similarly. The last inclusion follows from the induction hypothesis.

The proof of the second part of the Claim is by induction in $k = |w|$. For $k = 1$ the Claim considering the distance between $J_0$ and $J_1$ is a reformulation of the previous Proposition. Assume, the Claim is proved up to some $k \geq 1$. Let $w^1$ and $w^2$ be two words of length $k+1$, say $w^1 = \tilde{w}^1 x$ and $w^2 = \tilde{w}^2 y$ with $|\tilde{w}^1| = |\tilde{w}^2| = k$.

If $\tilde{w}^1$ differs from $\tilde{w}^2$ then the Claim follows because

$$J_{w^1} \subset J_{\tilde{w}^1}, J_{w^2} \subset J_{\tilde{w}^2}$$

and the induction hypothesis. So we may assume that

$$w^1 = w0, w^2 = w1.$$

Apply Proposition 3.3 again to conclude that $J_{w^1}$ and $J_{w^2}$ are disjoint with the appropriate distance. □

**Proof of Theorem 1.6:**

The proof of Theorem 1.6 relies on the relation between the two iterated function systems generated by respectively $\{R_0, R_1\}$ and $\{F_0, F_1\}$ as formulated in Lemma 3.1. Notice, the only difference between $\Sigma$ and $\overline{T}$ is that they carry different orders. This order does not play a role during the proof of Theorem 1.6. We will use the symbol $w$ for words which are in $\Sigma = \overline{T}$. In section 2 we constructed the continuous map $h : \Sigma \to A$. Namely, for $w \in \Sigma$, let

$$h(w) = \lim_{n\to\infty} R_{w_{\{n\}}}(f_0).$$

In particular, this map is onto. It is left to show that $h$ is injective.

Observe, for every word $w$ with $|w| = n$

$$R_{0w}(f_0) = R_0(R_w(f_0)).$$

In particular,

$$q_{n+1}(0w) = |V_{R_w(f_0)}|.$$

Recall that for $w \in \Sigma$ we denote the word consisting of the first $n$ symbols of $w \in \Sigma$ by $w_{\{n\}}$. Let $w^1, w^2 \in \Sigma$ be such that $h(w^1) = h(w^2)$. Then

$$\begin{aligned}
|q(0w^1) - q(0w^2)| &= \lim_{n\to\infty} |q_{n+1}(0w^1_{\{n\}}) - q_{n+1}(0w^2_{\{n\}})| \\
&= \lim_{n\to\infty} ||V_{R_{w^1_{\{n\}}}(f_0)}| - |V_{R_{w^2_{\{n\}}}(f_0)}|| \\
&\leq \text{Const} \lim_{n\to\infty} \text{dist}(R_{w^1_{\{n\}}}(f_0), R_{w^2_{\{n\}}}(f_0)) \\
&= \text{Const} \cdot \text{dist}(h(w^1), h(w^2)) = 0.
\end{aligned}$$

The strict monotonicity of the scaling function, Theorem 1.8, implies $w^1 = w^2$. We proved that $h : \Sigma \to A$ is a homeomorphism.

# 4 The Convexity Condition

In this section the Convexity Condition will be studied.

**Lemma 4.1.** *Let $f : (-1, 1) \to (-1, 1)$ be $C^2$. If*

- $f(0) = 0$,
- $Df(0) < -1$,
- $D^2 f(0) < 0$

*then*

$$D^2(f^2)(0) < 0.$$

*Proof.* The chain rule applied to $f^2$ gives

$$D^2(f^2)(x) = D^2f(f(x)) \cdot (Df(x))^2 + Df(f(x)) \cdot D^2f(x).$$

Using the properties of $f$ in $x = 0$ we get

$$D^2(f^2)(0) = D^2f(0) \cdot Df(0) \cdot [Df(0) + 1] < 0.$$

□

**Lemma 4.2.** *Let $C \subset W$ consisting of unimodal maps $f \in W$, with negative Schwarzian derivative, see [dMvS], and the following property: $f|[0,c]$ is convex, where $c$ is the critical point of $f$, and $f|[c,1]$ is strictly convex (The derivative of $f$ is decreasing over $[0,1]$ but strictly decreasing on $[c,1]$). Then*

$$R_0(C) \subset C.$$

*Proof.* Let $f \in C$ with critical point $c \in [0,1]$ and $p_f$ its fixed point. Let $V_f = P \cup Q$, where $P, Q$ are the two intervals on which $R_0 f$ is monotone. Choose $Q \subset V_f$ such that $f(Q) \subset [0,c]$. The convexity property of $f$ implies directly the strict convexity of $R_0(f)|Q$.

The Schwarzian derivative of $f$ is negative. This implies that $p_f$ is an expanding fixed point, otherwise it would attract the critical point. Hence, $Df(p_f) < -1$. The convexity condition of $f$ allows us to apply the previous Lemma:

$$D^2R_0(f)(p_f) < 0,$$

the derivative of $f^2$ is decreasing in $p_f$. Now, the Minimum Principle for maps with negative Schwarzian derivative, [dMvS], implies that $Df^2$ is decreasing monotonically to zero on the interval $[p_f, P]$: $R_0 f \in C$. □

**Lemma 4.3.** *The convexity condition* **C1** *holds true for any even critical exponent $\alpha$, the map $f_0|[p_{f_0}, 1]$ is strictly convex.*

*Proof.* Let $q_t \in W$ be a standard folding map. Clearly, $q_t \in C$. From [S2] we have

$$\lim_{n\to\infty} R_1^n q_t = f_1.$$

Let $\hat{f}$ be the unique fixed point of $R$ (with $O_{1^\infty}(\hat{f}) = f_1$). As in the proof of Claim 2.5 we get for every $\tilde{f} \in U$ with $O_{1^\infty}(\tilde{f}) = q_t$ that

$$\lim_{n\to\infty} R^n \tilde{f} = \hat{f}.$$

Hence,

$$\begin{aligned}
\lim_{n\to\infty} R_0^n q_t &= \lim_{n\to\infty} O_{0^\infty} R^n \tilde{f} \\
&= O_{0^\infty}(\hat{f}) \\
&= f_0\,,
\end{aligned}$$

where $f_0$ is the fixed point of $R_0$. This implies that the derivative of $f_0$ is decreasing because of the previous Lemma . The renormalization fixed point $f_0$ is real analytic. Hence, the set $E \subset [0,1]$ consisting of the flat points of $f_0$, points where $D^2 f_0$ vanishes, is finite.

The map $f_0$ is the fixed point of $R_0$. Hence, $s(E)$, the map $s$ is the affine scaling of the interval $[0,1]$ to $V_{f_0}$, is the set of flat points of $R_0 f_0 (= f_0)$. Let $Q \subset V_{f_0}$ be the maximal interval such that $f_0(Q) \subset [0,c]$. Any non-flat point $x \in Q$ will be a non-flat point of $R_0 f_0$, this follows from the convexity, maybe not strict, of $f_0$. Hence

$$s(E) \cap Q \subset E.$$

Assume, $E \cap [c,1) \neq \emptyset$ and let $x \in E \cap [c,1)$ be the right most point. The fact that $f_0$ is a renormalization fixed point implies that $s(c)$ is the left boundary point of $Q$. In particular we get

$$x < s(x) \in s(E) \cap s([c,1]) \subset E \cap Q,$$

contradicting the fact that $x$ was chosen to be the right most point in $E \cap [c,1)$. We proved that $f_0$ does not have flat points in $[p_{f_0}, 1) \subset [c,1)$. □

**Lemma 4.4.** *The convexity condition* **C2** *holds true for* $\alpha = 2$.

*Proof.* In the case $\alpha = 2$, an approximation of $f_1$ can be found in [La]. It is of the form $g = h(x^2)$, where $h$ is a degree 40 polynomial. It follows from Lemma 2.4 that $g'(x) = (h(x))^2$ is an approximation of the fixed point $f_0$. A numerical analysis shows the convexity condition **C2**. In particular,

$$D^2 R_1(f_0)(x) \leq -1.0\,,$$

for $x \in [(R_1(f_0))^3(c_{R_1(f_0)}), 1]$. □

The quadratic case, $\alpha = 2$, as described in Theorem 1.10 follows from Theorem 1.6, 1.8, and Lemma 4.3 and 4.4.

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