RATIONAL MAPS WITH DECAY OF GEOMETRY: RIGIDITY, THURSTON'S ALGORITHM AND LOCAL CONNECTIVITY.

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Abstract. *We study dynamics of rational maps that satisfy a decay of geometry condition. Well known conditions of non-uniform hyperbolicity, like summability condition with exponent one, imply this condition. We prove that Julia sets have zero Lebesgue measure, when not equal to the whole sphere, and in the polynomial case every connected component of the Julia set is locally connected.*

We show how rigidity properties of quasi-conformal maps that are conformal in a big dynamically defined part of the sphere, apply to dynamics. For example we give a partial answer to a problem posed by Milnor about Thurston's algorithm and we give a proof that the Mandelbrot set, and its higher degree analogues, are locally connected at parameters that satisfy the decay of geometry condition. Moreover we prove a theorem about similarities between the Mandelbrot set and Julia sets.

In an appendix we prove a rigidity property that extends a key situation encountered by Yoccoz in his proof of local connectivity of the Mandelbrot set at at most finitely renormalizable parameters.

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Introduction.

We study dynamics of complex rational maps $R \in \mathbb{C}(z)$, especially in the *Julia set* $J(R) \subset \hat{\mathbb{C}}$ of R, which is the closure of the repelling cycles of R. The set Crit of *critical points* of R plays a special role; we denote by $\mathrm{CV} = R(\mathrm{Crit})$ the set of *critical values* of R. We also denote by $B_\delta(z) \subset \hat{\mathbb{C}}$ the ball of radius δ centered at $z \in \hat{\mathbb{C}}$.

For our purposes is more natural to measure distances to critical values, rather than to critical points. So for given $c \in \mathrm{Crit} \cap J(R)$ and $\delta > 0$ we denote by $\tilde{B}_\delta(c)$ the connected component of $R^{-1}(B_\delta(R(c)))$ that contains c.

We study dynamics of rational maps $R \in \mathbb{C}(z)$ that satisfy the following condition.

Decay of Geometry Condition. *There is a function r_0 such that $r_0(\delta) \to \infty$ as $\delta \to 0$ and such that for any $c \in \mathrm{Crit} \cap J(R)$, any $n \geq 0$ and any connected component W of $R^{-n}(\tilde{B}_{\delta r_0(\delta)}(c))$ such that $dist(W, \mathrm{CV}) \leq \delta$, we have $diam(W) \leq \delta$.*

We denote by $\mathcal{S}$ the class of rational maps that satisfy this condition. This condition is vacuous for rational maps without critical points in the Julia set. The results that we present here are either well known or vacuous for these rational maps.

Well known conditions of non-uniform hyperbolicity, like summability condition with exponent one, imply the Decay of Geometry condition. A rational map $R \in \mathbb{C}(z)$ is said to satisfy the *summability condition* with exponent $\beta > 0$ if

$$\sum_{n \geq 0} |(R^n)'(v)|^{-\beta} < \infty,$$

for every critical value $v \in J(R)$ not in the backward orbit of a critical point; see also [NS], [Pr2], [GS2] and [PU2]. Derivatives are taken with respect to the spherical metric.

A rational map is said to satisfy the stronger *Collet-Eckmann* condition if $|(R^n)'(v)|$ is exponentially big in $n \geq 0$ for every such critical value v; see also [CE], [Pr2], [Pr3], [GS1], [PRS] and references therein.

The following table indicates how $r_0(\delta)$ growths as $\delta \to 0$, when one of these conditions is satisfied.

- *Summability condition exp.* 1 — $r_0(\delta) \to \infty$ as $\delta \to 0$.
- *Summability condition exp.* $\beta \in (0,1)$ — $\int_0^{\delta_0} (r_0(\delta))^{-\alpha} \frac{d\delta}{\delta} < \infty$, $\alpha = \frac{\beta}{1-\beta}$.
- *Collet-Eckmann* — $r_0(\delta) \geq C\delta^{-\alpha}$ *for some* $\alpha \in (0,1]$.
- *Non-Recurrent* — *Equivalent to* $r_0(\delta) \sim \delta^{-1}$.

Where *Non-Recurrent* stands for the condition that no critical point in the Julia set accumulates on a critical point under forward iteration.

We remark that the summability and Collet-Eckmann conditions allow the existence of parabolic cycles; compare with [PU2] and [GS2]. It is interesting to remark that there are rational maps in $\mathcal{S}$ that do not satisfy any summability condition.

Local connectivity and measure of Julia sets.

Our first result is that every connected component of a Julia set of a polynomial in $\mathcal{S}$ is locally connected. In fact we prove a stronger fact about combinatorial descriptions of dynamics done by Thurston for quadratic polynomials; see [T] and also [D]. We consider the work of Kiwi [Ki] that is valid in any degree.

To each polynomial $P \in \mathbb{C}[z]$ without irrationally indifferent cycles we can associate a ramified covering P^{top} of $\mathbb{C}$ and a semi-conjugacy $\pi : \mathbb{C} \longrightarrow \mathbb{C}$ from P to P^{top}. The dynamics of P^{top} (in particular in $J^{top}(P) = \pi(J(P))$) can be described in purely combinatorial terms.

Thus if π is an homeomorphism, the dynamics of P corresponds to the well understood dynamics of P^{top}. We remark that polynomials in $\mathcal{S}$ do not have irrationally indifferent cycles so these considerations apply to them (Corollary 7.2).

Theorem A. *Let $P \in \mathbb{C}[z]$ be a polynomial in $\mathcal{S}$. Then the following equivalent properties hold.*

(1) *The projection $\pi : \mathbb{C} \longrightarrow \mathbb{C}$ is a homeomorphism.*
(2) *(No wandering continua) For every non-trivial connected set $\xi \subset J(P)$ there are $l > 0$ and $k \geq 0$ such that $P^k(\xi) \cap P^{k+l}(\xi) \neq \emptyset$.*
(3) *Every non-trivial connected subset of $J(P)$ contains a pre-periodic point.*

Moreover these properties imply that every connected component of $J(P)$ is locally connected.

The equivalence of properties $1-3$ and that these imply the final statement, is proved in [Ki]. The following corollary follows immediately from the last assertion of Theorem A.

Corollary. *The Julia set of a polynomial in $\mathcal{S}$ is locally connected when it is connected.*

Our next result is about measure of Julia sets; see also [GS2], [Pr3] and [Lyu2].

Theorem B. *Let $R \in \mathbb{C}(z)$ be a rational map in $\mathcal{S}$ such that $J(R) \neq \hat{\mathbb{C}}$. Then $J(R)$ has zero Lebesgue measure.*

Rigidity and Thurston's algorithm.

With the area estimates needed for the proof of Theorem B we prove a rigidity property, which is our main technical result (Section 5.1). We refer to it as *Rigidity.* We also include a related rigidity property on an appendix, that is stated in an abstract setting. This generalizes a key situation encountered by Yoccoz in his unpublished proof of the local connectivity of the Mandelbrot set at non-infinitely renormalizable parameters.

Now we describe applications of rigidity. The first application is about the convergence of Thurston's algorithm that we proceed to describe; see [DH3], [HS] and [Pil].

Consider a topological ramified covering $\tilde{R}$ of the sphere S^2. That is $\tilde{R} : S^2 \longrightarrow S^2$ is locally of the form $\xi(z^n)$, where ξ is an homeomorphism and $n \geq 1$. There is at most a finite number of points for which $n > 1$; such points are called *ramification* points of $\tilde{R}$.

Define inductively coordinates $h_k : S^2 \longrightarrow S^2$, for $k \geq 0$, as follows. Let h_0 be equal to the identity and given the coordinate h_{k-1}, for some $k > 0$, let h_k be such that $Q_k = h_{k-1} \circ \tilde{R} \circ h_k^{-1}$ is a rational map; see Figure 1 (we identify S^2 with $\hat{\mathbb{C}}$). If $\tilde{R}$ is *quasiregular* such coordinate is uniquely determined up to normalization. The

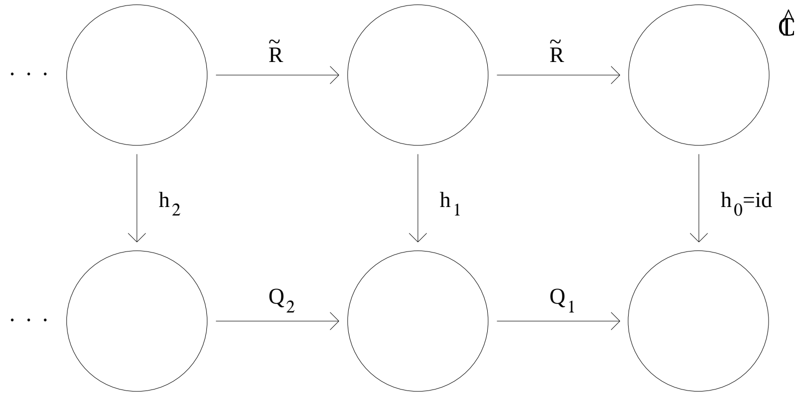

FIGURE 1. For $k > 0$ the coordinate h_k is chosen so that $Q_k = h_{k-1} \circ \tilde{R} \circ h_k^{-1}$ is a rational map.

ramified covering $\tilde{R}$ is said to be quasiregular if the homeomorphisms ξ as above are *quasi-conformal*. (A normalization is to fix three preferred points of S^2.)

In this way we obtain a sequence of rational maps Q_k related to $\tilde{R}$. Note that, if the rational maps Q_k converge uniformly to a rational map Q and the coordinate h_k converge uniformly to a continuous map $h : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$, then $\tilde{R} \circ h = Q \circ h$. That is $\tilde{R}$ is semi-conjugated to Q by h.

Thurston considered quasiregular maps $\tilde{R}$ for which the set

$$P(\tilde{R}) = \{\tilde{R}^n(r) \mid \text{ for } n \geq 1 \text{ and } r \text{ ramification point of } \tilde{R}\},$$

is finite. He determined (except for some specific cases) when the rational maps Q_k converge uniformly to a rational map. In this case $h_k|_{P(\tilde{R})}$ converges to an injective map. Milnor posed the following problem.

Problem (Milnor [Mil1]**).** *Under what conditions will the sequence of rational maps Q_k converge uniformly to a limiting map Q? Under what conditions, and on what subset of $\hat{\mathbb{C}}$, will the maps h_k converge uniformly to a limit h?.*

The next theorem gives a partial answer to this problem and will allows us to make a wide variety of small perturbations of a given rational map $R \in \mathcal{S}$, with $J(R) \neq \hat{\mathbb{C}}$.

Theorem C. *Let $R \in \mathbb{C}(z)$ be a rational map in $\mathcal{S}$ such that $J(R) \neq \hat{\mathbb{C}}$. Then for $\delta > 0$ small the following assertions hold.*

Non-Recurrent Quasiregular Perturbation: *There is a quasiregular map $\tilde{R}$ of $\hat{\mathbb{C}}$, of the same degree as R, such that the following properties hold.*
- *$\tilde{R}$ coincides with R outside $\tilde{B}_\delta(\mathrm{Crit} \cap J(R))$.*
- *For every ramified point $r \in \tilde{B}_\delta(\mathrm{Crit} \cap J(R))$ of $\tilde{R}$, we have $\tilde{R}^k(r) \notin \tilde{B}_{2\delta}(\mathrm{Crit} \cap J(R))$ for $k \geq 1$.*

Convergence of Thurston's Algorithm: *Fix $\tilde{R}$ given by the first part and consider Q_k and h_k as above, for an appropriated normalization of the coordinates h_k. Then there is a rational map $Q \in \mathbb{C}(z)$ and a continuous map $h : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$*

such that $Q_k \to Q$ and $h_k \to h$ uniformly; in particular Q has the same degree as R. Moreover Q is close to R and h is close to the identity as δ is close to 0.

We remark that the limit h of the coordinates h_k is *not* injective in general (Remark 6.1).

The proof of Thurston's theorem (mentioned above) requires subtle considerations in an appropriated Teichmuller space and it uses in an essential way the finiteness of the set $P(\tilde{R})$ of post ramification points. In our situation the proof of the convergence of Thurston's algorithm only requires basic facts about quasi-conformal homeomorphisms.

Polynomials with only one critical point.

The rest of this work is dedicated to the families of *unicritical polynomials*. That is, given $d \geq 2$, we restrict our attention to the family of monic polynomials $P_c(z) = z^d + c$, for $c \in \mathbb{C}$, whose unique finite critical point is 0.

Our first result is about the variation of the dynamics of P_c for c near a parameter c_0, such that $P_{c_0} \in \mathcal{S}$. One of the simplest dynamical properties of a polynomial P_c is the recurrence of the critical point 0. A point $w \in \mathbb{C}$ is said to be *recurrent* under P_c if the forward orbit of w under P_c accumulates on w. The following theorem shows a strong instability in the dynamics at a polynomial $P_{c_0} \in \mathcal{S}$ with recurrent critical point. The non-recurrent case is simpler to study; see for example [R-L1].

Theorem D (Instability in the Parameter). *Consider a polynomial $P_{c_0} \in \mathcal{S}$ with recurrent critical point. Then every non-trivial connected set of parameters containing c_0, also contains c such that the critical point of P_c is not recurrent.*

This theorem is the parameter analogue of part 3 of Theorem A.

A polynomial P_c is said to be *hyperbolic* if P_{c_0} has a finite attracting cycle or if $P_{c_0}^n(0) \to \infty$ as $n \to \infty$. The dynamics of hyperbolic polynomials is well understood.

The *Hyperbolicity Conjecture* for the family $P_c(z) = z^d + c$, for $c \in \mathbb{C}$, asserts that the set of parameters $c \in \mathbb{C}$ such that P_c is hyperbolic is dense in $\mathbb{C}$. It is well known that if the critical point of P_{c_0} is not recurrent, then c_0 is approximated by parameters c such that P_c is hyperbolic. Thus the Hyperbolicity Conjecture is equivalent to the following statement.

Re-statement of Hyperbolicity Conjecture. *Let $c_0 \in \mathbb{C}$ be such that the critical point of P_{c_0} is recurrent. Then c_0 is approximated by parameters $c \in \mathbb{C}$ such that the critical point of P_c is not recurrent.*

In particular Theorem D implies that a parameter c_0 such that $P_{c_0} \in \mathcal{S}$ has recurrent critical point is accumulated by parameters $c \in \mathbb{C}$ such that P_c is hyperbolic. (This also follows from Theorem B and [MSS].)

Douady and Hubbard made a remarkable conjecture that implies the density of hyperbolicity in the quadratic family (that is when $d = 2$); see [DH1]. Consider the *connectedness locus*

$$\mathcal{M}_d = \{c \in \mathbb{C} \mid J_c \text{ is connected}\}.$$

For $d = 2$ it is just denoted by $\mathcal{M}$ and it is also called the *Mandelbrot* set. Douady and Hubbard showed that $\mathcal{M}_d$ is compact and connected.

Conjecture (MLC). *The Mandelbrot set is locally connected.*

As was mentioned, a positive answer to this conjecture implies density of hyperbolicity in the quadratic family. The situation is analogous in any degree. So, if $\mathcal{M}_d$ is locally connected, then hyperbolicity is dense in the family $P_c(z) = z^d + c$, for $c \in \mathbb{C}$.

It follows easily from [Sch] that MLC is equivalent to the following statement.

Re-statement of MLC. *Let $c_0 \in \mathbb{C}$ be such that the critical point of P_{c_0} is recurrent. Then every non-trivial connected set of parameters containing c_0 also contains a parameter c such that the critical point of P_c is not recurrent.*

Again the situation is analogous any degree. In fact if $c_0 \in \mathbb{C}$ satisfies the property above, then $c_0 \in \partial\mathcal{M}_d$ and $\mathcal{M}_d$ is locally connected at c_0; see Appendix 9. Thus we obtain the following immediate corollary of Theorem D.

Corollary of Theorem D. *Consider a polynomial $P_{c_0} \in \mathcal{S}$ with recurrent critical point. Then $c_0 \in \partial\mathcal{M}_d$ and $\mathcal{M}_d$ is locally connected at c_0.*

After Theorem A it is easy to prove that polynomials $P_c \in \mathcal{S}$ such that $c \in \mathcal{M}_d$ are at most finitely renormalizable. Hence, for $d = 2$, Corollary of Theorem D follows from a result of Yoccoz that states that $\mathcal{M}$ is locally connected at parameters that are at most finitely renormalizable. The proof of Yoccoz does not apply for $d > 2$ and it is not clear if this corollary follows from his proof. In any case the technique in the proof of Theorem D is different from that of Yoccoz. In particular in the proof of Theorem D we do not mention puzzle pieces.

Our last theorem is about similarities between $\mathcal{M}_d$ and J_{c_0} and near c_0. We consider the notion of *asymptotic similarity* introduced by T. Lei in [Lei]. Given a compact subset X of $\mathbb{C}$ and $r > 0$ small, let

$$X_r = (\{\frac{1}{r}w \mid w \in X\} \cap \mathbb{D}) \cup \partial\mathbb{D},$$

that is, to get X_r consider the intersection of X with the disc of radius r centered at 0, scale it to the unit disk and for a technical reason add $\partial\mathbb{D}$. Moreover, for $\lambda \in \mathbb{C} - \{0\}$ and $\zeta \in \mathbb{C}$ we denote $\lambda X = \{\lambda w \mid w \in X\}$ and $X - \zeta = \{w - \zeta \mid w \in X\}$.

Theorem E. *Let $c_0 \in \partial\mathcal{M}_d$ be such that $P_{c_0} \in \mathcal{S}$ and such that the function r_0, in the definition of the class $\mathcal{S}$, satisfies*

$$\int_0^{\delta_0} (r_0(\delta))^{-\frac{2}{d}} \frac{d\delta}{\delta} < \infty, \text{ for some } \delta_0 > 0.$$

Then $\mathcal{M}_d$ and J_{c_0} are asymptotically similar at c_0. That is, there is $\lambda \in \mathbb{C} - \{0\}$ such that

$$\lim_{r \to 0} d_H((\mathcal{M}_d - c_0)_r, (\lambda(J_{c_0} - c_0))_r) = 0,$$

where d_H denotes the Hausdorff distance. If Moreover P_{c_0} satisfies the Collet-Eckmann condition, then there are $\alpha > 0$ and $C > 0$ such that for small $r > 0$,

$$d_H((\mathcal{M}_d - c_0)_r, (\lambda(J_{c_0} - c_0))_r) \leq Cr^{\alpha}.$$

The following corollary follows immediately from the previous theorem and from the table at the beginning of this introduction.

Corollary. *Let $c_0 \in \partial\mathcal{M}_d$ be such that P_{c_0} satisfies the summability condition with exponent $\frac{2}{d+2}$. Then $\mathcal{M}_d$ and J_{c_0} are asymptotically similar at c_0.*

For any $P_{c_0} \in \mathcal{S}$ for which Theorem E works we can prove that the similarity factor $\lambda \in \mathbb{C} - \{0\}$ is equal to

$$\lambda = \sum_{n \geq 0} \frac{1}{(P_{c_0}^n)'(c_0)}.$$

So we obtain the curious fact that the sum above is finite and different form zero for such parameters.

Smirnov proved in [Sm] that the set of parameters in $\partial \mathcal{M}_d$ satisfying the Collet-Eckmann condition has full harmonic measure in $\partial \mathcal{M}_d$. So we obtain the following corollary.

Corollary. *There is a set of full harmonic measure of parameters $c \in \partial \mathcal{M}_d$ such that $\mathcal{M}_d$ and J_c are asymptotically similar at c.*

A stronger statement was obtained in [R-L2]. T. Lei proved in [Lei] that $\mathcal{M}$ and J_c are asymptotically similar at c, for parameters c for which the critical point of P_c is strictly pre-periodic. The set of such parameters is countable. Moreover the corresponding polynomials satisfy the Collet-Eckmann condition, since the critical point must be mapped to a repelling periodic point. See also [W], and [R-L2] and [R-L1] for a finer notion of similarity and related results.

The idea of the proof of Theorem E is that for any $P_{c_0} \in \mathcal{S}$ we can construct a *parameter map*, which is a quasi-conformal homeomorphism form a neighborhood of c_0 in the dynamical plane to a neighborhood of c_0 in the parameter plane, that maps the Julia set to a set approximating $\mathcal{M}_d$ near c_0. Then asymptotic similarity easily reduces to proving that the parameter map is conformal at c_0. This parameter map is conformal in a big part of its domain. If the integral condition in Theorem E is satisfied, then the parameter map satisfies the hypothesis of a conformality criterion. In the Collet-Eckmann case we prove that this parameter map is $C^{1+\alpha}$-conformal for some $\alpha > 0$, using McMullen's measurable deep points. This yields the stronger conclusion in the theorem.

Organization of the paper. Now we describe the contents of each section.

In *Section* 1 we state the *Univalent Pull-back Condition* and we prove that it is equivalent to the Decay of Geometry Condition; see Proposition 1.2. In Section 1.1 we prove the relations between summability and Collet-Eckmann conditions and the class $\mathcal{S}$; see Proposition 1.4.

In *Section* 2 we prove some lemmas about expansion and we introduce Martens property in Section 2.1. In Section 2.2 we construct neighborhoods of critical points with Martens property at every scale; see Proposition 2.7.

Section 3 is concerned with Theorem A about topological models of dynamics of polynomials. The proof relies in a landing lemma for hyperbolic sets (proved in Section 3.1) and in Thurston-Kiwi Finiteness Theorem, stated in Section 3.2. In Section 3.2 we prove Theorem A under a technical assumption and assuming that there are no parabolic periodic points. In Section 7 we prove that this technical assumption is automatically satisfied for polynomials in the class $\mathcal{S}$. The proof of Theorem A in the general case (when there are parabolic periodic points) is in Section 7.2.

In *Section* 4 we prove the area estimates needed in the proof of Theorem B and in the proof of Rigidity. Moreover we prove Proposition B in Section 4.2, which is the essential part of Theorem B. To complete the proof of Theorem B we need again a mild assumption. In Section 7 we prove that this assumption is automatically satisfied.

In *Section* 5 we prove Rigidity. The proof is very simple, given the area estimates of Section 4; see Lemma 4.7 in Section 4.2. It relais in basic facts about qc maps that are stated in Appendix 10.

Section 6 contains the proof of Theorem C about Thurston's algorithm. We first state Proposition C which is a more complete version of Theorem C, including the existence of pseudo-conjugacies to the limiting rational map.

In *Section* 7 we exploit pseudo-conjugacies given by Proposition C, to prove that rational maps in the class $\mathcal{S}$ whose Julia set in not the whole sphere, enjoy several expansion properties at a global level (away from critical points). For example we prove that such rational maps do not have irrationally indifferent cycles nor Herman rings. For this purposes we state Corollary C of Theorem and Proposition C. As a consequence we complete the proof of Theorem B. Moreover, in Section 7.2 we complete the proof of Theorem A in the presence of parabolic periodic points. The main step is to construct neighborhoods with Martens property taking into account parabolic points; see Proposition 7.11.

In *Section* 8 we consider the family $P_c(z) = z^d + c$, for $c \in \mathbb{C}$ and for some fixed $d \geq 2$. After simple facts about dynamically defined holomorphic motions in Section 8.1, we prove Theorem D in Section 8.2. The proof is independent of Section 6, except that we need to know that the mild property EAC holds for polynomials; this is also proven in [R-L3]. The rest of Section 8 is dedicated to the proof of Theorem E. We build a parameter map in Sections 8.3 and 8.4 and in Section 8.5 we reduce Theorem E to prove that this parameter map is conformal at the critical value. Then we apply a conformality criterion to prove this. We also prove the stronger assertion in the Collet-Eckmann case.

In Appendix 9 we prove the re-statement of MLC, which follows easily from [Sch].

Appendix 10 contains some basic facts about quasi-conformal homeomorphisms.

In Appendix 11 we prove a rigidity property related to Rigidity of Section 5. The conclusion is weaker, but the hypothesis are simple enough to be presented in an abstract setting. This extends an argument of Yoccoz in his unpublished proof of the local connectivity of the Mandelbrot set at non-infinitely renormalizable parameters.

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Preliminaries.

Given a rational map $R \in \mathbb{C}(z)$ we denote by Crit the set of critical points of R. For a critical point $c \in$ Crit we denote by μ_c the multiplicity of R at c and $\mu_{max} = \max_c \mu_c$

denotes the maximal multiplicity of critical points in $J(R)$. In the case that a critical point in $J(R)$ is eventually mapped into another critical point, we treat a whole block c_1, ..., c_k of critical points, such that one is eventually mapped into the next one and maximal with this properties; as a single critical point of multiplicity $\mu_{c_1} \cdot ... \cdot \mu_{c_k}$. For example the critical value of this block means $R(c_k)$. With this convention we assume that no critical point in $J(R)$ is eventually mapped into some other critical point. Moreover we denote by CV the set of critical values of R, so with the convention above $\mathrm{CV} \cap J(R)$ is disjoint from Crit.

Given $z \in \hat{\mathbb{C}}$ we denote by $\omega(z)$ the *omega limit set* of z, that is the set of accumulation points of the forward orbit of z. For two numbers A and B, $A \sim B$ and $A = \mathcal{O}(B)$ means $C^{-1}B < A < CB$ and $A < CB$ for some implicit constant $C > 0$, respectively. Distances and derivatives are taken with respect to the spherical metric. For $z \in \mathbb{C}$ and $\delta > 0$ we denote by $B_\delta(z)$ the ball centered at z of radius δ. Recall that, for a critical point $c \in J(R)$ and $\delta > 0$ small $\tilde{B}_\delta(c)$ is the connected component of $R^{-1}(B_\delta(R(c)))$ that contains c, so $R(\tilde{B}_\delta(c)) = B_\delta(R(c))$ and $diam(\tilde{B}_\delta(c)) \sim \delta^{\frac{1}{\mu_c}}$.

Fix periodic orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ of period at least 2, so there is $r_K > 0$ such that for all $z \in \hat{\mathbb{C}}$, $B_{r_K}(z)$ is disjoint from $\mathcal{O}_1$ or from $\mathcal{O}_2$. Hence for every $n \geq 0$, every connected component of $R^{-n}(B_{r_K}(z))$ avoids $\mathcal{O}_1$ or $\mathcal{O}_2$. Thus we have the following spherical version of Koebe Distortion Theorem; see [Pom].

Koebe Distortion Theorem. *For $\varepsilon \in (0,1)$ there is $D = D(\varepsilon) > 1$ such that for any $z \in \hat{\mathbb{C}}$ and $r \in (0, r_K)$, if $\hat{W} \subset W$ are univalent pull-backs of $B_{\varepsilon r}(z) \subset B_r(z)$ by R^n, then the distortion of R^n in $\hat{W}$ is bounded by D. That is,*

$$D^{-1} \leq \frac{|(R^n)'(z_1)|}{|(R^n)'(z_2)|} \leq D, \text{ for } z_1, z_2 \in \hat{W}.$$

Moreover $D(\varepsilon) = 1 + \mathcal{O}(\varepsilon)$.

It is easy to see that, if $R \in \mathcal{S}$, then CV is disjoint from the parabolic points of R. Hence, by the Fatou-Sullivan classification of Fatou components, there is a neighborhood of $\mathrm{CV} \cap J(R)$ that avoids the forward orbits of critical points not in $J(R)$. We will implicitly assume that neighborhoods of points in $\mathrm{CV} \cap J(R)$ or $\mathrm{Crit} \cap J(R)$ are sufficiently small to avoid these forward orbits.

1. Decay of geometry and univalent pull-back conditions.

In this section we prove that the Decay of Geometry Condition, stated in the introduction, is equivalent to a *Univalent Pull-back Condition.* Moreover in Section 1.1 we prove Proposition 1.4 relating various conditions of non-uniform hyperbolicity with the asymptotics of the function r_0, involved in the Decay of Geometry Condition.

Definition 1.1. *The class $\mathcal{S}$ is the class of rational maps satisfying the Decay of Geometry Condition. Moreover, for $R \in \mathcal{S}$ and $\delta > 0$ small, we denote by $r(\delta)$ the biggest possible value of $r_0(\delta)$.*

Univalent Pull-back Condition. *There is a function r_1, defined for $\delta > 0$, such that $r_1(\delta) \to \infty$ as $\delta \to 0$ and such that for all $z \in \hat{\mathbb{C}}$ and $n \geq 0$, such that $R^j(z) \notin \tilde{B}_\delta(\mathrm{Crit} \cap J(R))$ for $0 \leq j < n$ and $R^n(z) \in \tilde{B}_{\delta r_1(\delta)}(c)$, for some $c \in \mathrm{Crit} \cap J(R)$, we have that the pull-back of $\tilde{B}_{\delta r_1(\delta)}(c)$ to z by R^n is univalent.*

Proposition 1.2. *If R satisfies the Decay of Geometry Condition then R satisfies the Univalent Pull-back condition with function $r_1 = r$. Furthermore, if R satisfies the Univalent Pull-back condition with function r_1, then there is $\kappa \in (0,1)$ only depending in R such that R satisfies the Decay of Geometry Condition and $r(\delta) \geq \kappa r_1(\delta)$, for $\delta > 0$ small. Moreover, for $0 < \delta < \delta_0$ small*

$$\kappa\delta r(\delta) \leq \delta_0 r(\delta_0),$$

and therefore $r(\delta) = \mathcal{O}(\delta^{-1})$.

Proof. 1.– Let $R \in \mathbb{C}(z)$ satisfying the Decay of Geometry Condition. Let $z \in \hat{\mathbb{C}}$ and $n \geq 0$ be as in the Univalent Pull-back Condition and consider a pull-back W_0, ..., $W_n = \tilde{B}_{\delta r(\delta)}(c)$ along the orbit of z, so $z \in W_0$. If $W_i \cap \text{Crit} \neq \emptyset$, for some $0 \leq i < n$, then by the Decay of Geometry Condition, $diam(W_{i+1}) < \delta$. Since $R^{i+1}(z) \in W_{i+1}$ does not belong to $B_\delta(\text{CV})$ we have that $W_{i+1} \cap \text{CV} = \emptyset$, which contradicts our assumption. So the W_i are disjoint from Crit and therefore the pull-back is univalent.

2.– Let $R \in \mathbb{C}(z)$ satisfying the Decay of Geometry Condition with function r_1. Let $\kappa \in (0,1)$ to be determined later and let $c \in \text{Crit} \cap J(R)$. Consider the special case of a pull-back W_0, ..., $W_n = B_{\kappa\delta r_1(\delta)}$ so that $W_i \cap B_\delta(\text{CV}) = \emptyset$ for $0 < i \leq n$ and so that $W_0 \cap B_\delta(c_0) \neq \emptyset$, for some $c_0 \in \text{Crit} \cap J(R)$. If $W_0 \subset B_\delta(c_0)$, there is nothing to prove, so assume the contrary. Consider the respective pull-back W_0', ..., $W_n' = \tilde{B}_{\delta r_1(\delta)}(c)$. By the Univalent Pull-back Property applied to a point z whose image lies in $W_0 - B_\delta(c_0)$, we obtain that the pull-back W_0' is univalent. So $mod(W_0' - \overline{W_0})$ depends in $\kappa \in (0,1)$ and it is big as κ is small. Note that W_0' avoids one of the two fixed periodic orbits $\mathcal{O}_1$ and $\mathcal{O}_2$, as in the Preliminaries. Thus, considering that $W_0 \cap B_\delta(c_0) \neq \emptyset$ and $c_0 \notin W_0$ we can choose κ, independent of δ, such that $diam(W_0) \leq \delta$. If $\delta > 0$ is small enough we have, $W_0' \subset B_{2\delta}(c_0) \subset B_{\kappa\delta r_1(\delta)}(c_0)$. Then the general case follows by induction.

3.– For $\delta > 0$ let $\tilde{r}_1(\delta)$ be the biggest possible value of $r_1(\delta)$. It follows by 2 that $\kappa\tilde{r}_1(\delta) \leq r(\delta) \leq \tilde{r}_1(\delta)$. It follows from the definition of the Univalent Pull-back condition that, if $0 < \delta < \delta_0$, then $\delta\tilde{r}_1(\delta) \leq \delta_0\tilde{r}_1(\delta_0)$. Thus $\kappa\delta r(\delta) \leq \delta_0 r(\delta_0)$ and $r(\delta) \leq C\delta^{-1}$, where $C = \kappa^{-1}\delta_0 r(\delta_0)$. □

Remark 1.3. *In the Decay of Geometry and Univalent Pull-back conditions we measure distances to critical values, rather than critical points. That is why we consider the balls $\tilde{B}$. This is because we are interested in conditions in the forward dynamics such as summability and Collet-Eckmann conditions. Our considerations apply to conditions in backward dynamics, by measuring distances to critical points, instead of critical values. This is the case of the Topological Collet-Eckmann condition and its equivalent formulations; see* [PRS].

1.1. **Summability and Collet-Eckmann conditions.** In this section we establish relations between different conditions of non-uniform hyperbolicity with the asymptotics of the function r, involved in the definition of the class $\mathcal{S}$.

Recall that by Proposition 1.2, for $R \in \mathcal{S}$ we have $r(\delta) = \mathcal{O}(\delta^{-1})$. Moreover, it is easy to see that $r(\delta) \sim \delta^{-1}$ if and only if no critical point in R is accumulated by other critical point. That is $r(\delta) \sim \delta^{-1}$ if and only if R is Non Recurrent; see Introduction. Together with Proposition 1.2 the following proposition proves the table in the Introduction.

Proposition 1.4. *Let R be a rational map of degree at least 2. Then,*

(1) *If R satisfies summability condition with exponent one, then $R \in \mathcal{S}$.*

(2) *If R satisfies summability condition with exponent $\beta \in (0,1)$, then letting $\alpha = \frac{\beta}{1-\beta}$,*
$$\int_0^{\delta_0} (r(\delta))^{-\alpha} \frac{d\delta}{\delta} < \infty \text{ for } \delta_0 > 0.$$
(3) *If R satisfies the Collet-Eckmann condition, then $r(\delta) > C\delta^{-\alpha}$ for some $\alpha \in (0,1]$ and $C > 0$.*

Recall that by Proposition 1.2 there is $\kappa \in (0,1)$ such that for $0 < \delta < \delta_0$ small we have $\kappa\delta r(\delta) \leq \delta_0 r(\delta_0)$. Thus, for given $\alpha > 0$ the following conditions are equivalent.

(1) $\int_0^{\delta_0} (r(\delta))^{-\alpha} \frac{d\delta}{\delta} < \infty$ for $\delta_0 > 0$.
(2) For all $\theta \in (0,1)$ we have, $\sum_{n \geq 1} (r(\theta^n))^{-\alpha} < \infty$.
(3) For some $\theta \in (0,1)$ we have, $\sum_{n \geq 1} (r(\theta^n))^{-\alpha} < \infty$.

The proof of Proposition 1.4 is based in Przytycki's shrinking neighborhoods, which is a tool to control distortion of backward iterates; see [Pr2]. It consists in the following: choose a sequence of positive numbers $\{d_n\}_{n \geq 1}$ such that $\Pi_{n \geq 1}(1-d_n) = \frac{1}{2}$, and put $D_n = \Pi_{k \leq n}(1 - d_k)$. Then given $r \in (0, r_K)$ and $w \in R^{-n}(z)$ let $U_n' \subset U_n$ be the connected components of $R^{-n}(B_{rD_n}(z))$ and $R^{-n}(B_{rD_{n+1}}(z))$ that contain w, respectively, so that $U_0 = B_r(z)$. Then, there is a constant $K > 1$ independent of n such that, if U_j does not contain a critical point, for $0 \leq j \leq n$, then

(1) $$K^{-1} d_n < dist(w, \zeta) |(R^n)'(\zeta)| < K d_n^{-1}, \text{ for } \zeta \in U_n';$$

see Lemma 1.2 of [Pr2]. For a critical point $c \in \text{Crit} \cap J(R)$ we will consider the shrinking neighborhoods with balls $\tilde{B}_{rD_n}(c)$, instead of $B_{rD_n}(c)$. In this case we have the same distortion estimate, but with other constant.

Let us consider some definitions for the proof of Proposition 1.4. Given $v \in \text{CV} \cap J(R)$ and $c \in \text{Crit} \cap J(R)$ consider all the times $0 < k_1(v,c) < k_2(v,c) < ...$ so that the pull-back of the closure of $\tilde{B}_{r_i}(c)$ by $R^{k_i(v,c)}$ to v is univalent, where $r_i > 0$ is the smallest number so that the closure of $\tilde{B}_{r_i}(c)$ contains $R^{k_i(c,v)}(v)$. Denote by $\xi_i(v,c)$ the respective preimage of c by $R^{k_i(c,v)}$.

Lemma 1.5. *Suppose that R satisfies the summability condition with exponent $\beta \in (0,1]$. If $\beta = 1$ let $\{\eta_n\}_{n \geq 1}$ be such that $\eta_n \to \infty$ as $n \to \infty$ and,*
$$\sum_{n \geq 1} \frac{\eta_n}{|(R^n)'(v)|^\beta} < \infty, \text{ for } v \in \text{CV} \cap J(R).$$
If $\beta > 1$ let $\eta_n = 1$. Let $C > 0$ and for $\delta > 0$ small put,
$$\rho(\delta) = C \inf_{dist(\xi_i(v,c),v) \geq \delta} \left(\frac{dist(\xi_i(v,c), v)}{\delta} \right) |(R^{k_i(v,c)+1})'(v)|^{1-\beta} \eta_{k_i(v,c)+1}.$$
Then, if C is small enough, for every $c \in \text{Crit} \cap J(R)$ and $z \in R^{-n}(c)$, such that $R^i(z) \notin \tilde{B}_\delta(\text{Crit})$ for $0 \leq i < n$, the pull-back of $\tilde{B}_{\delta\rho(\delta)}(0)$ to z by R^n is univalent.

Proof. Let d_k, as in the shrinking neighborhoods, be proportional to
$$\eta_{k+1} \max_{v \in \text{CV} \cap J(R)} |(R^{k+1})'(v)|^{-\beta},$$
for $k \geq 1$ and let $K > 0$ be the constant involved is the distortion estimate for the shrinking neighborhoods; see (1).

Put $U_0 = \tilde{B}_{2\delta r}(c)$ for some $r > 0$ and consider shrinking neighborhoods (U_i', U_i), for $0 \le i \le n$, so that $R^{n-i}(z) \in U_i'$. Let k be the least integer, if any, so that $U_k \cap \mathrm{CV} \neq \emptyset$ and let $v \in \mathrm{CV}$ such that $v \in U_k$. So R^k is univalent in U_k. Note that we may assume that $v \in J(R)$; see Preliminaries. Moreover, $k = k_i(v, c)$ for some $i \ge 1$ and $\xi = \xi_i(v, c) \in U_k'$ is the k-th preimage of c in U_k'. By property (1), of shrinking neighborhoods, and considering that d_k^{-1} is proportional to $\eta_{k+1}^{-1} \min_{v \in \mathrm{CV} \cap J(R)} |(R^{k+1})'(v)|^{\beta}$, we have that for some $C_0 > 0$ only depending in R,

$$
\begin{aligned}
\frac{dist(\xi, v)}{(\delta r)^{\frac{1}{\mu_{c_0}}}} &\le C_0 \frac{dist(\xi, v)}{dist(R^k(v), c_0)} \\
&= C_0 \left(\frac{dist(\xi, v)}{dist(R^k(v), c_0)} \right)^{\frac{\mu_{c_0}-1}{\mu_{c_0}}} |(R^k)'(v)|^{-\frac{1}{\mu_{c_0}}} \left(K \eta_{k+1}^{-1} |(R^{k+1})'(v)|^{\beta} \right)^{\frac{1}{\mu_{c_0}}} \\
&= C_1 \left(dist(\xi, v) \right)^{\frac{\mu_{c_0}-1}{\mu_{c_0}}} |(R^{k+1})'(v)|^{\frac{\beta-1}{\mu_{c_0}}} \eta_{k+1}^{-\frac{1}{\mu_{c_0}}},
\end{aligned}
$$

for some constant $C_1 > 0$. Thus

$$
\delta r > C_1^{-\frac{1}{\mu_{c_0}}} dist(\xi, v) |(R^{k+1})'(v)|^{1-\beta} \eta_{k+1}.
$$

By hypothesis $dist(\xi, v) \ge \delta$, so letting $C = \frac{1}{2} C_1^{-\frac{1}{\mu_{c_0}}}$ in the definition of ρ, we have $r > \rho(\delta)$. Hence, if we take $r = \rho(\delta)$, the neighborhoods U_k' avoid critical values and therefore the respective pull-back of $\tilde{B}_{\delta\rho(\delta)}(c)$ is univalent. □

Lemma 1.6. *Let $\kappa \in (0, 1)$ be the constant as in 2 of the proof of Lemma 1.2. Then there is $C_0 > 0$ such that R satisfies the Decay of Geometry Condition with the function r_0 given, for $\delta > 0$ small, by the minimum between $\frac{\kappa}{2}\rho(\delta)$ and,*

$$
C_0 \inf_{dist(\xi_i(v,c), v) < \delta} \left(\frac{\delta}{dist(\xi_i(v, c), v)} \right)^{\mu_c - 1} |(R^{k_i(v,c)+1})'(v)|.
$$

Proof. Define r_0 as above, for some constant $C_0 > 0$ to be determined. Given $c \in \mathrm{Crit} \cap J(R)$ consider a pull-back $U_0 = \tilde{B}_{\delta r_0(\delta)}(c)$, U_1, ..., U_k such that $U_k \cap B_\delta(\mathrm{CV}) \neq \emptyset$. We suppose by contradiction that $diam(U_k) > \delta$. Consider the respective pull-backs U_i' and U_i'' of $B_{\kappa^{-1}\delta r_0(\delta)}(c_0)$ and $B_{2\kappa^{-1}\delta r_0(\delta)}(c_0)$ respectively, so $U_i \subset U_i' \subset U_i''$. Since $r_0(\delta) \ge \kappa$, for small $\delta > 0$, arguing by induction is enough to consider the case when $U_i \cap \tilde{B}_\delta(\mathrm{CV}) = \emptyset$ for $0 \le i \le k$.

Since $r_0(\delta) \ge \frac{\kappa}{2}\rho(\delta)$, we have by the previous lemma that, $R^k : U_k'' \longrightarrow U_0''$ is univalent. If U_k' is disjoint from $B_\delta(\mathrm{CV})$, we have, by definition of κ, that $diam(U_k) \le \delta$.

So, let us assume that there is $v \in U_k' \cap \mathrm{CV}$. Hence $k = k_i(v, c)$, for some $i \ge 1$ and $\xi = \xi_i(v, c) \in U_k'$ is the k-th preimage of c in U_k'. Let $c_0 \in \mathrm{Crit} \cap J(R)$ be so that $R(c_0) = v$ and let U_{k+1}' be the connected component of $R^{-1}(U_k')$ that contains c_0. Since R^{k+1} is not univalent in U_{k+1}', we cannot apply Lemma 1.5 to a preimage z of ξ in U_{k+1}' and $n = k + 1$, therefore we have that $\xi \in B_\delta(v)$. By Koebe Distortion Theorem the distortion of R^k in U_k' is bounded by some definite constant $D > 1$, so there is $C_1 > 0$

only depending in R such that,

$$\begin{aligned}\frac{\delta}{(\delta r_0(\delta))^{\frac{1}{\mu_c}}} &\leq C_1\frac{diam(U_k)}{diam(U_0)} \leq C_1 D\frac{dist(\xi,v)}{dist(R^k(v),c)} \\ &\leq C_1 D^{1+\frac{1}{\mu_{c_0}}}\left(\frac{dist(\xi,v)}{dist(R^k(v),c)}\right)^{\frac{\mu_c-1}{\mu_c}} |(R^k)'(v)|^{\frac{1}{\mu_c}}, \text{ so,}\end{aligned}$$

$$r_0(\delta) > C_2\left(\frac{\delta}{dist(\xi,v)}\right)^{\mu_c-1}|(R^{k+1})'(v)|,$$

for some $C_2 > 0$ only depending in R. Since $dist(\xi,v) < \delta$, letting $C_0 = C_2$ we obtain a contradiction. So, $U_k' \cap \mathrm{CV} = \emptyset$ for this choice of C_0. □

Proof of Proposition 1.4. Note that by Lemma 1.2 is enough to prove the respective estimates for the function r_0, given by the previous lemma.

1.– It follows by the previous lemma, considering that $k_i(v,c)$, and hence $\eta_{k_i(v,c)}$ and $|(R^{k_i(v,c)+1})'(v)|$, are big, when $dist(\xi_i(v,c),v)$ is small.

2.– Choose $\theta \in (0,1)$ and note that, as observed after the statement of Proposition 1.4, it is enough to prove that the sum below is finite. Note that $\beta = \alpha(1-\beta)$ so,

$$\begin{aligned}\sum_{n\gg 1}(r_0(\theta^n))^{-\alpha} &\leq C\sum_{i,v,c}\Bigg(\sum_{\theta^n < dist(\xi_i(v,c),v)}\left(\frac{dist(\xi_i(v,c),v)}{\theta^n}\right)^{-\alpha}|(R^{k_i(v,c)+1})'(v)|^{-\alpha(1-\beta)} \\ &\quad + \sum_{\theta^n > dist(\xi_i(v,c),v),\ n\geq 0}\left(\frac{\theta^n}{dist(x_i(v,c),v)}\right)^{-\alpha(\mu_c-1)}|(R^{k_i(v,c)+1})'(v)|^{-\alpha}\Bigg) \\ &\leq \tilde{C}\sum_{i,v,c}|(R^{k_i(v,c)+1})'(v)|^{-\beta} < \infty.\end{aligned}$$

3.– Since R satisfies the Collet-Eckmann condition, there are $C_0 > 0$ and $\lambda > 1$ such that for every $v \in \mathrm{CV} \cap J(R)$ we have $|(R^k)'(v)| \geq C_0\lambda^k$. Choose $\mu \in (0,1)$ and note that if $dist(\xi_i(v,c),v) > \delta^{1-\mu}$ then $\frac{dist(\xi_i(v,c),v)}{\delta} > \delta^{-\mu}$.

By [PU1] there is $C_1 > 0$ and $\theta \in (0,1)$ such that $dist(\xi_i(v,c),v) \geq C_1\theta^{k_i(v,c)}$. Therefore there is $C_2 > 0$ and $\gamma \in (0,1)$ such that,

$$|(R^{k_i(c,v)+1})'(v)| \geq C_0\lambda^{k_i(v,c)} \geq C_2(dist(\xi_i(v,c),v))^{-\gamma}.$$

Hence if $dist(\xi_i(v,c),v) \leq \delta^{1-\mu}$, we have

$$|(R^{k_i(v,c)+1})'(v)| \geq C_2 dist(\xi_i(v,c),v)^{-\gamma} \geq C_2\delta^{-\gamma(1-\mu)}.$$

Thus there is a constant $C_4 > 0$ such that $r_0(\delta) \geq C_4\delta^{-\alpha}$, where $\alpha = \min(\mu,\gamma(1-\mu))$. □

2. Martens property.

This section is concerned with some dynamical properties of rational maps in $\mathcal{S}$ which will be exploited in the next sections. In particular we prove the existence of arbitrarily small neighborhoods of critical values with the so called *Martens property*, see below. We begin by proving two lemmas about expansion.

Lemma 2.1. *Let $R \in \mathcal{S}$ and $\delta_0 > 0$. Then there is $N = N(\delta_0) > 0$ such that for all z and $n \geq N$ such that $R^n(z) \in \tilde{B}_{\frac{1}{2}r(\delta_0)\delta_0}(c)$, for some $c \in \mathrm{Crit} \cap J(R)$, and $R^j(z) \notin \tilde{B}_{\delta_0}(\mathrm{Crit})$, for $0 \leq j < n$, we have $|(R^n)'(z)| > 1$.*

Proof. Suppose not. Then there is $\{z_k\}_{k\geq 1}$ and $n_k \to \infty$ such that $R^{n_k}(z_k) \in \tilde{B}_{\delta r(\delta)}(c)$, for some $c \in \mathrm{Crit} \cap J(R)$, $dist(R^{j+1}(z_k), \mathrm{CV}) \geq \delta_0$, for $0 \leq j < n$, and $|(R^{n_j})'(z_j)| \leq 1$. By the Univalent Pull-back Condition the pull-back W_k of $\tilde{B}_{r(\delta_0)\delta_0}(c)$ to z_k is univalent. Thus, by Koebe $\frac{1}{4}$ Theorem, W_k contains a ball of definite radius. Taking a subsequence, if necessary, assume that $z_k \to z$, so there is a ball B of definite radius such that $B \subset W_k$ for big k. But this is not possible by the eventually onto property of Julia sets; see [CG]. □

Lemma 2.2. *Let $R \in \mathcal{S}$, then there is $C_0 > 0$ such that for $\delta > 0$ small, for $z \in \hat{\mathbb{C}}$ and $n \geq 1$ such that $R^n(z) \in \tilde{B}_{\frac{1}{2}r(\delta)\delta}(c)$, for some $c \in \mathrm{Crit} \cap J(R)$, and $R^j(z) \notin \tilde{B}_{\delta}(\mathrm{Crit} \cap J(R))$, for $0 \leq j < n$, we have*

(i): $|(R^n)'(z)| \geq C_0(\delta r(\delta))^{\frac{1}{\mu_c}} \min(\delta^{-1}, \rho^{-1})$, *where* $\rho = dist(z, \mathrm{CV})$.
(ii): $|(R^n)'(z)| \geq C_0 \delta^{\frac{1}{\mu_c} - \frac{1}{\mu_{max}}}$.

Proof. (i) By the Univalent Pull-back Condition, the pull-back W of $B_{\delta r(\delta)}(c)$ to z by R^n is univalent. There are two cases.

Case 1. $\rho < \delta$. By the Decay of Geometry Condition $diam(W) \leq \delta$. So by Schwartz lemma $|(R^n)'(z)| \geq C_0(\delta r(\delta))^{\frac{1}{\mu_c}} \delta^{-1}$ for some constant $C_0 > 0$.

Case 2. $\rho \geq \delta$. By the Decay of Geometry Condition $W \cap \mathrm{CV} = \emptyset$. Moreover, if $\hat{W}$ is the respective pull-back of $\tilde{B}_{\frac{1}{2}\delta r(\delta)}(c)$, then the distortion of R^n in $\hat{W}$ is bounded by some definite constant. Hence by Koebe $\frac{1}{4}$ Theorem there is a constant $C_1 > 0$ such that

$$diam(\tilde{B}_{\frac{1}{2}\delta r(\delta)}(c))|(R^n)'(z)|^{-1} \leq C_1 dist(z, \mathrm{CV}) = C_1\rho.$$

So $|(R^n)'(z)| \geq C_0(\delta r(\delta))^{\frac{1}{\mu_c}} \rho^{-1}$ for some definite $C_0 > 0$.

(ii) Let $c_0 \in \mathrm{Crit} \cap J(R)$ and consider $w \in \tilde{B}_\delta(c_0)$ such that there is $l > 0$ such that $R^l(w) \in \tilde{B}_\delta(c_1)$ for some $c_1 \in \mathrm{Crit} \cap J(R)$ and $R^j(w) \notin \tilde{B}_\delta(\mathrm{Crit} \cap J(R))$, for $0 < j < l$. By (i) there is a constant $C_0' > 0$ such that $|(R^{k-1})'(R(w))| \geq C_0'(\delta r(\delta))^{\frac{1}{\mu_{c_1}}} \delta^{-1}$. Since $|R'(w)| \sim \delta^{\frac{\mu_{c_0}-1}{\mu_{c_0}}}$ and since $r(\delta) \gg 1$ for $\delta > 0$ small, we have that for $\delta > 0$ small $|(R^k)'(w)| \geq \delta^{\frac{1}{\mu_{c_1}} - \frac{1}{\mu_{c_0}}}$.

Choose $\delta > 0$ small and put $\delta_0 = \frac{1}{2}\delta r(\delta)$. Let $n_1 \leq n$ be the least integer such that $R^{n_1}(z) \in \tilde{B}_{\delta_0}(\mathrm{Crit} \cap J(R))$. It follows by Lemma 2.1 that $|(R^{n_1})'(z)| \geq C_0$ for some definite $C_0 > 0$. Consider the closest approximation times $0 < n_1 < \ldots < n_k = n$ of the forward orbit of z to critical points (measuring distances with the balls $\tilde{B}$) and let $c_i \in \mathrm{Crit} \cap J(R)$ be the closest critical point to $R^{n_i}(z)$, so that $c_k = c$. Then $|(R^{n_{i+1}-n_i})'(R^{n_i})| \geq \delta^{\frac{1}{\mu_{c_{i+1}}} - \frac{1}{\mu_{c_i}}}$, for $1 \leq i < k$. Therefore $|(R^{n_k - n_1})'(R^{n_1}(z))| \geq \delta^{\frac{1}{\mu_{c_1}} - \frac{1}{\mu_{c_k}}}$ and,

$$|(R^n)'(z)| \geq C_0 \delta^{\frac{1}{\mu_{c_0}} - \frac{1}{\mu_{c_1}}} \geq C_0 \delta^{\frac{1}{\mu_c} - \frac{1}{\mu_{max}}}. \square$$

2.1. **Maximal invariant sets.**

Definition 2.3. *Let V be a neighborhood of* $\mathrm{Crit} \cap J(R)$ *such that every connected component of V contains exactly one critical point in $J(R)$. Then we define:*

$$K(V) = \{z \mid R^j(z) \notin V, \text{ for } j \geq 0\}.$$

Note that $K(V)$ is compact and forward invariant by R. Moreover by Montel's Theorem $int(K(V)) \subset \hat{\mathbb{C}} - J(R)$; see [CG]. If W is a connected component of $\hat{\mathbb{C}} - K(V)$ not intersecting $\mathrm{Crit} \cap J(R)$ then $R(W)$ is also a connected component of $\hat{\mathbb{C}} - K(V)$ and $R : W \longrightarrow R(W)$ is proper.

Moreover note that if $V' \subset V$, then $K(V) \subset K(V')$, and if the orbit of $c \in \mathrm{Crit} \cap J(R)$ accumulates $\mathrm{Crit} \cap J(R)$, then $c \notin K(V)$ for any V. In later sections it will be useful to impose the following property.

Definition 2.4. *For $c \in \mathrm{Crit} \cap J(R)$ consider a simply-connected neighborhood V^c of c disjoint form the forward orbits of critical points not in $J(R)$. Moreover suppose that the sets V^c are pairwise disjoint and put $V = \cup_{\mathrm{Crit} \cap J(R)} V^c$. Then we say that V has* **Martens property** *if for any $n \geq 1$, and any connected component W of $R^{-n}(V)$ we have either $\overline{W} \cap \overline{V} = \emptyset$ or $\overline{W} \subset V$.*

Martens defined the concept of *nice interval* in [Mar] for self-maps of the interval. A interval is said to be nice if the forward orbit of every point in its boundary is disjoint from the interval itself. As the following lemma shows, Martens property is an analogous property.

Lemma 2.5. *Suppose that $V = \cup_{\mathrm{Crit} \cap J(R)} V^c$ has Martens property. Then for every $z \in \partial V$ we have $R^n(z) \notin \overline{V}$, for $n \geq 1$. In particular, for each $c \in \mathrm{Crit} \cap J(R)$, the set V^c is equal to the connected component of $\hat{\mathbb{C}} - K(V)$ that contains c, so $\partial V^c \subset K(V)$. Furthermore, for every connected component W of $\hat{\mathbb{C}} - K(V)$ there is $m_W \geq 0$ and $c(W) \in \mathrm{Crit} \cap J(R)$ such that $R^{m_W} : W \longrightarrow V^{c(W)}$ is a biholomorphism. Thus W is simply-connected and therefore $K(V)$ is connected.*

Proof. Suppose that $z \in \partial V$ is such that for some $n \geq 1$ we have $R^n(z) \in \overline{V^c}$, for some $c \in \mathrm{Crit} \cap J(R)$. Then the pull-back W of V^c by R^n to z is such that $z \in \overline{W} \cap \partial V$. Thus $\overline{W} \cap \overline{V} \neq \emptyset$ but $\overline{W} \not\subset V$, which contradicts Martens property.

As remarked above, if W is a connected component of $\hat{\mathbb{C}} - K(V)$ different from V^c, for $c \in \mathrm{Crit} \cap J(R)$, then $R(W)$ is also a connected component of $\hat{\mathbb{C}} - K(V)$ and $R : W \longrightarrow R(W)$ is proper. It follows that $R : W \longrightarrow R(W)$ is a biholomorphism. Then the rest of the assertions follow. □

The following lemma will be useful to produce sets with Martens property.

Lemma 2.6. *Let W be a neighborhood of* $\mathrm{Crit} \cap J(R)$ *such that each connected component contains exactly one critical point in $J(R)$. For $c \in \mathrm{Crit} \cap J(R)$ let V^c be a simply-connected neighborhood of c disjoint from the forward orbits of critical points not in $J(R)$, such that the sets V^c, for $c \in \mathrm{Crit} \cap J(R)$, are pairwise disjoint. Put $V = \cup V^c$ and suppose that $\overline{V} \subset W$ and $R(\partial V) \subset K(W)$. Then V has Martens property.*

Proof. Let $W_0, W_1, ...$ be a pull-back such that W_0 is a connected component of V and let U_n be the connected component of $\hat{\mathbb{C}} - K(W)$ that contains W_n. Since $\partial V \subset R^{-1}(K(W))$ it follows that either $W_n \subset U$ or $W_n \cap U = \emptyset$, so is enough to prove that $\overline{W_n} \subset U_n$.

We proceed by induction. For $n = 0$ just note that $\overline{W_0}$ is contained in W by hypothesis, so $\overline{W_0} \subset U_0$. So suppose that $\overline{W_n} \subset U_n$, for some $n > 0$. If U_n does not intersect $\mathrm{Crit} \cap J(R)$ then $R : U_n \longrightarrow U_{n-1}$ is proper so $\overline{W_n} \subset U_n$ by the induction hypothesis. If U_n intersects $\mathrm{Crit} \cap J(R)$ then let U_n' be the connected component of $R^{-1}(U_{n-1})$ that contains W_n, so $U_n' \subset U_n$. By the induction hypothesis, $\overline{W_n} \subset U_n' \subset U_n$. □

2.2. **Neighborhoods with Martens property.** Fix $R \in \mathcal{S}$ for all this section. In this section we prove the existence of neighborhoods of $\mathrm{Crit} \cap J(R)$ with Martens property at every scale. This is one of the main properties of rational maps in the class $\mathcal{S}$.

Proposition 2.7. *Let $R \in \mathcal{S}$. Then there is a constant $C > 0$ such that for any $\delta > 0$ small and every $v \in \mathrm{CV} \cap J(R)$ there is a simply-connected neighborhood U^v of v such that*

$$B_\delta(v) \subset U^v \subset B_{\eta(\delta)\delta}(v),$$

where $\eta(\delta) = 1 + C(r(\delta))^{-\frac{1}{\mu_{max}}}$, and so that the union V of the connected components of $R^{-1}(U)$ intersecting $\mathrm{Crit} \cap J(R)$ has Martens property, where $U = \cup_{\mathrm{CV} \cap J(R)} U^v$.

The proof of Proposition 2.7 is at the end of this section and is based in the following lemma.

Lemma 2.8. *For $\delta > 0$ small put $V = \tilde{B}_\delta(\mathrm{Crit} \cap J(R))$ and for $c \in \mathrm{Crit} \cap J(R)$ and $n \geq 0$ let V_n^c be the connected component of $\cup_{0 \leq i \leq n} R^{-i}(V)$ that contains c. Then $V_n^c \subset \tilde{B}_{2\delta}(c)$.*

Proof. We proceed by induction in n. For $n = 0$ we have $V_0^c = \tilde{B}_\delta(c)$, so the assertion is trivial. Suppose that the assertion holds for $n \geq 0$ and let $w \in V_{n+1}^{c_0} - V$. For every point $z \in V_{n+1}^{c_0}$ there is $0 \leq m \leq n+1$ and $c \in \mathrm{Crit} \cap J(R)$ such that $R^m(z) \in \tilde{B}_\delta(c)$; let $m(z)$ be the least of such numbers. Let X be a connected component of $V_{n+1}^{c_0} - V$ containing w and let $z \in X$ minimizing $m(z)$. Let $c \in \mathrm{Crit} \cap J(R)$ be such that $R^{m(z)}(z) \in \tilde{B}_\delta(c)$. Considering that $m(z) > 0$, we have by induction hypothesis

$$R^{m(z)}(X) \subset V_n^c \subset \tilde{B}_{2\delta}(c_0).$$

By the Univalent Pull-back condition the pull-back of $\tilde{B}_{\delta r(\delta)}(c)$ by $R^{m(z)}$ to z is univalent; denote by g the respective inverse branch. Suppose that δ is small enough so that $r(\delta) \gg 2$ and so that the distortion of g in $\tilde{B}_\delta(c)$ is bounded by some definite constant $D > 1$. In particular $diam(X) \leq 2D diam(g(\tilde{B}_\delta(c)))$. By (i) of Lemma 2.2, for some $C_0 > 0$,

$$|(R^{m(z)-1})'(R(z))| \geq C_0(\delta r(\delta))^{\frac{1}{\mu_c}}\delta^{-1}.$$

So there is a constant $C_1 > 0$ such that

$$\begin{aligned} diam(g(V_0^c)) &\leq DC_0^{-1}|R'(z)|^{-1}\delta(\delta r(\delta))^{-\frac{1}{\mu_c}} diam(\tilde{B}_\delta(c))) \\ &\leq C_1\delta^{-\frac{\mu_{c_0}-1}{\mu_{c_0}}}\delta(r(\delta))^{-\frac{1}{\mu_c}}. \end{aligned}$$

Thus, if $\delta > 0$ is small,

$$diam(X) \leq 2D diam(g(\tilde{B}_\delta(c))) \leq 2DC_1\delta^{-\frac{1}{\mu_{c_0}}}(r(\delta))^{\frac{1}{\mu_c}} \ll diam(\tilde{B}_\delta(c_0)).$$

It follows that $w \in \tilde{B}_{2\delta}(c_0)$ and therefore $V_{n+1}^{c_0} \subset \tilde{B}_{2\delta}(c_0)$. □

Proof of Proposition 2.7. For $c \in \mathrm{Crit} \cap J(R)$ let V and V_n^c as in the previous lemma with 2δ instead of δ, so $V_n^c \subset \tilde{B}_{4\delta}(c)$. Note that $V^c = \cup_{n \geq 0} V_n^c \subset \tilde{B}_{4\delta}(c)$ is the connected component of $\hat{\mathbb{C}} - K(V)$ containing c.

Thus for any connected component W of $\hat{\mathbb{C}} - K(V)$ there is $m_W \geq 1$ and $c = c(W) \in \mathrm{Crit} \cap J(R)$ such that $R^{m_W} : W \longrightarrow V^c$ is a biholomorphism. By the Univalent Pull-back condition the inverse of this map extend in a univalent way to $\tilde{B}_{2\delta r(2\delta)}$. By Koebe Distortion Theorem we have that for $\delta > 0$ small the distortion of R^{m_W} in W is bounded by some definite constant independent of δ. It follows by Lemma 2.2 that there is a constant $C > 0$ such that if W is a connected component $\hat{\mathbb{C}} - K(V)$ intersecting $B_{2\delta}(v)$, for some $v \in \mathrm{CV} \cap J(R)$, then $diam(W) < C\delta(r(\delta))^{-\frac{1}{\mu_{\max}}}$. Put $\eta(\delta) = 1 + C(r(\delta))^{-\frac{1}{\mu_{max}}}$.

For $v \in \mathrm{CV} \cap J(R)$ let,

$$\hat{U}^v = \{B_\delta(v) \cup (\cup W) \mid W \text{ connected component of } \hat{\mathbb{C}} - K(V) \text{ intersecting } B_\delta(v)\},$$

so $B_\delta(v) \subset \hat{U}^v \subset B_{\eta(\delta)\delta}(v)$ if δ is small enough. Moreover let

$$U^v = \hat{U}^v \cup \{ \text{ connected component } W \text{ of } \hat{\mathbb{C}} - \hat{U}^v \text{ such that } diam(W) < diam(\hat{U}^v)\}.$$

Note that $diam(\hat{U}^v) \ll diam(\hat{\mathbb{C}})$ so there is only one connected component of $\hat{\mathbb{C}} - \hat{U}^v$ whose diameter is not smaller than $\hat{U}^v$. Thus U^v is simply-connected. Moreover $B_\delta(v) \subset U^v \subset B_{\eta(\delta)\delta}(v)$ and $\partial U^v \subset \partial \hat{U}^v \subset K(V) \cap B_{\eta(\delta)\delta}(v)$. Then the proposition follows by Lemma 2.6. □

3. Topological models of Julia sets.

In this section we indicate the proof of Theorem A, about topological models of polynomials. The proof is based in a Landing Lemma proved in Section 3.1 and in Thurston-Kiwi Finiteness Theorem stated in Section 3.2. We prove Theorem A under a technical assumption and assuming that there are no parabolic periodic points; see Section 3.2. The proof of this particular case of Theorem A contains the essential ideas involved in the general case. In Section 7 we prove that this technical assumption is automatically satisfied for polynomials in the class $\mathcal{S}$. The proof of Theorem A in the general case is in Section 7.2.

We begin with an easy consequence of Lemma 2.2.

Lemma 3.1. *Fix $\eta > 1$ and let $\delta > 0$ be small. Let $V = \cup V^c$ be a neighborhood of* $\mathrm{Crit} \cap J(R)$ *with Martens property such that $\tilde{B}_\delta(\mathrm{Crit} \cap J(R)) \subset V \subset \tilde{B}_{\eta\delta}(\mathrm{Crit} \cap J(R))$. Then there is a constant $C_1 > 0$, independent of δ, such that for every connected component W of $\hat{\mathbb{C}} - K(V)$ we have*

- **(i):** *$diam(W) \leq C_1 \delta^{\frac{1}{\mu_{max}}}$.*
- **(ii):** *If $\rho = dist(U, c_0)$ then $diam(W) \leq C_1 \max(\delta, \rho)(r(\delta))^{-\frac{1}{\mu_{max}}}$.*

Proof. Let $m = m_W \geq 1$ and $c = c(W) \in \mathrm{Crit} \cap J(R)$ be such that $R^m : W \longrightarrow V^c$ is a biholomorphism. By the Univalent Pull-back Property the inverse of this map extend in an univalent way to $\tilde{B}_{\delta r(\delta)}(c)$. We may suppose δ small enough so that $r(\delta) > 2\eta$, so by Koebe Distortion theorem the distortion of R^m in W is bounded by some definite constant independent of δ. Considering that $diam(V^c) \sim \delta^{\frac{1}{\mu_c}}$ we have $diam(W) \leq C_2 \delta^{\frac{1}{\mu_c}} |(R^m)'(z)|^{-1}$, for any $z \in W$ and some definite constant $C_2 > 0$. Then the lemma follows considering that by Lemma 2.2 there is a definite constant $C_0 > 0$ such that

$$|(R^n)'(z)|^{-1} \leq C_0^{-1} (r(\delta)\delta)^{-\frac{1}{\mu_c}} \max(\delta, \rho) \text{ and } |(R^n)'(z)|^{-1} \leq C_0^{-1} \delta^{\frac{1}{\mu_{max}} - \frac{1}{\mu_c}}. \square$$

3.1. **Landing Lemma.** The aim of this section is to prove a general Landing Lemma for hyperbolic sets of polynomials. An analogous property was proved in greater generality by Przytycki, in [Pr1]. The proof that we present here is essentially the same as the usual proof in the case of repelling periodic points; see for example [Mil3].

We need to introduce the notion of ray. We first introduce it in the more standard case when the Julia set is connected. In Section 3.1.1 we consider the disconnected case. For references to this section see [GM] and [CG].

Let $P \in \mathbb{C}[z]$ be a polynomial and let,

$$K(P) = \{z \mid \{P^n(z)\}_{n\geq 0} \text{ is bounded}\},$$

which is called the *filled in* Julia set of P. Then $P^{-1}(K(P)) = K(P)$ and $K(P)$ is a compact subset of $\mathbb{C}$. Moreover $J(P) = \partial K(P)$.

If $K(P)$ is connected there is a conformal representation $\varphi : \hat{\mathbb{C}} - K(P) \longrightarrow \hat{\mathbb{C}} - \overline{\mathbb{D}}$ fixing infinity, that conjugates the dynamics of P to that of $z \longrightarrow z^d$, where d is the degree of P. There are exactly $d-1$ such coordinates. The continuous function $G = G_P : \mathbb{C} \longrightarrow \mathbb{R}$ equal to $\ln|\varphi|$ in $\mathbb{C} - K(P)$ and equal to 0 in $K(P)$, is called the *potential* or *Green* function. This function does not depend in the choice of φ and satisfies $G \circ P = d$. The sets of the form $G^{-1}(p)$, for $p > 0$, are analytic Jordan curves called *equipotentials*.

The sets of the form $\varphi^{-1}(\{re^{2\pi i\theta} | \, r > 1\})$ are called *rays* and the set of *accumulation points* of this ray, is the set of accumulation points of $\varphi^{-1}(re^{2\pi i\theta})$, as $r \to 1$, which is a full compact set. If this set is the singleton z, then we say that the ray *lands* at z and that z is the *landing point* of the ray. Rays and its accumulation sets do not depend in the choice of φ.

Landing Lemma. *Let $P \in \mathbb{C}[z]$ and let $K \subset J(P)$ be a forward invariant set for P, such that P is uniformly expanding in K. Then every $\zeta \in K(P)$ is the landing point of some ray.*

The proof of this lemma is based in the following well known univalent pull-back property for such K: There is $r > 0$ such that for all $y \in K$ and $n \geq 0$ the pull-back B of $B_r(P^n(y))$ to y by P^n is univalent. Note that, taking r smaller, we may assume that the distortion of P^n in B is bounded by some definite constant.

Proof of the Landing Lemma in the connected case.

1.- By the property stated above it follows that there is $C_1 > 0$ such that for all $y \in K$ and $z \in B_{C_1|(P^k)'(y)|^{-1}}(y)$, we have that $G(z) \leq d^{-k}$.

2.- For $z \notin K(P)$ let $\delta(z) = dist(z, K(P))$. Note that for all $\varepsilon > 0$ there is $C = C(\varepsilon) > 1$ such that $\delta(z) \geq \varepsilon dist(z, y)$ for $y \in K$ implies,

$$C^{-1}|(P^k)'(y)|^{-1} < dist(z, y) < C|(P^k)'(y)|^{-1},$$

where $k = [\ln|G(z)|]$ is the integer part of $\ln|G(z)|$. In fact, let m be the least integer such that $dist(P^m(z), P^m(y))$ is comparable to r. By bounded distortion we have

$$\delta(P^m(z)) \geq D^{-1}\varepsilon dist(P^m(z), P^m(y)),$$

where $D > 1$ is a definite distortion constant. This implies that $G(P^m(z)) \sim 1$ for some implicit constant only depending in ε, or equivalently $|m - k| < K(\varepsilon)$. Thus $dist(z, y) \sim |(P^m)'(y)|^{-1} \sim |(P^k)'(y)|^{-1}$, for implicit constants only depending in ε.

3.- Given a point z_n with $d^{-n} \leq G(z_n) < d^{-(n-1)}$ let z_k be the point in the same ray as z_n but with equipotential $G(z_k) = d^{k-n}G(z_n)$. It follows by a well known theorem of Koebe (see Corollary 1.4 of [Pom]) that $\delta(z_k) \sim dist(z_k, z_{k-1})$. Let us prove that for all $\varepsilon > 0$ small there is $m = m(\varepsilon) \geq 1$ such that if $\delta(z_k) \geq \varepsilon dist(z_k, \zeta)$ then there is $k - m < k_1 < k$ such that $\delta(z_{k_1}) \geq \varepsilon dist(z_{k_1}, \zeta)$. In fact suppose that $\delta(z_{k+i}) < \varepsilon dist(z_{k+i}, \zeta)$, for $0 < i \leq n$. In this case,

$$\begin{aligned} dist(z_{k+i+1}, \zeta) &\leq dist(z_{k+i}, \zeta) + dist(z_{k+i+1}, z_{k+i}) \leq dist(z_{k+i}, \zeta) + K_0\delta(z_{k+i}) \\ &\leq (1 + K_0\varepsilon) dist(z_{k+i}, \zeta), \end{aligned}$$

for some definite $K_0 > 0$. Let $\lambda > 1$ be so that $|(P^l)'(w)| \geq K_1\lambda^l$ for all $w \in K$ and $l \geq 0$. Then, if ε is small enough so that $1+K_0\varepsilon < \lambda$, it follows that $dist(z_{k+n}, \zeta) \ll |(P^n)'(\zeta)|^{-1}$ if n is big enough. By 1 it follows that $n < m = m(\varepsilon)$, which proves the assertion.

4.- By the property stated above there is $\varepsilon > 0$ such that for any n big there is $z_n \notin K(P)$ such that $\delta(z_n) \geq \varepsilon dist(z_n, \zeta)$. Fix such z_n with $n \gg 1$. It follows by induction in 3 that there is ε_0 and a sequence $n = n_0 < n_1 < ...$ such that $n_{i+1} - n_i < m = m(\varepsilon_0)$ and such that $\delta(z_{n_i}) \geq \varepsilon_0 dist(z_{n_i}, \zeta)$. Hence by bounded distortion there is $\varepsilon_1 > 0$ such that for all $k \leq n$ we have $\delta(z_k) > \varepsilon_1 dist(z_k, \zeta)$, so by 2, $dist(z_k, \zeta) \sim |(P^k)'(\zeta)|^{-1}$.

5.- Consider a sequence z_n^n like above, so that the points z_i^n for $i \geq n$ belong to the same ray. Consider a sequence $n_i \to \infty$ such that $z_0^{n_i} \to z_0$. Then the ray containing z_0 lands at ζ, since by 4, $dist(z_i, \zeta) \sim |(P^i)'(\zeta)|^{-1}$ which decreases exponentially. □

3.1.1. *Disconnected case.* If the Julia set of a polynomial $P \in \mathbb{C}[z]$ is not connected then we cannot define rays as in the connected case because $\hat{\mathbb{C}} - K(P)$ is not represented by $\hat{\mathbb{C}} - \overline{\mathbb{D}}$. We consider a definition of ray following Goldberg and Milnor; see Appendix A of [GM].

We begin by defining the *potential* or *Green* function $G : \mathbb{C} \longrightarrow [0, \infty)$ of a polynomial $P \in \mathbb{C}[z]$ of degree $d \geq 2$; see [CG] for references. This function is given by,

$$G(z) = \lim_{n\to\infty} \frac{1}{d^n} \ln |P^n(z)|;$$

it is implicit that the limit always exists. Then $G(z) = 0$ if and only if $z \in K(P)$ and that G is a continuous function and harmonic in $\mathbb{C} - K(P)$. In the connected case the gradient flow $-\nabla G$ does not have singularities in $\mathbb{C} - K(P)$ and the rays are the flow lines of $-\nabla G$ in $\mathbb{C} - K(P)$. In the disconnected case $-\nabla G$ has singularities that are locally the preimage by z^m, with $m \geq 2$, of a constant flow. The singularities are exactly the preimages of critical points in $\hat{\mathbb{C}} - K(P)$.

As in the connected case there are coordinates φ conjugating the polynomial P to z^d near infinity. These coordinates satisfy $\ln |\varphi(z)| = G(z)$, so they map flow lines to pieces of straight lines passing through 0. So we can parameterize the different flow lines in a neighborhood of infinity, by angles in $\mathbb{R}/\mathbb{Z}$, just as in the connected case.

Given an angle $\theta \in \mathbb{R}/\mathbb{Z}$ there are two cases. Either the corresponding flow line is smooth and we call it a *ray* as in the connected case, or the corresponding flow line is not smooth and we consider two (broken) rays that are the limits of smooth rays with angles converging to θ. One is when the convergence to θ and the other is when the convergence is from the left. We associate them angles θ^+ and θ^- respectively. We can think of the ray with angle θ^+ (resp. θ^-) as the (broken) flow line of $-\nabla G$ the we obtaining by continuing with the rightmost (resp. leftmost) flow line at each time that we encounter a singularity of $-\nabla G$.

Note that the potential function gives a natural parameterization of a ray by the interval $(0, \infty)$. As in the connected case we define the set of *accumulation* points of a ray $r : (0, \infty) \longrightarrow \mathbb{C}$, parametrized by the potential, as the set of accumulation points of $r(t)$ as $t \to 0$. If the accumulation set of a ray is the singleton $\{z\} \subset J(P)$ we say that the ray *lands* at z.

The Landing Lemma also holds in the disconnected case. The proof is analogous to the connected case. The only adaptation is that in 5, of the proof of the connected case, we may choose z_n^n in a smooth ray, so the limit of these smooth rays is also a ray.

3.2. **Thurston-Kiwi Finiteness Theorem and Theorem A.** It will important to known, in order to apply the Landing Lemma, that polynomials in $\mathcal{S}$ satisfy the following property.

EAC. *Given $R \in \mathbb{C}(z)$ be a rational map and denote by $\mathcal{P}$ the set of parabolic points of R. Then for every neighborhood V of* $\mathrm{Crit} \cup P$ *the rational map R is uniformly expanding in,*

$$\{z \in J(R) \mid R^n(z) \notin V \text{ for } n \geq 0\}.$$

It is proven in Corollary 7.9 in Section 7 that polynomials in $\mathcal{S}$ satisfy this property. Przytycki proved in ([Pr2], Lemma 3.1) that rational maps satisfying the summability condition with exponent one satisfy this property. In the real setting Mañé proved in [Mañé] that an analogous property holds for any map of class C^2. However there are complex quadratic polynomials that do not satisfy this property; see example of Douady and Hubbard in [Mil2].

To state the Finiteness Theorem we need to introduce the so called *puzzle ends* (see [Lev]) or *fibers*; see [Sch]. We follow the approach of Kiwi; see [Ki].

Let $P \in \mathbb{C}[z]$ be a polynomial without irrationally indifferent periodic points. The *puzzle end* of a point $w \in J(P)$ is the intersection of all the sets that are the connected components of $J(P) - Z$ containing w, where $Z \subset J(R) - \{w\}$ is a finite set of pre-periodic points. Puzzle ends are full compact connected sets that partition $J(P)$. Moreover the image by P of a puzzle end is also a puzzle end.

Finiteness Theorem (Thurston-Kiwi). *Let $P \in \mathbb{C}[z]$ be a polynomial without irrationally indifferent periodic points. Then a puzzle end is either pre-periodic, in which case it is a singleton, or it intersects at most a finite number of accumulation sets of rays.*

This theorem was proved by Thurston for quadratic polynomials and by Kiwi in its full generality; see [Ki]. We remark that in fact the stronger assertion hold, that a puzzle end that is not pre-periodic can intersect at most a 2^d *ray impressions*, where $d \geq 2$ is the degree of P; see [Ki].

Now we give a proof of Theorem A assuming EAC and that there no indifferent periodic points. In Section 7 we prove that polynomials in the class $\mathcal{S}$ satisfy EAC and do not have irrationally indifferent periodic points. In the presence of parabolic periodic points we improve Martens neighborhoods given by Proposition 2.7; see Proposition 7.11. Then the proof is the same as the particular case presented here; see Section 7.2.

In [Ki] Kiwi proved that conditions $1-3$ in Theorem A are equivalent to the condition that every puzzle end is a singleton. Moreover these conditions imply the last assertion in Theorem A. So to prove Theorem A is enough to prove that every puzzle end is a singleton.

Proof of Theorem A assuming property EAC and that there are no indifferent periodic points. Since there are no irrationally indifferent periodic points the considerations of puzzle ends apply to P. Recall that by [Ki] is enough to prove that all puzzle ends are singletons.

Suppose, by contradiction, that the puzzle end ξ of a critical point is not a singleton, so by the Finiteness Theorem ξ intersects at most a finite number of accumulation sets of rays. By Proposition 2.7 there is a sequence $V_i = \cup_c V_i^c$ of neighborhoods of $\mathrm{Crit} \cap J(P)$ with Martens property so that the diameter of V_i^c goes to zero as $i \to \infty$. Since ξ is connected it follows that ξ intersects ∂V_i in a point x_i, for big i. We may suppose that the x_i are different.

By Lemma 2.5 $x_i \in \partial V_i \subset K(V_i)$ so x_i belongs to the set $K(V_i) \cap J(P)$. By hypothesis P satisfies EAC and it does not have parabolic periodic points, so P is uniformly expanding in $K(V_i) \cap J(P)$. So by the Landing Lemma each x_i is the landing point of some ray; see Section 3.1. But this contradicts the Finiteness Theorem since each $\{x_i\} \subset \xi$ is an accumulation set of a ray. Hence all puzzle ends containing critical points are singletons.

Let $w \in J(P)$ such that $\omega(w) \cap \mathrm{Crit} \neq \emptyset$. Let n_i be the least integer such that $P^{n_i}(w) \in V_i^{c_i}$ for some $c_i \in \mathrm{Crit} \cap J(P)$. We may univalently pull-back $V_i^{c_i}$ to w by P^{n_i} to obtain $V_i(z)$, that contains w. By Lemma 3.1 we have that, $diam(V_i(w)) \leq C diam(V_i^{c_i}) \to 0$ as $i \to \infty$, for some definite $C > 0$; hence we can prove that the puzzle end containing w is trivial just as before.

It remains to prove that the puzzle end of a point $z \in J(P)$ such that $\omega(z) \cap \mathrm{Crit} = \emptyset$ is a singleton. This is now standard given the triviality of the puzzle ends of critical points; see for example [H], [Lyu2] and [Mil2]. □

4. Area estimates.

In this section we prove area estimates for rational maps in $\mathcal{S}$. This is one of the main ingredients in the proof of Theorem B. It is also used in a essential way in the proof of Rigidity (sated in the next section) and in the proof of Theorems C, D and E.

Given $X \subset \hat{\mathbb{C}}$, we denote by $|X|$ the spherical area of X.

Area Estimates. *Let $R \in \mathcal{S}$ such that $J(R) \neq \hat{\mathbb{C}}$ and fix $\eta > 1$ close to one. Then there is $\varepsilon = \varepsilon(\eta) > 0$ and $A = A(\eta) > 0$ such that the following assertion is true. For $\delta > 0$ small consider simply connected neighborhoods V_δ^c of $c \in \mathrm{Crit} \cap J(R)$ such that $\tilde{B}_\delta(c) \subset V_\delta^c \subset \tilde{B}_{\eta\delta}(c)$ and such that $V_\delta = \cup_c V_\delta^c$ has Martens property. Then*

$$\frac{|\tilde{B}_{\varepsilon\delta r(\delta)}(c) - K(V_\delta)|}{|\tilde{B}_{\varepsilon\delta r(\delta)}(c)|} \leq A(r(\delta))^{-\frac{2}{\mu_{max}}}.$$

We first construct a special nest V_n of neighborhoods of $\mathrm{Crit} \cap J(R)$ with Martens property. Clearly it will be enough to prove that Area Estimates for $V_\delta = V_n$.

Fix $\eta > 1$ as in the Area Estimates and fix $\tau \in (0, \eta^{-1})$. Let $\varepsilon > 0$ be small and for $n \geq n_0$, for some big n_0, let $m(n)$ be the greatest integer such that $\tau^{m(n)} \leq 2\varepsilon\tau^n r(\tau^n)$. Since $\delta r(\delta) \leq \delta_0 r(\delta_0)$ for $\delta \leq \delta_0$ it follows that $m(n)$ is a non decreasing sequence.

Proposition 4.1. *There is $n_0 \gg 1$ such that for all $n \geq m(n_0)$ and $v \in \mathrm{CV} \cap J(R)$ there is a simply-connected neighborhood U_n^v of v, such that,*

$$B_{\tau^n}(v) \subset U_n^v \subset B_{\eta\tau^n}(v),$$

and so that if V_n^c is the connected component of $R^{-1}(U_n)$ that contains $c \in \mathrm{Crit} \cap J(R)$, then $V_n = \cup_{\mathrm{Crit} \cap J(R)} V_n^c$ has Martens property. Moreover, for $n \geq n_0$ we have $\partial U_n \subset K(V_{m(n)})$, where $U_n = \cup U_n^v$.

Put $L_n = R^{-1}(K(V_{m(n)})) - V_n$. Then for every connected component W of $\hat{\mathbb{C}} - L_n$ there is $m_W \geq 1$ and $c(W) \in \mathrm{Crit} \cap J(R)$ such that $R^{m_W} : W \longrightarrow V_{m(n)}^{c(W)}$ is a biholomorphism, with distortion bounded by $D = D(\varepsilon) \leq 1 + M\varepsilon$, for some constant M independent of n.

Note that is enough to prove the Area Estimates for $V_\delta = V_n$, for $n \geq n_0$. We will prove the Area Estimates hold for V_n, for $n \geq n_0$, for an appropriated choice of $\varepsilon > 0$; see Section 4.1.

For $c \in \mathrm{Crit} \cap J(R)$ and $n \geq n_0$ let

$$\xi_n^c = \frac{|V_{m(n)}^c - K(V_n)|}{|V_{m(n)}^c|} \text{ and } \xi_n = \max_c \xi_n^c.$$

Thus, to prove the Area Estimates, is enough to prove that there is a constant $A_0 > 0$ such that for n big $\xi_n \leq A_0 (r(\tau^n))^{-\frac{2}{\mu_{max}}}$. Finally note that, since $J(R) \neq \hat{\mathbb{C}}$, it follows that $K(V_n) \cap V_{m(n)}^c$ has non empty interior, so $\xi_n < 1$.

Proof of Proposition 4.1. Let $U_{m(n_0)}$ given by Proposition 2.7 for $\delta = \tau^{m(n_0)}$. For $v \in \mathrm{CV} \cap J(R)$ and for $m(n_0) \leq r \leq n_0 + 1$ let, $\tilde{U}_r^v$ be the connected component of,

$$B_{\tau^r}(v) \cup (\mathbb{C} - K(V_{m(n_0)})),$$

that contains v. Therefore $\partial \tilde{U}_r^v \subset K(V_{m(n_0)})$. Let

$$U_r^v = \tilde{U}_r^v \cup \{ \text{ connected components } U \text{ of } \hat{\mathbb{C}} - \tilde{U}_r^v \text{ such that } diam(U) < diam(\tilde{U}_r^v)\},$$

which is a simply connected neighborhood of v so that $\partial U_r^v \subset K(V_{m(n_0)})$. Moreover, by Lemma 3.1 we have $B_{\tau^r}(v) \subset U_r^v \subset B_{\eta_1 \tau^r}(v)$, where $\eta_1 = 1 + \mathcal{O}(\varepsilon)$.

For $c \in \mathrm{Crit} \cap J(R)$ let V_r^c be the connected component of $R^{-1}(U_n^{R(c)})$ that contains c. By Lemma 2.6, $V_r = \cup V_r^c$ has Martens property. Taking ε smaller if necessary one may suppose that $V_{m(n_0)} \supset V_{m(n_0)+1} \supset ... \supset V_{n_0}$. In a similar way we can define inductively neighborhoods U_j^v of $v \in \mathrm{CV} \cap J(R)$, for $j > n_0 + 1$, such that $\partial U_{j+1} \subset K(V_{m(j)})$ and $B_{\tau^r}(v) \subset U_r^v \subset B_{\eta_1 \tau^r}(v)$, that satisfy the conclusions of the Proposition.

If W is a connected component of $\hat{\mathbb{C}} - L_n$, then $R(W)$ is a connected component of $\mathbb{C} - K(V_n)$, so there is $m = m_W$ and $c = c(W) \in \mathrm{Crit} \cap J(R)$ such that $R^m : R(W) \longrightarrow V_{m(n)}^c$ is a biholomorphism. By the Univalent Pull-back Property the inverse of this map extends to $\tilde{B}_{\tau^{m(n)} r(\tau^{m(n)})}(c)$, so the distortion of R^m in $R(W)$ is bounded $1 + C_1 (r(\tau^{m(n)}))^{-\frac{1}{\mu_c}}$, for a constant $C_1 > 0$ independent of W and n. By Lemma 3.1 there is $C_2 > 0$ independent of W and n such that $diam(R(W)) \leq C_2 \varepsilon diam(U_n)$. By definition of L_n we have $R(W)\cap U_n=\emptyset$, so by Koebe Distortion Theorem, the distortion of R in W is $1+\mathcal{O}(\varepsilon)$. Thus the distortion of R^{m+1} in W is $1+\mathcal{O}(\varepsilon)$. □

Remark 4.2. *Note that the neighborhoods V_n of the nest are such that $\partial K(V_n)-J(R)$ has zero Lebesgue measure. In fact, by construction $\partial K(V_n)-J(R)$ is locally a finite union of analytic curves; cf. proof of Proposition 2.7.*

4.1. **First landing maps and decay of area.** In this section we proof the Area Estimates. We keep the previous notation. Recall that is enough to prove that there is a constant $A_0>0$ such that for big n we have

$$\xi_n=\max_{\mathrm{Crit}\cap J(R)}\left\{\xi_n^c=\frac{|V^c_{m(n)}-K(V_n)|}{|V^c_{m(n)}|}\right\}\le A_0(r(\tau^n))^{-\frac{2}{\mu_{max}}}.$$

We first prove that $\xi_n\to 0$, by showing that the ξ_n satisfy an inductive inequality; see Lemmas 4.3 and 4.4. Then we prove the desired bound for ξ_n, see Lemma 4.5.

Let us begin with a remark about distortion. Let W be a connected component of the complement of $L_n=R^{-1}(K(V_{m(n)}))-V_n$, different from V_n^c, for $c\in\mathrm{Crit}\cap J(R)$. By Proposition 4.1 there is $m_W\ge 1$ and $c=c(W)\in\mathrm{Crit}\cap J(R)$ such that $R^{m_W}:W\longrightarrow V^c_{m(n)}$ is a biholomorphism with distortion bounded by $D=D(\varepsilon)>1$. Therefore, letting $\xi_W=\frac{|W-K_n|}{|W|}$, we have that

$$\frac{\xi_W}{\xi_n}\le D^2\frac{1-\xi_W}{1-\xi_n},$$

or equivalently $\xi_W\le\tilde{\xi_n}=\frac{D^2\xi_n}{1+(D^2-1)\xi_n}\le D^2\xi_n$.

The following lemma is independent of dynamics.

Lemma 4.3. *Let $v_n\in(0,1)$ such that $v_n\to 1$ as $n\to\infty$ and let $\sigma\in(0,1)$. If $D>1$ is close to one, then every sequence $\xi_n\in(0,1)$ such that,*

$$1-\xi_n\ge\frac{1-\xi_{n-1}}{1+(D^2-1)\xi_{n-1}}\,v_n\frac{1}{1-D^{-2}\tilde{\xi}_{n-1}\sigma},$$

where $\tilde{\xi}_{n-1}=\frac{D^2\xi_{n-1}}{1-(D^2-1)\xi_{n-1}}$, is such that $\xi_n\to 0$ as $n\to\infty$.

Proof. For $\xi\in(0,1)$ put $\tilde{\xi}=\frac{D^2\xi}{1+(D^2-1)\xi}$. Note that,

$$(1+(D^2-1)\xi)(1-D^{-2}\tilde{\xi}\sigma)<1+(D^2-1)\xi-\frac{\xi\sigma}{1+(D^2-1)\xi}.$$

So, if $\rho\in(0,\sigma)$ and if D is close enough to one, there is $\xi_0\in(0,1)$ such that,

$$(1+(D^2-1)\xi)(1-D^{-2}\tilde{\xi}\sigma)<1-\rho\xi, \tag{2}$$

for $\xi\in(0,\xi_0)$. Moreover we may suppose that D is close enough to one, so that $(1+(D^2-1)\xi)(1-D^{-2}\xi\sigma)<1$ for $\xi\in[\xi_0,1]$. Thus, if $\rho>0$ is small enough, we have (2) for all $\xi\in(0,1)$. Hence $1-\xi_n\ge\frac{1-\xi_{n-1}}{1-\rho\xi_{n-1}}v_n$.

Consider $\varepsilon\in(0,1)$. If $\xi_{n-1}<\varepsilon$ we have $\frac{1-\xi_n}{1-\xi_{n-1}}\ge\frac{v_n}{1-\rho\varepsilon}$. Since $v_n\to 1$ as $n\to\infty$ it follows that there are infinitely many n for which $\xi_n<\varepsilon$. Note that $1-\xi_n\ge\frac{1-\xi_{n-1}}{1-\rho\xi_{n-1}}v_n$ implies,

$$\xi_n\le\frac{1-\rho}{1-\rho\xi_{n-1}}\xi_{n-1}v_n+(1-v_n).$$

Since $v_n \to 1$ as $n \to \infty$, it follows that for n big, $\xi_{n-1} < \varepsilon$ implies $\xi_n < \varepsilon$. Therefore there is $n(\varepsilon)$ such that $\xi_n < \varepsilon$ for every $n \geq n(\varepsilon)$. Since $\varepsilon \in (0,1)$ was arbitrary, it follows that $\xi_n \to 0$ as $n \to \infty$. □

Lemma 4.4. *$\xi_n \to 0$ as $n \to \infty$.*

Proof. We denote $K_l = K(V_l)$.

1.– Let U be a connected component of $V_{m(n)} - V_{n-1} - K_{n-1}$ and let $m_U \geq 0$ and $c(U) \in \mathrm{Crit} \cap J(R)$ be such that $R^{m_U} : U \longrightarrow V_{n-1}^{c(U)}$ is a biholomorphism. By Proposition 4.1 we have that,

$$D^{-2} \frac{|V_{n-1}^{c(U)} \cap K_n|}{|V_{n-1}^{c(U)}|} \leq \frac{|U \cap K_n|}{|U|},$$

so $|U - K_n| \leq |U| \left(1 - D^{-2} \frac{|V_{n-1}^{c(U)} \cap K_n|}{|V_{n-1}^{c(U)}|}\right)$.

Fix $c \in \mathrm{Crit} \cap J(R)$. Note that $V_{m(n)}^c - V_{n-1}^c - K_{n-1}$ is contained in $V_{m(n)} - V_{n-1} - L_{n-1}$. As remarked above, for each connected component W of this later set, we have

$$|W - L_{n-1}| \leq \tilde{\xi}_{n-1}|W|.$$

Therefore $\sum_U |U| \leq \sum_W \tilde{\xi}_{n-1}|W| \leq \tilde{\xi}_{n-1}|V_{m(n)}^c - V_{n-1}^c|$, where the sum is over all connected components U of $V_{m(n)}^c - V_{n-1}^c - K_{n-1}$, and all connected components W of $V_{m(n)}^c - V_{n-1}^c - L_{n-1}$, respectively.

Therefore we have,

$$\begin{aligned} |V_{m(n)}^c|\xi_n^c &= |V_{m(n)}^c - K_n| \leq |V_{n-1}^c| + \sum_W |W| \left(1 - D^{-2} \frac{|V_{n-1}^{c(W)} \cap K_n|}{|V_{n-1}^{c(W)}|}\right) \\ &\leq |V_{n-1}^c| + \tilde{\xi}_{n-1}|V_{m(n)}^c - V_{n-1}^c| \left(1 - D^{-2}(\min_{c_0} \frac{|V_{n-1}^{c_0} \cap K_n|}{|V_{n-1}^{c_0}|})\right), \text{ so} \end{aligned}$$

$$1 - \xi_n^c \geq \frac{|V_{m(n)}^c - V_{n-1}^c|}{|V_{m(n)}^c|} \left(1 - \tilde{\xi}_{n-1} + \tilde{\xi}_{n-1} D^{-2}(\min_{c_0} \frac{|V_{n-1}^{c_0} \cap K_n|}{|V_{n-1}^{c_0}|})\right).$$

2.– In a similar way we obtain

$$|V_{n-1}^c - K_n| \leq |V_n^c| + \tilde{\xi}_{n-1}|V_{n-1}^c - V_n^c| \left(1 - D^{-2}(\min_{c_0} \frac{|V_{n-1}^{c_0} \cap K_n|}{|V_{n-1}^{c_0}|})\right).$$

Therefore we have

$$\frac{|V_{n-1}^c \cap K_n|}{|V_{n-1}^c|} \geq \frac{|V_{n-1}^c - V_n^c|}{|V_{n-1}^c|} \left(1 - \tilde{\xi}_{n-1}\left(1 - D^{-2}(\min_{c_0} \frac{|V_{n-1}^{c_0} \cap K_n|}{|V_{n-1}^{c_0}|})\right)\right), \text{ and,}$$

$$\min_{c_0} \frac{|V_{n-1}^{c_0} \cap K_n|}{|V_{n-1}^{c_0}|} \geq \frac{(1 - \tilde{\xi}_{n-1})(\min_c \frac{|V_{n-1}^c - V_n^c|}{|V_{n-1}^c|})}{1 - D^{-2}\tilde{\xi}_{n-1}(\min_c \frac{|V_{n-1}^c - V_n^c|}{|V_{n-1}^c|})}$$

3.– Combining 2 and 3 it follows that,

$$\begin{aligned}
\frac{1-\xi_n^c}{1-\xi_{n-1}} &= \frac{1-\xi_n^c}{1-\tilde{\xi}_{n-1}}\frac{1}{1+(D^2-1)\xi_{n-1}} \\
&\geq \frac{1}{1+(D^2-1)\xi_{n-1}}\frac{|V_{m(n)}^c - V_{n-1}^c|}{|V_{m(n)}^c|}\left(1+\frac{D^{-2}\tilde{\xi}_{n-1}(\min_c \frac{|V_{n-1}^c - V_n^c|}{|V_n^c|})}{1-D^{-2}\tilde{\xi}_{n-1}(\min_c \frac{|V_{n-1}^c - V_n^c|}{|V_n^c|})}\right) \\
&= \frac{1}{1+(D^2-1)\xi_{n-1}}\frac{|V_{m(n)}^c - V_{n-1}^c|}{|V_{m(n)}^c|}\left(\frac{1}{1-D^{-2}\tilde{\xi}_{n-1}(\min_c \frac{|V_{n-1}^c - V_n^c|}{|V_n^c|})}\right).
\end{aligned}$$

Note that $\min_c \frac{|V_{n-1}^c - V_n^c|}{|V_n^c|} = 1-\tau^{\frac{2}{\mu_{max}}} + \mathcal{O}(\varepsilon)$, so if ε is small enough, there is $\sigma \in (0,1)$ so that $\sigma \leq \min_c \frac{|V_{n-1}^c - V_n^c|}{|V_n^c|}$ for big n. Moreover,

$$v_n = \min_c \frac{|V_{m(n)}^c - V_{n-1}^c|}{|V_{m(n)}^c|} \sim 1-(r(\tau^n))^{-\frac{2}{\mu_{max}}} \to 1 \text{ as } n \to \infty.$$

Hence the sequence ξ_n satisfies the hypothesis of the previous lemma. Since $D = D(\varepsilon) = 1 + \mathcal{O}(\varepsilon)$ we have that, if $\varepsilon > 0$ was chosen small enough then $\xi_n \to 0$. □

Lemma 4.5. *Let $n \gg 1$ and $k \geq 1$ be such that $m(n) = m(n+1) = ... = m(n+k)$. Letting $b_i = \xi_i \min_c \frac{|V_{m(n)}^c|}{|V_n^c|}$ we have*

$$b_{n+k} \leq 1 + D^9 \max_c \frac{|V_{m(n)}^c|}{|V_{m(n-1)}^c|} b_{n-1}.$$

Proof. By hypothesis $m(n+k) = m(n)$, so for any connected component W of $V_{m(n)} - V_{n+k} - R^{-1}(K_{m(n)})$ there is $m \geq 1$ and $c_0 \in \text{Crit} \cap J(R)$ so that $R^{m_W} : W \longrightarrow V_{m(n)}^{c_0}$ is a biholomorphism.

Since $m(n) \geq n-1$ it follows that, if U is a connected component of $V_{m(n)} - V_{n+k} - R^{-1}(K_{n-1})$, then there are $m \geq 1$ and $c_0 \in \text{Crit} \cap J(R)$ so that $R^m : U \longrightarrow V_m^{c_0}$ is a biholomorphism.

Fix $c \in \text{Crit} \cap J(R)$. It follows by the distortion property of Proposition 4.1 that

$$\begin{aligned}
\xi_{n+k}^c |V_{m(n+k)}^c| - |V_{n+k}^c| &= |V_{m(n)}^c - V_{n+k}^c - K_{n+k}| \\
&\leq D^2 |V_{m(n)}^c - V_{n+k}^c - R^{-1}(K_{n-1})| (\max_{c_0} \frac{|V_{n-1}^{c_0} - K_{n+k}|}{|V_{n-1}^{c_0}|}).
\end{aligned}$$

As in the previous lemma we have

$$|V_{m(n)}^c - V_{n+k}^c - R^{-1}(K_{n-1})| \leq D^2 \xi_{n-1} |V_{m(n)}^c - V_{n+k}^c|, \text{ and also}$$
$$|V_{n-1}^c - V_{n+k}^c - R^{-1}(K_{n-1})| \leq D^2 \xi_{n-1} |V_{n-1}^c - V_{n+k}^c|.$$

1.– Let us prove that for big n we have $\max_{c_0} \frac{|V_{n-1}^{c_0} - K_{n+k}|}{|V_{n-1}^{c_0}|} \leq D \max_c \frac{|V_{n+k}^c|}{|V_{n-1}^c|}$. In a similar way as above we obtain

$$\begin{aligned}
\frac{|V_{n-1}^c - V_{n+k}^c - K_{n+k}|}{|V_{n-1}^c|} &\leq D^2 \frac{|V_{n-1}^c - V_{n+k}^c - R^{-1}(K_{n-1})|}{|V_{n-1}^c - V_{n+k}^c|} (\max_{c_0} \frac{|V_{n-1}^{c_0} - K_{n+k}|}{|V_{n-1}^{c_0}|}) \\
&\leq D^4 \xi_{n-1} (\max_{c_0} \frac{|V_{n-1}^{c_0} - K_{n+k}|}{|V_{n-1}^{c_0}|}),
\end{aligned}$$

Therefore

$$\max_{c_0} \frac{|V^{c_0}_{n-1} - K_{n+k}|}{|V^{c_0}_{n-1}|} \leq (1 - D^4 \xi_{n-1})^{-1} (\max_c \frac{|V^c_{n+k}|}{|V^c_{n-1}|})$$

Considering that $\xi_n \to 0$ as $n \to \infty$ we have that, for big n,

$$\max_{c_0} \frac{|V^{c_0}_{n-1} - K_{n+k}|}{|V^{c_0}_{n-1}|} \leq D (\max_c \frac{|V^c_{n+k}|}{|V^c_{n-1}|}).$$

2.− Combining previous inequalities we obtain

$$\begin{aligned} \xi^c_{n+k} |V^c_{m(n+k)}| - |V^c_{n+k}| &\leq D^5 \xi_{n-1} |V^c_{m(n)} - V^c_{n+k}| (\max_{c_0} \frac{|V^{c_0}_{n+k}|}{|V^{c_0}_{n-1}|}) \\ &\leq D^5 \xi_{n-1} |V^c_{m(n)}| (\max_{c_0} \frac{|V^{c_0}_{n+k}|}{|V^{c_0}_{n-1}|}), \text{ so} \end{aligned}$$

$$\xi^c_{n+k} = \frac{|V^c_{m(n)} - K_{n+k}|}{|V^c_{n+k}|} \leq \frac{|V^c_{n+k}|}{|V^c_{m(n)}|} + D^5 \xi_{n-1} (\max_{c_0} \frac{|V^{c_0}_{n+k}|}{|V^{c_0}_{n-1}|}).$$

Thus $b_{n+k} \leq 1 + D^5 \min_c \frac{|V^c_{m(n)}|}{|V^c_{n+k}|} \xi_{n-1} (\max_{c_0} \frac{|V^{c_0}_{n+k}|}{|V^{c_0}_{n-1}|})$. Note that the minimum and the maximum are realized for c and c_0 with maximal multiplicity. Since for such critical points the respective quantities are comparable by a factor of D^2, it follows that if c minimizes $\frac{|V^c_{m(n)}|}{|V^c_{m(n-1)}|}$ we have

$$\begin{aligned} b_{n+k} &\leq 1 + D^7 \frac{|V^c_{m(n)}|}{|V^c_{m(n-1)}|} (\xi_{n-1} \frac{|V^c_{m(n-1)}|}{|V^c_{n-1}|}) \\ &\leq 1 + D^9 \max_c \frac{|V^c_{m(n)}|}{|V^c_{m(n-1)}|} b_{n-1}. \square \end{aligned}$$

Proof of the Area Estimates. Note that is enough to prove that the b_i are bounded. By the previous lemma is enough to bound the b_j for j so that $m(j+1) < m(j)$. If $j_1 < j_2$ are two such consecutive numbers it follows by the previous lemma that,

$$b_{j_2} \leq 1 + D^9 \max_c \frac{|V^c_{m(j_1+1)}|}{|V^c_{m(j_1)}|} b_{j_1}.$$

Letting $\varepsilon > 0$ smaller if necessary we may assume that $D^9 \max_c \frac{|V^c_{m(j_1+1)}|}{|V^c_{m(j_1)}|}$ is less than some definite constant less than one. Then the b_j are bounded and the Area Estimates follows. $\square$

4.2. **About Lebesgue measure of Julia sets.** In this section we prove the following.

Proposition B. *Let $R \in \mathcal{S}$ be a rational map such that $J(R) \neq \hat{\mathbb{C}}$ and consider a nest $V_n = \cup V^c_n$ of neighborhoods with Martens property given by Proposition 4.1. Then the area of $\hat{\mathbb{C}} - K(V_n)$ goes to zero as $n \to \infty$. In particular the set,*

$$\{z \in J(R) \mid \omega(z) \cap \text{Crit} \neq \emptyset\} = \cap_{n \gg 1} (\hat{\mathbb{C}} - K(V_n)),$$

has zero Lebesgue measure.

Recall that $\omega(z)$ denotes the omega limit set of $z \in \hat{\mathbb{C}}$, which is the set of accumulation points of the forward orbit of z.

In Section 7 we prove that, if $J(R) \neq \hat{\mathbb{C}}$, then the set of points in $J(R)$ that do not accumulate critical points under forward iteration has zero Lebesgue measure. This completes the proof of Theorem B asserting that in this case the Julia set has zero Lebesgue measure. The proof of this fact is somehow indirect and it depends in the proof of Theorem C. There is a more direct proof by Przytycki, under the stronger summability condition with exponent one; see Lemma 3.1 of [Pr2]. So now we obtain.

Corollary. *Let $R \in \mathbb{C}(z)$ be a rational map satisfying the summability condition with exponent one and so that $J(R) \neq \hat{\mathbb{C}}$. Then, the Lebesgue measure of $J(R)$ is zero.*

The proof of Proposition B depends in the following lemma, which is a consequence of the Area Estimates.

Lemma 4.6. *Fix $\eta > 1$ close to one. Given $\delta > 0$ small let $U_\delta = \cup_{v \in \mathrm{CV} \cap J(R)} U^v$ be as in Proposition 2.7, so that $B_\delta(v) \subset U^v \subset B_{\eta\delta}(v)$ and the union V_δ of the connected components of $R^{-1}(U_\delta)$ intersecting $\mathrm{Crit} \cap J(R)$ has Martens property. Then there is a constant $A_1 > 0$ independent of $\delta > 0$ such that,*

$$\frac{|U^v - K(V_\delta)|}{|U^v|} \leq A_1 (r(\delta))^{-\frac{2}{\mu_{max}}}.$$

Proof. Denote by V_δ^c the connected component of V_δ containing $c \in \mathrm{Crit} \cap J(R)$ so $V_\delta = \cup_c V_\delta^c$. Since V_δ has Martens property, for every connected component W of $\hat{\mathbb{C}} - K(V_\delta)$ there are $m = m_W \geq 0$ and $c = c(W) \in \mathrm{Crit} \cap J(R)$ such that $R^m : W \longrightarrow V_\delta^c$ is a biholomorphism. Denote by R^{-m_W} the respective inverse branch of R^{m_W}. By the Univalent Pull-back Property R^{-m_W} extends univalently to $\tilde{B}_{r(\delta)\delta}(c)$.

For $v \in \mathrm{CV} \cap J(R)$ let $S^v = \{W \mid W \cap U^v \neq \emptyset\}$ and let $\varepsilon > 0$ given by the Area Estimates. For $W \in S$ and $\nu \in (0,1)$ let $W(\nu) = R^{-m_W}(\tilde{B}_{\nu\varepsilon r(\delta)\delta}(c(W)))$. By Koebe Distortion Theorem there is $\nu \in (0,1)$ independent of δ such that if $W_0, W_1 \in S$ are such that $W_0(\nu) \cap W_1(\nu) \neq \emptyset$, then either $W_0(\nu) \subset W_1(1)$ or $W_1(\nu) \subset W_0(1)$. Let

$$T^v = \{W \in S^v \mid W(\nu) \not\subset W_0(1) \text{ for all } W_0 \in S^v - \{W\}\}.$$

Thus the sets $W(\nu)$, for $W \in T^v$, are pairwise disjoint. By Koebe Distortion Theorem the distortion of R^{m_W} in $W(1)$ is bounded by some constant $D > 1$, independent of $\delta > 0$. It follows by Lemma 3.1 that if $\delta > 0$ is small enough there is a constant $C > 0$ independent of δ such that

$$U^v - K(V_\delta) \subset (\cup_{T^v} W(1)) \subset B_{C\delta}(v).$$

By the Area Estimates we have

$$|W - K(V_\delta)| \leq D^2 A (r(\delta))^{-\frac{2}{\mu_{max}}} |W|.$$

Therefore

$$|U^v - K(V_\delta)| \leq D^2 A (r(\delta))^{-\frac{2}{\mu_{max}}} \sum_{T^v} |W(1)| \leq \nu^{-2} D^4 A (r(\delta))^{-\frac{2}{\mu_{\max}}} \sum_{T^v} |W(\nu)|.$$

Considering that the sets $W(\nu)$, for $W \in T^v$, are pairwise disjoint we have $\sum_{T^v} |W(\nu)| \leq |B_{C\delta}(v)|$. Thus there is a constant $A_1 > 0$ such that $|U^v - K(V_\delta)| \leq A_1 (r(\delta))^{-\frac{2}{\mu_{max}}} |U^v|$.
□

Proof of Proposition B. Let $\tau \in (0,1)$ and $\eta \in (1, \tau^{-1})$ be as in Proposition 4.1, so that $\tilde{B}_{\tau^i}(c) \subset V_i^c \subset \tilde{B}_{\eta\tau^i}(c)$. Then, for $i \gg 1$,

$$|\hat{\mathbb{C}} - K(V_{i+1})| \leq D_i^2 \left(\max_c \frac{|V_i^c - K(V_{i+1})|}{|V_i^c|} \right) |\hat{\mathbb{C}} - K(V_i)|,$$

where the distortion constant $D_i > 1$ satisfies $D_i = 1 + \mathcal{O}((r(\tau^n))^{-\frac{1}{\mu_{max}}})$. Applying the previous lemma for $\delta = \tau^i$ it follows that $\frac{|V_i^c - K(V_{i+1})|}{|V_i^c|}$ is less than some definite constant less than one, so the proposition follows. □

The following lemma is an easy consequence of Lemma 4.6 and it will be used in Section 5 to prove Rigidity.

Lemma 4.7. *Let $R \in \mathcal{S}$ be such that $J(R) \neq \hat{\mathbb{C}}$. Then for every $\varepsilon_0 > 0$ there is $\delta_0 = \delta_0(\varepsilon_0) > 0$ and $\rho > 1$ such that for any $\delta \in (0, \delta_0)$ the set $V = \cup V^c$ with Martens property given by Proposition 2.7, for this choice of δ, is such that for any $c \in \mathrm{Crit} \cap J(R)$,*

$$\frac{|\tilde{B}_{\rho\delta}(c) - K(V)|}{|\tilde{B}_{\rho\delta}(c)|} < \varepsilon_0.$$

Proof. Let $\rho > 1$ be such that for every $c \in \mathrm{Crit} \cap J(R)$ and $\delta > 0$ small, $|\tilde{B}_{2\delta}(c)| < \frac{\varepsilon_0}{2}|\tilde{B}_{\rho\delta}(c)|$. By Lemma 4.6, applied to $\rho\delta$ instead of δ, and by Koebe Distortion Theorem, we have that for every $c \in \mathrm{Crit} \cap J(R)$ and every $\delta > 0$ small

$$|\tilde{B}_{\rho\delta}(c) - V^c - K(V)| < \frac{\varepsilon_0}{2} |\tilde{B}_{\rho\delta}(c)|,$$

Hence,

$$|\tilde{B}_{\rho\delta}(c) - K(V)| = |\tilde{B}_{\rho\delta}(c) - V^c - K(V)| + |V^c| < (\frac{\varepsilon_0}{2} + \frac{\varepsilon_0}{2})|\tilde{B}_{\rho\delta}(c)|. \square$$

5. Mostly conformal maps and Rigidity.

Let us fix throughout all this section a rational map $R \in \mathcal{S}$ such that $J(R) \neq \hat{\mathbb{C}}$. In this section we prove a rigidity property of qc maps that are conformal in a big dynamically defined set. Such maps appear naturally in pull-back procedures like Thurston's algorithm (see Section 6), in pseudo-conjugacies (see [Lyu2] and Section 8.2) and in parameter maps; see [Lyu3], [R-L2] and Section 8.4.

Before the statement of Rigidity, let us recall a simplified version of Proposition 4.1, that provide us with a nest of neighborhoods with Martens property.

Lemma 5.1. *Fix $\tau \in (0,1)$ and $\eta \in (1, \tau^{-1})$. Then there are neighborhoods U_n^v, for $n \gg 1$ and $v \in \mathrm{CV} \cap J(R)$ so that*

$$B_{\tau^n}(v) \subset U_n^v \subset B_{\eta\tau^n}(v),$$

and so that, if V_n^c is the connected component of $R^{-1}(U_n^{R(c)})$ that contains $c \in \mathrm{Crit} \cap J(R)$, then $V_n = \cup V_n^c$ has Martens property. Moreover there is a sequence $m(n) \leq n$ so that $n - m(n) \to \infty$ as $n \to \infty$ such that and $\partial U_n \subset K(V_{m(n)})$, where $U_n = \cup U_n^v$, $n \gg 1$.

Note that this lemma implies that for any $N > 1$ and any $n \gg 1$, we have that for every connected component W of $\hat{\mathbb{C}} - K(V_n)$ there is $m_W \geq 0$ and $c = c(W) \in \mathrm{Crit} \cap J(R)$ such that $R^{m_W} : W \longrightarrow V_n^c$ is a biholomorphism whose inverse extends univalently to $V_{m(n)}^c$. By Koebe Distortion Theorem it follows that, for $n \gg 1$, the distortion of R^{m_W}

in W is bounded by some definite constant $D = D(n) > 1$; in fact D only depends in $n - m(n)$ and $D = D(n) \to 1$ as $n \to \infty$.

5.1. **Statement of Rigidity.** The following is our main technical result.

Rigidity. *Let $R \in \mathcal{S}$ such that $J(R) \neq \hat{\mathbb{C}}$ and consider the nest $V_n = \cup V_n^c$ as in Lemma 5.1. There are constants $K > 1$ and $C > 0$ only depending in $\min_{c,n\gg 1} mod(V_n^c - \overline{V_{n+1}^c})$, so that for n big enough and for every qc-homeomorphism $\chi : V_n^c \longrightarrow \chi(V_n^c) \subset \hat{\mathbb{C}}$ that is conformal Lebesgue almost everywhere in $V_n^c \cap K(V_{n+1})$, there is a K-qc homeomorphism $\hat{\chi} : V_n^c \longrightarrow \chi(V_n^c)$ at hyperbolic distance from χ at most C.*

Remark 5.2.
(1) *The property that $\hat{\chi}$ is at a bounded hyperbolic distance from χ implies that, if χ extends continuously to the boundary of V_n^c, then so does $\hat{\chi}$ and their extensions coincide. This property will be important to apply the Gluing Lemma, stated in Appendix 10.*
(2) *Note that given a preferred point $w \in V_n^c$ we may assume that $\hat{\chi}(w) = \chi(w)$, by changing the constants $K > 1$ and $C > 0$.*
(3) *As the proof of Rigidity shows, for each $c \in \mathrm{Crit} \cap J(R)$ the annulus $A_n^c = V_{n-1}^c - \overline{V_n^c}$ also has this Rigidity property. In fact, for any qc map $\chi : A_n^c \longrightarrow \mathbb{C}$ that is conformal Lebesgue almost everywhere in $A_n^c \cap K(V_n)$, there is a K_0-qc homeomorphism $\hat{\chi} : A_n^c \longrightarrow \chi(A_n^c)$ at a hyperbolic distance at most C from χ, for definite constants $K_0 > 1$ and $C > 0$. In particular $mod(\chi(A_n^c))$ is at least $K_0^{-1} mod(A_n^c)$, which is independent of χ; compare with the rigidity property in Appendix 11. It follows from the proof that we can take $K_0 > 1$ arbitrarily close to 1 by letting n big.*

Rigidity holds because $V_n - \overline{V_{n+1}} - K(V_{n+1})$ has well distributed and small area. With the following Lemma 5.3 we will make this precise. First let us consider some notation. Given $U \subset \hat{\mathbb{C}}$ conformaly equivalent to $\mathbb{D}$ and $E \subset U$ put,

$$\|E\|_U = \sup |\varphi(E)|,$$

where the supremum is taken over all biholomorphisms $\varphi : U \longrightarrow \mathbb{D}$. Note that $\|E\|_U$ is invariant by biholomorphisms.

Lemma 5.3. *Given $\varepsilon > 0$ small, there is $N > 1$ and $n_0 \gg 1$, so that if $n \geq n_0$ then for all $c \in \mathrm{Crit} \cap J(R)$, we have $\|V_{n-N}^c - K(V_n)\|_{V_{n-N}^c} < \varepsilon$.*

Proof. By Koebe Distortion Theorem there is $D > 1$ that bounds the distortion in $V_{m+1}^{c_0}$, for $m \gg 1$, of any univalent map defined in V_m^c. By Lemma 4.7, in Section 4.2, there is $N > 1$ such that for $n \gg 1$,

$$\frac{|V_{n+1-N}^{c_0} - K(V_n)|}{|V_{n+1-N}^{c_0}|} < D^{-2}\varepsilon.$$

Consider a biholomorphism $\varphi : V_{n-N}^c \longrightarrow D$. For each connected component W of $V_{n-N}^c - K(V_{n+1-N})$ there is $m_W \geq 0$ and $c(W) \in \mathrm{Crit} \cap J(R)$ such that $R^{m_W} : W \longrightarrow V_{n+1-N}^{c(W)}$ is a biholomorphism whose inverse g_W extends univalently to $V_{n-N}^{c(W)}$. So the distortion of $\varphi \circ g_W$ in V_{n+1-N} is bounded by D. Therefore,

$$\frac{|\varphi(W - K(V_n))|}{|\varphi(W)|} \leq D^2 \frac{|V_{n+1-N}^{c(W)} - K(V_n)|}{|V_{n+1-N}^{c(W)}|} < \varepsilon.$$

Since such W cover $V_{n-N}^c - K(V_n)$ the lemma follows. $\square$

Observe that for $n \gg 1$ the connected components of $V_n - \overline{V_{n+1}} - K(V_{n+1})$ are organized by the connected components of $V_n - \overline{V_{n+1}} - K(V_{n-N})$, which are mapped biholomorphic to the V_{n-N}^c, for $c \in \mathrm{Crit} \cap J(R)$, by appropriated iterates of R. By the previous lemma we have $\|W - K(V_n)\|_W < \varepsilon$, for $n \gg 1$. It is in this sense that $V_n^c - \overline{V_{n+1}^c} - K(V_n)$ has well distributed and small area.

5.2. **Smoothing.** In this section we consider smoothing lemmas used in the proof of Rigidity; see Lemmas 5.4 and 5.6

Lemma 5.4. *Given $\delta_0 > 0$ and K_0, $K_1 > 1$ there is $\varepsilon > 0$ such that if W_0, $W \subset \mathbb{C}$ are biholomorphic to $\mathbb{D}$ and $E \subset W_0$ satisfies $\|E\|_{W_0} < \varepsilon$, then for any K_1-qc homeomorphism $\chi : W_0 \longrightarrow W$ conformal Lebesgue almost everywhere outside E, then there is a K_0-qc homeomorphism $\hat{\chi} : W_0 \longrightarrow W$ such that,*

$$hypd_W(\chi, \hat{\chi}) \leq \delta_0.$$

Proof. Consider a tilling of $\mathbb{D}$ by hyperbolic hexagons H_i as in Figure 2, so that every such hexagon is isometric to a model hexagon H. We define $\hat{\chi}$ equal to χ in the vertices of the hexagons. For a given hexagon we consider coordinates in $\mathbb{D}$ so that it becomes the model hexagon H, centered at 0, and so that χ fixes 0. By hypothesis χ is conformal except for a set of Lebesgue measure at most ε, where $\varepsilon > 0$ is to be chosen.

By Lemma 10.3 in Appendix 10, χ is close to some map $z \longrightarrow \lambda z$, with $|\lambda| = 1$, as $\varepsilon \to 0$. We suppose ε small enough so that the image of the vertices of H by χ form a hyperbolic hexagon $\mathcal{H}$. We let $\tilde{\chi}$ map the sides of H to the sides of $\mathcal{H}$, in such a way that the hyperbolic length is preserved, up to a multiplicative constant in each side. If $\varepsilon > 0$ is small enough $\hat{\chi}$ extends to a K_0-qc homeomorphism between the interior of H and the interior of $\mathcal{H}$ and so that the hyperbolic distance between χ and $\hat{\chi}$ in H is at most δ_0.

Doing this for every hexagon H_i we obtain an homeomorphism $\hat{\chi}$ of $\mathbb{D}$ at a distance at most δ_0 from χ and that is K_0-qc in the complement of $\cup \partial H_i$. Since $\cup \partial H_i$ has σ finite length, it is a qc removable set and therefore $\hat{\chi}$ is a K_0-qc homeomorphism; see Appendix 10. $\square$

Lemma 5.5. *For $r \in (0,1)$ there is $K(r) \geq 1$ such that for every homeomorphism χ of $\mathbb{D}$ conformal in $\{r < |z| < 1\}$, there is a $K(r)$-qc homeomorphism $\hat{\chi}$ of $\mathbb{D}$ that coincides with χ in $\{r^{\frac{1}{2}} < |z| < 1\}$.*

Proof. For $s > 0$ denote $\{|z| < s\}$ by $\mathbb{D}_s$. Consider an uniformization

$$\varphi : \chi(\mathbb{D}_{r^{\frac{1}{2}}}) \longrightarrow \mathbb{D}_{r^{\frac{1}{2}}},$$

such that $\varphi \circ \chi(0) = 0$. By Schwartz reflection principle applied to $\partial \mathbb{D}_{r^{\frac{1}{2}}}$, we have that φ extends to a biholomorphism $\varphi : \chi(\mathbb{D}) \longrightarrow V$ for some neighborhood V of $\mathbb{D}_{r^{\frac{1}{2}}}$. Let $\hat{\chi}$ be the homeomorphism of $\mathbb{D}$ that coincides with χ in $\{r^{\frac{1}{2}} < |z| < 1\}$ and so that

$$\varphi \circ \hat{\chi}(se^{i\theta}) = se^{ih(\theta)}, \tag{3}$$

in $\mathbb{D}_{r^{\frac{1}{2}}}$. Note that the function $h : \mathbb{R} \longrightarrow \mathbb{R}$ is analytic.

Since φ is conformal, is enough to estimate the dilatation of $\varphi \circ \hat{\chi}$ in terms of r only. This distortion is clearly equal to $\max(\sup(e^{ih})', (\inf(e^{ih})')^{-1})$, which is bounded by the distortion of e^{ih}. Since $\varphi \circ \chi$ is holomorphic and univalent in $\{r < |z| < 1\}$ and by (3), it

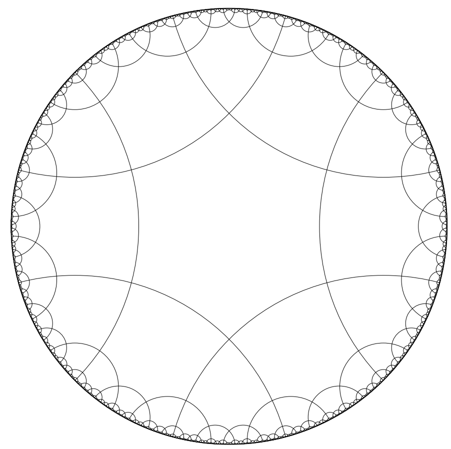

FIGURE 2. The vertices of the hexagons from a uniformly distributed set in $\mathbb{D}$.

follows by Koebe Distortion Theorem that the distortion of e^{ih} can be bounded in terms of r only. $\square$

Lemma 5.6. *Let $U \subset \mathbb{C}$ be biholomorphic to $\mathbb{D}$, let $K \subset U$ be a full compact set and let $K_0 \geq 1$. Then there is $K_1 \geq 1$, only depending in K_0 and in the modulus of the annulus $A = U - K$, so that for any homeomorphism $\chi : U \longrightarrow \chi(U) \subset \mathbb{C}$ that is K_0-qc in A, there is a K_1-qc homeomorphism $\hat{\chi} : U \longrightarrow \chi(U)$ at a hyperbolic distance to χ bounded in terms of $K_0^{-1} mod(A)$ only.*

Proof. We may suppose that $U = \mathbb{D}$ and note that there is $r \in (0, 1)$ only depending in $mod(A)$ so that $\{r < |z| < 1\} \subset A$. For χ given, consider a K_0-qc homeomorphism $\psi : \chi(U) \longrightarrow \mathbb{D}$ so that $\psi \circ \chi$ is conformal in A. Then the lemma follows by Lemma 5.5 with $K_1 = K_0 K(r)$ and considering that $mod(\{r^{\frac{1}{2}} < |z| < 1\}) \geq \frac{1}{2} K_0^{-1} mod(A)$. $\square$

5.3. **Proof of Rigidity.** Let $N > 1$ and fix $n \gg 1$. We organize the connected components of $V_n - K(V_{n+1})$ in *levels* as follows; we will call this connected components just *components*. For $c \in \text{Crit} \cap J(R)$ let V_{n+1}^c, have level 0. Note that for each connected component W of $V_n - \overline{V_{n+1}} - R^{-1}(K(V_n))$ there is $m_W \geq 1$ and $c = c(W) \in \text{Crit} \cap J(R)$ such that $R^{m_W} : W \longrightarrow V_n^c$ is a biholomorphism. We denote its inverse by g_W.

By Lemma 5.1 g_W extends univalently to V_{n-N}^c. Then $g_W(V_{n+1}^c)$ is a connected component of $V_n - K(V_{n+1})$, we assign it level 1. In general a connected component W of $V_n - K(V_{n+1})$ determines $c = c(W) \in \text{Crit} \cap J(R)$ and connected components W_1, ..., W_l of $V_n - \overline{V_{n+1}} - R^{-1}(K(V_n))$ so that,

$$W = g_{W_1} \circ ... \circ g_{W_l}(V_{n+1}^c).$$

We assign level l to W and we denote $g_W = g_{W_1} \circ ... \circ g_{W_l}$, which extends in a univalent way to V_{n-N}^c. Note that if W_0 and W_1 are components of the same level, then $g_{W_0}(V_n^{c(W_0)})$ and $g_{W_1}(V_n^{c(W_1)})$ are disjoint.

Lemma 5.7. *It is enough to prove Rigidity for maps χ that are conformal Lebesgue almost everywhere outside components of level at most $k(\chi) \geq 0$.*

Proof. Let χ be as in Rigidity. By Ahlfors-Bers Integration Theorem, for every $k \geq 0$, we can write $\chi = h_k \circ \chi_k$, where $\chi_k : V_n^c \longrightarrow \chi(V_n^c)$ is a qc homeomorphism, that is conformal Lebesgue almost everywhere in the components of level at most k and h_k is a normalized qc homeomorphism of $\chi(V_n^c)$, that is conformal Lebesgue almost everywhere outside the components of level at least k.

Moreover we may suppose that the dilatation of h_k is bounded by the dilatation of χ. Since the area of the union of the components of level at least k, goes to zero as $k \to \infty$, it follows that h_k converges uniformly to the identity of $\chi(V_n^c)$; see part 1 of Lemma 10.3.

Suppose that Rigidity holds for χ_k, for definite constants $K > 1$ and $C > 0$, and let $\hat{\chi}_k$ be the respective K-qc homeomorphisms. By the compactness of normalized K-qc homeomorphisms there is $k_i \to \infty$ as $i \to \infty$ so that $\hat{\chi}_{k_i}$ converges to a K-qc homeomorphism $\hat{\chi} : V_n^c \longrightarrow \chi(V_n^c)$. Since the hyperbolic distance between $\hat{\chi}_k$ and χ is at most C, it follows that the hyperbolic distance of $\hat{\chi}$ to χ is also bounded by C. □

Note that the connected components of $V_n - \overline{V_{n+1}} - K(V_{n-N})$ cover the components of level one. We call these connected components *pieces* of level 0. Moreover for any component W of level l the image by g_W of a piece of level 0 will be called a *piece of level* l. Thus a piece of level $l+1$ is contained in a unique piece of level l and every piece of level l is contained in $g_W(V_{n+1}^{c(W)})$ for some component W of level l.

Proof of Rigidity. Fix $K_0 > 1$ and let $K_1 > 1$ be as in Lemma 5.6 for $U = V_n^c$ and $K = \overline{V_{n+1}^c}$, where $n \gg 1$ and $c \in \text{Crit} \cap J(R)$. Note that, by the proof of Lemma 5.6 we can take $K_1 = K_0 K(r)$, where r only depends in $mod(V_n^c - \overline{V_{n+1}^c})$. Thus K_1 can be taken arbitrarily close to $K(r)$ by letting K_0 close to 1. Hence $K = K_1$ depends only in $\min_c mod(V_n^c - \overline{V_{n+1}^c})$ for big n.

Let $\varepsilon > 0$ as in Lemma 5.4 for these choices of K_0 and K_1 and consider $N = N(\varepsilon)$ as in Lemma 5.3. Fix $n \gg 1$ and we consider the notation and terminology above.

1.– Consider a qc homeomorphism $\chi : V_n^c \longrightarrow \chi(V_n^c) \subset \hat{\mathbb{C}}$ conformal in $V_n^c \cap K(V_{n+1})$. By Lemma 5.7 we may suppose that χ is conformal outside the components of level at most k.

We will prove by induction in $0 \leq m \leq k$ the two following properties.

(1) There is a qc homeomorphism $\chi_m : V_n^c \longrightarrow \chi(V_n^c)$ that coincides with χ outside the pieces of level m and it is K_0-qc in each piece of level m.
(2) There is a qc homeomorphism $\tilde{\chi}_m$ that coincides with χ outside $\cup g_W(V_n^{c(W)})$ and it is K_1-qc in $\cup g_W(V_n^{c(W)})$, where the union is over all components W of level m.

Note that Rigidity follows from 2 for $m = 0$, with $K = K_1$. Moreover, by hypothesis 1 is satisfied for $m = k$.

2.– Suppose that 1 holds for level $0 \leq m \leq k$. Then for every component W of level m the homeomorphism χ_m is K_0-qc in $g_W(V_n^{c(W)} - \overline{V_{n+1}^{c(W)}})$. Thus by Lemma 5.6 there is a K_1-qc homeomorphism $\chi_W : g_W(V_n^{c(W)}) \longrightarrow \chi(g_W(V_n^{c(W)}))$ at a bounded hyperbolic distance from χ_m, where the bound is independent of W. Thus replacing χ_m in each

$g_W(V_n^{c(W)})$, for all components W of level m, we obtain $\tilde{\chi}_m$ by the Gluing Lemma, as in 2.

3.− Suppose that 2 holds for level $0 < m \leq k$. Let W be a piece of level $m-1$, so $\tilde{\chi}_m$ is K_1-qc in W. Since there is an iterate of R which is a biholomorphism between W and $V_{n-N}^{c(W)}$ for some $c = c(W) \in \mathrm{Crit} \cap J(R)$; it follows that $\|W - K(V_n)\|_W < \varepsilon$. Since χ and $\tilde{\chi}_m$ coincide in $W \cap K(V_n)$ and χ is conformal there, it follows that $\tilde{\chi}_m$ is also conformal in $W \cap K(V_n)$ (cf. part 2 of Gluing Lemma); so Lemma 5.4 applies to $\tilde{\chi}_m$ to give us a K_0-qc homeomorphism $\chi_W : W \longrightarrow \chi(W)$ at a bounded hyperbolic distance from $\tilde{\chi}_m$. Replacing $\tilde{\chi}_m$ in each such W, we obtain by the Gluing Lemma a qc homeomorphism χ_{m-1} that is K_0-qc in the pieces of level $m-1$. So 1 holds for $m-1$ and the induction is complete. □

6. Thurston's algorithm.

In this section we prove Theorem C about Thurston's algorithm. The first part of Theorem C follows from the existence of a nest as in Lemma 5.1. We give a more detailed version of the second part of Theorem C as Proposition C below. The proof of Proposition C is in Sections 6.1 and 6.2.

Let us recall Thurston's algorithm. We say that a map $\tilde{R} : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ is *quasiregular* if it is locally of the form $\xi(z^m)$, for some $m \geq 1$ and some qc homeomorphism ξ; see Appendix 10 for background in quasi-conformal maps. In particular a quasiregular map $\tilde{R}$ is a ramified covering of $\hat{\mathbb{C}}$. Let σ_0 be the standard complex structure of $\hat{\mathbb{C}}$ and let $\sigma_k = (\tilde{R}^*)^k(\sigma_0)$ be its pull-back under $\tilde{R}^k$. Consider the unique biholomorphism $h_k : (\hat{\mathbb{C}}, \sigma_k) \longrightarrow (\hat{\mathbb{C}}, \sigma_0)$ with an appropriated normalization, so $Q_k = h_{k-1} \circ \tilde{R} \circ h_k^{-1}$ is a rational map of the same degree as R; see Figure 1.

Proposition C. *Let $R \in \mathcal{S}$ be a rational map such that $J(R) \neq \hat{\mathbb{C}}$ and consider nests $V_n = \cup V_n^c$ and $U_n = \cup_v U_n^v$, for $c \in \mathrm{Crit} \cap J(R)$ and $v \in \mathrm{CV} \cap J(R)$, given by Lemma 5.1. Let n big and consider a quasiregular map $\tilde{R}$ of the same degree as R, that coincides with R outside V_{n+1} and so that for any ramification point r of $\tilde{R}$ we have $\tilde{R}(r) \in K(V_n)$. Consider Q_k and h_k as above, so that h_k fixes three preferred points of $K(V_n)$. Then the following assertions hold.*

- **Thurston's algorithm:** *There is a rational map Q and a continuous map $h : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$, so that $Q_k \to Q$ and $h_k \to h$ uniformly. Then h maps ramification points of $\tilde{R}$ to critical points of Q, preserving local degrees. Furthermore Q is close to R and h is close to the identity as n is big.*
- **Pseudo-conjugacy:** *Let $K \geq 1$ be given by Rigidity and let Q and h as above. Then there is a K-qc homeomorphism $\tilde{h}$ of $\hat{\mathbb{C}}$ that coincides with h in $K(V_n)$ and so that $h \circ R(w) = Q \circ h(w)$ for all $w \notin V_n$.*

Remark 6.1. (1) *In general h is not injective. For example, by making a small perturbation to some $\tilde{R}_0$, we may obtain $\tilde{R}$ with a saddle periodic point. So the semi-conjugacy h cannot be injective in this case.*
(2) *We remark that $\tilde{R}$ may have several ramification points in V_n^c, with the appropriated multiplicities. Thus the number of ramification points of $\tilde{R}$ may be strictly bigger than the number of critical points of R.*

Proof of Theorem C given Proposition C. Choose $\tau \in (0, 2^{-\frac{1}{3}})$ and positive integers N and N_0 such that for $\delta > 0$ small there is n such that for $v \in \mathrm{CV} \cap J(R)$,

$$U^v_{n+N_0} \subset B_\delta(v) \subset U^v_{n+1} \subset U^v_n \subset B_{2\delta}(v) \subset U^v_{n-N}.$$

Thus the part of Theorem C about Thurston's algorithm follows from the respective part of Proposition C. It reminds to prove the first part of Theorem C. By Lemma 5.1 the connected component W^v of $\hat{\mathbb{C}} - K(V_{n-N})$ containing $v \in \mathrm{CV} \cap J(R)$ is well inside $U^v_{n+N_0}$; see also Lemma 3.1. Consider a qc homeomorphism ξ^v of $U^v_{n+N_0}$ that extends to the identity in $\partial U^v_{n+N_0}$ and so that $\xi^v(v) \in \partial W^v$. Let ξ be the homeomorphism of $\hat{\mathbb{C}}$ that coincides with ξ^v in $U^v_{n+N_0}$ and is the identity otherwise. By the Gluing Lemma ξ is qc.

Put $\tilde{R} = \xi \circ R$ which is a quasiregular map. Note that $\tilde{R}$ coincides with R outside $V_{n+N_0} = \cup V^c_{n+N_0} \subset B_\delta(\mathrm{Crit} \cap J(R))$ and the ramification points of $\tilde{R}$ in V_{n+N_0} are the points in $\mathrm{Crit} \cap J(R)$. Let $c \in \mathrm{Crit} \cap J(R)$ and put $v = R(c)$, so

$$\tilde{R}(c) = \xi^v(v) \in \partial W^v \subset K(V_{n-N}).$$

Since $\tilde{R}$ coincides with R in $K(V_{n-N})$ it follows that the orbit of c by $\tilde{R}$ is contained in $K(V_{n-N})$. In particular it is disjoint from V_{n-N}, which contains $\tilde{B}_{2\delta}(\mathrm{Crit} \cap J(R))$. □

6.1. **Compactness and stabilization.** Consider a quasiregular map $\tilde{R}$ as in Proposition C, but with the stronger assumption that for every ramification point r of $\tilde{R}$ we have $\tilde{R}(r) \in K(V_{n-1})$, instead of $\tilde{R}(r) \in K(V_n)$. This is just a formally stronger property: consider a nest $\tilde{V}_m = \cup V^c_m$ so that $\tilde{V}_{2n} = V_n$ and then replace $n+1$, n and $n-1$ by $2n+2$, $2n+1$ and $2n$ respectively in the argument.

The proof of Proposition C is as follows. We first prove that, for $k \geq m$, the qc homeomorphism $h_k^{-1} \circ h_m$ is conformal almost everywhere in $h_m(\mathcal{F}^+_m)$, where $\mathcal{F}_m = \tilde{R}^{-m}(K(V_{n+1}))$; see Lemma 6.2. Given $m \geq 1$ we apply Rigidity to restrictions of the maps $h_k \circ h_m^{-1}$, for $k \geq m$, and using compactness of K-qc maps we construct a K-qc homeomorphism χ_m so that for $k \geq m$ the maps $\chi_k \circ h_k$ and $\chi_m \circ h_m$ coincide in $\mathcal{F}_m$; see Figure 3. Moreover χ_m is conformal Lebesgue almost everywhere in $h_m(\mathcal{F}_m)$, where $\mathcal{F}_m = \tilde{R}^{-m}(K(V_n))$.

In Section 6.2 we study the geometry of the sets $\hat{\mathcal{F}}_m = \chi_m \circ h_m(\mathcal{F}_m)$. We prove that the connected components of $\hat{\mathbb{C}} - \hat{\mathcal{F}}_m$ have small diameter as m is big; see Lemma 6.5. As consequence we obtain that the maps $\chi_{k-1} \circ Q_k \circ \chi_k^{-1}$ and $\chi_k \circ h_k$ converge uniformly to maps $Q : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ and $h : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ respectively; see Lemma 6.6. Then we prove that the Lebesgue measure of $\hat{\mathbb{C}} - \hat{\mathcal{F}}_m$ is small as m is big; see Lemma 6.7. Since χ_m^{-1} is conformal Lebesgue almost everywhere in $\hat{\mathcal{F}}_m = \chi_m(h_m(\mathcal{F}_m))$ it follows that χ_m converges to the identity as $m \to \infty$. We conclude that Q_m and h_m converge uniformly to Q and h respectively, so Q is a rational map of the same degree as R.

Lemma 6.2. *For $m \geq 0$ let $\mathcal{F}^+_m = \tilde{R}^{-m}(K(V_{n+1}))$. Then, for every $k \geq m$ the qc homeomorphism $h_k \circ h_m^{-1}$ of $\hat{\mathbb{C}}$ is conformal Lebesgue almost everywhere in $h_m(\mathcal{F}^+_m)$.*

Proof. By construction $K(V_{n+1})$ is forward invariant by $\tilde{R}$ and $\tilde{R}$ is conformal almost everywhere there. Therefore σ_0 and σ_{k-m} coincide almost everywhere in $K(V_{n+1})$. Thus $\sigma_m = (\tilde{R}^m)^*(\sigma_0)$ and $\sigma_k = (\tilde{R}^m)^*(\sigma_{k-m})$ coincide almost everywhere in $\mathcal{F}^+_m = \tilde{R}^{-m}(K(V_{n+1}))$. Since $h_m^{-1} : (\hat{\mathbb{C}}, \sigma_0) \longrightarrow (\hat{\mathbb{C}}, \sigma_m)$ and $h_k : (\hat{\mathbb{C}}, \sigma_k) \longrightarrow (\hat{\mathbb{C}}, \sigma_0)$ are holomorphic, it follows that $h_k \circ h_m^{-1}$ is conformal almost everywhere in $h_m(\mathcal{F}^+_m)$. □

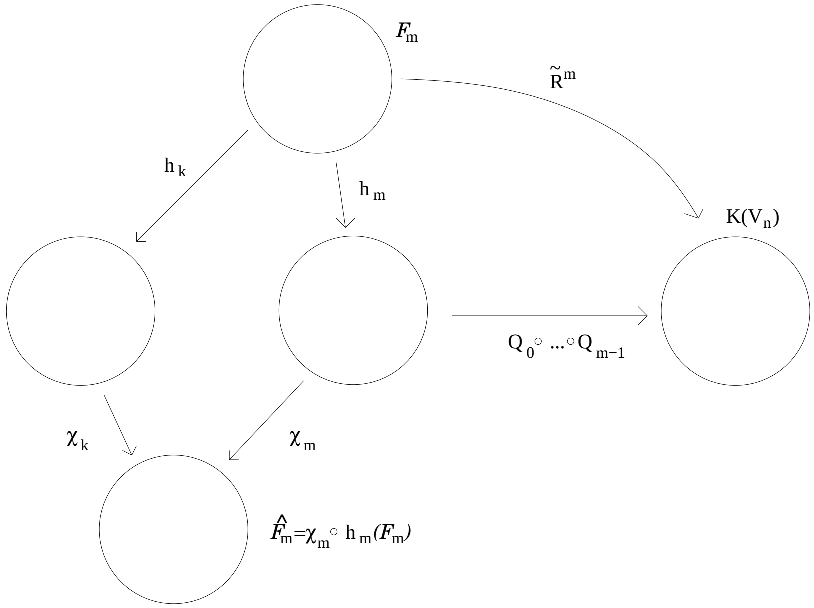

FIGURE 3. The limit maps χ_i.

Since $\tilde{R}(r) \in K(V_{n-1})$ for every ramification point of $\tilde{R}$, it follows that any pull-back of V_{n-1}^c, for $c \in \text{Crit} \cap J(R)$, by $\tilde{R}$ is univalent. Let $m \geq 0$ and let W be a connected component of $\hat{\mathbb{C}} - \mathcal{F}_m$, where $\mathcal{F}_m = \tilde{R}^{-m}(K(V_n))$. So there is $l \geq m$ and $c \in \text{Crit} \cap J(R)$ so that W is a pull-back of V_n^c by $\tilde{R}^l$. By the remark above it follows that the respective pull-back of V_{n-1}^c is univalent. We denote it by W^-.

Lemma 6.3. *Put $\mathcal{F}_m = \tilde{R}^{-m}(K(V_n))$ and let W be a connected component of $\hat{\mathbb{C}} - \mathcal{F}_m$. Then there is a biholomorphism $\varphi_W : V_{n-1}^c \longrightarrow h_m(W^-)$ such that $\varphi_W(V_n^c) = h_m(W)$,*

$$\varphi_W(V_{n-1}^c \cap K(V_n)) = h_m(W^- \cap \mathcal{F}_m) \text{ and}$$
$$\varphi_W(V_{n-1}^c \cap K(V_{n+1})) = h_m(W^- \cap \mathcal{F}_m^+).$$

Proof. Note that $\tilde{R}^m(W)$ (resp. $\tilde{R}^m(W^-)$) is a connected component of $\hat{\mathbb{C}} - K(V_n)$ (resp. $\hat{\mathbb{C}} - K(V_{n-1})$) and $\tilde{R}^m$ is univalent in W (resp. W^-). Therefore,

$$Q_1 \circ ... \circ Q_m : h_m(W^-) \longrightarrow \tilde{R}^m(W),$$

is a biholomorphism that maps $h_m(W)$, $h_m(\mathcal{F}_m)$ and $h_m(\mathcal{F}_m^+)$ to $\tilde{R}^m(W)$, $K(V_n)$ and $K(V_{n+1})$ respectively; see Figure 3. Since $\tilde{R}^m(W^-)$ is a connected component of $\hat{\mathbb{C}} - K(V_{n-1})$, there is $l \geq 1$ such that $R^l : W^- \longrightarrow V_{n-1}^c$ is a biholomorphism that maps $\tilde{R}^m(W)$, $W^- \cap K(V_n)$ and $W^- \cap K(V_{n+1})$ to V_n^c, $V_{n-1}^c \cap K(V_n)$ and $V_{n-1}^c \cap K(V_{n+1}^c)$ respectively. Then the lemma follows with $\varphi_W^{-1} = R^l \circ Q_1 \circ ... \circ Q_m$. $\square$

Let $K \geq 1$ and $C > 0$ be the constants given by Rigidity, that are independent of n; see Section 5.1. Let W be a connected component of $\hat{\mathbb{C}} - \mathcal{F}_m$ and let φ_W as in Lemma 6.3. By the previous lemma it follows that for every $k \geq m$,

$$h_k \circ h_m^{-1} \circ \varphi_W : V_n^c \longrightarrow h_k(W)$$

is a qc homeomorphism that is conformal almost everywhere in $V_n^c \cap K(V_{n+1})$. By Rigidity there is a K-qc homeomorphism at hyperbolic distance from $h_k \circ h_m^{-1} \circ \varphi_W$ bounded by C; see Section 5.1. It follows by the Gluing Lemma that there is a normalized K-qc homeomorphism $\chi_{k,m}$ of $\hat{\mathbb{C}}$ that coincides with $h_k \circ h_m^{-1}$ in $h_m(\mathcal{F}_m)$; see 1 of Remark 5.2.

By compactness of normalized K-qc homeomorphisms of $\hat{\mathbb{C}}$, it follows that there is a sequence $l_i \to \infty$ such that for every $m \geq 0$, the homeomorphisms $\chi_{l_i,m}$ converge uniformly to a normalized K-qc homeomorphism χ_m. Note that $\chi_{k,m}$ coincides with $h_k \circ h_m^{-1}$ in $h_m(\mathcal{F}_m)$ and it is holomorphic there. Therefore $\chi_{l_i,k}^{-1} \circ \chi_{l_i,m}$, and hence $\chi_k^{-1} \circ \chi_m$, coincides with $h_k \circ h_m^{-1}$ in $h_m(\mathcal{F}_m)$ and it is holomorphic there. Hence for $k \geq m$ the homeomorphisms $\chi_k \circ h_k$ and $\chi_m \circ h_m$ coincide in $\mathcal{F}_m$.

Considering that $\tilde{R} = h_{m-1}^{-1} \circ Q_m \circ h_m$, it follows that for $k \geq m$ the maps $\chi_{k-1} \circ Q_k \circ \chi_k^{-1}$ and $\chi_{m-1} \circ Q_m \circ \chi_m^{-1}$ coincide in $\hat{\mathcal{F}}_m = \chi_m \circ h_m(\mathcal{F}_m)$.

6.2. **Geometry in the limit.** In this section we complete the proof of Proposition C.

For $m \geq 0$ let $\hat{\mathcal{F}}_m = \chi_m \circ h_m(\mathcal{F}_m)$. We call a connected component $\hat{W}$ of $\hat{\mathbb{C}} - \hat{\mathcal{F}}_m$, for some $m \geq 0$, *component*. Note that there is a connected component W of $\hat{\mathbb{C}} - \mathcal{F}_m$ such that $\hat{W} = \chi_m \circ h_m(W)$. We denote $\hat{W}^- = \chi_m \circ h_m(W^-)$.

The *level* of $\hat{W}$ is the integer $l \geq 0$ such that $\tilde{R}^l : W \longrightarrow V_n^c$ is an homeomorphism, for some $c \in \text{Crit} \cap J(R)$. Since in this case $\tilde{R}^m(W)$ is a connected component of the complement of $\mathcal{F}_0 = K(V_n)$, it follows that $l \geq m$. Thus a connected component of $\hat{\mathbb{C}} - \hat{\mathcal{F}}_m$ has level at least l. Moreover a component of level l is a connected component of $\hat{\mathbb{C}} - \mathcal{F}_l$, but is not a connected component of $\hat{\mathbb{C}} - \hat{\mathcal{F}}_{l+1}$.

For $c \in \text{Crit} \cap J(R)$ put $\hat{V}^c = \chi_0(V_n^c)$ and $\hat{V}_-^c = \chi_0(V_{n-1}^c)$. We begin with a lemma analogous to Lemma 6.3.

Lemma 6.4. *Let $\hat{W}$ be a component of level l. Then there is $c = c(\hat{W}) \in \text{Crit} \cap J(R)$ and a K-qc homeomorphism $\psi_{\hat{W}} : \hat{W^-} \longrightarrow \hat{V}_-^c$ such that $\psi_{\hat{W}}(\hat{W}) = \hat{V}^c$ and*

$$\psi_{\hat{W}}(\hat{W}^- \cap \mathcal{F}_{l+1}) = \hat{V}_-^c \cap \hat{\mathcal{F}}_1.$$

Proof. By definition of level there is W and $c \in \text{Crit} \cap J(R)$ such that $\tilde{R} : W \longrightarrow V_n^c$ is an homeomorphism and such that $\chi_l \circ h_l(W) = \hat{W}$. Then

$$Q_1 \circ ... \circ Q_l : \chi_l^{-1}(W^-) \longrightarrow V_{n-1}^c$$

As in Lemma 6.3 the K-qc homeomorphism

$$\psi_{\hat{W}} = \chi_m^{-1} \circ Q_1 \circ ... \circ Q_l : \hat{W^-} \longrightarrow V_{n-1}^c$$

has the required properties. □

Note that the property $\partial U_n \subset K(V_{n-1})$ implies that if W_i is a connected component of $\hat{\mathbb{C}} - \mathcal{F}_{m_i}$, for $i = 0, 1$, such that $W_1 \subset W_0$ but $W_1 \neq W_0$, then $\overline{W_1^-} \subset W_0$. Hence, if $\hat{W}_0$ and $\hat{W}_1$ are different components such that $\hat{W}_1 \subset \hat{W}_0$, then $\overline{\hat{W}_1} \subset \hat{W}_0$.

Lemma 6.5. *For every $\varepsilon > 0$ there is $M > 0$ such that if $m \geq M$, then every connected component W of $\hat{\mathcal{F}}_m$ has diameter less than ε.*

Proof. 1.– Since a connected component of $\hat{\mathbb{C}} - \hat{\mathcal{F}}_m$ has level at least m, is enough to prove that components of high level have small diameter.

Let $\hat{W}$ be a component of big level. Let $\hat{W}_k = \hat{W} \subset \hat{W}_{k-1} \subset ... \subset \hat{W}_0$ be all different components that contain $\hat{W}_k$. Thus $\hat{W}_0$ is a connected component of $\hat{\mathbb{C}} - \hat{\mathcal{F}}_0$, let l_0 be the level of $\hat{W}_0$.

For $0 \leq i \leq k$ let $\psi_i : \hat{W}_i^- \longrightarrow V_-^{c_i}$ be the map given by Lemma 6.4 for $\hat{W}_i$, where $c_i \in \mathrm{Crit} \cap J(R)$. Thus, for $i < k$, we have that $\psi_i(\hat{W}_{i+1})$ is a connected component of $\hat{\mathbb{C}} - \mathcal{F}_1$. Let l_{i+1} be the level of this component. Thus $l = l_0 + l_1 + ... + l_k$. Then there are three cases.

Case 1.- l_0 is big. Since $\hat{\mathcal{F}}_0 = \chi_0(K(V_n))$, it follows that $\hat{W}_0$ has small diameter. Hence $W = W_k \subset W_0$ has small diameter.

Case 2.- There is $0 < i \leq k$ such that l_i is big. Then it follows that the diameter of $\hat{U}_i = \psi_{i-1}(\hat{W}_i) \subset \hat{V}^{c_{i-1}}$ is small. Thus the modulus of $\hat{V}_-^{c_{i-1}} - \overline{\hat{U}_i}$ is big. Since ψ_{i-1} is K-qc it follows that the modulus of $\hat{W}_{i-1}^- - \overline{\hat{W}_i}$ is big and therefore the diameter of $\hat{W}_i$ is small.

Case 3.- k is big. As remarked above for all $1 \leq i \leq k$ we have that $\hat{W}_i^- \subset \hat{W}_{i-1}$. Since

$$\psi_{i-1}(\hat{W}_i^- - \overline{\hat{W}_i}) = \hat{V}_-^{c_i} - \overline{\hat{V}^{c_i}}$$

has definite modulus and ψ_{i-1} is K-qc it follows that $\hat{W}_i^- - \overline{\hat{W}_i}$ has definite modulus. By Grotzsch inequality it follows that the modulus of $\hat{W}_0 - \overline{\hat{W}_k}$ is big, so $\hat{W} = \hat{W}_k$ has small diameter. □

Lemma 6.6. *The maps $\chi_{m-1} \circ Q_m \circ \chi_m^{-1}$ and $\chi_m \circ h_m$ converge uniformly to maps $Q : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ and $h : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ respectively.*

Proof. We prove the assertion about $\chi_m \circ h_m$, the other being similar. Let $\varepsilon > 0$ and let $M = M(\varepsilon)$ be given by the previous lemma. Recall that for $k \geq M$ we have that $\chi_k \circ h_k$ coincides with $\chi_M \circ h_M$ in $\mathcal{F}_M$. Thus, if W is a connected component of $\hat{\mathcal{F}}_M$, then $\chi_k \circ h_k(W) = \hat{W} = \chi_M \circ h_M(W)$, for $k \geq M$. By the previous lemma $diam(\hat{W}) < \varepsilon$, so the distance between $\chi_{k_0} \circ h_{k_0}$ and $\chi_{k_1} \circ h_{k_1}$, for k_0, $k_1 \geq M$, is at most ε. Thus $\chi_k \circ h_k$ is a Cauchy sequence, and hence a convergent one. □

Lemma 6.7. *The set $\cup_{m \geq 0} \hat{\mathcal{F}}_m$ has full Lebesgue measure in $\hat{\mathbb{C}}$. Hence the Lebesgue measure of $\hat{\mathbb{C}} - \hat{\mathcal{F}}_m$ is small as m is big.*

This lemma is based in the following one.

Lemma 6.8. *Fix $c \in \mathrm{Crit} \cap J(R)$ and fix $p_c \in A^c = \hat{V}_-^c - \overline{\hat{V}^c}$ contained in the interior of $\hat{\mathcal{F}}_1$. Then there is $\varepsilon > 0$ such that for every K-qc homeomorphism $\xi : \hat{A}^c \longrightarrow \xi(\hat{A}^c) \subset \mathbb{C}$ such that $\xi(\hat{A}^c)$ encloses 0 and $\xi(p_c) = 1$, the ball $\{|z-1| < \varepsilon\}$ is contained in $\xi(\hat{A}^c \cap \hat{\mathcal{F}}_1)$.*

Proof. Suppose not, so there is a sequence of K-qc homeomorphisms $\xi_i : \hat{A}^c \longrightarrow \xi_i(\hat{A}^c) \subset \mathbb{C}$ so that $\xi_i(\hat{A}^c)$ encloses 0, $\xi_i(p_c) = 1$ and so that the ball $\{|z-1| < \frac{1}{k_i}\}$ is not contained in $\xi_i(\hat{A}^c \cap \hat{\mathcal{F}}_1)$, where $k_i \to \infty$ as $i \to \infty$. By compactness of K-qc maps we may suppose that ξ_{k_i} converges uniformly to a K-qc homeomorphism $\xi : \hat{A}^c \longrightarrow \xi(\hat{A}^c)$. Hence $1 = \xi(p_c)$ does not belong to the interior of $\xi(\hat{A}^c \cap \hat{\mathcal{F}}_1)$, which is a contradiction since p_c was chosen in the interior of $\hat{\mathcal{F}}_1$. □

Proof of Lemma 6.7. Let $\hat{w} \notin \cup\hat{\mathcal{F}}_m$. Is enough to prove that $\hat{w}$ is not a Lebesgue density point of the complement of $\cup\hat{\mathcal{F}}_m$.

Let $... \subset \hat{W}_k \subset \hat{W}_{k-1} \subset ... \subset \hat{W}_0$ be all different components that contain $\hat{w}$. Note that there is infinitely many of them. Recall that for $i > 0$ we have $\overline{\hat{W}_i^-} \subset \hat{W}_{i-1}$. Hence by Grotzsch inequality $diam(\hat{W}_i)$ goes to zero as $i \to \infty$. By Lemma 6.4 for each $i \geq 0$ there is a K-qc map ψ_i that maps the annulus $\hat{A}_i = \hat{W}_i^- - \overline{\hat{W}_i}$ to the annulus $\hat{A}^{c_i} = \hat{V}_-^{c_i} - \overline{\hat{V}^{c_i}}$, for some $c_i \in \text{Crit} \cap J(R)$, and that maps points in $\hat{\mathcal{F}}_{l_i+1}$ to points in $\hat{\mathcal{F}}_1$, where l_i is the level of $\hat{W}_i$.

It follows by the previous lemma that for each $i \geq 0$ there is a ball in $\hat{\mathcal{F}}_{l_i+1}$ centered at $\psi_i^{-1}(p_{c_i})$ with radius at least of the order of $dist(\hat{w}, \psi_i^{-1}(p_{c_i}))$. Thus $\hat{w}$ is not a Lebesgue density point of $\hat{\mathbb{C}} - \cup\hat{\mathcal{F}}_m$. □

Proof of Proposition C. Recall that χ_m is a normalized K-qc homeomorphism of $\hat{\mathbb{C}}$ such that χ_m^{-1} is conformal Lebesgue almost everywhere in $\hat{\mathcal{F}}_m$. Since the Lebesgue measure of $\hat{\mathbb{C}} - \hat{\mathcal{F}}_m$ is small as m is big, it follows that χ_m^{-1} converges uniformly to the identity; see Lemma 10.3. By Lemma 6.6 the maps $\chi_{m-1} \circ Q_m \circ \chi_m^{-1}$ and $\chi_m \circ h_m$ converge uniformly to a maps $Q : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ and $h : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ respectively. Hence Q_m and h_m converges uniformly to Q and h respectively. In particular Q is a rational map of the same degree as R.

Note that h coincides with $\chi_1 \circ h_1$ in $\mathcal{F}_1 = \tilde{R}^{-1}(K(V_n))$ and this set contains all ramification points of $\tilde{R}$. Moreover $\mathcal{F}_1$ is connected; see Proposition 2.5. Since $\chi_1 \circ h_1$ is an homeomorphism, it follows that for every ramification point r of $\tilde{R}$ the local degree of Q at $\chi_1 \circ h_1(r)$ is at least equal to the local degree of $\tilde{R}$ at r. Since this is true for every ramification point r of $\tilde{R}$, it follows that $h(r)$ is a critical point of Q with the same local degree.

It reminds to prove the pseudo-conjugacy part. Note that by construction χ_0 is a normalized K-qc homeomorphism of $\hat{\mathbb{C}}$ that conjugates $\tilde{R}$, and therefore R, in $K(V_n)$ to Q in $\chi_0(K(V_n))$. But every connected component of $\hat{\mathbb{C}} - K(V_n)$ is univalently mapped by some iterate of R to some V_n^c. Since R is conformal we can redefine χ_0 in these connected components (different from the V_n^c) to obtain a normalized K-qc homeomorphism $\tilde{h}$ of $\hat{\mathbb{C}}$ that conjugates R to Q outside V_n. Then the proposition follows considering that $\tilde{h}$ coincides with χ_0 (and therefore with h) in $K(V_n)$. □

7. Measure and expansivity.

In this section we prove that rational maps in $\mathcal{S}$, whose Julia set in not $\hat{\mathbb{C}}$, have several expanding properties. For example we prove that these rational maps do not have irrationally indifferent cycles nor Herman rings (see Corollary 7.2 below) and we complete the proof of Theorem B by proving that the set of points that do not accumulate critical points under forward iteration has zero Lebesgue measure; see Proposition B in Section 4.2. Moreover in Section 7.1 we deal with parabolic periodic points and we prove property EAC stated in Section 3.2; see Proposition 7.8. In Section 7.2 we complete the proof of Theorem A by improving sets with Martens property.

The difficulty proving this properties is that the Decay of Geometry Condition (defining the class $\mathcal{S}$) imposes a condition on forward orbits that accumulate critical points.

But the properties that we prove in this section are related to forward orbits that do not accumulate critical points. So is not clear how to prove these facts directly.

The basic idea in the proofs is to use the pseudo-conjugacies given by Proposition C, of the previous section. Theorem C and Proposition C provide us with pseudo-conjugacies between the rational map in question and a rational map that does not have critical points in its Julia set; see Corollary C below. This later rational map satisfies the properties that we want to prove, and we use the pseudo-conjugacy to transport these properties.

So the results in this section will depend in the following immediate corollary of Proposition and Theorem C.

Corollary C. *Let $R \in \mathcal{S}$ be such that $J(R) \neq \hat{\mathbb{C}}$ and consider the nest $V_n = \cup V_n^c$ as in Lemma 5.1. Then, for every $n \gg 1$ there is a rational map Q_n with no critical points in $J(Q_n)$ and a qc homeomorphism χ_n of $\hat{\mathbb{C}}$ that conjugates the dynamics of R in $K(V_n)$ to the dynamics of Q_n in $\chi_n(K(V_n))$. Moreover $\chi_n(K(V_n))$ does not contains critical points of Q_n.*

Rational maps with with no critical points in the Julia set are well behaved. In fact by a Theorem of Fatou such a rational map cannot have irrationally indifferent cycles nor Herman rings; see for example [CG] or [Mil3]. So these rational maps can have at worst parabolic cycles.

These rational maps are also referred as *parabolic rational maps*. The Julia set of a parabolic rational map has zero Lebesgue measure; see [Lyu1]. Furthermore they admit an expanding metric defined in a neighborhood of the Julia set minus the parabolic points; see [LY]. It follows that parabolic rational maps satisfy condition EAC stated in Section 3.2.

Proposition 7.1. *Let $R,\ Q \in \mathbb{C}(z)$ be rational maps of degree at least two, let V be a neighborhood of* $\mathrm{Crit} \cap J(R)$ *and suppose that there is a qc homeomorphism χ of $\hat{\mathbb{C}}$ that conjugates R and Q outside V. If $\{p_1, ..., p_m\}$ is a repelling, attracting, parabolic, Siegel or Cremer cycle of R that is disjoint from $\overline{V}$, then $\{\chi(p_1), ..., \chi(p_m)\}$ is a cycle of Q of the same kind.*

Proof. Replacing R and Q by R^m and Q^m we may suppose that $m = 1$, so $p = p_1$ is a fixed point of R. By hypothesis χ conjugates R in a neighborhood of p to Q in a neighborhood of $\chi(p)$. Hence p is an attracting (resp. repelling) fixed point of R if and only if Q is an attracting (resp. repelling) fixed point of Q. Parabolic fixed points are characterized as non-attracting fixed points that attract an open set. Moreover Siegel fixed points are characterized as having arbitrarily small neighborhoods that are invariant by the map and a local inverse of the map. Since these properties are preserved by topological conjugacy it follows that p is parabolic (resp. Siegel) if and only if $\chi(p)$ is. It follows that p is Cremer if and only if $\chi(p)$ is. □

Corollary 7.2. *A rational map $R \in \mathcal{S}$ such that $J(R) \neq \hat{\mathbb{C}}$, does not have irrationally indifferent cycles nor Herman rings.*

Proof. Let p be a periodic point of R and let $n \gg 1$ so that the orbit of p is disjoint from $\overline{V_n}$. Let Q_n and χ_n be as in Corollary C, so Q_n does not have critical points in $J(Q_n)$ and therefore Q_n does not have irrationally indifferent cycles nor Herman rings; see [CG] or [Mil3]. By Proposition 7.1 p has the same nature as $\chi_n(p)$, so p is not irrationally indifferent.

Suppose that R has a Herman ring H with period k. Choose $n \gg 1$ so that the interior of $K(V_n)$ intersects H. Since $K(V_n)$ is forward invariant by R, it follows that $K(V_n)$ contains a sub annulus $A \subset H \cap K(V_n)$ so that $R^k : A \longrightarrow A$ is conjugated to an irrational rotation. Then $Q_n^k : \chi_n(A) \longrightarrow \chi_n(A)$ is conjugated to an irrational rotation. Thus $\chi_n(A)$ is contained in a Herman ring or in a Siegel disc of Q_n, but Q_n has neither of them. □

Corollary 7.3. *Consider a V_n pseudo-conjugacy χ between R and a rational map Q. Then $\chi(K(V_n) \cap J(R)) = \chi(K(V_n)) \cap J(Q)$.*

Proof. Since $K(V_n)$ is forward invariant and by the previous corollary it follows that a point $w \in K(V_n)$ belongs to the Fatou set of R if and only if $R^n(w)$ converges to an attracting or parabolic cycle contained in $K(V_n)$. As in the previous corollary $\chi(K(V_n))$ is disjoint form the Siegel discs and Herman rings of Q. So $\chi(w)$ belong to the Fatou set of Q if and only if its forward orbit converges to an attracting or parabolic cycle in $\chi(K(V_n))$. Thus $w \in K(V_n)$ belongs to the Fatou set of R if and only if $\chi(w)$ belongs to the Fatou set of Q. □

Proof of Theorem B. In Proposition B we proved that the set $\{z \in J(R) \mid \omega(z) \cap \mathrm{Crit} \neq \emptyset\}$ has zero Lebesgue measure, so it is enough to prove that the set of points in $J(R)$ that do not accumulate critical points under forward iteration has zero Lebesgue measure; see Section 4.2. Hence it is enough to prove that the sets $K(V_n) \cap J(R)$ have zero Lebesgue measure. If Q_n and χ_n are as in Corollary C, then by the previous corollary $\chi_n(K(V_n) \cap J(R)) \subset J(Q_n)$. Since Q_n has no critical points in $J(Q_n)$ and $J(Q_n) \neq \hat{\mathbb{C}}$ it follows that $J(Q_n)$ has zero Lebesgue measure; see [Lyu1]. Since qc homeomorphisms preserve sets of zero Lebesgue measure, the theorem follows. □

The following corollary will by needed in Section 8.

Corollary 7.4. *The set $\partial K(V_n)$ has zero Lebesgue measure.*

Proof. By Theorem B it follows that $K(V_n) \cap J(R)$ has 0 Lebesgue measure and by construction $\partial K(V_n) - J(R)$ has 0 Lebesgue measure; see Remark 4.2 in Section 4. □

The following lemma will be used in the next section to prove property EAC.

Lemma 7.5. *Let Q and R be rational maps and let $K \subset \tilde{K} \subset \hat{\mathbb{C}}$ be compact sets forward invariant by Q, such that $\tilde{K}$ is a non-trivial connected set and Q is uniformly expanding in K. Moreover suppose that χ is a qc homeomorphism of $\hat{\mathbb{C}}$ that conjugates the dynamics of Q in $\tilde{K}$ to the dynamics of R in $\chi^{-1}(\tilde{K})$ and $\chi^{-1}(K)$ does not contains critical points of R. Then R is uniformly expanding in $\chi^{-1}(K)$.*

Proof. Since $\chi^{-1}(K)$ does not contains critical points of R, there is $\varepsilon_0 > 0$ such that for all $w \in \chi(K)$ the distortion of R in $B_{\varepsilon_0}(w)$ is bounded by some $D > 1$, close to 1. Since Q is uniformly expanding in K there is $\varepsilon_1 > 0$ and $\lambda > 1$ such that for all $z \in K$ the pull-back W of $B_{\varepsilon_1}(Q^n(z))$ by Q^n to z is univalent and $diam(W) \leq C\lambda^{-n}$ for some definite $C > 0$; we assume $\varepsilon_1 > 0$ small enough so that $diam(\chi^{-1}(W)) < \varepsilon_0$. By the Hölder property of qc homeomorphisms there are $\alpha \in (0, 1)$ and $C_0 > 0$ be such that,

$$dist(\chi^{-1}(z_0), \chi^{-1}(z_1)) \leq C_0 dist(z_0, z_1)^{\alpha},\ z_0, z_1 \in \hat{\mathbb{C}};$$

see Appendix 10. Suppose that $\varepsilon_1 \ll diam(\tilde{K})$ and fix $z \in K$. Since $\tilde{K}$ is connected, for n big there is $w_n \in \partial B_{\frac{\varepsilon_1}{2}}(Q^n(z)) \cap \tilde{K}$, so $dist(\chi^{-1}(Q^n(z)), \chi^{-1}(w_n)) \sim 1$.

Consider the preimage y_n of w_n by Q^n in the connected component of $Q^{-n}(B_{\varepsilon_1}(Q^n(z)))$ that contains z. So $dist(z, y_n) \leq C\lambda^{-n}$ and therefore $dist(\chi(z), \chi(y_n)) \leq C_0 C^\alpha \lambda^{-\alpha n}$. So there is $C_1 > 0$ such that $|(R^n)'(z)| \geq C_1(\lambda^\alpha D^{-1})^n$. By taking ε_0 smaller if necessary we may assume that $\lambda^\alpha D^{-1} > 1$, so R is uniformly expanding in $\chi(K)$. ☐

7.1. **Parabolic periodic points and property EAC.** In this section we deal with parabolic points and we prove that rational maps $R \in \mathbb{C}(z)$, for which $J(R) \neq \hat{\mathbb{C}}$, satisfy property EAC stated in Section 3.2; see also Proposition 7.8 below. This property is used in the proof of Theorem A.

We begin with some concepts and notation; see Appendix A of [Sh] or [DH1] for references. Given $R \in \mathbb{C}(z)$ let $\mathcal{P} \subset J(R)$ be the set of parabolic points of R. Every parabolic point $p \in \mathcal{P}$ has associated a set $\Pi(p)$ of *repelling petals* $\pi \subset \hat{\mathbb{C}}$ that are open sets containing p in the boundary, for which there is a so called *Fatou coordinate* $\varphi_\pi : \pi \longrightarrow \{Rez < 0\}$ with the property that for every positive integer n such that $R^n(\pi) \cap \pi \neq \emptyset$ the map R^n is given by $z \longrightarrow z + n$ in the coordinate φ_π. We denote by $\Pi = \cup_{\mathcal{P}} \Pi(p)$ the set of all repelling petals. For $\pi \in \Pi$ and $k \geq 0$ we denote

$$\tilde{B}_{-k}(\pi) = \varphi_\pi(\{Rez < -k\}),$$

and for $p \in P$,

$$\tilde{B}_{-k}(p) = \{p\} \cup \left(\cup_{\Pi(p)} \tilde{B}_{-k}(\pi)\right),$$

which is a neighborhood of p in $J(R)$; usually k will be an integer. We may suppose that, if $\pi_0, \pi_1 \in \Pi$ are such that $R(\pi_0) \cap \pi_1 \neq \emptyset$, then $R(\tilde{B}_{-1}(\pi_0)) = \tilde{B}_0(\pi_1) = \pi_1$.

Lemma 7.6. *Let $R \in \mathcal{S}$ and fix $\delta > 0$ small. Let $r_K > 0$ be as in Koebe Distortion Theorem and let $r \in (0, r_K)$ and $w \in J(R)$. Then there are $\varepsilon_0 > 0$ and $C_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ and any univalent pull-back W of $B_r(w)$ disjoint from $\mathcal{P}$, such that the respective pull-back W_ε of $B_{\varepsilon r}(w)$ intersects $\tilde{B}_{-1}(\pi)$ for some $\pi \in \Pi$ (resp. $\tilde{B}_\delta(c)$ for some $c \in \mathrm{Crit} \cap J(R)$), then we have that $W_\varepsilon \subset \pi$ and $diam(\varphi_\pi(W_\varepsilon)) \leq C_0\varepsilon$ (resp. $diam(W_\varepsilon) \leq C_0 \varepsilon diam(\tilde{B}_\delta(c))$).*

Proof. The case when W_ε intersects $\tilde{B}_\delta(c)$ is simple is similar to part (i) of Lemma 3.1, so we restrict our attention to the other case.

Let $p \in \mathcal{P}$ be the parabolic point such that $\pi \in \Pi(p)$. By the local description of dynamics of parabolic points we know that the boundary of π makes a definite angle at p and that the image of $J(R)$ by φ_π lies in a strip of the form,

$$\{z \mid |Im(z)| < R_0 \text{ and } Re(z) < 0\};$$

see figure 4 and see [CG] or [Mil3] for references.

Hence the distance from any point $w_0 \in J(R) \cap \tilde{B}_{-1}(\pi)$ to the boundary of $\pi = \tilde{B}_0(\pi)$ is comparable to the distance from w_0 to p. Iterating if necessary we may assume that W_ε intersects $\tilde{B}_{-1}(\pi) - \tilde{B}_{-2}(\pi)$. Since W_ε intersects $J(R)$ there is a definite $\kappa > 0$ independent of ε such that $W_\kappa \subset \tilde{B}_{-1}(\pi)$, where W_κ is the respective pull-back of $B_{\kappa r}(w)$. We assumed that $\varphi_\pi(W_\varepsilon)$ intersects $\{z| -2 < Re(z) < -1\}$, so by Koebe Distortion Theorem there is $C_0 > 0$ such that $diam(\varphi_\pi(W_\varepsilon)) \leq C_0\varepsilon$. ☐

Lemma 7.7. *Let $R \in \mathcal{S}$ and let $\delta > 0$ be small. Consider a neighborhood $V = \cup_{\mathrm{Crit} \cap J(R)} V^c$ with Martens property given by Proposition 2.7, so that $diam(V^c) \sim \delta^{\frac{1}{\mu_c}}$. Let U be a connected component of $\hat{\mathbb{C}} - K(V)$ intersecting $\tilde{B}_{-1}(\pi)$ for some $\pi \in \Pi$. Then*

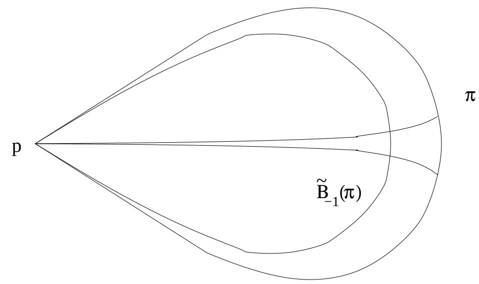

FIGURE 4. Local structure of petals. The cusp contains the part of Julia set in the petal.

$U \subset \pi$ and $diam(\varphi_\pi(U)) \le C_1(r(\delta))^{-\frac{1}{\mu_{max}}}$, for some $C_1 > 0$ independent of $\delta > 0$. In particular $diam(U) = o(dist(U, \mathcal{P}))$.

Proof. Let $m = m_W$ and $c = c(W) \in \mathrm{Crit} \cap J(R)$ be such that $R^m : U \longrightarrow V^c$ is a biholomorphism. By the Univalent Pull-back Condition the respective pull-back of $\tilde{B}_{\delta r(\delta)}(c)$ by R^m is univalent. Then we just apply the previous lemma to $w = c$, to $\tilde{B}_{\delta r(\delta)}(c)$ instead of $B_r(c)$, and with $\varepsilon = (r(\delta))^{-\frac{1}{\mu_c}}$. □

Proposition 7.8. *Let $R \in \mathcal{S}$ such that $J(R) \ne \hat{\mathbb{C}}$ and denote by $\mathcal{P}$ the set of parabolic points of R. Then R satisfies property EAC. That is, for every neighborhood V of $(\mathrm{Crit} \cap J(P)) \cup \mathcal{P}$ the rational map R is uniformly expanding in,*

$$K(V) = \{z \in J(P) \mid P^i(z) \notin V \text{ for } i \ge 0\}.$$

Proof. Recall that $K(V_n)$ is forward invariant by R and $K(V_n)$ is connected; see Lemma 2.5. For $n \gg 1$ let Q_n and χ_n as in Corollary C. So Q_n does not have critical points in $J(Q_n)$ and therefore it satisfies property EAC.

It follows by Lemma 7.7 that for every $p \in \mathcal{P}$ we may choose an arbitrarily small neighborhood V^p so that $\partial V^p \subset K(V_n)$. Put $V = V_n \cup (\cup_{\mathcal{P}} V^p)$. Since Q_n does not have critical points in $J(Q_n)$ and $\chi_n(K(V_n) \cap J(R)) \subset J(Q_n)$ it follows that Q_n is uniformly expanding in $\chi_n(K(V)) \subset \chi_n(K(V_n))$. By Lemma 7.5 applied to $K = \chi(K(V))$ and to $\tilde{K} = \chi(K(V_n))$ it follows that R is uniformly expanding in $K(V)$. Since this is for any n big, the proposition follows. □

The following immediate consequence of Proposition 7.8 was used in the proof of Theorem A in Section 3.2 (in the absence of parabolic periodic points).

Corollary 7.9. *Polynomials in the class $\mathcal{S}$ satisfy property EAC.*

7.2. **Improvement of Martens sets and proof of Theorem A.** In this section we complete the proof of Theorem A. Recall that in Section 3 we indicated the proof of Theorem A in the absence of parabolic points. The proof in the general case is the same, only that we have to improve Martens neighborhoods to include parabolic periodic points; see Proposition 7.11 below.

The following property is analogous to the Univalent Pull-back Condition, but for parabolic points; see Section 1.

Univalent Pull-back Property For Parabolic Points. *Let $R \in \mathcal{S}$ and let $\delta > 0$ be small. Then there is $\delta_{\mathcal{P}} > 0$ such that for any parabolic point $p \in \mathcal{P}$, any $k \geq 1$ and any $w \in R^{-k}(p)$ such that $R^{k-1}(w) \notin \mathcal{P}$ and such that $R^j(w) \notin \tilde{B}_\delta(\text{Crit} \cap J(R))$, for $0 \leq j \leq k$, we have that the pull-back of $\tilde{B}_{\delta_{\mathcal{P}}}(p)$ by R^k to w is univalent.*

Proof. We assume that $\mathcal{P} \neq \emptyset$, otherwise the statement is vacuous. So $J(R) \neq \hat{\mathbb{C}}$ and the considerations of Section 7 apply to R. Consider a nest $V_n = \cup V_n^c$ for $n \gg 1$ as in Lemma 5.1 and let n be such that $V_{n-1} \subset B_\delta(\text{Crit} \cap J(R))$. Consider the pseudo-conjugacy χ_n and the rational map Q_n given by Corollary C. So Q_n does not contains critical points in its Julia set.

1.– The following property of Q_n is a consequence of the existence of a expanding metric for Q_n, defined in some neighborhood of $J(Q_n) - \mathcal{P}$; see for example [LY].

Let W be a neighborhood of the set $\hat{\mathcal{P}}$ of parabolic periodic points of Q. Then for every $\delta_0 > 0$ there is $\hat{\varepsilon}(\delta_0) > 0$ such that for every set U intersecting $J(Q_n)$, disjoint from W and with diameter less than $\hat{\varepsilon}(\delta_0)$, we have that every pull-back of U has diameter less than δ_0.

2.– Let $\hat{p}$ be a parabolic point of Q_n and consider $\hat{w} \in Q_n^{-k}(\hat{p})$ such that $Q_n^{k-1}(\hat{w}) \notin \hat{\mathcal{P}}$. Let $\hat{\varepsilon} > 0$ small, to be chosen later. Consider the respective pull-back $\hat{W}_0$, ..., $\hat{W}_k = B_\varepsilon(\hat{p})$ by Q_n along the orbit of $\hat{w}$. Assume $\hat{\varepsilon} > 0$ small enough so that $\hat{W}_{k-1}$ is disjoint from a fixed neighborhood W of $\hat{\mathcal{P}}$. Choosing $\delta_0 < dist(\partial\chi_n(V_{n-1}), \chi_n(V_n))$ and assuming $\hat{\varepsilon} \leq \hat{\varepsilon}(\delta_0)$ (where $\hat{\varepsilon}(\delta_0)$ is as in 1) we have that, if $\hat{W}_0$ intersects $\partial\chi_n(V_{n-1})$, then $\hat{W}_0 \cap \chi_n(V_n) = \emptyset$.

3.– By Lemma 7.7 we may choose a neighborhood $\hat{U}$ of $\hat{p}$ contained in $B_\varepsilon(\hat{p})$ so that $\partial\hat{U} \subset \chi_n(K(V_n))$. Since χ_n is a V_n pseudo-conjugacy it follows from 2 that $U = \chi_n^{-1}(\hat{U})$ is such that for every pull-back W_0, W_1, ..., $W_k = U$, such that W_{k-1} does not contains $p = \chi_n^{-1}(\hat{p})$, we have that $W_0 \cap \partial V_{n-1} \neq \emptyset$ implies $W_0 \cap V_n = \emptyset$.

4.– Choose $\delta_{\mathcal{P}} > 0$ small enough so that $B_{\delta_{\mathcal{P}}}(p) \subset U$ for every $p \in \mathcal{P}$, where $U = U(p)$ as in 3. Fix $p \in \mathcal{P}$ and let $w \in R^{-k}(p)$ be such that $R^{k-1}(w) \notin \mathcal{P}$ and consider the respective pull-back W_0, W_1, ..., $W_k = B_{\delta_{\mathcal{P}}}(p)$, so that W_{k-1} does not contains p. Note that is enough to prove that the W_i are disjoint from V_n. Suppose that for some i the pull-back W_i intersects V_n. Since $R^i(w) \in W_i$ lies outside of V, that contains V_{n-1}, we have that W_i must intersect ∂V_{n-1}. Hence by 3 we have $W_i \cap V_n \neq \emptyset$, which contradicts our assumption. So the statement follows. □

We have the consequent decay of geometry; compare with part (ii) of Lemma 3.1.

Corollary 7.10. *There is $C_2 > 0$ such that for every $\delta > 0$ small there is $l = l(\delta) \gg 1$ such that the following property holds. For any parabolic periodic point $p \in \mathcal{P}$, any $k \geq 1$ and any $w \in R^{-k}(p)$ such that $R^i(w) \notin \tilde{B}_\delta(\text{Crit} \cap J(R))$, for $0 \leq i < k$, and such that the connected component W_0 of $R^{-k}(\tilde{B}_{-l}(p))$ containing w intersects $\tilde{B}_{-l}(\pi)$, for some $\pi \in \Pi$ (resp. $\tilde{B}_\delta(c)$ for some $c \in$ Crit $\cap J(R)$) we have that $W_0 \subset \pi$ and $diam(\varphi_\pi(W_0)) \leq C_2(r(\delta))^{-\frac{1}{\mu_{max}}}$ (resp. $diam(W_0) \leq C_2(r(\delta))^{-\frac{1}{\mu_{max}}} diam(V^c)$).*

Proof. Put $\varepsilon = (r(\delta))^{-\frac{1}{\mu_{max}}}$ and let $l = l(\delta)$ be big enough so that for every $p \in \mathcal{P}$ we have $\tilde{B}_{-l}(p) \subset B_{\varepsilon\delta_P}(p)$, where $\delta_{\mathcal{P}}$ is as above. Consider the pull-back W_0, ..., $W_k = \tilde{B}_{-l}(p)$ and the respective pull-back $\hat{W}_0$, ..., $W_k = B_{\delta_{\mathcal{P}}}(p)$. If $R^{k-1}(w)$ belongs to the orbit of p, then we have $W_{k-1} = \tilde{B}_{-(l-1)}(R^{k-1}(w))$, so we may assume that $R^{k-1}(w)$ is not in the orbit of p. Therefore, by the previous, the pull-back $\hat{W}_0$ is univalent. Then the corollary follows from Proposition 7.6. $\square$

The following proposition is an improvement of Proposition 2.7, that takes into account parabolic points.

Proposition 7.11. *Let $R \in \mathcal{S}$, let $\delta > 0$ be small and let $k = k(\delta)$ be given by the previous corollary. Then there is $C > 0$ only depending in R such that for every $c \in (\mathrm{Crit} \cap J(R))$ there are open sets V^c satisfying the properties of Proposition 2.7 and for every $p \in \mathcal{P}$ there is a set V^p such that*

$$\tilde{B}_{-k}(p) \subset V^p \subset \tilde{B}_{-k+\ln \eta(\delta)}(p),$$

where $\eta(\delta) = 1 + C(r(\delta))^{-\frac{1}{\mu_{max}}}$ such that if W is a pull-back of V^c, for some $c \in (\mathrm{Crit} \cap J(R)) \cup \mathcal{P}$, then either $\overline{W} \cap \overline{V} = \emptyset$ or $\overline{W} \subset V$, where $V = \cup_{(\mathrm{Crit} \cap J(R)) \cup \mathcal{P}} V^c$.

Proof. The proof is exactly the same as the proof of Proposition 2.7in Section 2.2, since we have univalent pull-back property (stated above) and decay of geometry given by Lemma 7.7 and Corollary 7.10. $\square$

Consider $V = \cup_{(\mathrm{Crit} \cap J(R)) \cup \mathcal{P}} V^c$ as in the previous proposition. As in Lemma 2.5 we have $\partial V \subset K(V)$. By Corollary 7.9, if $P \in \mathcal{S}$ is a polynomial, then P is uniformly expanding in $K(V) \cap J(R)$ and therefore every point in $K(V) \cap J(R)$ is the landing point of some ray; see Landing Lemma in Section 3.1. In particular every point in ∂V is the landing point of some ray.

Proof of Theorem A. We proceed as in the proof of the case when there are no parabolic cycles; see Section 3.2. We just replace the neighborhoods with Martens property given by Proposition 2.7 by the neighborhoods given by the previous Proposition 7.11 and we replace the set $\mathrm{Crit} \cap J(P)$ by $(\mathrm{Crit} \cap J(P)) \cup \mathcal{P}$ in the argument, to prove that the puzzle ends corresponding to critical points in $J(P)$ are singletons.

Then the proof follows as before, that is, we use Lemma 3.1 to prove that puzzle ends of points whose orbit accumulate critical points are singletons and we use the standard argument to prove that the rest of the puzzle ends are singletons; see [H], [Mil2], [Lyu2] and see [Ki] for puzzles in the presence of parabolic points. $\square$

8. Unicritical polynomials.

This section is dedicated to the family of polynomials $P_c(z) = z^d + c$, for $c \in \mathbb{C}$ and for a given $d \geq 2$. We prove Theorem D about instability in the parameter and Theorem E about similarities between the connectedness locus $\mathcal{M}_d$ and Julia sets.

We begin with Section 8.1 where we review some basic properties of holomorphic motions compatible with dynamics. Basic facts about holomorphic motions are stated in Appendix 10. Section 8.2 contains the proof of Theorem D. The proof is based in the concept of pseudo-conjugacies that appears in [Lyu2] and we use Rigidity several times.

The proof of Theorem D is independent of Sections 6 and 7, except that we use that polynomials in $\mathcal{S}$ satisfy property EAC stated in Section 3.2. Recall that by Corollary 7.9 polynomials in $\mathcal{S}$ satisfy this property. This is also proven in [R-L3]. Moreover Przytycki proved this property for rational maps that satisfy the summability condition with exponent one; see proof of Lemma 3.1 of [Pr2].

The remaining sections are dedicated to the proof of Theorem E. The idea is to transport dynamical data to parameter plane via holomorphic motions. This part is largely inspired by [Lyu3]. In Sections 8.3 and 8.4 we construct, for each polynomial $P_{c_0} \in \mathcal{S}$, a parameter map from the dynamical plane to the parameter plane. Then in Section 8.5 we prove Theorem E by proving that, under the hypothesis of Theorem E, this parameter map is conformal at the critical value (and C^{1+}-conformal in the Collet-Eckmann case). For this we use a conformality criterion stated in [LV] and McMullen's measurable deep points to obtain C^{1+}-conformality.

8.1. **Holomorphic motions compatible with dynamics.** In this section we consider some basic properties of holomorphic motions compatible with dynamics. Basic facts about holomorphic motions are stated in Appendix 10.

First let us recall the notion of Botcher coordinates. For every $c \in \mathbb{C}$ there is a coordinate φ_c that conjugates P_c near infinity to the map z^d and that is tangent to the identity at infinity. Such coordinate is called *Botcher coordinate* and it can be defined so that it depends holomorphically in c. In the case that $J(P_c)$ is connected, this coordinate can be extended to $\mathbb{C} - K(P_c)$ and it is determined as the unique biholomorphism between $\hat{\mathbb{C}} - K(P_c)$ to $\hat{\mathbb{C}} - \overline{\mathbb{D}}$ that is tangent to the identity at infinity.

Fix a parameter $c_0 \in \mathbb{C}$. Then an holomorphic motion $i : W \times K \longrightarrow \mathbb{C}$ with base point $c_0 \in W$ it is said to be *compatible with dynamics* if for all $z \in K$ such that $P_{c_0}(z) \in K$ we have $i_c(P_{c_0}(z)) = P_c(i_c(z))$, for all $c \in W$.

Lemma 8.1. *Let $W \subset \mathbb{C}$ be a connected open set, $c_0 \in W$ and $U \subset \mathbb{C} - K(P_{c_0})$ an open set invariant by P_{c_0}. If $i : W \times U \longrightarrow \mathbb{C}$ is an holomorphic motion compatible with dynamics, with base point c_0, then $i_c : U \longrightarrow \mathbb{C}$ is holomorphic for every $c \in W$.*

Proof. Note that we may suppose W bounded. Consider a neighborhood U_∞ of ∞ forward invariant by P_{c_0} so that φ_{c_0} is defined in U_∞ and φ_c^{-1} is defined in $\varphi_{c_0}(U_\infty)$, for all $c \in W$. Since W is connected it follows that for any $(c, z) \in W \times (U \cap U_\infty)$ we have $i_c(z) = \varphi_c^{-1} \circ \varphi_{c_0}(z)$, so i_c is holomorphic in U_∞.

For $(c, z) \in W \times U$ we can find a neighborhood $\hat{U}$ of z so that there is n for which $P_{c_0}^n(\hat{U}) \subset U$ and $P_{c_0}^n : \hat{U} \longrightarrow P_c^n(\hat{U})$ is a biholomorphism. It follows that P_c^n is univalent in $i_c(\hat{U})$, denote by g_c its inverse branch. Then i_c is given by the holomorphic map $g_c \circ i_c \circ P_{c_0}^n$ in $\hat{U}$. □

Lemma 8.2. *Let $c_0 \in \mathbb{C}$ and suppose that $K \subset \mathbb{C}$ is closed and forward invariant by P_{c_0} and that P_{c_0} is uniformly expanding in K. Then there is $r > 0$ and an holomorphic motion $i : B_r(c_0) \times K \longrightarrow \mathbb{C}$ compatible with dynamics, so that $i_{c_0} \equiv id$.*

Note that K is not assumed to be compact.

Proof. For the compact subset of K of points with bounded orbit, this follows by the expansive property; see [Sh]. For the unbounded case fix an invariant neighborhood U_∞ of ∞, so there is an holomorphic motion of U_∞ defined in a neighborhood of c_0, that is compatible with dynamics. Considering that P_{c_0} is uniformly expanding in K, there is a unique way to pull-back this holomorphic motion to an holomorphic motion of $U_\infty \cup K$, reducing the domain of definition if necessary. □

Lemma 8.3. *Let $i : W \times K \longrightarrow \mathbb{C}$ by an holomorphic motion. Moreover suppose that $(c_0, z_0) \in W \times K$ is such that $c_0 = i_{c_0}(z_0)$, but $i_c(z_0) \not\equiv c$. Then for $z \in K$ close to z_0 there is $c \in W$ close to c_0 such that $c = i_c(z)$.*

Proof. Consider the holomorphic motion $j_c(w) = i_c(w) - i_c(z_0)$, so $j_c(z_0) \equiv 0$. Reducing W and K if necessary we may assume that $j(W \times K) \subset B_R(0)$, for some $R > 0$. Let $\psi : W \longrightarrow \mathbb{C}$ be defined by $\psi(c) = c - i_c(z_0)$, so $\psi(c_0) = 0$ but $\psi \not\equiv 0$ by hypothesis; let $m \geq 1$ be such that $\psi(c) \sim (c - c_0)^m$. Note that for $z \in K$ close to z_0 we have to find c close to c_0 such that $j_c(z) = \psi(c)$.

Let $\gamma \subset W$ be the path around c_0 which is the preimage of $\{w \mid |w| = 2|j_{c_0}|\}$ by ψ. So $\psi : \gamma \longrightarrow \{w| \, |w| = 2|j_{c_0}(z)|\}$ is of degree m and there is $C > 0$ such that $\gamma \subset B_{C|j_{c_0}(z)|^{\frac{1}{m}}}(c_0)$. Since $|j_{c_0}(z)|^{\frac{1}{m}} \ln |j_{c_0}(z)| \ll 1$ it follows by Lemma 10.4 that there is $\kappa > 0$ such that for all $c \in \gamma \subset B_{C|j_{c_0}(z)|^{\frac{1}{m}}}(c_0)$,

$$|j_c(z) - j_{c_0}(z)| \leq \kappa |c - c_0| |j_{c_0}(z)| \ln(|j_{c_0}(z)|^{-1}) \leq \kappa_1 |j_{c_0}(z)|^{1+\frac{1}{m}} \ln(|j_{c_0}(z)|^{-1}),$$

for some $\kappa_1 > 0$. So if z is close enough to z_0 we have that $|j_c(z)| < 2|j_{c_0}(z)|$ for all $c \in \gamma$. By Rouche's theorem it follows that there are $m \geq 1$ parameters c inside γ for which $j_c(z) = \psi(c)$. □

8.2. **Pseudo-conjugacies and instability.** In this section we prove Theorem D asserting that, if $P_{c_0} \in \mathcal{S}$ has recurrent critical point, then for every non-trivial connected set $\xi \subset \mathbb{C}$ containing c_0 there is a parameter $c \in \xi$ such that the critical point of P_c is not recurrent.

We fix a parameter $c_0 \in \partial \mathcal{M}_d$ in the class $\mathcal{S}$ throughout all this section. Consider the nest V_n as in Lemma 5.1. Since 0 is the only critical point in $J(P_{c_0})$ we have that $V_n = V_n^0$ and $U_n = U_n^0$ are connected and moreover $V_n = P_{c_0}^{-1}(U_n)$.

All qc homeomorphisms of $\mathbb{C}$ that we will consider will be conformal in a definite neighborhood of infinity. The normalization of such homeomorphisms will be to be tangent to the identity at infinity and to fix 0; see Appendix 10.

Definition 8.4. *Let $n \gg 1$. Then a V_n* **pseudo-conjugacy** *between P_{c_0} and P_c is a normalized qc homeomorphism χ of $\mathbb{C}$ that is conformal in int$(K(V_n))$ and that conjugates the dynamics of P_{c_0} in $K(V_n)$ to that of P_c in $\chi(K(V_n))$.*

By redefining a V_n pseudo conjugacy in the complement of $K(V_n)$, we may suppose that it conjugates the dynamics of P_{c_0} and P_c outside V_n; without increasing its dilatation.

If χ is a V_n pseudo conjugacy between P_{c_0} and P_c, then $P_c : \chi(\partial V_n) \longrightarrow \chi(\partial U_n)$ is of degree d. Thus the critical point of P_c belongs to $\chi(V_n)$ or equivalently $c \in \chi(U_n)$. In particular $\chi(K(V_n))$ does not contains 0 and there for the forward orbit under P_c of every point in $\chi(K(V_n))$ does not accumulate 0. It follows that, if P_c is such that $c \in \chi(K(V_n))$, then the critical point of P_c is not recurrent.

Lemma 8.5. *Let $K \geq 1$ be given by Rigidity for the nest V_n. Then for every V_n pseudo-conjugacy χ there is a K-qc V_{n-1} pseudo-conjugacy that coincides with χ in $K(V_{n-1})$.*

Proof. By Corollary 7.4, $\partial K(V_n)$ has zero Lebesgue measure, thus a V_n pseudo-conjugacy is conformal Lebesgue almost everywhere in $K(V_n)$. Hence, we can apply Rigidity to χ

restricted to every connected component of $\mathbb{C}-K(V_{n-1})$. Applying the Gluing Lemma we obtain that there is a K-qc V_{n-1} pseudo-conjugacy $\hat{\chi}$ that coincides with χ in $K(V_{n-1})$. We assumed that $\hat{\chi}(0)=0=\chi(0)$, which is not a priori warranted by Rigidity, but by part 2 of Remark 5.2 in Section 5.1 we may assume so. $\square$

Lemma 8.6. *Let $W_n \subset \mathbb{C}$ be the set of all parameters c for which there is a V_n pseudo-conjugacy between P_{c_0} and P_c. Then $diam(W_n) \to 0$ as $n \to 0$.*

Proof. By the previous lemma, for $c \in W_n$ there is a K-qc V_{n-1} pseudo-conjugacy χ between P_{c_0} and P_c that is conformal in $K(V_{n-1})$. By Proposition B the Lebesgue measure of $\mathbb{C}-K(V_n)$ goes to zero as $n \to \infty$; see Section 4.2. Since $K \geq 1$ is independent of n, it follows by Lemma 10.3 that for every $\varepsilon > 0$ there is $N \gg 1$ such that if $n \geq N$ then $c \in \chi(U_n) \subset B_\varepsilon(U_n)$. Thus $W_n \subset B_\varepsilon(U_n)$. Since $diam(U_n) \to 0$ as $n \to \infty$ we have that $diam(W_n) \to 0$ as $n \to \infty$. $\square$

Lemma 8.7. *Let $c \in W_n$ and χ_c be a V_n pseudo-conjugacy between P_{c_0} and P_c. Then there is $r > 0$ and a holomorphic motion*

$$i : B_r(c_0) \times \mathbb{C} \longrightarrow \mathbb{C}$$

compatible with dynamics in $B_r(c_0) \times K(V_n)$, so that for all $\hat{c} \in W_n$ the map $i_{\hat{c}} : \mathbb{C} \longrightarrow \mathbb{C}$ is a V_n pseudo-conjugacy between P_{c_0} and P_c. Moreover $i_c \equiv \chi_c$ in $K(V_n)$. In particular W_n is an open set.

Proof. By Corollary 7.9, P_{c_0} is uniformly expanding in $K(V_n)$ and by Lemma 7.5 P_c is uniformly expanding in $P_c(K(V_n))$. Thus it follows by Lemma 8.2 there is an holomorphic motion $i : B_r(c) \times K(V_n) \longrightarrow \mathbb{C}$ which is compatible with dynamics and such that $i_c = \chi_c|_{K(V_n)}$. Then the lemma follows by Slodkowsky Extension Theorem; see Appendix 10. $\square$

The following lemma is an immediate consequence of compactness of normalized qc homeomorphisms.

Lemma 8.8. *Let $c \in \overline{W_n}$ and $\{c_i\}_{i \geq 1} \subset W_n$ such that $c_i \to c$ as $i \to \infty$. Moreover let χ_i be a K-qc V_{n-1} pseudo-conjugacy between P_{c_0} and P_{c_i}. Taking a sub sequence if necessary we may assume that χ_i converges uniformly to some χ. Then χ is a K-qc V_{n-1} pseudo-conjugacy between P_{c_0} and P_c.*

Lemma 8.9. *We have that $c \in W_n$ (resp. ∂W_n) if and only if there is a K-qc V_{n-1} pseudo-conjugacy between P_{c_0} and P_c such that $c \in \chi(U_n)$ (resp. $c \in \chi(\partial U_n)$).*

It follows by this lemma that if $c \in \partial W_n$, then the critical point 0 is not recurrent under P_c. Indeed if χ is a V_{n-1} pseudo conjugacy given by the lemma, then $P(0) = c \in \chi(\partial U_n) \subset \chi(K(V_n))$.

Proof. If $c \in W_n$ then the lemma follows by previous observations. On the other hand, if χ is a K-qc V_{n-1} pseudo-conjugacy between P_{c_0} and P_c with $c \in \chi(U_n)$, then consider the holomorphic motion $i : B_r(c_0) \times \mathbb{C} \longrightarrow \mathbb{C}$, given by Lemma 8.7. Taking r smaller if necessary we may assume that for all $\hat{c} \in B_r(c)$, we have $\hat{c} \in i_{\hat{c}}(U_n)$. Consider the restriction j of i to $B_r(c) \times K(V_{n-1})$. Extend j to $B_r(c) \times K(V_n)$, by a pull-back procedure. By Slodkowsky Extension Theorem, we may suppose that j is defined in $B_r(c) \times \mathbb{C}$, so for all $\hat{c} \in B_r(c)$, the map $j_{\hat{c}}$ is a V_n pseudo-conjugacy between P_{c_0} and P_c. Hence $c \in W_n$.

If $c \in \partial W_n$, the existence of a K-qc pseudo-conjugacy χ so that $c \in \chi(\overline{U_n})$ follows from Lemma 8.8. By the previous $\chi(c) \notin \chi(U_n)$ since $c \notin W_n$, so $\chi(c) \in \chi(\partial U_n)$. On the other hand suppose that c is such that there is a K-qc V_{n-1} pseudo-conjugacy χ between P_{c_0} and P_c such that $c \in \chi(\partial(U_n))$. By the previous $c \notin W_n$. Consider $r > 0$ and the holomorphic motion $i : B_r(c) \times \mathbb{C} \longrightarrow \mathbb{C}$ as in Lemma 8.7. By Lemma 8.3, for all $\hat{z} \in U_n$ close to $\chi^{-1}(c) \in \partial U_n$ there is $\hat{c} \in B_r(c)$ close to c so that $\hat{c} = i_{\hat{c}}(z) \in i_{\hat{c}}(U_n)$. By the previous $\hat{c} \in W_n$. Hence $c \in \overline{W_n} - W_n = \partial W_n$ as claimed. □

Proof of Theorem D. Let ξ be a connected set containing c_0. By Lemma 8.6 $diam(W_n) \to 0$ as $n \to \infty$ and since W_n contains c_0, it follows that there is $n \gg 1$ such that $\xi \cap \partial W_n \neq \emptyset$. By the previous lemma, if $c \in \xi \cap \partial W_n$, then the critical point of P_c is not recurrent. □

8.3. **Full neighborhoods.** Recall that the open set W_n is the set of all parameters $c\mathbb{C}$ for which there is a V_n pseudo-conjugacy between P_{c_0} and P_c. The aim of this section is to prove the following proposition.

Proposition (Full Neighborhoods). *There is an homomorphic motion $i_n : W_n \times \mathbb{C} \longrightarrow \mathbb{C}$ compatible with dynamics in $W_n \times K(V_n)$, so that $i_{c_0} \equiv id$ and so that for every $c \in W_n$ the homeomorphism i_c is a V_n pseudo-conjugacy between P_{c_0} and P_c.*

It follows by this proposition that the sets V_n are *full* in the sense that W_n is the maximal domain where $K(V_n)$ can be extended in such holomorphic motion. In fact, if such an holomorphic motion i_n can be extended to $c \in \partial W_n$, then by Lemma 8.9 we should have $c \in (i_n)_c(\partial U_n)$. But this implies every one of the $d > 1$ preimages w of $(i_n)_c^{-1}(c)$ by P_{c_0} is mapped to 0 by i_n, which is not possible since $(i_n)_c$ should be injective.

The proof of this proposition is at the end of this section and it depends in two lemmas.

Lemma 8.10. *Fix n big. Suppose that $c_1, c_2 \in W_n$ are such that there are V_n pseudo-conjugacies χ_i between P_{c_0} and P_{c_i} such that $\chi_1^{-1}(c_1) = \chi_2^{-1}(c_2) \in K(V_n)$. Then $c_1 = c_2$.*

Proof. Let $V^i = \chi_i(V_n)$ for $i = 1, 2$ and put $\chi = \chi_2 \circ \chi_1^{-1}$. By hypothesis V^i is disjoint from the postcritical set of P_{c_i} and therefore, for every $m \geq 0$ and any connected component W of $P_{c_i}^{-m}(V^i)$ we have that $P_{c_i}^m : W \longrightarrow V^i$ is univalent. As before, by redefining χ outside V^1, we may suppose that χ conjugates P_{c_1} to P_{c_2} outside V^1.

For $k \geq 0$ we will define inductively a qc homeomorphism χ^k that conjugates P_{c_1} to P_{c_2} outside $P_{c_1}^{-k}(V^1)$, whose dilatation is bounded by that of χ. Put $\chi^0 = \chi$ and suppose that χ^k is already defined for some $k \geq 0$. For any connected component W of $P_{c_1}^{-(k+1)}(V^1)$, the polynomial P_{c_1} is univalent in W, so we may redefine χ^k in W so that the conjugacy equation $P_{c_2} \circ \chi^{k+1} = \chi^{k+1} \circ P_{c_1}$ is satisfied in W. By the Gluing Lemma such χ^{k+1} is qc and its dilatation is bounded by that of χ^k.

It follows that the dilatation of the χ^k is bounded by that of χ and therefore there is a subsequence $k_i \to \infty$ so that χ_{k_i} converges uniformly to a qc homeomorphism χ^∞. Since the diameters of the connected components of $P_{c_1}^{-k}(V^1)$ converge to zero as $k \to \infty$ it follows that χ^∞ is a qc conjugacy between P_{c_1} and P_{c_2}. Moreover χ^∞ is conformal outside J_{c_1}.

Since $\chi_1(c_1) \in K(V_n)$ it follows that the critical point of P_{c_1} is not recurrent, so J_{c_1} has zero Lebesgue measure. Thus χ^∞ is conformal Lebesgue almost everywhere and therefore χ^∞ must be the identity. Hence $c_1 = c_2$. □

Lemma 8.11. *Given n big and $z \in \overline{U_n} \cap K(V_{n-1})$ there is a unique $c = \psi(z) \in \overline{W_0}$ for which there is a K-qc V_n pseudo-conjugacy χ_c between P_{c_0} and P_c with $c = \chi_c(z)$.*

Moreover ψ is continuous in $\overline{U_n} \cap K(V_{n-1})$ and $\psi(\partial U_n) = \partial W_n$. In particular $\partial W_n = \psi(\partial U_n)$ is connected and therefore W_n is simply-connected.

Proof. The existence of $c = \psi(z)$ for $z \in U_n \cap K(V_{n-1})$ is given by Proposition C. For $z \in \partial U_n \subset K(V_{n-1})$ follows considering that z is accumulated by points in $U_n \cap K(V_{n-1})$ and then by considering the limiting pseudo-conjugacy as in Lemma 8.8.

Uniqueness of c follows by the previous lemma and that $\psi(\partial U_n) = \partial W_n$ follows by Lemma 8.9. It reminds to prove that ψ is continuous. Let $z \in \overline{U_n} \cap K(V_{n-1})$ put $c = \psi(z)$ and consider $r > 0$ and the holomorphic motion $i : B_r(c) \times K(V_{n-1}) \longrightarrow \mathbb{C}$ given by Lemma 8.7. If $\hat{z} \in B_r(c)$ is close to c, by Lemma 8.3 there is $\hat{c}$ close to c so that $i_{\hat{c}}(\hat{z}) = \hat{c}$. So, if $\hat{z} \in \overline{U_n} \cap K(V_{n-1})$, then $\psi(\hat{z}) = \hat{c}$ is close to $\psi(z) = c$. □

Proof of the proposition. Let us measure distances in W_n with respect to the hyperbolic distance of W_n, so for $r > 0$ the set $B_r(c_0) \subset W_n$ denotes the hyperbolic ball with radius r centered at c_0. Let r be the supremum of the numbers r for which there is an holomorphic motion, as in the proposition, defined in $B_r(c_0)$. So we want to prove that $r = \infty$.

Suppose by contradiction that $r < \infty$. By Lemma 8.7 there is $\varepsilon > 0$ such that for all $c \in \partial B_r(c_0)$ there is an holomorphic motion j defined in $B_\varepsilon(c) \times K(V_n)$ compatible with dynamics and such that $j_c \equiv i_c|_{K(V_n)}$. Since $B_\varepsilon(c) \cap B_r(c_0)$ is connected it follows that i coincides with j in $B_\varepsilon(c) \cap B_r(c_0)$ and therefore, using Slodkowsky Extension Theorem, i extends to $B_\varepsilon(c)$. Repeating this argument we see that if we take a $\frac{\varepsilon}{2}$-dense set c_1, ..., c_n in $\partial B_r(c_0)$, then i extend to $B_{r+\frac{\varepsilon}{2}}(c_0) \subset B_r(c_0) \cup (\cup_{1 \le i \le n} B_\varepsilon(c_i))$. This contradicts the definition of r, so we must have $r = \infty$ and the proposition follows. □

8.4. **Parameter map.** The objective of this section is to construct a qc map ψ from the dynamical plane to the parameter plane, so that it *almost* maps the Julia set to $\mathcal{M}_d$. The main ingredient is the following easy consequence of the previous section. So let us consider the holomorphic motion i_n as in the previous section.

Proposition 8.12. *Fix n so that i_n is defined. Then for every $z \in U_n$ there is a unique $\psi_n(z) \in W_n$ such that $\psi_n(z) = i_{\psi_n(z)}(z) \in W_n$. The map $\psi_n : U_n \longrightarrow W_n$ is a locally qc homeomorphism that is conformal Lebesgue almost everywhere in $U_n \cap K(V_n)$. In particular a point $z \in U_n \cap K(U_n)$ belongs to $J(P_{c_0})$ if and only if $\psi_n(z) \in \mathcal{M}_d$.*

The proof of this proposition is at the end of this section. Now we define a *parameter map.*

Note that for every connected component W of $U_n - K(V_{n-1})$ there is $m \ge 1$ such that $P_{c_0}^m : W \longrightarrow V_n$ is a biholomorphism. Moreover the closure of such W is contained in U_n, so ψ is qc in such W and therefore we can apply Rigidity to ψ_n restricted to W. It follows by the Gluing Lemma and by compactness of K-qc homeomorphisms that there is a K-qc homeomorphism $\tilde{\psi}_n : U_n \longrightarrow W_n$ that coincides with ψ in $U_n \cap K(V_{n-1})$. In particular $\tilde{\psi}_n(U_{n+1}) = W_{n+1}$.

Let $n_0 \ge 0$ so that i_n is defined for $n \ge n_0$ and put $U = U_{n_0}$. By the Gluing Lemma and by compactness of K-qc homeomorphisms there is a K-qc map $\psi : U \longrightarrow W_{n_0}$ that coincide with $\tilde{\psi}_n$ in $(U_n - U_{n+1}) \cap K(V_{n-1})$ for $n \ge n_0$. It follows that the diameters of $W_n = \psi_n(U_n)$ go to zero as $n \to 0$, so $\psi(c_0) = c_0$; compare with Lemma 8.6. By Corollary 7.4 the set $\partial K(V_n)$ has zero Lebesgue measure, so ψ is conformal Lebesgue almost everywhere outside the sets $(U_n - U_{n+1}) - K(V_{n-1})$ for $n \ge n_0$.

Proof of Proposition 8.12. The proof follows the lines of [Lyu3]. Consider $z \in U_n$. Note that for $c \in W_n$ close to ∂W_n we have that $(i_n)_c(z)$ is close to the boundary of $(i_n)_c(U_n)$, so $c \neq i_c(z)$. Hence the number of solutions $\psi(z)$ of the equation $\psi_n(z) = i_{\psi_n(z)}(z)$ in W_n is equal to the winding number of $c - i_c(z)$, when c makes one turn around a closed curve close to ∂W_n. This number depends continuously in z, so it is independent of z. By Lemma 8.10 this number is equal to 1 for all $z \in U_n \cap K(V_{n-1})$, so it is equal to 1 for all $z \in U_n$.

In other words we proved that the graph $G = \{(c, c) \mid c \in W_n\}$ is a global transversal to the foliation $\mathcal{F}$ with leaves $\mathcal{F}_z = \{(c, (i_n)_c(z)) \mid c \in W_n\}$. Then ψ_n is the holonomy between the global transversal $T = \{c_0\} \times U_n$ and G. It follows (just as the Qc Lemma) that this holonomy is locally qc. Moreover by Lemma 8.1 the foliation with leaves $\mathcal{F}_z$ for $z \in U_n \cap int(K(V_n))$ is holomorphic, so the holonomy ψ_n is holomorphic in $U_n \cap int(K(V_n))$. □

8.5. **Conformality of the parameter map and asymptotic similarity.** In this section we prove Theorem E about asymptotic similarity between $\mathcal{M}_d$ and Julia sets. We first reduce Theorem E to prove that the parameter map, of the previous section, is conformal; and to prove that it is C^{1+}-conformal in the Collet-Eckmann case.

A map $\psi : U \longrightarrow \mathbb{C}$ is said to be *conformal* at a point $c_0 \in U$ if there is $\lambda \in \mathbb{C} - \{0\}$ such that $\psi(w) = \psi(c_0) + \lambda(w - c_0) + o(|w - c_0|)$. If $o(|w - c_0|)$ is replaced by $\mathcal{O}(|w - c_0|^{1+\alpha})$ for some $\alpha > 0$, we say that ψ is C^{1+}-conformal at c_0.

Recall that by Proposition 1.4, if P_{c_0} satisfies the Collet-Eckmann condition, then there are $\alpha_0 > 0$ and $C_0 > 0$ such that $r(\delta) \geq C_0 \delta^{-\alpha_0}$. Consider the *flat metric* $|\frac{dz}{z}|$ in $\mathbb{C} - \{0\}$ that makes $\mathbb{C} - \{0\}$ isometric to the straight cylinder $S^1 \times \mathbb{R}$.

Lemma 8.13. *If the parameter map ψ of the previous section is conformal at c_0, then $\mathcal{M}_d$ and J_{c_0} are asymptotically similar at c_0. Moreover suppose that there are $\alpha_0 > 0$ and $C_0 > 0$ such that $r(\delta) \geq C_0 \delta^{-\alpha_0}$ and suppose that the parameter map ψ is C^{1+}-conformal at c_0. Then there are $\alpha > 0$ and $C > 0$ such that, if $\lambda \in \mathbb{C} - \{0\}$ is the derivative of ψ at c_0, then*

$$d_H((\mathcal{M}_d - c_0)_r, (\lambda(J_{c_0} - c_0))_r) \leq Cr^\alpha.$$

Recall that d_H denotes the Hausdorff distance and for $X \subset \mathbb{C}$ and $r > 0$ the set X_r denotes $\frac{1}{r}(X \cap \{|z| \leq r\}) \cup \partial\mathbb{D}$.

Proof. Diameter will be taken with respect to the flat metric. Let $E = \cup_{n \geq n_0}(U_n - U_{n+1} - K(V_{n-1}))$. By Lemma 3.1 every connected component W of E has diameter $o(dist(W, c_0))$ in $\mathbb{C} - \{c_0\}$. Let $\tilde{J} = J_{c_0} \cup E$, so $d_H((J_{c_0} - c_0)_r, (\tilde{J} - c_0)_r) \to 0$ as $r \to 0$.

Since $\psi(c_0) = c_0$ and since ψ is qc it follows that for every connected component W of E the diameter of $\psi(W)$ in $\mathbb{C} - \{c_0\}$ is $o(dist(W), c_0)$. Moreover the boundary of W intersects J_{c_0}. Since for $z \in U - E$ we have that $\psi(z) \in \mathcal{M}_d$ if and only if $z \in J_{c_0}$, it follows that $d_H((\psi(\tilde{J}) - c_0)_r, (\mathcal{M}_d - c_0)_r) = o(1)$.

Suppose that ψ is conformal at c_0 and let $\lambda \in \mathbb{C} - \{0\}$ be the derivative of ψ at c_0. Then $d_H((\lambda(\tilde{J} - c_0))_r, (\psi(\tilde{J}) - c_0)_r) = o(1)$, so it follows that $d_H((\lambda(J_{c_0} - c_0))_r, (\mathcal{M}_d - c_0)_r) = o(1)$ which by definition is that $\mathcal{M}_d$ and J_{c_0} are asymptotically similar at c_0. The second part follows in a similar way. □

To prove the conformality of the parameter map we use the following conformality criterions; see [LV] and [McM].

Conformality Criterion. *Let $\psi : U \longrightarrow \psi(U) \subset \mathbb{C}$ be a qc homeomorphism. Suppose that $c_0 \in U$ is such that ψ is conformal outside a set of finite measure in $\mathbb{C} - \{c_0\}$, with respect to the flat metric of $\mathbb{C} - \{c_0\}$. Then ψ is conformal at c_0.*

C^{1+}-Conformality Criterion (McMullen [McM]**).** *Let $\psi : U \longrightarrow \psi(U) \subset \mathbb{C}$ be a qc homeomorphism. Suppose that there are α_1 and C_1 such that for $r > 0$ small, ψ is conformal in $B_r(c_0)$ except for a set of Lebesgue measure at most $C_1 r^{2+\alpha_1}$. Then ψ is C^{1+}-conformal at c_0.*

Recall that $\tau \in (0, 1)$ and $\eta \in (1, \tau^{-1})$ are such that $B_{\tau^n}(c_0) \subset U_n \subset B_{\eta\tau^n}(c_0)$ and recall that the parameter map ψ is conformal Lebesgue almost every where outside the sets $U_n - U_{n+1} - K(U_{n+1})$. Moreover by Lemma 4.6 there is $A_1 > 0$ such that $|U_n - U_{n+1} - K(U_n)| < A_1 |U_n| r(\tau^n)^{-\frac{2}{d}}$.

Proof of Theorem E. By the above there is $K_0 > 0$ such that the cylindrical area of $U_n - U_{n+1} - K(U_n)$ is bounded by $K_0 r(\tau^n)^{-\frac{2}{d}}$. By the remark after Proposition 1.4 we have by hypothesis that $\sum_{n \geq n_0} r(\tau^n)^{-\frac{2}{d}} < \infty$. Thus ψ satisfies the hypothesis of the conformality criterion, so ψ is conformal at c_0 and by Lemma 8.13, $\mathcal{M}_d$ and J_{c_0} are asymptotically similar at c_0.

Collet-Eckmann case. Suppose that P_{c_0} satisfies the Collet-Eckmann case. So by Proposition 1.4 there are $\alpha_0 > 0$ and $C_0 > 0$ such that $r(\delta) \geq C_0 \delta^{-\alpha_0}$. By Lemma 8.13 is enough to prove that the parameter map ψ is C^{1+}-conformal at c_0. By the above there is $K_1 > 0$ independent of n, such that,

$$|U_n - K(V_n)| \leq A_1 |U_n| r(\tau^n)^{-\frac{2}{d}} \leq A_1 |U_n| C_0^{-1} (\tau^n)^{\frac{2}{d}\alpha_0} \leq K_1 (\tau^n)^{2+\frac{2}{d}\alpha_0}.$$

Thus ψ satisfies the hypothesis of the C^{1+}-conformality criterion, with $\alpha_1 = \frac{2}{d}\alpha_0$ and with some constant $C_1 > 0$. Hence ψ is C^{1+}-conformal at c_0 and Theorem E follows by Lemma 8.13. □

9. Appendix: Re-statement of MLC.

Now we prove the re-statement of MLC stated in the introduction. This follows easily from the theory of *parapuzzle* ends or *fibers* in [Sch]. Now we state some properties of parapuzzle ends that can be found in [Sch]. Fix $d \geq 2$ and let $P_c(z) = z^d + c$, for $c \in \mathbb{C}$.

For each $c_0 \in \mathbb{C}$ such that P_{c_0} is not hyperbolic, we associate its *parapuzzle end*. Parapuzzle ends are full compact sets that are either disjoint or equal. As mentioned in the introduction, if the parapuzzle end of a parameter c_0 is trivial, then $c_0 \in \partial\mathcal{M}_d$ and $\mathcal{M}_d$ is locally connected at c_0.

Lemma (parapuzzle ends). *Let $c_0 \in \partial\mathcal{M}_d$ such that the critical point of P_{c_0} is recurrent and such that all its periodic points are repelling. In particular P_{c_0} is not hyperbolic. Then the parapuzzle end of c_0 is equal to the maximal connected set $\xi \subset \mathbb{C}$ of parameters that contains c_0 and such that for every $c \in \xi_{c_0}$ the critical point of P_c is recurrent.*

The parapuzzle end of a parameter whose respective polynomial has an indifferent periodic point is trivial. The parapuzzle end of a parameter $c_0 \in \mathbb{C}$ such that the critical point of P_{c_0} is not recurrent, and such that P_{c_0} is not hyperbolic, is trivial; see [H] and [Sch]. Thus this lemma implies that the re-statement of MLC is in fact equivalent to MLC.

Let us define parapuzzle ends of parameters $c_0 \in \mathbb{C}$ such that P_{c_0} is not hyperbolic and such that all its periodic points are repelling.

Douady and Hubbard proved that $\mathcal{M}_d$ is a full connected set, and moreover there is a biholomorphic map $\varphi_{\mathcal{M}_d} : \hat{\mathbb{C}} - \mathcal{M}_d \longrightarrow \hat{\mathbb{C}} - \overline{\mathbb{D}}$ which is tangent to the identity at infinity. The preimage by $\varphi_{\mathcal{M}_d}$ of $\{re^{2\pi i\theta} \mid r > 0, \theta \in \mathbb{R}\}$ is called the *ray* with angle θ; and the preimage of $\{|z| = r \mid r > 1\}$ is an analytic Jordan curve called *equipotential*. The ray of angle θ is said to *land* at some point c_0 if $\varphi^{-1}_{\mathcal{M}_d}(re^{2\pi i\theta}) \to c_0$ as $r \to 1$. It is known that all rays with rational angle land and the landing parameter is such that the critical point is not recurrent. More precisely the critical point is either strictly pre-periodic or it converges to a parabolic cycle.

A *parapuzzle* of $\mathcal{M}_d$ is a disc bounded by a finite number of rays with rational angles and an equipotential, so that the intersection with $\mathcal{M}_d$ is a non-empty connected set; see [H]. Then the parapuzzle end of c_0 is the intersection of all parapuzzles containing c_0.

Proof of the lemma. Recall that the parapuzzle end of a parameter c with all cycles repelling and with non-recurrent critical point in $J(P_c)$, is trivial; see [H] and [Sch]. Since parapuzzle ends are disjoint or equal, it follows that for all c in the parapuzzle end of c_0, the critical point of P_c is recurrent. Thus the parapuzzle end of c_0 is contained in ξ.

If Π is a parapuzzle containing c_0 then, as remarked above, a parameter $c \in \partial\Pi \cap \mathcal{M}_d$ is such that the critical point of P_c is not recurrent. Since for parameters c not in $\mathcal{M}_d$, the critical point of P_c is not recurrent (the critical point escapes to infinity), it follows that $\xi \subset \Pi$. Thus ξ is contained in the parapuzzle end of c_0. □

10. Appendix. Quasi-conformal homeomorphisms and holomorphic motions.

In this appendix we review some properties of quasi-conformal maps. See [LV] and [A] for references.

Given $K \geq 1$ we say that an homeomorphism χ is *K-quasi-conformal*, or *K-qc* for short, if the following equivalent conditions hold.

(1) For every annulus $A \subset \mathbb{C}$ we have $K^{-1} mod(A) \leq mod(\chi(A)) \leq K mod(A)$.
(2) χ has a distributional derivative that is locally in L^2 and $\|D\chi\|^2 \leq K Jac(D\chi)$ Lebesgue almost everywhere.

By 1 the inverse of a K-qc homeomorphism is also a K-qc homeomorphism. In this case χ is differentiable (in the usual sense) Lebesgue almost everywhere and this derivative coincides with the distributional derivative almost everywhere. The constant $K \geq 1$ is called the *dilatation* of χ. If we do not want to specify the dilatation we just say that χ is *quasi-conformal* or *qc*. Conformal maps coincide with 1-qc maps.

Qc homeomorphisms preserve sets of Lebesgue measure 0 and sets of σ-finite sets are qc removable: if $\chi : U \longrightarrow \chi(U) \subset \mathbb{C}$ is an homeomorphism that is K-qc outside a set of σ-finite length, then χ is K-qc. Moreover K-qc homeomorphisms of $\mathbb{C}$ are Hölder with constants only depending in K. The same is true for K-qc homeomorphisms of the sphere that fix three prescribed points and for K-qc homeomorphisms of the disc with respect to the hyperbolic metric.

An *ellipse field* σ is to associate to Lebesgue almost every point $z \in \hat{\mathbb{C}}$ an ellipse in the tangent plane to $\hat{\mathbb{C}}$ at z, up to scale. All ellipse fields considered will have *bounded* *dilatation*, that is the dilatation of Lebesgue almost every ellipse is bounded by a constant $K \geq 1$. The *dilatation* of an ellipse is the ratio of its major axis to its minor axis. In this case we say the dilatation of the ellipse field is bounded by K. Moreover all ellipse fields will be *measurable* in the sense that the dilatation and the direction of the major axis of the ellipses are measurable functions.

We denote by σ_0 the standard complex structure, which is the one formed just by circles. For any K-qc homeomorphism h, the pull-back $\chi^*\sigma_0$ is an ellipse field with dilatation bounded by K; cf. characterization 2 of K-qc homeomorphisms. The following theorem is a remarkable converse of this property.

Ahlfors-Bers Integration Theorem. *Let σ be a measurable ellipse field of $\hat{\mathbb{C}}$ with dilatation bounded by $K \geq 1$. Then there is a unique K-qc homeomorphism h, up to postcomponing with an automorphism of $\hat{\mathbb{C}}$, so that $\sigma = h^*\sigma_0$.*

Thus an ellipse field in $\hat{\mathbb{C}}$ induces a complex structure in $\hat{\mathbb{C}}$ using the homeomorphism given by the Ahlfors-Bers Integration Theorem as a single chart. So ellipse fields will also be called *complex structures*.

An important property is *compactness of normalized* K-qc homeomorphisms: for any $K \geq 1$ and any sequence of normalized K-qc homeomorphisms there is a subsequence that converges uniformly to a normalized K-qc homeomorphism. A *normalization* is a restriction that can be satisfied for every homeomorphism after post-composing by a uniquely determined automorphism of $\hat{\mathbb{C}}$. A normalization in $\hat{\mathbb{C}}$ is to fix three points. If we only consider homeomorphisms that are holomorphic in a definite neighborhood of ∞, a normalization is being tangent to the identity at infinity and fixing 0.

All previous considerations apply to general Riemann surfaces. A normalizations in $\mathbb{C}$ and $\mathbb{D}$ is to fix two preferred points.

One of the main points of this paper is to use rigidity properties of qc maps that are conformal in a big set, where big can be taken in several senses; cf. Sections 5, 6 and 8 and see also Appendix 11. Lemmas 10.3 below is a basic property of this kind that is used several times in this paper; see also Lemma 10.1.

Let us consider some concepts. We begin by recalling the definition of the modulus of an annulus. Every topological annulus $A \subset \hat{\mathbb{C}}$ is either conformally equivalent to $\mathbb{C} - \{0\}$ or to $\{z|\, 1 < |z| < R\}$, where $R \in (1, \infty]$ is then uniquely determined. In this case $\ln(R)$ is called the *modulus* of A and is denoted by $mod(A)$. This is not completely standard, some authors prefer to call $\frac{1}{2\pi}\ln R$ the modulus of A; see for example [A]. We follow [LV].

Other definition of $mod(A)$ is given by,

$$mod(A) = \left(\frac{1}{2\pi}\inf_h \int\int_A |\nabla h|^2 dxdy\right)^{-1},$$

where the infimum is taken over all C^1 functions $h : A \to (0,1)$ such that $h(z) \to 1$ as z approaches a determined end of A and $h(z) \to 0$ as z approaches the other end. The infimum of this Dirichlet integral is realized by an harmonic function.

Consider the *flat metric* $|\frac{dz}{z}|$ in $\mathbb{C} - \{0\}$ that makes $\mathbb{C} - \{0\}$ isometric to $S^1 \times \mathbb{R}$. The restriction of this metric a round annulus $\{1 < |z| < R\}$ will also be called the *flat metric* of this annulus. Then one can define such a metric in every annulus, by means of a biholomorphism into a round annulus.

Lemma 10.1. *Let A be an annulus endowed with the flat metric and let $\chi : A \longrightarrow \chi(A)$ be a K-qc homeomorphism conformal in $A - \mathcal{N}$ for some $\mathcal{N} \subset A$. Then,*

$$mod(A)\left(1 + \frac{(K-1)|\mathcal{N}|}{2\pi mod(A)}\right)^{-1} \leq mod(\chi(A)) \leq mod(A) + \frac{(K-1)|\mathcal{N}|}{2\pi},$$

where $|\mathcal{N}|$ is the area of $\mathcal{N}$ with respect to the flat metric.

Remark 10.2. *This lower bound is somehow pessimistic, it can be attained only for $\mathcal{N}$ with a very particular geometry. In Appendix 11 we give conditions in the sizes and distribution of the components of $\mathcal{N}$, so that there is a lower bound of $mod(\chi(A))$* **independent of the dilatation** *of χ; see also Remark 5.2 in Section 5.1*

Proof. We just prove the lower bound, the other inequality is stated in [LV]; see (6.6) p. 221. Suppose that A is the straight cylinder $S^1 \times (0, mod(A))$ and denote by π the projection $\pi : A \longrightarrow (0,1)$ given by $\pi(\theta, t) = \frac{t}{mod(A)}$. By the definition of modulus above,

$$\frac{1}{mod(\chi(A))} \leq \frac{1}{2\pi}\int\int_{\chi(A)} |\nabla(\pi \circ \chi^{-1})|^2 dxdy = \frac{1}{2\pi}(mod(A))^{-2}\int\int_{\chi(A)} \|D\chi^{-1}\|^2 dxdy$$

$$\begin{aligned} &\leq \frac{1}{2\pi}(mod(A))^{-2}\left(\int\int_{\chi(A)} Jac(\chi^{-1})dxdy + (K-1)\int\int_{\chi(\mathcal{N})} Jac(\chi^{-1})dxdy\right) \\ &= \frac{2\pi mod(A) + (K-1)|\mathcal{N}|}{2\pi(mod(A))^2}. \square \end{aligned}$$

Lemma 10.3. *Let $K \geq 1$ be given. Then the following assertions hold.*

1. *Let $\{\chi_k\}_{k\geq 1}$ be a sequence of K-qc normalized homeomorphisms such that χ_k is conformal outside a set of Lebesgue measure ε_k, so that $\varepsilon_k \to 0$ as $k \to \infty$. Then χ_k converges uniformly to the identity.*
2. *Then for every $\varepsilon > 0$ there is δ such that any normalized K-qc homeomorphism of $\hat{\mathbb{C}}$ that is conformal except for a set of Lebesgue measure δ, is ε close to the identity.*

Proof. 1.– Consider a subsequence χ_{k_n} that converges uniformly to a normalized K-qc homeomorphism χ. If A is an annulus we have by the previous lemma that $mod(\chi_n(A)) \to mod(A)$ so $mod(\chi(A)) = mod(A)$. Thus χ preserves the modulus of annuli and by the characterization 1 of qc homeomorphisms, χ is 1-qc. Thus χ is conformal. Since χ is normalized we have that $\chi = id$. Since this is for an arbitrary convergent subsequence, the assertion follows.

2.– If this is not true then there is a sequence of normalized K-qc homeomorphisms χ_k that are ε away from the identity and so that χ_k is conformal outside a set of measure $\frac{1}{k}$. This contradicts 1. □

The following lemma can be found in [DH2].

Gluing Lemma. *Let $U \subset \mathbb{C}$ be a bounded open set and $\chi : \mathbb{C} \longrightarrow \mathbb{C}$ a K-qc homeomorphism. Suppose that $\chi_0 : U \longrightarrow \chi(U)$ so that the map $\tilde{\chi}$ that is equal to χ outside U and is equal to χ_0 in U. Then :*

1. *$\tilde{\chi}$ is a qc homeomorphism of $\mathbb{C}$.*
2. *The derivatives of $\tilde{\chi}$ and χ coincide Lebesgue almost everywhere in $\mathbb{C} - U$.*

10.1. **Holomorphic motions.** In this section we review the concept of holomorphic motions and some of its properties; see [MSS] for references. Let W be an open subset of $\mathbb{C}$, often biholomorphic to the unit disc $\mathbb{D}$. Then an *holomorphic motion* i of a set $K \subset \hat{\mathbb{C}}$ defined over W, is a map $i : W \times K \longrightarrow \hat{\mathbb{C}}$, so that for all $c \in W$ the map $i_c : K \longrightarrow \hat{\mathbb{C}}$ is injective and for all $z \in K$ the function $c \longrightarrow i_c(z)$ is holomorphic. Usually there is a *base point* $c_0 \in W$ for which i_{c_0} is the identity.

Slodkowsky Extension Theorem. *Any holomorphic motion $i : \mathbb{D} \times K \longrightarrow \hat{\mathbb{C}}$ of $K \subset \hat{\mathbb{C}}$, can be extended to an holomorphic motion of $\hat{\mathbb{C}}$.*

See [Sl]. The following lemma was proven in [MSS].

Qc Lemma. *Let $W \subset \mathbb{C}$ and $U \subset \hat{\mathbb{C}}$ be open sets and let $i : W \times U \longrightarrow \hat{\mathbb{C}}$ be an holomorphic motion with base point. Then, for every $c \in W$ the map i_c is quasi-conformal.*

We use the following lemma in Section 8.1.

Lemma 10.4. *Let $W \subset \mathbb{C}$ be an open set and let $K \subset \mathbb{C}$. Consider an holomorphic motion $j : W \times K \longrightarrow B_R(0) \subset \mathbb{C}$ for some $R > 0$. Moreover suppose there is $z_0 \in K$ is such that $j_c(z_0) \equiv 0$ and fix $c_0 \in W$. Then there is a constant $\kappa > 0$ such that for every $(c, z) \in W \times K$ with $|c - c_0| \ln(|j_c(z) - j_{c_0}(z)|^{-1}) \ll 1$ we have*

$$|j_c(z) - j_{c_0}(z)| \leq \kappa |c - c_0| |j_{c_0}(z)| \ln(|j_{c_0}(z)|^{-1}).$$

Proof. Dividing j par R, we may suppose that $R = 1$. Moreover we may suppose that $W = \mathbb{D}$ and $c_0 = 0$. Fix $z \in K - \{z_0\}$ and put $w_0 = j_0(z)$. Note that the function $c \longrightarrow j_c(z)$ goes from $\mathbb{D}$ to $\mathbb{D} - \{0\}$. Moreover the map $w \longrightarrow w_0 e^{\frac{-2w}{1+w} \ln |z_0|}$ is a local isometry between $\mathbb{D}$ and $\mathbb{D} - \{0\}$, with respect to the hyperbolic metrics $\rho_{\mathbb{D}}$ and $\rho_{\mathbb{D}-\{0\}}$, that maps 0 to w_0. By Schwartz lemma,

$$j_c(z) \in \{\zeta \in \mathbb{D} - \{0\} \mid \rho_{\mathbb{D}-\{0\}}(\zeta, z_0) \leq \rho_{\mathbb{D}}(c, 0)\}.$$

Hence,

$$|j_c(z) - j_{c_0}(z)| \leq \sup_{|w|<|c|} |z_0 e^{\frac{-2w}{1+w} \ln |z_0|} - z_0| = |z_0| |e^{\frac{-2|c|}{1+|c|} \ln |z_0|} - 1|.$$

By hypothesis $|c| \ln |z_0| \ll 1$ so there is a constant $\kappa > 0$ such that

$$|j_c(z) - j_{c_0}(z)| \leq C|c||z_0| \ln(|z_0|^{-1}).\square$$

11. Appendix. Rigid Annuli.

In his unpublished proof of local connectivity of the Mandelbrot set at non-infinitely renormalizable parameters, J. C. Yoccoz encountered the following situation. There is a quasi-conformal map χ defined in the annulus $A = \{z \in \mathbb{C} \mid \frac{1}{3} < |Re(z)| \text{ and } |Im(z)| < 1\}$ that is conformal Lebesgue almost everywhere outside the set $\mathcal{N}$, which is defined as the least set containing the square $\{z \in \mathbb{C} \mid |Re(z)|, |Im(z)| < \frac{1}{3}\}$ and the images of itself under the affine maps $z \longrightarrow \frac{z+1}{3}$ and $z \longrightarrow \frac{z-3}{3}$; see Figure 5.

An important step in his proof was to prove that there is a bound *independent of the map* χ for the modulus of the annulus $\chi(A)$. Note that there is no restriction in the dilatation of χ.

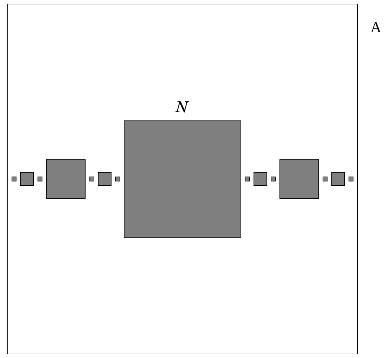

FIGURE 5. The annulus A and the set $\mathcal{N}$ in black.

There are various ways to prove this property. Probably the easiest one is an extremal length argument, observing that the set $(A-\mathcal{N})\cap\mathbb{R}$ has positive capacity. There is also a proof using Dirichlet integrals in [Se] and there is a much stronger result in [Ka].

Unfortunately this part of the argument, in Yoccoz theorem, was replaced by Hubbard in [H], which is to the best of my knowledge the only complete published proof of Yoccoz theorem.

Here we extend this rigidity property to some other pairs $(A,\mathcal{N})$, where $\mathcal{N}$ is obtained by an iterative function system with possibly infinitely many branches. This situation appears naturally in dynamics as fist return maps, and this rigidity property has applications to make estimates in parameter planes (as in Yoccoz's theorem; see also [Lyu3]) or in pull-back procedures like in Theorem C; see also [R-L3].

Theorem (Rigid Annuli). *Let $U \subset \mathbb{C}$ be a bounded disc and $K \subset U$ be a non-trivial compact set, so that $A = U - K$ is an annulus. Moreover consider a collection of discs $\{U_i\}_{i\geq 1}$ with pairwise disjoint closures such that $\overline{U_i} \subset A$ and let $g_{U_i} : U \longrightarrow U_i$ be biholomorphisms. Let $\mathcal{N}$ be the smallest set containing K and all the images of itself by the g_{U_i}. Then there are $m_0 > 0$ and $\varepsilon_0 > 0$ such that if:*

> **Modulus:** *The maps g_{U_i} extend in a univalent way to a disc U', such that $m = mod(U' - \overline{U}) \geq m_0$;*
> **Area:** *$\varepsilon = \frac{area(\cup U_i)}{area(U)}$ is smaller than ε_0;*

then there is a constant $M > 0$ such that for any qc homeomorphism $\chi : A \longrightarrow \chi(A)$ conformal Lebesgue almost everywhere in $A - \mathcal{N}$, we have

$$mod(\chi(A)) \geq M.$$

Remark 11.1. (1) *Note that the U_i can accumulate the boundary of A. Moreover K can have non-empty interior and for example ∂K may have positive Lebesgue measure.*
(2) *Since the U_i are disjoint and $mod(U_i' - \overline{U_i})$ is definite, with $U_i' \subset A$, it follows that $diam(U_i) \to 0$.*

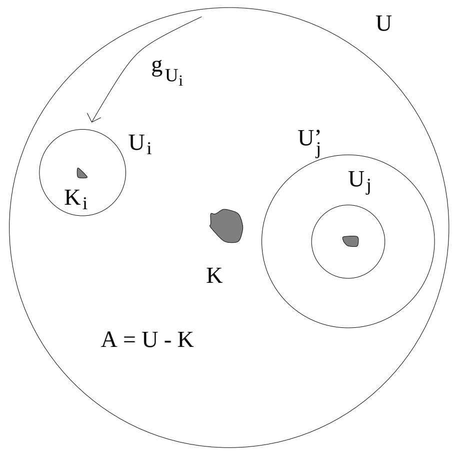

FIGURE 6. Illustration for $(A, \mathcal{N})$.

The conclusion of this theorem is weaker than Rigidity of Section 5.1 (see part 3 of Remark 5.2) but the hypothesis are simple enough to be presented in a more abstract setting. The proofs is via Dirichlet integrals, following the proof in [Se] of Yoccoz's situation. This theorem appeared in the unpublished [R-L3] to prove some results weaker that those stated in Section 7.

11.1. **Collapsing map in Sobolev space.** In this section we will reduce the theorem to the existence of a function in an appropriated Sobolev space, that collapses all the connected components of $\mathcal{N}$ to a point. For an open set $\mathcal{U} \subset \mathbb{C}$ denote by $W^{1,2}(\mathcal{U})$ the Sobolev space of functions $f : \mathcal{U} \to \mathbb{C}$ with distributional derivatives in $L^2(\mathcal{U})$ and norm,

$$\|f\|^2_{\mathcal{U}} = \|f\|^2_{\mathcal{U},1,2} = \int\int_{\mathcal{U}} \|Df\|^2 dxdy.$$

This norm is invariant by conformal maps. That is if $\chi : \mathcal{U} \longrightarrow \mathcal{V}$ is a conformal map then $\|f \circ \chi^{-1}\|_{\mathcal{V}} = \|f\|_{\mathcal{U}}$, this is easy to see by the change of variable formula and considering that $\|D\chi^{-1}\|^2 = Jac(\chi^{-1})$, since χ^{-1} is conformal.

We will reduce the Theorem to the following lemma.

Lemma 11.2. *Let $(A, \mathcal{N})$ be as in the theorem. Then, if m is big enough and $\varepsilon > 0$ is small enough, there is a continuous function*

$$f : A \longrightarrow S^1 \times (0, 1) \subset S^1 \times [0, 1],$$

satisfying the following properties.

1. *If $\{z_i\}_{i\geq 1} \subset A$ is such that $z_i \to \partial U$ then $f(z_i) \to S^1 \times \{1\}$ and if $z_i \to \partial K$ then $f(z_i) \to S^1 \times \{0\}$.*
2. *We have $f(x) = f(y)$, with $x \neq y$ if and only if x and y belong to the closure of the same connected component of $\mathcal{N}$.*
3. *We have $f \in W^{1,2}(A)$ and the norm of f can be bounded in terms of mod(A) only.*

Let us deduce deduce the theorem assuming the previous lemma. Let $\chi : A \longrightarrow \mathbb{C}$ be a qc homeomorphism into its image, conformal in $A - \mathcal{N}$ in the distributional sense. Hence

χ^{-1} and $f\circ\chi^{-1}$ belong locally to $W^{1,2}$ and χ^{-1} is conformal in the distributional sense in $\chi(A-\mathcal{N})$; see Remark 11.1. Let $\pi: S^1\times(0,1)\longrightarrow(0,1)$ the projection $\pi((\theta,t))=t$. Considering that $Df|_{\mathcal{N}}\equiv 0$ we have

$$\begin{aligned}
\int\int_{\chi(A)}|\nabla(\pi\circ f\circ\chi^{-1})|^2dxdy &\leq \int\int_{\chi(A)}|\nabla\pi(f\circ\chi^{-1})|^2\|D(f\circ\chi^{-1})\|^2dxdy\\
&\leq \int\int_{\chi(A-\mathcal{N})}\|D(f\circ\chi^{-1})\|^2dxdy\\
&\quad+\int\int_{\chi(\mathcal{N})}\|Df(\chi^{-1})\|^2\|D\chi^{-1}\|^2dxdy\\
&= \int\int_{A-\mathcal{N}}\|Df\|^2dxdy=\|f\|^2_A,
\end{aligned}$$

By the definition of modulus with Dirichlet integrals, this gives a lower bound of $\|f\|_A^{-1}$ for $mod(\chi(A))$; see [A] or Appendix 10.

11.2. **Model.** In next section we will construct inductively a sequence $f_n\in W^{1,2}(A)$ converging to the desired f of Lemma 11.2. In this section we will describe a model function $h: U\longrightarrow U$ in $W^{1,2}(U)$ that will serve to construct f_n from f_{n-1}.

This model function h will be continuous and such that $h(x)=h(y)$, with $x\neq y$, if and only if $x,y\in K$. Moreover h will be C^2 in $A=U-K$ and it will be equal to the identity in a neighborhood of ∂U. Furthermore $|U|^{-1}\|h\|^2_U$ will be bounded in terms of $mod(A)$ only, where $|X|$ denotes the area of $X\subset\mathbb{C}$.

We will construct h that is C^2 in A, except for an analytic curve. We can obtain h to be C^2 in by using a bump function. Fix $z_0\in K$ and let $\varphi: U\longrightarrow\mathbb{D}$ be a biholomorphism such that $\varphi(z_0)=0$. Consider $\delta>0$ small and let $U_{1-\delta}=\varphi^{-1}(|D_{1-\delta})$. We will choose δ depending in $mod(A)$ only. Consider a conformal representation,

$$\psi: U_{1-\delta}-K\longrightarrow\mathbb{D}_{1-\delta}-\overline{\mathbb{D}}_{r_0}$$

sending the end corresponding to $\partial U_{1-\delta}$ to the end corresponding to $\partial\mathbb{D}_{1-\delta}$ and for $t\in[r_0,1-\delta)$ let $U_t=K\cup\psi^{-1}(\{|z|<t\})$.

Let $\tilde{h}_0:[r_0,1-\delta]\longrightarrow[0,1-\delta]$ be the quadratic diffeomorphism tangent to the identity at $1-\delta$. Note that $\tilde{h}_0$ has a non-zero derivative at r_0. Consider $h_0:\overline{\mathbb{D}}_{1-\delta}-\mathbb{D}_{r_0}\longrightarrow\overline{\mathbb{D}}_{1-\delta}$ given in polar coordinates by $h_0(\theta,r)=(c(\theta),\tilde{h}_0(r))$ so that $h_0\circ\psi=\varphi$ in $\partial U_{1-\delta}$. So the function $c:\mathbb{R}\longrightarrow\mathbb{R}$ is analytic and its distortion is bounded in terms of $mod(A)$ only. Moreover note that $\|Dh_0\|$ is bounded and it can be bounded in terms of $mod(A)$ only. Then h is defined as the identity in $U-U_{1-\delta}$, equal to $\varphi^{-1}\circ h_0\circ\psi$ in $U_{1-\delta}-K$ and constant equal to z_0 in K; see Figure 7.

Note that h is C^2 in $A=U-K$, except for the analytic curve $\partial U_{1-\delta}$.

1.– Let us prove that $h\in W^{1,2}(U)$. Note that $h\in L^2(U)$ as bounded function. For t small let $h_t:\overline{\mathbb{D}}_{1-\delta}-\mathbb{D}_{r_0+t}\longrightarrow\overline{\mathbb{D}}_{1-\delta}$ be defined like h_0 but with $\tilde{h}_0$ replaced with $\tilde{h}_t:[r_0+t,1-\delta]\longrightarrow[0,1-\delta]$ which is the unique homeomorphism which coincides with $\tilde{h}_0$ in $[r_0+2t,1-\delta]$ and that is affine in $[r_0+t,r_0+2t]$. So the derivative of $\tilde{h}_t$ in $[r_0+t,r_0+2t]$ is close to $2\tilde{h}_0'(r_0)$.

In an analogous way define continuous maps $\hat{h}_t: U\longrightarrow U$ which are the identity in $U-U_{1-\delta}$ and constant in $\overline{U}_{r_0+t}$. Like h, the functions $\hat{h}_t$ and h_t belong to L^2 as bounded

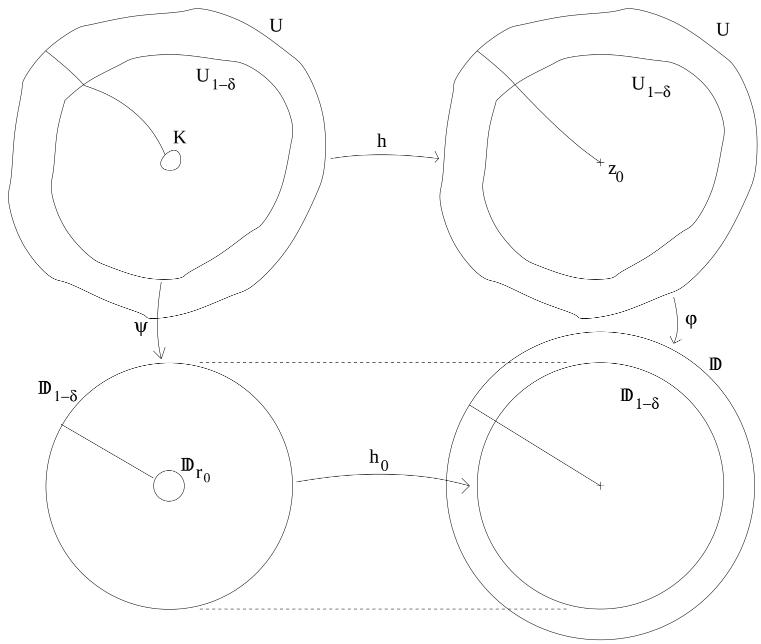

FIGURE 7. The model function h is defined radially with $\tilde{h}_0$.

functions and moreover $\hat{h}_t$ is C^2 in $A = U - K$ except in $\partial U_{1-\delta}$ and ∂U_{r_0+t} which are analytic curves. Extend h_t to $\mathbb{D}_{1-\delta}$ by defining it constant equal to 0 in $\overline{\mathbb{D}}_{r_0+t}$.

By conformality of ψ it follows that $\|\hat{h}_t\|_{U_{1-\delta}} = \|\varphi^{-1} \circ h_t\|_{\mathbb{D}_{1-\delta}}$. By Koebe Distortion Theorem the distortion of φ^{-1} in $\mathbb{D}_{1-\delta}$ is bounded by some constant $D > 1$, only depending in δ. Thus

$$\|\varphi^{-1} \circ h_t\|_{\mathbb{D}_{1-\delta}} \leq D\|h_t\|_{\mathbb{D}_{1-\delta}}.$$

But $\|h_t\|_{\mathbb{D}_{1-\delta}}$ is uniformly bounded, since $\|Dh_t\|$ is uniformly bounded in $\mathbb{D}_{1-\delta}$ (for small t). Hence $\hat{h}_t \in W^{1,2}(U)$.

Note that $\hat{h}_t \to h$ uniformly. Let us prove that the convergence is also in $W^{1,2}(U)$. Fix t_0 small. Note that $Dh_t \equiv 0$ in $\mathbb{D}_{r_0}$ and $\|Dh_t\|$ is bounded in $\mathbb{D}_{r_0+t_0} - \mathbb{D}_{r_0}$, independently of $t < t_0$. Thus, for all $s, t < t_0$ we have

$$\int\int_{\mathbb{D}_{r_0+t_0}} \|D(h_t - h_s)\|^2 dxdy \leq C_1|\mathbb{D}_{r_0+t_0} - \mathbb{D}_{r_0}| = C_1\pi t_0(2r_0 + t_0).$$

By the conformal invariance of the norm in $W^{1,2}(U)$ it follows that $\hat{h}_t$ is a Cauchy sequence in $W^{1,2}(U)$ and therefore $h \in W^{1,2}(U)$.

2.– Now let us prove that $|U|^{-1}\|h\|_U^2$ can be bounded in terms of $mod(A)$ only. In fact

$$\|h\|_U^2 = |U - U_{1-\delta}| + \|h\|_{U_{1-\delta}}^2 \text{ and}$$

$$\|h\|_{U_{1-\delta}} = \|\varphi^{-1} \circ h_0 \circ \psi\|_{U_{1-\delta}} = \|\varphi^{-1} \circ h_0\|_{\mathbb{D}_{1-\delta}}.$$

Note that $|U_{1-\delta}| \leq D^2|(\varphi^{-1})'(0)|^2|\mathbb{D}|_{1-\delta}$. So $\|\varphi^{-1} \circ h_0\|_{\mathbb{D}_{1-\delta}} \leq D|(\varphi^{-1})'(0)|\|h_0\|_{\mathbb{D}_{1-\delta}}$ and $\|\varphi^{-1} \circ h_0\|_{\mathbb{D}_{1-\delta}}^2 \leq K|U'_{1-\delta}|\|h_0\|_{\mathbb{D}_{1-\delta}}^2$, where $K > 0$ only depends in δ. Since h only depends in δ and r_0 (which depend in $mod(A)$ only) it follows that $|U|^{-1}\|h\|_U^2$ can be bounded in terms of $mod(A)$ only.

3.− The following lemma will be used in Section 11.4.

Lemma 11.3. *For every $\kappa > 0$ there is $\epsilon_1 = \epsilon_1(\delta, mod(A)) > 0$ such that if the relative area of $E \subset U$ in U is less than ϵ_1 then,*

$$\int\int_E \|Dh\|^2 dxdy \leq \kappa \|h\|_U.$$

Proof. All implicit constants and dependences will by in terms of $mod(A)$ only. Note that $\|h\|_U \geq |h(U)| = |U|$. Since $h = id$ in $U - U_{1-\delta}$, we have $\int\int_{E-U_{1-\delta}} \|Dh\|^2 dxdy = |E - U_{1-\delta}|$. On the other hand, since ψ is conformal,

$$\begin{aligned} \int\int_{E\cap U_{1-\delta}} \|Dh\|^2 dxdy &= \int\int_{\psi(E\cap U_{1-\delta})} \|D(\varphi^{-1}\circ h_0)\|^2 dxdy \\ &\leq C|(\varphi^{-1})'(0)|^2 |\psi(E\cap U_{1-\delta})|, \end{aligned}$$

the last considering that the distortion of φ^{-1} in $\mathbb{D}_{1-\delta}$ is bounded in terms of δ only and $\|Dh_0\|$ is bounded in terms of $mod(A)$ only. Then the lemma follows considering that if the relative area of $E \cap U_{1-\delta}$ in $U_{1-\delta}$ is small then the relative area of $\psi(E \cap U_{1-\delta})$ in $\mathbb{D}_{1-\delta}$ is also small. □

11.3. **The quasi-affine property.** Let us describe the procedure to construct the function f of Lemma 11.2. We organize the connected components of $\mathcal{N}$ (or just *components* for short) in levels as follows. By definition the $K_i = g_{U_i}(K)$ are connected components of $\mathcal{N}$ and we assign them level 1. In general a component L can be written in a unique way as,

$$L = g_{U_{i_1}} \circ g_{U_{i_2}} \circ ... \circ g_{U_{i_n}}(K).$$

We assign to L level n and denote $g_{U_{i_1}} \circ ... \circ g_{U_{i_n}}$ by g_L. Note that such L has naturally associated $W = g_L(U)$, then we denote $g_W = g_L$ and $W' = g_W(U')$ and we say that W is of level n.

We will define inductively functions $f_n : A \longrightarrow S^1 \times (0,1)$ such that $f_n(z) = f_{n-1}(z)$ unless $z \in L$ for some L of level n. Moreover f_n will satisfy properties 1 and 3 of Lemma 11.2 and property 2 for all connected components of $\mathcal{N}$ of level less than or equal to n. In Section 11.4 we prove that $\{f_n\}_{i\geq 1}$ is a Cauchy sequence in $W^{1,2}(A)$.

Furthermore f_n will satisfy the following property by induction,

Quasi-affine property. *There is $D > 0$ such that for all W of level $n+1$, there is a real affine map A_W of $\mathbb{R}^2$ and a function $f_W : U \to \mathbb{C}$ that is C^2 such that $\|f_W - id\|_{C^2,W} < D$ and*

$$f_n \circ g_W = A_W \circ f_W : U \longrightarrow f_n(U_{i_{n+1}}).$$

where $W = g_{U_{i_1}} \circ ... \circ g_{U_{i_{n+1}}}(U)$.

Let $f_0 : A \longrightarrow S^1 \times (0,1)$ be an homeomorphism resulting by composing a conformal representation of A into a straight cylinder, with an appropriated real affine map. Then the quasi-affine property for f_0 follows by Koebe Distortion Theorem for some constant $D = D_0$, which is small as m is big.

Inductive step. Suppose by induction that we are given f_{n-1} with the properties above, satisfying the quasi-affine property with some small constant $D = D_{n-1}$. We will construct f_n satisfying the properties above and the quasi-affine property with some small constant D_n. For W of level n define f_n in W as,

$$f_n|_W = f_{n-1} \circ g_W \circ h \circ g_W^{-1},$$

and for points not in any W of level n define f_n as f_{n-1}. Then f_n is continuous, since the diameters of the components of level n go to 0; see Remark 11.1. Moreover f_n satisfies 1 and 3 of Lemma 11.2 and 2 of the same lemma for components of level less than or equal to n.

Lemma 11.4. *Given $d > 0$ there is $m_1 = m_1(mod(A)) > 0$ such that if $m \geq m_1$ then for all U_i,*

$$\|A_{U_i}^{-1} \circ h \circ g_{U_i} - id\|_{C^2,U} < d.$$

where $A_{U_i} = D(h \circ g_{U_i})(z_0)$.

Proof. Since h is C^2 in $A = U - K$ is enough to prove the lemma for components U_i close to K. If $m \leq mod(U' - \overline{U})$ is big enough we may suppose that $\tilde{U}_i \subset U_{1-\delta}$, where $\tilde{U}_i \subset U_i'$ is uniquely determined by

$$mod(U_i' - \overline{\tilde{U}}_i) = mod(\tilde{U}_i - \overline{U}_i) = \frac{1}{2} mod(U_i' - \overline{U}_i) \geq \frac{m}{2}.$$

Since $\psi \circ g_{U_i}$ is conformal $mod(\psi(\tilde{U}_i - U_i)) \geq \frac{m}{2}$ and ψ is C^2 close to a conformal affine map in U_i by Koebe Distortion Theorem. Moreover there are $\theta(m) = \mathcal{O}(e^{-m})$ and $\eta(m) = 1 + \mathcal{O}(e^{-m})$ such that

$$\psi(U_i) \subset (r_0 + t_0, r_0 + \eta(m)t_0) \times (\theta_0, \theta_0 + \theta(m))$$

in polar coordinates, for some θ_0 and t_0. Moreover $h_0(r, \theta) = (\tilde{h}_0(r), c(\theta)(r - r_0))$, where $c(\theta)$ is analytic in θ and with distortion bounded in terms of $mod(A)$ only. Since $\tilde{h}_0$ is differentiable at r_0, $\partial_r h_0$ is almost constant in $\psi(U_i)$. Furthermore,

$$\sup_{z_0, z_1 \in \psi(U_i')} \frac{\partial_\theta h_0(z_0)}{\partial_\theta h_0(z_1)} = \mathcal{O}(e^{-m}),$$

hence,

$$\|A_{U_i}^{-1} \circ h \circ g_{U_i} - id\|_{C^2,U} = \mathcal{O}(e^{-m}).$$

This considering that $diam(h_0(\psi(U_i)))$ is small and therefore the distortion of φ in this set is small by Koebe distortion. $\square$

Now lets prove the quasi-affine property for f_n. So fix W of level n as above and $W_1 \subset W$ of level $n + 1$. Note that every component of level $n + 1$ is contained in some $\tilde{W}$ of level n, so we are in the general situation. Let U_i uniquely determined by $g_{W_1} = g_W \circ g_{U_i}$. By induction hypothesis we have $f_{n-1} \circ g_W = A_W \circ f_W$, where $\|f_W - id\|_{C^2,U} < D_{n-1}$ and A_W is affine. Note that by definition $f_n = A_W \circ f_W \circ h \circ g_W^{-1}$. Let A_{U_i} as in the lemma and let $A_{W_1} = A_W \circ A_{U_i}$, then if m is big enough,

$$\|A_{U_i}^{-1} \circ h \circ g_{U_i} - id\|_{C^2,U} < d,$$

for some fixed small d. Put

$$f_{W_1} = A_{W_1}^{-1} \circ A_W \circ f_W \circ h \circ g_{U_i} = (A_{U_i}^{-1} \circ f_W \circ A_{U_i}) \circ (A_{U_i}^{-1} \circ h \circ g_{U_i}).$$

Note that $\|A_{U_0}\| = \mathcal{O}(e^{-m})$ so A_{U_0} is a definite contraction for big m. By the quasi-affine property

$$\|A_{U_0}^{-1} \circ f_W \circ A_{U_0} - id\|_{C^2} = \mathcal{O}(D_{n-1}\|A_{U_0}\|),$$

so if m is big enough we get a bound τD_{n-1} for the norm above, for some $\tau = \tau(m, mod(A)) \in (0,1)$ independent of n. Considering that by hypothesis e^{-m} and D_{n-1} are small the quasi-affine property follows for f_n for some $D_n = \tau' D_{n-1} + \mathcal{O}(e^{-m})$, where $\tau' = \tau'(m, mod(A))$ is such that fixed $mod(A)$, $\tau' \to 0$ as $m \to \infty$. Hence, fixed $mod(A)$, if m is big enough the $\{D_n\}$ are small and in particular uniformly bounded.

11.4. **Sobolev estimates.** For a real affine map $\hat{A}$ denote $Dil(\hat{A}) = \sup_{|u|=|v|=1} \frac{|\hat{A}u|}{|\hat{A}v|}$. Let W be of level n and A_W, f_W as in the quasi-affine property, so $f_{n-1} \circ g_W = A_W \circ f_W : U \longrightarrow f_n(U_{i_n})$, where $g_W = g_{U_{i_1}} \circ ... \circ g_{U_{i_n}}$. By the quasi-affine property,

$$\begin{aligned}
\|f_{n-1}\|_W^2 &= \int\int_W \|D(A_W \circ f_W \circ g_W^{-1})\|^2 dxdy \\
&= \int\int_U \|D(A_W \circ f_W)\|^2 dxdy \\
&= \int\int_U Dil(A_W \circ f_W) Jac(A_W \circ f_W) dxdy \\
&\sim Dil(A_W)|f_{n-1}(W)|.
\end{aligned}$$

Considering that $f_n \circ g_W = A_W \circ f_W \circ h$ and by the quasi-affine property,

$$\begin{aligned}
\|f_n\|_W^2 &= \int\int_W \|D(A_W \circ f_W \circ h \circ g_W^{-1})\|^2 dxdy \\
&= \int\int_U \|D(A_W \circ f_W \circ h)\|^2 dxdy \\
&\leq \int\int_U Dil(D(A_W \circ f_W)) \circ h \, Jac(A_W \circ f_W) \circ \, h \|Dh\|^2 dxdy \\
&\sim Dil(A_W)|f_n(W)||U|^{-1}\|h\|_U^2.
\end{aligned}$$

By the previous it follows that $\|f_n\|_W^2 \leq K\|f_{n-1}\|_W^2$ for some definite $K > 0$ which only depends in m and $mod(A)$. Let W_0 be a component of level $n-1$, then

$$\begin{aligned}
\sum_{W \subset W_0, \text{ level n}} \|f_n\|_W &\leq K \int\int_{\cup W} \|Df_{n-1}\|^2 dxdy \\
&= K \int\int_{\cup W} \|D(A_{W_0} \circ f_{W_0} \circ h \circ g_{W_0}^{-1})\|^2 dxdy \\
&= K \int\int_{\cup g_{W_0}^{-1}(W)} \|D(A_{W_0} \circ f_{W_0} \circ h)\|^2 dxdy \\
&\leq K_2 Dil(A_{W_0}) \frac{|f_{n-1}(W_0)|}{|U|} \int\int_{\cup U_i} \|Dh\| dxdy \\
&\leq K_1 \kappa \|f_{n-2}\|_{W_0}
\end{aligned}$$

Where $K_1, K_2 > 0$ only depend in $mod(A)$ and κ can be taken arbitrarily small by letting ε small enough; see Lemma 11.3. Thus if ε is small enough it follows that there is $\gamma \in (0,1)$ such that for all n,

$$\sum_{W \text{ level } n} \|f_n\|_W \leq \gamma \sum_{W_1 \text{ level } n-2} \|f_{n-2}\|_{W_1}.$$

Thus $\|f_{n+1} - f_n\|_A = \mathcal{O}(\gamma^n)$ so $\{f_n\}$ is a Cauchy sequence in $W^{1,2}(A)$ and hence a convergent one. This proves Lemma 11.2 considering that the f_n converge uniformly to some f, by Remark 11.1 and the quasi-affine property.

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