NONDIVERGENCE OF HOROCYCLIC FLOWS ON MODULI SPACE

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Abstract. The earthquake flow and the Teichmüller horocycle flow are flows on bundles over the Riemann moduli space of a surface, and are similar in many respects to unipotent flows on homogeneous spaces of Lie groups. In analogy with results of Margulis, Dani and others in the homogeneous space setting, we prove strong nondivergence results for these flows. This extends previous work of Veech. As corollaries we obtain that every closed invariant set for the earthquake (resp. Teichmüller horocycle) flow contains a minimal set, and that almost every quadratic differential on a Teichmüller horocycle orbit has a uniquely ergodic vertical foliation.

Contents

1. Introduction

We consider two flows on noncompact moduli spaces associated with a surface S: Thurston's earthquake flow and the Teichmüller horocycle flow. In recent years many parallels between the dynamics of these flows and those of unipotent flows on homogeneous spaces of Lie groups have been discovered. A fundamental series of results in the homogeneous space setting originated with G. A. Margulis' 1971 result that there are no divergent orbits (i.e., orbits which eventually leave every compact subset of the space) for a unipotent flow. This nondivergence result was improved substantially in subsequent work of many authors, notably S. G. Dani, and has been important in many applications.

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In this paper we prove analogous results in the geometric setting, extending earlier work of W. A. Veech. For three statements in the homogeneous space setting (see below, Theorems U1, U2, U3) we obtain analogues for the earthquake flow (Theorems E1, E2, E3) and the Teichmüller horocycle flow (Theorems H1, H2, H3). These results are stated in §2. As a simple special case of our results, we obtain:

> *There are no divergent orbits for the earthquake (resp., Teichmüller horocycle) flow.*

Even this statement, recently proved by Veech for the Teichmüller horocycle flow, appears to be new for the earthquake flow.

In analogy with the developments in the homogeneous space setting following Margulis' original result, the nondivergence results we obtain are stronger in that they describe the relative amount of time that an orbit spends within a compact set K in terms of the size of K. Take the third statement for example. Theorem U3 (see §2 for a precise formulation), proved in 1998 by D. Kleinbock and Margulis [KlMa], gives for any point p in the space, any sufficiently large compact set K, and *any* $T > 0$, an upper bound on the relative time, up to time T, that the orbit of p under a unipotent flow spends in K. For various applications, it is important that the bound can in some sense be made independent of p and T. Our corresponding results are (see Theorems E3, H3):

> *There are constants C, α, depending only on S, such that for every p in the flow-space, there is $\rho > 0$ such that for every $T > 0$ and every $0 < \epsilon < \rho$ we have:*
>
> $$\frac{|\{t \in [0,T] : \phi_t p \notin K_\epsilon\}|}{T} < C \cdot \left(\frac{\epsilon}{\rho}\right)^\alpha,$$
>
> *where $|\cdot|$ denotes Lebesgue measure on $\mathbb{R}$, ϕ_t denotes the earthquake (resp., Teichmüller horocycle) map at time t, and $\{K_\epsilon\}_{\epsilon>0}$ is a natural exhaustion of the flow-space by compact sets.*

Our results have applications of two kinds. The first concerns dynamical properties of the earthquake flow and the Teichmüller horocycle flow. We obtain the following (Corollaries 2.6 and 2.7):

- *Any closed subset of the flow-space which is invariant under the earthquake (resp. Teichmüller horocycle) flow contains a minimal closed invariant subset. This subset is compact.*
- *Any locally finite invariant ergodic measure for the action of the earthquake (resp. Teichmüller horocycle) flow is finite.*

The second statement was proved previously by Veech in [Ve] for the Teichmüller horocycle flow.

The second kind of application concerns the Teichmüller *geodesic* flow and unique ergodicity of foliations. As in [KlMa] (and also implicitly in

[KeMasSm]), it is possible to use nondivergence along certain paths (e.g., horocycles, orbits of the circle group, or polynomial curves) to obtain information about the behavior of a typical geodesic near infinity. This makes it possible to extend some previous results of Veech, Masur, and Kerckhoff–Masur–Smillie. Thus we obtain (Corollary 2.8):

> *Let q be a quadratic differential, and let $\{h_s q : s \in \mathbb{R}\}$ be its Teichmüller horocycle orbit. Then for almost every s, the vertical foliation associated to $h_s q$ is uniquely ergodic.*

In the above result, 'almost every s' can be understood with respect to either Lebesgue measure, or any measure on the real line satisfying a certain decay condition similar to one introduced in [Ve].

To conclude this introduction, we briefly describe the relation of our work to that of other authors. Many of the ideas used in our proofs for the Teichmüller horocycle flow were introduced by Kerckhoff–Masur–Smillie [KeMasSm] and developed further by Veech [Ve]. A nonessential difference between these papers and our work is that they deal with the action of the circle group $\mathrm{SO}(2, \mathbb{R})$, and use it to obtain information about horocycles, while we work directly with horocycles. An essential difference is that, in order to obtain our quantitative nondivergence results, we must avoid an argument by contradiction and a passage to a limit used by previous authors. This yields an effective proof, on which Theorem H3 and Corollary 2.8 depend.

For the earthquake flow, the questions considered here do not seem to have been considered previously. In fact it appears that many rather obvious questions regarding the dynamical properties of the earthquake flow (e.g. description of invariant measures, ergodicity, etc.) await a systematic study.

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1.1. **Organization of the Paper.** In §2 we describe the results, obtained by Margulis, Dani, and others for homogeneous spaces, which motivate this paper, and state our results. To give an idea of our arguments, in §3 we have isolated a simple idea, which is a component of our proofs for both the earthquake flows and Teichmüller horocycle flow, and appears in the work of previous authors. We call this the "sparse cover argument". Since the preliminaries required for formulating and proving our results are rather involved, they are deferred to a separate section, §4. In the hope of making this paper understandable to those who are not experts in Teichmüller theory, and given the lack of a suitable survey paper, we have tried in §4 to give systematic definitions and pointers to the somewhat intimidating literature. In §5 and §6 we prove the nondivergence results for the earthquake flow and Teichmüller horocycle flow respectively. §7 is devoted to the proof of the applications, and §8 to a result which demonstrates (in the case that S is a punctured torus or a quadruply punctured sphere) that while they share many properties, the earthquake flow and Teichmüller horocycle flow are quite different.

2. Strong nondivergence

A homogeneous space flow is given by a Lie group G, a closed subgroup Γ, and a one–parameter subgroup $U = \{u_t : t \in \mathbb{R}\}$, where U acts on the quotient G/Γ by $u(g\Gamma) = (ug)\Gamma$. For example, taking $G = \mathrm{SL}(2, \mathbb{R})$, $\Gamma = \mathrm{SL}(2, \mathbb{Z})$, the flow-space G/Γ is the unit tangent bundle to the modular surface and the flow is the geodesic flow when U is the diagonal group. It is the horocycle flow when U is the group of upper triangular unipotent (all eigenvalues 1) matrices.

2.1. **A short history of the 'Margulis Lemma'.** A series of fundamental theorems in the study of the dynamics of unipotent subgroups acting on homogeneous spaces originated with the following result, proved by Margulis [Ma1] in 1971:

Lemma 2.1 (Margulis)**.** *There are no divergent trajectories for the action of a one–parameter unipotent subgroup on* $\mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{Z})$.

(A subgroup $U \subset \mathrm{SL}(n, \mathbb{R})$ is *unipotent* if every element $u \in U$ is unipotent).

The result, sometimes referred to as the 'Margulis Lemma',[1] was considerably strengthened in subsequent work of Margulis and many other authors, notably Dani. We now give a brief survey of these developments, which motivate this paper. In all of the results stated below, $G = \mathrm{SL}(n, \mathbb{R})$, $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ and $\{u_t\}$ is a unipotent one–parameter subgroup of G (the results are actually valid in a more general setting). Letting $|\cdot|$ denote Lebesgue measure on $\mathbb{R}$, for $K \subset G/\Gamma$, $x \in G/\Gamma$ and $T > 0$, we define

$$\mathrm{Avg}_{T,x}(K) = \frac{|\{t \in [0, T] : u_t x \in K\}|}{T}. \tag{1}$$

In the 1979 paper [Da1], Dani obtained the following strengthening of Lemma 2.1, which shows not only that the orbit returns to a compact set infinitely often, but that it returns along a set of times with positive lower density:

Theorem 2.2. [Da1, Theorem 2.10] *For any* $x \in G/\Gamma$ *and any* $\{u_t\}$*, there is a compact subset* $K \subset G/\Gamma$ *such that*

$$\liminf_{T \to \infty} \mathrm{Avg}_{T,x}(K) > 0.$$

In the same paper, Dani obtained the following corollary:

Corollary 2.3. [Da1, Theorem 0.1] *Every locally finite* $\{u_t\}$*-invariant,* $\{u_t\}$*-ergodic measure on* G/Γ *is finite.*

[1] A wealth of other results by Margulis render this terminology somewhat ambiguous.

In 1986 Dani obtained two results strengthening Theorem 2.2. The first shows that for large enough K, the time averages can be made arbitrarily close to one, and that K can be chosen uniformly for points in a compact:

Theorem U1. [Da3, Theorem 6.2] *For any compact $K \subset G/\Gamma$ and any $\epsilon > 0$ there is a (larger) compact $K' \subset G/\Gamma$ such that for any $x \in K$,*

$$\liminf_{T\to\infty} \mathrm{Avg}_{T,x}(K') \geq 1 - \epsilon.$$

The second shows that K can in fact be chosen uniformly for all points in the space, outside of a natural class of potential counterexamples. Let us say that an orbit $\{u_t x\}$ is *constant on a subspace* if there is a proper linear subspace $W \subset \mathbb{R}^n$, defined over $\mathbb{Q}$, such that $t \mapsto \|u_t \mathbf{x} \mathbf{p}_W\|$ is a constant function. Here $\mathbf{x} \in G$ is a pre-image of x and $\mathbf{p}_W \in \bigwedge^{\dim W} \mathbb{R}^n$ is a nontrivial element in the one–dimensional subspace corresponding to W.

Theorem U2. [Da2, Theorem 3.1] *For any $\epsilon > 0$ there is a compact $K \subset G/\Gamma$ such that for any $x \in G/\Gamma$, one of the following holds:*

(1) $\liminf_{T\to\infty} \mathrm{Avg}_{T,x}(K) \geq 1 - \epsilon$.
(2) $\{u_t x\}$ *is constant on a subspace.*

In [DaMa], Dani and Margulis deduced the following:

Corollary 2.4. *Any closed $\{u_t\}$-invariant set contains a minimal closed $\{u_t\}$-invariant set. Any minimal closed $\{u_t\}$-invariant subset is compact.*

In 1998, in the course of their work on Diophantine approximation on manifolds, D. Kleinbock and Margulis obtained a quantitative version of Theorem U1, relating $\mathrm{Avg}_{T,x}(K)$ with the size of K. To state it, define for every $\epsilon > 0$,

$$K_\epsilon = \{y\Gamma \in G/\Gamma : \forall v \in \mathbb{Z}^n - 0, \|yv\| \geq \epsilon\}.$$

By Mahler's compactness criterion, each K_ϵ is compact, and each compact K is contained in K_ϵ for some ϵ.

Theorem U3. [KlMa, Theorem 5.3] *There exists $C > 0$ such that for any $x \in G/\Gamma$ there exists ρ such that for any $0 < \epsilon < \rho$ and any $T > 0$,*

$$\mathrm{Avg}_{T,x}(K_\epsilon) \geq 1 - C\left(\frac{\epsilon}{\rho}\right)^{1/n^2}.$$

It is essential for number–theoretic applications that the constants C and $\frac{1}{n^2}$ in the above statement do not depend on the point x.

See also the papers [Sh, EsMoSh, KlMa] for more work extending these results in various directions and for various applications.

2.2. **Strong nondivergence for earthquakes.** We introduce only the notation which will be required for stating our results, referring the reader to §4 where the notation is re-introduced with more care and detail.

Let S be a surface, $\widetilde{\mathcal{P}}$ (resp. $\widetilde{\mathcal{P}_1}$) the associated bundle (resp. moduli space) of geodesic measured laminations over hyperbolic metrics, and π :

$\widetilde{\mathcal{P}} \to \widetilde{\mathcal{P}_1}$ the quotient map. For every $t \in \mathbb{R}$, $E_t : \widetilde{\mathcal{P}_1} \to \widetilde{\mathcal{P}_1}$ denotes the earthquake map at time t. Let Γ^S be the equivalence classes of simple closed curves and for each $\gamma \in \Gamma^S$ and $\mathbf{p} \in \widetilde{\mathcal{P}}$ let $\ell_{\mathbf{p},\gamma}$ be the length of γ with respect to $\pi(\mathbf{p})$ and let $\ell_{\mathbf{p},\gamma}(t)$ be the length of γ with respect to the $E_t\pi(\mathbf{p})$. Averaging measures refer to averages along orbits of the earthquake flow (that is, in (1), u_t is replaced by E_t).

For $\epsilon > 0$ we let:

$$K_\epsilon = \pi(\{p \in \widetilde{\mathcal{P}_1} : \forall \gamma \in \Gamma^S, \ell_{p,\gamma} \geq \epsilon\}). \tag{2}$$

By a result of D. Mumford [Mu], K_ϵ is compact, and each compact $K \subset \mathcal{P}_1$ is contained in K_ϵ for some ϵ.

We prove the following:

Theorem E1. *For any $\epsilon > 0$ and any compact $K \subset \mathcal{P}_1$ there is a (larger) compact $K' \subset \mathcal{P}_1$ such that for every $p \in K$ and every $T > 0$ we have:*

$$\mathrm{Avg}_{T,p}(K') \geq 1 - \epsilon.$$

In order to state the next theorem, say that an earthquake orbit $\{E_t p\}$ is *constant on a subsurface* if, fixing $\mathbf{p} \in \pi^{-1}(p)$, there is an element $\gamma \in \Gamma^S$ such that $t \mapsto \ell_{\mathbf{p},\gamma}(t)$ is a constant function.

Theorem E2. *For any $\epsilon > 0$ there is a compact subset $K \subset \mathcal{P}_1$ such that for any $p \in \mathcal{P}_1$, one of the following statements holds:*

(1) $\liminf_{T\to\infty} \mathrm{Avg}_{T,p}(K) \geq 1 - \epsilon$.
(2) $\{E_t p\}$ *is constant on a subsurface.*

Theorem E3. *There is $C > 0$ such that for every $p \in \mathcal{P}_1$ there is $\rho > 0$ such that for every $T > 0$ and every $0 < \epsilon < \rho$ we have:*

$$\mathrm{Avg}_{T,p}(K_\epsilon) \geq 1 - \frac{C\epsilon}{\rho}. \tag{3}$$

2.3. **Strong nondivergence for Teichmüller horocycles.** Once again, we introduce only the notation required for stating the results and refer the reader to §4 for more details.

Let $\widetilde{\mathcal{Q}_1}$ (resp. $\mathcal{Q}_1$) be the bundle (resp. moduli space) of unit area holomorphic quadratic differentials on S and let $\pi : \widetilde{\mathcal{Q}_1} \to \mathcal{Q}_1$ be the quotient map. Let $h_t : \mathcal{Q}_1 \to \mathcal{Q}_1$ denote the Teichmüller horocycle map at time t. Averaging measures refer to averages along orbits of this flow (that is, in (1), u_t is replaced with h_t).

For $\mathbf{q} \in \widetilde{\mathcal{Q}_1}$ let $\mathcal{L}_{\mathbf{q}}$ be the set of saddle connections of $\mathbf{q}$ on S, and for $\delta \in \mathcal{L}_{\mathbf{q}}$, let $l_{\mathbf{q},\delta}$ denote the length of δ in the flat metric corresponding to $\mathbf{q}$ and let $l_{\mathbf{q},\delta}(t)$ be the length of δ in the flat metric corresponding to $h_t\pi(\mathbf{q})$.

Define, for $\epsilon > 0$,

$$K_\epsilon = \pi(\{q \in \widetilde{\mathcal{Q}_1} : \forall \delta \in \mathcal{L}_q, l_{q,\delta} \geq \epsilon\}).$$

As before, this is a compact subset of $\mathcal{Q}_1$. Also, $\mathcal{Q}_1 = \bigcup_{\epsilon>0} K_\epsilon$.

Caution: In this setting it is no longer true that every compact $K \subset \mathcal{Q}_1$ is contained in K_ϵ for some ϵ. This accounts for the somewhat weaker formulation of Theorem H1 below.

We prove the following:

Theorem H1. *For any $\epsilon > 0$ and any $\eta > 0$ there is $\eta' > 0$ such that for every $q \in K_\eta$ and $T > 0$ we have:*

$$\mathrm{Avg}_{t,q}(K_{\eta'}) \geq 1 - \epsilon.$$

The special case of Theorem H1 in which K_η is replaced with a single point was proved by Veech [Ve, Theorem 5.28].

We say that a Teichmüller horocycle $\{h_t q\}$ is *constant on a saddle connection* if, fixing $\mathbf{q} \in \pi^{-1}(q)$, there is an element $\delta \in \mathcal{L}_{\mathbf{q}}$ such that $t \mapsto l_{\mathbf{q},\delta}(t)$ is a constant function.

Theorem H2. *For any $\epsilon > 0$ there is a compact subset $K \subset \mathcal{Q}_1$ such that for any $q \in \mathcal{Q}_1$, one of the following statements holds:*

1. $\liminf_{T\to\infty} \mathrm{Avg}_{T,q}(K) \geq 1 - \epsilon$.
2. $\{h_t q\}$ *is constant on a saddle connection.*

Theorem H3. *There are constants $C > 0$, $\alpha > 0$ such that for every $q \in \mathcal{Q}_1$ there is $\rho > 0$ such that for every $T > 0$ and every $0 < \epsilon < \rho$ we have:*

$$\mathrm{Avg}_{T,q}(K_\epsilon) \geq 1 - C \cdot \left(\frac{\epsilon}{\rho}\right)^\alpha.$$

Remark 2.5. *We obtain explicit bounds on the constants C, α appearing in the statement of the theorem (see the remark following Theorem 6.3).*

2.4. **Applications.** We now state some applications of our nondivergence results. The first two are analogous to Corollaries 2.3 and 2.4:

Corollary 2.6. *For both the earthquake flow and the Teichmüller horocycle flow, any locally finite ergodic invariant measure is finite.*

In the flow-space $\mathcal{Q}_1$, any locally finite invariant measure which is ergodic for the action of $\mathrm{SL}(2,\mathbb{R})$ *is finite.*

(A measure is locally finite if it is finite on compact sets).

For the Teichmüller horocycle flow this was proved by Veech (see [Ve, Theorem 0.4]).

Corollary 2.7. *For both the earthquake flow and the Teichmüller horocycle flow, any closed invariant set contains a minimal closed invariant subset. Any minimal closed invariant set is compact.*

Whereas Corollary 2.6 follows from our nondivergence results by an argument of Dani (which is included for the reader's convenience), the proof of Corollary 2.7 requires some additional geometric arguments.

Our results also yield information regarding the Teichmüller *geodesic* flow. Recall that a quadratic differential $q \in \widetilde{\mathcal{Q}_1}$ is said to be *uniquely ergodic* if there is a unique (up to scaling) transverse invariant measure for the vertical foliation defined by q. It was proved by Masur [Mas1] that almost every $q \in \mathcal{Q}_1$ is uniquely ergodic. A natural question is how the set of uniquely ergodic points intersects lower–dimensional subsets of $\mathcal{Q}_1$, for example submanifolds, Cantor sets, etc. Instances of this general question have been considered in a number of papers. The main result of [KeMasSm] is that for every $q \in \mathcal{Q}_1$ and almost every θ (with respect to Lebesgue measure on $\mathbb{T}$), $e^{i\theta}q$ is uniquely ergodic. The main result of [Ve] is that for a measure μ satisfying a certain decay condition (see below), for every $q \in \mathcal{Q}_1$, and $\mu-$almost every θ, $e^{i\theta}q$ is uniquely ergodic. Our results yield similar conclusions, in which the orbit of the circle group is replaced with an orbit for the Teichmüller horocycle flow $\{h_s : s \in \mathbb{R}\}$:

Corollary 2.8. *Let $q \in \mathcal{Q}_1$. For almost every $s \in \mathbb{R}$, $h_s q$ is uniquely ergodic.*

To state our results for general measures, we make some definitions. Let $B(x,r)$ denote the interval $(x-r, x+r) \subset \mathbb{R}$, let μ be a Borel measure on $\mathbb{R}$, and let $F : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function. We say that μ is *F-decaying* if for every $x \in \mathbb{R}$, every $r > 0$ and every $0 < \epsilon < 1$ we have

$$\mu(B(x, \epsilon r)) \leq F(\epsilon)\mu(B(x,r)). \tag{4}$$

Note that Lebesgue measure is F-decaying for $F(x) = x$. Another example of an F-decaying measure is the standard measure on the middle–thirds Cantor set, which is F-decaying with $F(x) = Cx^{\frac{\log 2}{\log 3}}$. See [Ve, §2] for more details.

We say that μ satisfies *Veech's decay condition* if it is F-decaying for some F satisfying $\lim_{x \to 0} F(x) = 0$.

Corollary 2.9. *If μ is a measure on $\mathbb{R}$ which satisfies Veech's decay condition, then for μ-almost every s, $h_s q$ is uniquely ergodic.*

3. Good families and sparse covers

In order to explain our ideas, let us begin with a simple argument, which we call the 'Sparse Cover argument', and which will be useful in the sequel. It illustrates how a polynomial–like property of length functions gives a quantitative estimate of the amount of time in which there is at least one small length function.

Let $\mathcal{F}$ be a collection of continuous functions $\mathbb{R} \to \mathbb{R}_+$. For $\theta > 0$, $f \in \mathcal{F}$ and $I \subset \mathbb{R}$ an interval, we let

$$I_{f,\theta} = \{s \in I : f(s) < \theta\},$$
$$I_{\mathcal{F},\theta} = \{s \in I : \exists f \in \mathcal{F}, f(s) < \theta\},$$

and

$$\|f\|_I = \sup_{s \in I} f(s).$$

Definition 3.1. *Let C, α, ρ be positive constants. We say that $\mathcal{F}$ is (C, α, ρ)-* **good** *if for every interval $I \subset \mathbb{R}$ and every $f \in \mathcal{F}$ we have, for $0 < \epsilon < \rho$,*

$$\frac{|I_{f,\epsilon}|}{|I|} \leq C \left(\frac{\epsilon}{\|f\|_I} \right)^{\alpha} \tag{5}$$

(where $|\cdot|$ denotes Lebesgue measure on $\mathbb{R}$).

*We say that $\mathcal{F}$ is (C, α)-***good** *if it is (C, α, ρ)-good for every ρ.*

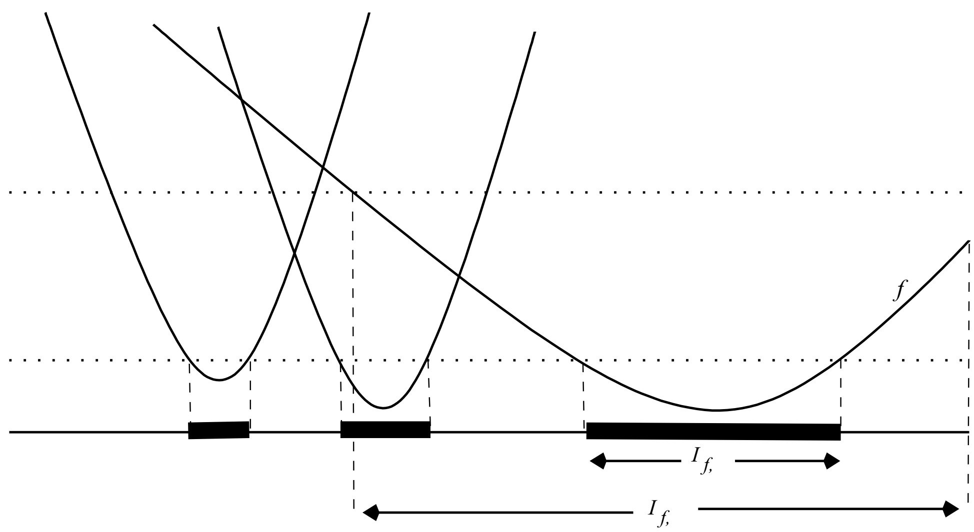

FIGURE 1. A few functions from a good family. $I_{\mathcal{F},\epsilon}$ is thickened on the x axis.

Definition 3.1 is motivated by work of Kleinbock and Margulis. An example of a (C, α)-good collection is any collection of polynomials with a uniform bound on the degree. See [KlMa, §3] for more details.

We are interested in estimating the measure of $I_{\mathcal{F},\epsilon}$.

Proposition 3.2 (Sparse Cover)**.** *Let $\mathcal{F}$ and I be as above. Suppose that there are positive C, α, ρ, M such that:*

1. *$\mathcal{F}$ is (C, α, ρ)-good.*
2. *For every $f \in \mathcal{F}$, $\|f\|_I \geq \rho$.*
3. *For every $s \in I$,*

$$\#\{f \in \mathcal{F} : f(s) < \rho\} \leq M.$$

Then for every $0 < \epsilon < \rho$,

$$\frac{|I_{\mathcal{F},\epsilon}|}{|I|} \leq MC \left(\frac{\epsilon}{\rho} \right)^{\alpha}. \tag{6}$$

Proof: Let

$$J = \int_I \#\{f \in \mathcal{F} : f(s) < \rho\} ds.$$

The integrand in J is bounded above by M, and hence

$$J \leq M|I|. \tag{7}$$

For any $f \in \mathcal{F}$, write $I_{f,\rho}$ as a countable union of disjoint intervals $\bigcup_{V \in \mathcal{V}} V$. For each $V \in \mathcal{V}$ we have by (5) that

$$|V_{f,\epsilon}| \leq C|V| \left(\frac{\epsilon}{\rho}\right)^{\alpha}$$

and hence

$$|I_{f,\epsilon}| \leq C|I_{f,\rho}| \left(\frac{\epsilon}{\rho}\right)^{\alpha}.$$

Using continuity of each $f \in \mathcal{F}$ and assumption 3, we see that the set

$$\{f \in \mathcal{F} : I_{f,\rho} \neq \emptyset\}$$

is countable, so the sum $\sum_{f \in \mathcal{F}} |I_{f,\rho}|$ is well–defined. Thus we obtain:

$$\begin{aligned} J &= \sum_{f \in \mathcal{F}} |I_{f,\rho}| \\ &\geq \left[C\left(\frac{\epsilon}{\rho}\right)^{\alpha}\right]^{-1} \sum_{f \in \mathcal{F}} |I_{f,\epsilon}| \\ &\geq \left[C\left(\frac{\epsilon}{\rho}\right)^{\alpha}\right]^{-1} |I_{\mathcal{F},\epsilon}|. \end{aligned} \tag{8}$$

where the last line follows from the fact that $I_{\mathcal{F},\epsilon} = \bigcup_{f \in \mathcal{F}} I_{f,\epsilon}$.

Combining (7) and (8) yields (6). $\square$

3.1. **Ideas for proving non-divergence.** We explain our strategy for proving our non-divergence results. By a result of Mumford, if a sequence $p_n \in \mathcal{P}_1$ (resp. $q_n \in \mathcal{Q}_1$) leaves compact sets, then there are simple closed curves γ_n (resp. saddle connections δ_n) such that $\ell_{p_n,\gamma_n} \to 0$ (resp. $l_{q_n,\delta_n} \to 0$). From this one sees that our non-divergence results would follow if the conditions of Proposition 3.2 hold for the collection $\mathcal{F}$ of all length functions $\ell_\gamma(t)$ (resp. $l_\delta(t)$).

For the *hyperbolic* length functions $\ell_{p,\gamma}(t)$ this turns out to be true. More precisely, the verification that $\mathcal{F}$ is (C, α, ρ)-good for appropriate C, α, ρ is the main result of §5. The remaining conditions are easily verified.

For the *flat* length functions $l_{q,\delta}(t)$, the fact that $\mathcal{F}$ is good is easily verified, but condition 3 does not hold, and the idea of the proof is to find a subcollection $\mathcal{F}_0 \subset \mathcal{F}$ for which condition 3 does hold, and such that, for a large set of t, if there is $\delta \in \mathcal{F}$ for which $l_\delta(t)$ is small then there is also $\delta' \in \mathcal{F}_0$ for which $l_{\delta'}(t)$ is small. Finding this subset $\mathcal{F}_0$ is the main result of §6 (proof of Theorem 6.3).

3.2. **μ-good functions.** We record a variant of Proposition 3.2 which will be useful when dealing with measures satisfying a certain decay condition (see §6.5). To this end, let ρ be a positive constant, let $I \subset \mathbb{R}$ be an interval, let μ be a regular Borel measure on $\mathbb{R}$, and let $F : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function.

Definition 3.3. *We say that $\mathcal{F}$ is (μ, F, ρ)*-**good on** *I if for every $f \in \mathcal{F}$ and $0 < \epsilon < \rho$ we have*

$$\frac{\mu(I_{f,\epsilon})}{\mu(I_{f,\rho})} \le F(\epsilon/\rho).$$

The following more general form of Proposition 3.2 follows using the same proof:

Proposition 3.4. *Let $\mathcal{F}$ and I be as above. Suppose that there is a positive M, and μ, F, ρ as above such that:*

1. *$\mathcal{F}$ is (μ, F, ρ)-good on I.*
2. *For every $s \in I$,*

$$\#\{f \in \mathcal{F} : f(s) < \rho\} \le M.$$

Then for any $0 < \epsilon < \rho$,

$$\frac{\mu(I_{\mathcal{F},\epsilon})}{\mu(I)} \le MF(\epsilon/\rho). \tag{9}$$

□

4. The flow spaces

In this section we will give the necessary definitions for working with the earthquake flow-space $\widetilde{\mathcal{P}_1}$ and the Teichmüller flow-space $\widetilde{\mathcal{Q}_1}$. Below we let S be an orientable surface of genus g with n punctures, where $\chi(S) = 2 - 2g - n < 0$ so that S admits a hyperbolic structure.

Let $\mathrm{Homeo}_+(S)$ denote the group of orientation-preserving homeomorphisms of S, $\mathrm{Homeo}_0(S)$ its identity component, and let

$$\mathrm{Mod}(S) = \mathrm{Homeo}_+(S)/\,\mathrm{Homeo}_0(S)$$

denote the mapping class group.

The Teichmüller space of S, denoted $\mathrm{Teich}(S)$, is the space of all finite-area complete hyperbolic metrics (equivalently analytically finite complex structures) on S, modulo the action of $\mathrm{Homeo}_0(S)$. "Analytically finite" means that punctures have neighborhoods holomorphically equivalent to a disk minus a point. Another description is the set of all faithful representations $\pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ with discrete image, such that loops around punctures are mapped to parabolic elements, modulo conjugation of the image in $\mathrm{PSL}(2,\mathbb{R})$. The image group is called a Fuchsian group, and the quotient of $\mathbb{H}^2$ by its action yields an identification of S with a hyperbolic surface. See e.g. [Ab] or [Gar] for details.

The Riemann moduli space is $\mathrm{Teich}(S)/\mathrm{Mod}(S)$, or in other words the space of all hyperbolic (or complex) structures up to homeomorphisms.

4.1. **Measured Laminations.** Let σ be a complete hyperbolic metric on S. A *geodesic lamination* λ on (S,σ) is a closed subset of S foliated by geodesics. A *measured lamination* λ is a geodesic lamination together with a Borel measure on every closed arc transverse to λ, so that this family of measures is invariant under restriction and isotopy through transverse arcs. We also assume that λ is the support of its measure – i.e. no arc crossing λ has zero measure.

The measured laminations with compact support make up a space denoted by $\mathcal{ML}(S,\sigma)$, which has a natural topology coming from weak-* convergence of the measures on transversals. Thurston introduced this space and proved that it is homeomorphic to $\mathbb{R}^{6g-6+n}$. (For more about laminations, see [CasBle], [Bon1], or [Bon2]).

Let Γ^S be the set of free homotopy classes of unoriented simple closed curves on S, and let us write $\mathbb{R}_+\Gamma^S = \mathbb{R}_+ \times \Gamma^S$. Given σ, $\mathbb{R}_+\Gamma^S$ can be identified with a subset of $\mathcal{ML}(S,\sigma)$, by associating to $w\gamma \equiv (w,\gamma)$ the lamination with support the geodesic representative of γ, and transverse measure assigning mass w to every intersection point of an arc with the geodesic. Thurston showed that this subset, which we call the *simple laminations*, is dense in $\mathcal{ML}(S,\sigma)$.

There is an action of $\mathbb{R}_+$ on all of $\mathcal{ML}(S,\sigma)$, given by multiplying the transverse measure by positive numbers. The quotient

$$\mathcal{PML}(S,\sigma) = (\mathcal{ML}(S,\sigma) - 0)/\mathbb{R}_+,$$

known as the *projective measured lamination space*, is homeomorphic to a sphere.

If a different metric σ' is used the space $\mathcal{ML}(S,\sigma')$ is naturally homeomorphic to $\mathcal{ML}(S,\sigma)$, via a map that restricts to the identity on $\mathbb{R}_+\Gamma^S$. Thus it is justifiable to write $\mathcal{ML}(S)$ and $\mathcal{PML}(S)$, omitting the metric.

Length and intersection number: The length of $\lambda \in \mathcal{ML}(S,\sigma)$ is defined by integrating over the support of λ the product of the transverse measure and the arclength measure on the leaves. On simple laminations $w\gamma$, this reduces to the σ-length of γ, multiplied by w. Denote this length by $\ell(\lambda,\sigma)$.

The geometric intersection number of two geodesics, which is just the number of their transverse intersections, extends to a function $i : \mathcal{ML}(S) \times \mathcal{ML}(S) \to \mathbb{R}$, which is the total mass of the product of the transverse measures.

Both i and ℓ are continuous. Both are homogeneous in their lamination parameters, i.e. $i(a\lambda, b\mu) = ab\,i(\lambda,\mu)$, and $\ell(c\lambda,\sigma) = c\ell(\lambda,\sigma)$. In particular note that for each $\sigma \in \mathrm{Teich}(S)$, $\mathcal{PML}(S)$ can be identified with $\{\lambda \in \mathcal{ML}(S) : \ell(\lambda,\sigma) = 1\}$.

The lamination bundles: Let

$$\widetilde{\mathcal{P}} = \mathrm{Teich}(S) \times \mathcal{ML}(S)$$

and let $\widetilde{\mathcal{P}_1} \subset \widetilde{\mathcal{P}}$ be the subset of *unit-length laminations*, namely the set of pairs (σ, λ) with $\ell(\lambda, \sigma) = 1$. We can make the identification

$$\widetilde{\mathcal{P}_1} = \mathrm{Teich}(S) \times \mathcal{PML}(S)$$

by the above discussion. If $\mathbf{p} \in \widetilde{\mathcal{P}}$ we denote its coordinates by $\sigma_{\mathbf{p}}$ and $\lambda_{\mathbf{p}}$.

The mapping class group $\mathrm{Mod}(S)$ acts naturally on both metrics and laminations. The action on $\widetilde{\mathcal{P}}$ is discrete, and we define $\mathcal{P} = \widetilde{\mathcal{P}}/\operatorname{Mod}(S)$ and $\mathcal{P}_1 = \widetilde{\mathcal{P}_1}/\operatorname{Mod}(S)$. Let $\pi : \widetilde{\mathcal{P}_1} \to \mathcal{P}_1$ be the quotient map.

4.2. **The earthquake flow.** Thurston introduced earthquakes in the late 70's as a generalization of Dehn twists. Kerckhoff [Ker2] applied them to prove the Nielsen realization conjecture. A very readable and general development can be found in [Th]. We will give a simplified account suitable for our purposes.

Let us first define an earthquake along a simple closed geodesic λ in a hyperbolic surface (S, σ). Intuitively this is a change of hyperbolic structure produced by cutting along λ, shifting the two sides a relative distance t along λ, and re-gluing.

To keep track of the structure as a point in Teichmüller space, it is helpful to first lift to the universal cover $\mathbb{H}^2$ of S, where $\pi_1(S)$ acts as a Fuchsian group G_σ and λ lifts to an invariant discrete family of disjoint geodesics $\Lambda = \cup_i \lambda_i$, which are called the "fault lines". Given $t \in \mathbb{R}$, an earthquake of measure t on this picture is a bijective (but not necessarily continuous) map $\mathcal{E} : \mathbb{H}^2 \to \mathbb{H}^2$ whose restriction $\mathcal{E}_X$ to each component X of $\mathbb{H}^2 - \Lambda$ is an orientation-preserving isometry, and such that, if X and Y are adjacent components separated by λ_i, then $\mathcal{E}_X$ and $\mathcal{E}_Y$ differ by a translation of t along λ_i. That is, $\mathcal{E}_X^{-1}\mathcal{E}_Y$ is a translation by $|t|$ along λ_i, leftward as viewed from X if $t > 0$ and rightward if $t < 0$. Note that the definition of "leftward" depends on fixing an orientation of $\mathbb{H}^2$, and is invariant under interchange of X and Y.

Λ and the number t determine $\mathcal{E}$ uniquely on $\mathbb{H}^2 - \Lambda$ up to post-composition by isometries, and Λ is G_σ-invariant. One can verify from this that for each $g \in G_\sigma$ there is an isometry g^t such that $\mathcal{E}g = g^t\mathcal{E}$ on $\mathbb{H}^2 - \Lambda$, and the map $g \mapsto g^t$ is an isomorphism of G_σ to a new Fuchsian group G^t_σ. We write $G^t_\sigma = \mathcal{E}G_\sigma\mathcal{E}^{-1}$, which certainly holds on $\mathbb{H}^2 - \mathcal{E}(\Lambda)$, and can be made to hold everywhere with an appropriately invariant definition of $\mathcal{E}$ on Λ. The resulting representation $\pi_1(S) \to G^t_\sigma$ gives us a point of $\mathrm{Teich}(S)$, which we denote by $\mathcal{E}(t, \lambda, \sigma)$.

We can generalize this to an earthquake along an arbitrary measured lamination λ, as follows. First if we multiply λ by a weight $m > 0$ we can define $\mathcal{E}(t, m\lambda, \sigma) = \mathcal{E}(mt, \lambda, \sigma)$. Since $\mathbb{R}_+\Gamma^S$ is dense in $\mathcal{ML}(S)$, we can take a sequence $m_i\gamma_i \in \mathbb{R}_+\Gamma^S$ converging to any $\lambda \in \mathcal{ML}(S)$. Kerckhoff

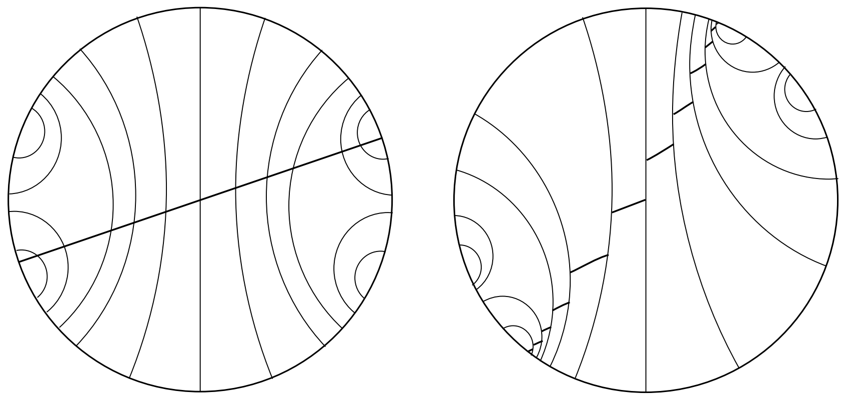

FIGURE 2. An earthquake in the Poincaré disk

shows in [Ker2] that the corresponding structures $\mathcal{E}(m_i t, \gamma_i, \sigma)$ converge in Teich(S), and the limit is independent of the sequence (Thurston in [Th] gives a definition without a limiting step).

Now given a point $\mathbf{p} \in \widetilde{\mathcal{P}_1}$, we define

$$E_t(\mathbf{p}) = (\mathcal{E}(t, \lambda_{\mathbf{p}}, \sigma_{\mathbf{p}}), \lambda_{\mathbf{p}}).$$

It is not hard to see that E_t preserves $\widetilde{\mathcal{P}_1}$, and defines a flow. The action of E_t clearly commutes with Mod(S), and so we obtain a flow on $\mathcal{P}_1$, also denoted E_t.

For $\gamma \in \mathcal{ML}(S)$, let $\ell_{\mathbf{p},\gamma}(t)$ denote the length $\ell(\gamma, \sigma_{E_t(\mathbf{p})})$. Kerckhoff proved in [Ker2] that, if $\mathbf{p}_i \to \mathbf{p}$, then

$$\ell_{\mathbf{p}_i,\gamma} \to \ell_{\mathbf{p},\gamma} \tag{10}$$

uniformly on compact subsets of $\mathbb{R}$. Kerckhoff further showed:

Proposition 4.1. *For every $\mathbf{p} \in \widetilde{\mathcal{P}_1}$ and every $\gamma \in \Gamma^S$, the function $t \mapsto \ell_{\mathbf{p},\gamma}(t)$ is C^1 and convex.*

If $i(\gamma, \lambda_{\mathbf{p}}) = 0$ then $\ell_{\mathbf{p},\gamma}(t)$ is constant, and if $i(\gamma, \lambda_{\mathbf{p}}) > 0$ then $\ell_{\mathbf{p},\gamma}$ is proper, with derivative strictly increasing and given by

$$\frac{d}{dt}\ell_{\mathbf{p},\gamma}(t) = \int_{\gamma_t} \cos(\theta_{y,t}) d\lambda(y). \tag{11}$$

Here γ_t is the $\sigma_{E_t(\mathbf{p})}$-geodesic representative of γ, and $\theta_{y,t}$ is the angle of intersection between $\lambda_{\mathbf{p}}$ and γ_t at an intersection point y (note that the support of the transverse measure of λ on γ_t is the set of intersection points).

In particular we note that for all t,

$$-i(\lambda_p, \gamma) < \frac{d}{dt}\ell_{p,\gamma}(t) < i(\lambda_p, \gamma). \tag{12}$$

Let us record here some well-known properties of $\mathcal{P}_1$:

Proposition 4.2 (Mumford's Compactness Criterion)**.** [Mu] *Let $P \subset \widetilde{\mathcal{P}_1}$ be given. Then $\pi(P) \subset \mathcal{P}_1$ is precompact if and only if*

$$\inf\{\ell_{\mathbf{p},\gamma} : \mathbf{p} \in P, \gamma \in \Gamma^S\} > 0.$$

Note, the standard version of this is that compact subsets of the moduli space $\mathrm{Teich}(S)/\operatorname{Mod}(S)$ correspond to lower bounds on lengths of geodesics; this extends immediately to $\mathcal{P}_1$ because $\mathcal{PML}(S)$ is compact.

The following standard fact may be derived from the Collar Lemma (see e.g. [Bu, Chapters 4–5]):

Proposition 4.3. *For every $\rho > 0$ there exists M such that for every $\sigma \in \mathrm{Teich}(S)$,*

$$\#\{\gamma \in \Gamma^S : \ell(\gamma, \sigma) \leq \rho\} \leq M. \tag{13}$$

For ρ sufficiently small, one may take $M = 3g - 3 + n$.

4.3. **Quadratic differentials and measured foliations.** A holomorphic quadratic differential on S, for the purposes of this paper, is a singular Euclidean metric with a foliation by straight lines, with singularities of cone angle a multiple of π.

More precisely, following Kerckhoff-Masur-Smillie [KeMasSm] let $\mathbf{Q} = \mathbf{Q}(S)$ be the set of all atlases of charts $\mathbf{q}$ of the following type. Consider first the case that S is closed. Away from a finite set $\Sigma \subset S$, every point on S has a neighborhood with a chart to $\mathbb{R}^2$, so that transition maps are of the form $z \mapsto \pm z + c$. Thus the Euclidean metric and the set of lines of any given slope in $\mathbb{R}^2$ are preserved by the transitions, and make sense on $S - \Sigma$ as well. The preimages of the horizontal and vertical lines in $\mathbb{R}^2$ are called the horizontal and vertical foliations of $\mathbf{q}$, respectively.

Around a singular point $x \in \Sigma$ there is a neighborhood U and a k-fold branched cover ($k \geq 3$) from U to $\mathbb{R}^2/\pm$ (branched over x and taking x to 0), which when restricted to a small neighborhood in the complement of x, and lifted back to $\mathbb{R}^2$, gives a chart compatible with the charts in the previous paragraph. Thus the horizontal foliation has a k-pronged singularity at x, and the metric in a neighborhood is inherited from k Euclidean halfplanes glued cyclically together along rays.

If S has punctures let $\hat{S}$ be the compactification obtained by filling the punctures, and require of $\mathbf{q}$ that it extend to $\hat{S}$, so that a puncture has a singular neighborhood as above, but with $k \geq 1$ instead of $k \geq 3$ (see Figure 3). We still denote by $\Sigma(\mathbf{q})$ the set of singularities in S, not including the punctures (but see Lemma 4.9 to see how to turn punctures into regular singularities with a branched cover).

The horizontal and vertical foliations have natural transverse measures, in which transverse arcs to one of them are assigned the length of their projection to the other.

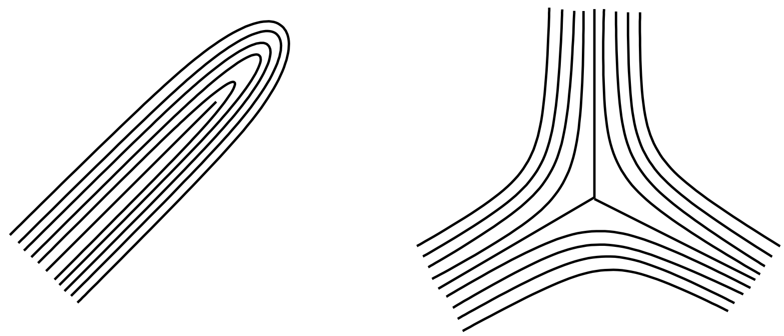

FIGURE 3. k-pronged singularities for $k = 1$ and $k = 3$.

The natural action of $\mathrm{Homeo}_+(S)$ gives us quotients

$$\widetilde{\mathcal{Q}} = \mathbf{Q}/\,\mathrm{Homeo}_0(S) \tag{14}$$

$$\mathcal{Q} = \mathbf{Q}/\,\mathrm{Homeo}_+(S) = \widetilde{\mathcal{Q}}/\,\mathrm{Mod}(S). \tag{15}$$

Let $|\mathbf{q}|$ denote the total area of $\mathbf{q}$, let $\mathbf{Q}_1 = \{\mathbf{q} \in \mathbf{Q} : |\mathbf{q}| = 1\}$, let $\widetilde{\mathcal{Q}_1}$ and $\mathcal{Q}_1$ be the corresponding quotients, and let $\pi : \widetilde{\mathcal{Q}_1} \to \mathcal{Q}_1$ be the projection.

There is a standard description of $\widetilde{\mathcal{Q}}$ as a bundle of holomorphic tensors over the Teichmüller space, as follows. Identifying $\mathbb{R}^2$ with $\mathbb{C}$ we see that the charts of any $\mathbf{q} \in \mathbf{Q}$ give a complex structure on S. The tensor $dz^2 \equiv dz \otimes dz$ on $\mathbb{C}$ pulls back via the charts to give a quadratic differential that is holomorphic with respect to this structure. Its zero set is exactly Σ (the order of a zero is 2 less than the number k of prongs), and it may have up to first-order poles at the punctures (corresponding to $k = 1$). Taking the quotient by the action of $\mathrm{Homeo}_+(S)$, we obtain a projection $\widetilde{\mathcal{Q}} \to \mathrm{Teich}(S)$ whose fibres are the spaces of holomorphic quadratic differentials at each point. This description endows $\widetilde{\mathcal{Q}}$ with a natural bundle topology.

Convergence $\mathbf{q}_i \to \mathbf{q}$ has the following properties: there is a sequence of homeomorphisms $f_i : S \to S$ isotopic to the identity, so that

- *Singularities converge:* The sets $f_i(\Sigma(\mathbf{q}_i))$ converge, in the Hausdorff topology, to $\Sigma(\mathbf{q})$.

  Furthermore, every singular horizontal leaf of $\mathbf{q}$ is a limit of f_i-images of singular horizontal leaves of $\mathbf{q}_i$.
- *Flat structure converges:* Any $x \notin \Sigma(\mathbf{q})$ has a neighborhood $U \subset S - \Sigma(\mathbf{q})$ with $\mathbf{q}$-chart $h : U \to \mathbb{C}$ so that, for large enough i, there are $\mathbf{q}_i$-charts h_i for which $h_i \circ f_i^{-1} \circ h^{-1}$ converge in C^∞ to a map of the form $z \mapsto \pm z + c$.

These properties follow from the fact that in the universal cover $\mathbb{H}^2$ the differentials are represented by holomorphic functions and convergence in $\widetilde{\mathcal{Q}}$ corresponds to convergence of these functions in the compact-open topology.

4.4. **The spaces are equivalent:** Hubbard-Masur showed [HubMas] that $\widetilde{\mathcal{Q}}$ can be expressed as a product

$$\widetilde{\mathcal{Q}} = \mathrm{Teich}(S) \times \mathcal{MF}(S),$$

where $\mathcal{MF}(S)$ is Thurston's space of measured foliations. This is the space of all foliations on S with transverse measures and singularity structure as described above, modulo the equivalence relation generated by $\mathrm{Homeo}_0(S)$ and "Whitehead moves", which collapse together singularities joined by a leaf, or do the inverse (see [FLP] or [Lev]). For any $\mathbf{q} \in \widetilde{\mathcal{Q}}$, its projection to the $\mathcal{MF}(S)$ factor is just the horizontal measured foliation defined by its charts. The projection to the $\mathrm{Teich}(S)$ factor is just the bundle map defined above.

Thurston showed that $\mathcal{MF}(S)$ is naturally identified with $\mathcal{ML}(S)$, by "straightening" the leaves: each non-singular leaf of a measured foliation (i.e. a leaf that does not meet Σ), lifted to the universal cover $\mathbb{H}^2$, meets the circle at infinity in two points and so determines a geodesic joining those two points. The closure of the union of these geodesics, projected back to S, forms a geodesic lamination (see [Lev]). As with $\mathcal{PML}(S)$ we let $\mathcal{PMF}(S)$ denote the projectivization.

Thus we obtain an identification of $\widetilde{\mathcal{Q}}$ with $\widetilde{\mathcal{P}}$. Since each fiber of $\widetilde{\mathcal{Q}_1}$ over $\mathrm{Teich}(S)$ may be identified with $\mathcal{PMF}(S)$, and each fiber of $\widetilde{\mathcal{P}_1}$ may be identified with $\mathcal{PML}(S)$, we obtain an identification of $\widetilde{\mathcal{Q}}_1$ with $\widetilde{\mathcal{P}}_1$. These identifications respect the $\mathrm{Mod}(S)$-action, and hence give an identification of $\mathcal{P}_1$ with $\mathcal{Q}_1$.

4.5. $\mathrm{SL}(2,\mathbb{R})$ **action on** $\widetilde{\mathcal{Q}_1}$ **and flows.** Again following [KeMasSm], there is a natural action on $\widetilde{\mathcal{Q}_1}$ by $\mathrm{SL}(2,\mathbb{R})$: For any $\mathbf{q} \in \widetilde{\mathcal{Q}_1}$ and $M \in \mathrm{SL}(2,\mathbb{R})$, replace each chart ϕ of $\mathbf{q}$ by $M \circ \phi$ where M acts linearly on $\mathbb{R}^2$. This preserves the compatibility condition. (In fact this action factors through $\mathrm{PSL}(2,\mathbb{R})$ but this will not matter for us.)

Let U denote the unipotent subgroup of $\mathrm{SL}(2,\mathbb{R})$ represented by

$$\left\{ h_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}.$$

The action of this group on $\widetilde{\mathcal{Q}_1}$ is called the Teichmüller horocyclic flow. The action of the subgroup

$$\left\{ g_l = \begin{pmatrix} e^{l/2} & 0 \\ 0 & e^{-l/2} \end{pmatrix} : l \in \mathbb{R} \right\}.$$

is the Teichmüller geodesic flow.

4.6. **Action on paths and saddle connections.** Let α be a path in S (smooth in the differentiable structure of $\mathbf{q}$). Because of the form of the transition maps of $\mathbf{q}$, the local projections of $d\alpha$ to the x and y axes in charts are well-defined up to sign, and integrating them we obtain a vector, denoted $(x(\alpha, \mathbf{q}), y(\alpha, \mathbf{q}))$, well-defined up to a multiple of ± 1. $\mathrm{SL}(2, \mathbb{R})$ transforms these linearly, and in particular

$$\begin{pmatrix} x(\alpha, h_t(\mathbf{q})) \\ y(\alpha, h_t(\mathbf{q})) \end{pmatrix} = \pm \begin{pmatrix} x(\alpha, \mathbf{q}) + t y(\alpha, \mathbf{q}) \\ y(\alpha, \mathbf{q}) \end{pmatrix}. \tag{16}$$

A *saddle connection* with respect to $\mathbf{q}$ is a path $\delta : (0, 1) \to S - \Sigma$ whose image in any chart is a Euclidean straight line, and which extends continuously to $\bar{\delta} : [0, 1] \to \hat{S}$ mapping the endpoints to singularities or punctures. We will sometimes confuse the reader by writing δ (resp. $\bar{\delta}$) for $\delta((0, 1))$ (resp. $\bar{\delta}([0, 1])$). We say that two saddle connections δ_1, δ_2 are *disjoint* if $\delta_1(s_1) \neq \delta_2(s_2)$ for any $s_1, s_2 \in (0, 1)$.

Let $\mathcal{L}_{\mathbf{q}}$ denote all the saddle connections for $\mathbf{q}$. Since linear maps do not change the property of being a straight line, we may identify $\mathcal{L}_{\mathbf{q}}$ with $\mathcal{L}_{M\mathbf{q}}$ for any $M \in \mathrm{SL}(2, \mathbb{R})$.

For $\delta \in \mathcal{L}_{\mathbf{q}}$ let

$$l(\delta, \mathbf{q}) = \max(|x(\delta, \mathbf{q})|, |y(\delta, \mathbf{q})|).$$

(Note it is more common to take Euclidean length, but this will be more convenient for us). Now considering the horocyclic flow h_t we define

$$l_{\mathbf{q},\delta}(t) \equiv l(\delta, h_t(\mathbf{q})).$$

When $\mathbf{q}$ is clear from the context we abbreviate $l_{\mathbf{q},\delta}(t)$ by $l_\delta(t)$ and we sometimes write $l_{\mathbf{q},\delta}$ for $l_{\mathbf{q},\delta}(0)$. An immediate consequence of (16) is:

Lemma 4.4. *For each $\mathbf{q} \in \widetilde{\mathcal{Q}_1}$ and each $\delta \in \mathcal{L}_{\mathbf{q}}$, either $l_{\mathbf{q},\delta}(t)$ is a constant function of t or there are t_0 and $c > 0$ such that*

$$l_{\mathbf{q},\delta}(t) = \max\{c, c|t - t_0|\}.$$

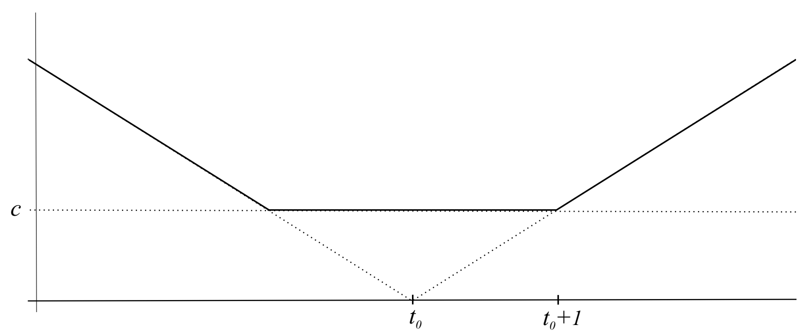

FIGURE 4. The (flat) length of a saddle connection along a horocycle.

From this it is easy to see that

Proposition 4.5. *The collection*

$$\mathcal{F} = \{l_{\mathbf{q},\delta} : \mathbf{q} \in \mathcal{Q}, \delta \in \mathcal{L}_{\mathbf{q}}\}$$

is $(2, 1)$-good.

See figure 4. □

In analogy with Proposition 4.2, we have:

Proposition 4.6. *Let $Q \subset \widetilde{\mathcal{Q}_1}$ be such that*

$$\inf\{l_{\mathbf{q},\delta} : \mathbf{q} \in Q, \delta \in \mathcal{L}_{\mathbf{q}}\} > 0.$$

Then $\pi(Q) \subset \mathcal{Q}_1$ is compact.

Proof: See e.g. [KeMasSm, Prop. 1]. □

The direct analogue of Proposition 4.3 does not hold for flat lengths of saddle connections. We will use the following weaker statements:

Proposition 4.7. *There exists M (depending only on S) such that for all $\mathbf{q} \in \widetilde{\mathcal{Q}_1}$, if $\delta_1, \ldots, \delta_r \in \mathcal{L}_{\mathbf{q}}$ are disjoint then $r \leq M$.*

Proof: See e.g. [KeMasSm, Lemma, pg. 302]. □

We will also require the following:

Proposition 4.8. *For every $\rho > 0$ and every $\mathbf{q} \in \widetilde{\mathcal{Q}_1}$,*

$$\#\{\delta \in \mathcal{L}_{\mathbf{q}} : l_{\mathbf{q},\delta} \leq \rho\} < \infty.$$

The proof is an easy consequence of compactness and the uniqueness of saddle connections in a homotopy class. □

4.7. **Punctures and branched covers.** There is a simple reduction that allows us to ignore the case of punctures in most of the arguments regarding quadratic differentials.

If S has punctures, represented as a subset P of the compactified surface $\widehat{S}$, suppose that $\beta : \widetilde{S} \to \widehat{S}$ is a finite-sheeted cover branched over P. Conformal structures and holomorphic quadratic differentials (viewed as charts) naturally pull back via β. If $\mathbf{q} \in \mathbf{Q}(S)$ has a k-pronged singularity at $x \in P$, then at a preimage branched b times, the pullback $\beta^*\mathbf{q}$ has a bk-pronged singularity. In particular if $b \geq 3$ then the preimage of a puncture is always a zero of $\beta^*\mathbf{q}$, even if the puncture is a pole ($k = 1$). With this in mind we state:

Lemma 4.9. *For any oriented closed surface $\widehat{S}$ and nonempty finite set $P \subset \widehat{S}$, if $S = \widehat{S} - P$ is hyperbolic then there is a finite branched covering $\beta : \widetilde{S} \to \widehat{S}$, branched of degree at least 3 at each preimage of each point in P, and unbranched elsewhere.*

This covering induces a map $\beta^* : \widetilde{\mathcal{Q}}(S) \to \widetilde{\mathcal{Q}}(\widetilde{S})$, which commutes with the action of $\mathrm{SL}(2,\mathbb{R})$, and so that for each $\mathbf{q} \in \widetilde{\mathcal{Q}}(S)$ the set $\{l_{\mathbf{q},\delta} : \delta \in \mathcal{L}_{\mathbf{q}}\}$ of length functions of saddle connections is the same as the corresponding set $\{l_{\beta^*\mathbf{q},\delta'} : \delta' \in \mathcal{L}_{\beta^*\mathbf{q}}\}$.

Proof: For any integer $d > 1$ and any nonempty finite subset Z of a surface X whose cardinality $|Z|$ is even, there is a connected cover $X_{d,Z} \to X$ of degree d so that each point of Z has a single preimage branched with degree d over it. (Make cuts along disjoint segments connecting pairs of points in Z, take d copies and glue appropriately).

Thus, if $|P|$ is even we take the cover $\widehat{S}_{3,P} \to \widehat{S}$.

If $|P|$ is odd and greater than 1, fix one $p_0 \in P$ and let $P' = P - \{p_0\}$. In the cover $R = \widehat{S}_{2,P'}$, the preimage $\widetilde{P}$ of P has an even number of points, one for each point in P' and two for p_0. We therefore take the cover $R_{3,\widetilde{P}}$.

Finally if $|P| = 1$ then $\widehat{S}$ cannot be a sphere since S is hyperbolic. $\widehat{S}$ therefore admits a double (unbranched) cover R and we form $R_{3,\widetilde{P}}$ with $\widetilde{P}$ the preimage of P.

In all cases call the final branched cover $\beta : \widetilde{S} \to \widehat{S}$. Since, for any $\mathbf{q} \in \mathbf{Q}(S)$, every β-preimage of a puncture is a zero of $\beta^*\mathbf{q}$, we obtain a map $\beta^* : \mathbf{Q}(S) \to \mathbf{Q}(\widetilde{S})$. The homotopy lifting property implies that any homeomorphism $\hat{S} \to \hat{S}$ that is isotopic to the identity fixing P lifts to such a map on $\widetilde{S}$. Thus β^* induces a map $\beta^* : \widetilde{\mathcal{Q}}(S) \to \widetilde{\mathcal{Q}}(\widetilde{S})$.

It is also clear that the map commutes with the $\mathrm{SL}(2,\mathbb{R})$ action, since this action is by post-composition on charts, which commutes with pullback.

Finally, fixing $\mathbf{q} \in \mathbf{Q}(S)$, any saddle connection in S (including those terminating in punctures) clearly lifts to a union of saddle connections for $\beta^*\mathbf{q}$, each of the same length. Conversely, a saddle connection for $\beta^*\mathbf{q}$ projects to a geodesic arc for $\mathbf{q}$, which cannot have self-intersections since it makes constant angle with the horizontal foliation, and thus is a saddle connection. The statement about length functions follows. $\square$

5. Nondivergence for earthquakes

5.1. **Length Functions are good.** Our proofs of theorems E1-E3 will be based on the fact that, as a consequence of Kerckhoff's theorem and a little geometry, length functions along earthquake paths form a good family.

Proposition 5.1. *There are c, ρ such that the family of length functions*

$$\mathcal{F} = \{\ell_{\mathbf{p},\gamma}(t) : \mathbf{p} \in \widetilde{\mathcal{P}_1}, \gamma \in \Gamma^S\}$$

is $(c, 1, \rho)$-good.

This proposition will follow from Lemma 5.2 below, which states that each length function is a bounded distance from a function of the form $c|t - t_0|$, in an appropriately uniform way.

For $\mathbf{p} \in \widetilde{\mathcal{P}_1}$ and $\gamma \in \Gamma^s$, we let

$$\mu = i(\lambda_{\mathbf{p}}, \gamma).$$

If $\mu \neq 0$ then $\ell_\gamma(t) \equiv \ell_{\mathbf{p},\gamma}(t)$ is nonconstant, and by Kerckhoff's convexity theorem (Proposition 4.1), the minimum

$$\epsilon_\gamma = \min_{t \in \mathbb{R}} \ell_\gamma(t) \tag{17}$$

is attained at a unique point $t_\gamma \in \mathbb{R}$, and for every $\rho > 0$,

$$J(\rho) = \{t \in \mathbb{R} : \ell_\gamma(t) \le \rho\}$$

is an interval.

With this notation we can state:

Lemma 5.2. *There are constants ρ and C, depending only on S, such that for any $\mathbf{p} \in \widetilde{\mathcal{P}_1}$, $\gamma \in \Gamma^S$ and all $t \in J(\rho)$,*

$$\mu|t - t_\gamma| - C\epsilon_\gamma \le \ell_{\mathbf{p},\gamma}(t) \le \mu|t - t_\gamma| + \epsilon_\gamma. \tag{18}$$

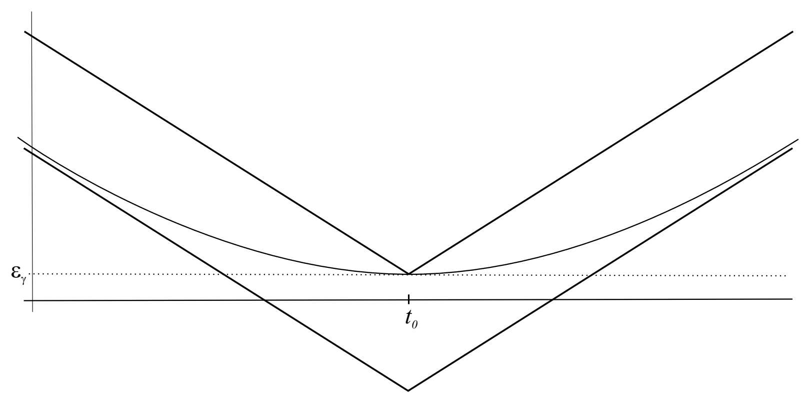

FIGURE 5. Lemma 5.2 shows that $\ell_{\mathbf{p},\gamma}$ is sandwiched between two V-shaped functions.

Proof: Note first that the right-hand inequality in (18) follows immediately from Kerckhoff's theorem (12) and the mean value theorem (see [St, §4.3]). Thus it remains to prove the left-hand inequality.

By replacing $\mathbf{p}$ by $E_{t_\gamma}\mathbf{p}$ we may (and will) assume that $t_\gamma = 0$. If $\mu = 0$ then $\ell_\gamma(t) = \epsilon_\gamma$ for all t and there is nothing to prove. So we may assume that $\mu > 0$.

The rest of the lemma will follow from this claim:

Claim 5.3. *There are constants C and ρ_0, depending only on S, such that whenever $\epsilon_\gamma \le \rho_0$ we have:*

$$1 - C\epsilon_\gamma \le \ell_\gamma(\pm 1/\mu). \tag{19}$$

Proof that Claim 5.3 implies Lemma 5.2: Set $\rho < \min\{\rho_0, \frac{1}{1+C}\}$. By (19), the function $\ell_{\mathbf{p},\gamma}(t)$ is at least $\mu|t| - C\epsilon_\gamma$ for $t = 0$ and $t = \pm 1/\mu$. If it falls below $\mu|t| - C\epsilon_\gamma$ for $t \in [-1/\mu, 1/\mu]$, then by the mean value theorem $|\frac{d}{dt}\ell_{\mathbf{p},\gamma}(t)| \ge \mu$ for some t, contradicting (12). Thus if $J = J(\rho)$ is contained in $[-1/\mu, 1/\mu]$ we are done. Otherwise, since $0 \in J$ and J is an interval, we obtain that either $1/\mu \in J$ or $-1/\mu \in J$. Applying (19) we get that $1 - C\rho \le \rho$, and this contradicts the assumption that $\rho < \frac{1}{1+C}$. $\square$

Proof of Claim 5.3: The idea is that, if ϵ_γ is sufficiently small then at time $t_\gamma = 0$ the leaves of λ are squeezed close together where they cross the geodesic representative of γ. Thus the $\pm 1/\mu$-earthquake along λ can be approximated by a single shift of size 1 along just one leaf, and an $O(\epsilon_\gamma)$-correction. We shall take some care in the proof in order to be sure that our estimates are uniform. Some standard estimates used in the argument will be deferred to §5.3.

For a number x the notation $x = O(\epsilon)$ means that there is $C > 0$, *depending only on* S, such that $|x| \leq C\epsilon$. For a matrix G, $G = O(\epsilon)$ means that $\|G\| = O(\epsilon)$, where we use the matrix norm $\|G\| = \sup_{v \neq 0} |Gv|/|v|$.

Because of the density of simple laminations in $\mathcal{ML}(S)$, and the convergence property (10) for length functions, there is no loss of generality in assuming that $\lambda_{\mathbf{p}}$ is a simple closed curve.

We start with $\rho_0 = 1$ (we will have to make ρ_0 smaller later on). Abbreviate $\epsilon = \epsilon_\gamma$. So from now on we assume that $|\mu t| = 1$, and that $\epsilon \leq 1$.

As in §4.2, let $G_{\sigma_{\mathbf{p}}}$ be a Fuchsian group uniformizing the metric $\sigma_{\mathbf{p}}$ and let $\widetilde{\lambda}$ be the lift of $\lambda_{\mathbf{p}}$ to $\mathbb{H}^2$. Let $\mathcal{E}$ denote a t-earthquake map of $\mathbb{H}^2$ with fault locus $\widetilde{\lambda}$ (determined up to post-composition with isometries), and $G^t_{\sigma_{\mathbf{p}}}$ the Fuchsian group $\mathcal{E} G_{\sigma_{\mathbf{p}}} \mathcal{E}^{-1}$ uniformizing $\sigma_{E_t \mathbf{p}}$.

Let $A \in G_{\sigma_{\mathbf{p}}}$ be an element representing γ, so that its axis L projects to the $\sigma_{\mathbf{p}}$-geodesic representative of γ in S. If $A_t = \mathcal{E} A \mathcal{E}^{-1}$ denotes the image of A in $G^t_{\sigma_{\mathbf{p}}}$, then our length $\ell_\gamma(t)$ is the translation distance $d(A_t)$ of A_t along its axis.

Fixing the orientation on L which makes A a positive translation, let $\{\lambda_i\}_{i \in \mathbb{Z}}$ be the leaves of $\widetilde{\lambda}$ which intersect L, ordered by the position of the intersection points $z_i = \lambda_i \cap L$ on L. Because $\widetilde{\lambda}$ is $G_{\sigma_{\mathbf{p}}}$-invariant, there is some $n \in \mathbb{N}$ such that $A(\lambda_i) = \lambda_{i+n}$ for all i. Let $\epsilon_i = \mathrm{dist}(z_i, z_{i+1})$. We have $\epsilon = \sum_{i=1}^n \epsilon_i$.

Let X_i be the complementary component of $\widetilde{\lambda}$ containing the interval (z_i, z_{i+1}). Up to postcomposing $\mathcal{E}$ with a hyperbolic isometry, we may assume that $\mathcal{E}_{X_0}$ is the identity. With this normalization, A_t is given by the Möbius transformation $\mathcal{E}_{X_n} \circ A$.

By the definition of earthquakes, we can express $\mathcal{E}_{X_n}$ as a composition of shifts along the λ_i,

$$\mathcal{E}_{X_n} = B_1 \circ \cdots \circ B_n \tag{20}$$

where

$$B_i = \mathcal{E}^{-1}_{X_{i-1}} \circ \mathcal{E}_{X_i}$$

is the hyperbolic translation with axis λ_i and translation distance $1/n = |t\mu|/n$ in the direction to the left of L.

Let $\overrightarrow{\lambda_i}$ denote the ray on λ_i with basepoint z_i, pointing to the left of L. Notice that B_i maps $\overrightarrow{\lambda_i}$ into itself. There is a unique isometry G_i mapping $\overrightarrow{\lambda_i}$ onto $\overrightarrow{\lambda_{i+1}}$.

Normalizing so that $z_0 = \mathbf{i} = \sqrt{-1} \in \mathbb{H}^2$, we have:

Claim 5.4. *For each i and k between 0 and n,*

$$G_i = I + O(\epsilon_i),$$

and

$$B_i^k = O(1).$$

In fact, we can express G_i as a composition $C_{i+1} \circ R_i \circ C_i^{-1}$, where C_i is a translation along L taking $\mathbf{i}$ to z_i and R_i is a rotation around $\mathbf{i}$ of angle $\theta_{i+1} - \theta_i$. We have $|\theta_{i+1} - \theta_i| < \epsilon_i$ because the leaves λ_i and λ_{i+1} are disjoint (see Proposition 5.7 in §5.3). Thus $R_i = I + O(\epsilon_i)$. The distance from $\mathbf{i}$ to each of the z_i is at most $\sum \epsilon_i = \epsilon \le 1$, so $C_i = O(1)$ and hence $C_i \circ R_i \circ C_i^{-1} = I + O(\epsilon_i)$. Finally, $C_{i+1}C_i^{-1} = I + O(\epsilon_i)$ since it is a translation along L of distance ϵ_i. This establishes the first part of the claim.

To obtain the second part, express B_i^k as $C_i R_i T_i^k C_i^{-1}$, where T_i is a translation by distance $1/n$ along an axis passing through $\mathbf{i}$. □

We have $B_{i+1} = G_i B_i G_i^{-1}$. Therefore for any s and any $i = 1, \ldots, n-1$ we have

$$B_i B_{i+1}^s = B_i G_i B_i^s G_i^{-1} = B_i^{s+1}[B_i^{-s}, G_i], \tag{21}$$

where $[a, b]$ denote the commutator $aba^{-1}b^{-1}$. Applying induction on n we obtain:

$$B_1 \cdots B_n = B_1^n \cdot \prod_{i=1}^{n-1} [B_i^{i-n}, G_i]. \tag{22}$$

Using the bounds in Claim 5.4, we obtain that that for each i, $[B_i^{i-n}, G_i] = I + O(\epsilon_i)$. Lemma 5.6 implies that

$$\prod_{i=1}^{n-1} [B_i^{i-n}, G_i] = I + O(\epsilon). \tag{23}$$

We conclude that $A_t = B_1^n(I + O(\epsilon))$. The translation distance of B_1^n is 1, and B_1^n is in a compact subset of $\mathrm{SL}(2, \mathbb{R})$ (since its axis runs through z_1). Thus, since translation distance is a differentiable function on $\mathrm{SL}(2, \mathbb{R})$ we conclude that

$$\ell_\gamma(t) = d(A_t) = 1 + O(\epsilon),$$

completing the proof of Claim 5.3 and hence of Lemma 5.2. □

We can now complete the proof that length functions form a good family:

Proof of Proposition 5.1: Let ρ and C be as in Lemma 5.2, and let ϵ_γ and μ be as above. Let $c > 4(1 + C)$, and let $I \subset \mathbb{R}$, $f = \ell_\gamma \in \mathcal{F}$ and $0 < \epsilon < \rho$ be given. Let $I_{f,\epsilon} = \{t \in I : f(t) \le \epsilon\}$. If $I_{f,\epsilon} = \emptyset$ there is nothing to prove, so we may assume that $\epsilon \ge \epsilon_\gamma$ and $\mu > 0$.

Assume first that $\epsilon \ge \|f\|_I/2$. Then $c\left(\frac{\epsilon}{\|f\|_I}\right) \ge 4 \cdot 1/2 > 1 \ge \frac{|I_{f,\epsilon}|}{|I|}$.

Now suppose that $\epsilon < \|f\|_I/2$. Let $t_0 \in I_{f,\epsilon}$. There is $t \in I_{f,\epsilon}$ such that $|t-t_0| \geq \frac{|I_{f,\epsilon}|}{2}$. Hence, by the left hand side of (18):

$$\mu \frac{|I_{f,\epsilon}|}{2} \leq \mu|t-t_0| \leq f(t) + C\epsilon_\gamma \leq f(t) + C\epsilon \leq \epsilon(1+C). \tag{24}$$

Also, by Proposition 4.1, $f(t') = \|f\|_I$ where t' is one of the endpoints of I. Therefore, using the right hand side of (18):

$$|I| \geq |t'-t_0| \geq \frac{f(t')-\epsilon_\gamma}{\mu} \geq \frac{f(t')-\epsilon}{\mu} = \frac{\|f\|_I - \epsilon}{\mu} \geq \frac{\|f\|_I}{2\mu}. \tag{25}$$

Now putting together equations (24) and (25) we obtain

$$\frac{|I_{f,\epsilon}|}{|I|} < \frac{c\epsilon}{\|f\|_I}.$$

□

5.2. **Proofs of the Theorems.** Throughout this subsection, we let $\rho, c > 0$ be as in Proposition 5.1, $M = M(\rho)$ be as in Proposition 4.3.

Now applying Propositions 3.2 and 5.1 we immediately obtain:

Theorem 5.5. *Let $\rho_0 \in (0,\rho], \mathbf{p} \in \widetilde{\mathcal{P}_1}$ and the interval $I \subset \mathbb{R}$ satisfy:*

$$\text{for every } \gamma \in \Gamma^S, \ \|\ell_{\mathbf{p},\gamma}\|_I \geq \rho_0. \tag{26}$$

Then for all $0 < \epsilon' < \rho_0$,

$$\frac{|\{t \in I : E_t p \notin K_{\epsilon'}\}|}{|I|} < \frac{cM\epsilon'}{\rho_0}.$$

□

Proof of Theorem E2: Let $\epsilon > 0$ be given. Making ϵ smaller if necessary, assume that $\epsilon' = \frac{\epsilon\rho}{cM} < \rho$. Now let $K = K_{\epsilon'}$. By Mumford's compactness criterion, K is a compact subset of $\mathcal{P}_1$.

Let $p \in \mathcal{P}_1$ and $\mathbf{p} \in \pi^{-1}(p)$. Assume that the second alternative in the statement of the theorem does not hold. Thus $\ell_{\mathbf{p},\gamma}(t)$ is a nonconstant function of t for any $\gamma \in \Gamma^S$.

Let

$$\Gamma_0 = \{\gamma \in \Gamma^S : \ell_{\mathbf{p},\gamma}(0) < \rho\}. \tag{27}$$

Γ_0 is finite by Proposition 4.3. Each of the functions

$$\{\ell_{\mathbf{p},\gamma}(t) : \gamma \in \Gamma_0\}$$

is nonconstant by our assumption, and hence by Proposition 4.1, is proper. Thus for t_0 large enough we will have $\ell_{\mathbf{p},\gamma}(t_0) \geq \rho$ for all $\gamma \in \Gamma_0$. Then it follows that for all $T \geq t_0$, (26) is satisfied for $I = [0,T]$ and $\rho_0 = \rho$, and hence

$$\mathrm{Avg}_{T,p}(K) \geq 1 - \frac{cM\epsilon'}{\rho} = 1 - \epsilon. \tag{28}$$

□

Proof of Theorem E1: Given ϵ and K as in the statement of the theorem, let

$$\theta = \inf\{\ell_{\mathbf{p},\gamma} : \mathbf{p} \in \pi^{-1}(K), \gamma \in \Gamma^S\}.$$

By Proposition 4.2, $\theta > 0$. Let

$$\rho_0 = \min\{\rho, \theta\}.$$

Let $\epsilon' = \frac{\rho_0 \epsilon}{CM}$. By making ϵ' smaller if necessary, assume that $\epsilon' < \rho_0$. Now define $K' = K_{\epsilon'}$.

Since $\rho_0 \leq \theta$, for $\gamma \in \Gamma^S$ and any $\mathbf{p} \in \pi^{-1}(K)$ we have $\ell_{\mathbf{p},\gamma}(0) \geq \rho_0$ and hence (27) is satisfied for $I = [0, T]$ for any $T > 0$. Hence,

$$\mathrm{Avg}_{T,p}(K') \geq 1 - \frac{cM\epsilon'}{\rho_0} \geq 1 - \epsilon.$$

□

Proof of Theorem E3: Let $C = c(3g - 3 + n)$, where g is the genus of S and n is the number of punctures. Given p, let ρ_0 be small enough so that:

- $\rho_0 \leq \rho$.
- $M = M(\rho_0) \leq 3g - 3 + n$ (see Proposition 4.3).
- $\rho_0 \leq \inf_{\gamma \in \Gamma^S} \ell_{p,\gamma}$.

Then for any $T > 0$, $\epsilon < \rho_0$, (26) is satisfied for $I = [0, T]$, and hence

$$\mathrm{Avg}_{T,p}(K_\epsilon) \geq 1 - \frac{cM\epsilon}{\rho_0} \geq 1 - \frac{C\epsilon}{\rho_0}.$$

□

5.3. **Auxilliary results.** In this subsection we complete the proofs of two estimates which were used in the proof of Lemma 5.2.

The first is the following matrix inequality.

Lemma 5.6. *Let $A_1, \ldots, A_n \in M_d(\mathbb{R})$ be matrices satisfying $\sum \|A_i\| < C_1$. Then*

$$\prod_{i=1}^{n} (I + A_i) = I + O\left(\sum \|A_i\|\right)$$

where the implicit constant depends only on C_1.

Proof. Let $\epsilon_i = \|A_i\|$ and $\sum \epsilon_i < C_1$. Note first the inequality

$$\left\| \prod_{i=1}^{n} (I + A_i) - I \right\| \leq \prod_{i=1}^{n} (1 + \epsilon_i) - 1$$

which comes from expanding both sides and applying the triangle inequality and submultiplicativity of the matrix norm.

Next, we note that $1 + x \leq e^x \leq 1 + C_2 x$ for $x \in [0, C_1]$, where C_2 depends on C_1, and conclude that

$$\prod (1 + \epsilon_i) \leq \prod e^{\epsilon_i} = e^{\sum \epsilon_i} \leq 1 + C_2 \sum \epsilon_i.$$

This establishes the desired inequality. □

The following lemma is a well-known basic fact in hyperbolic geometry. See [CaEpGr, 5.2.6] for a similar statement.

Lemma 5.7. *Let L be a geodesic in $\mathbb{H}^2$. For $i = 1, 2$, let z_i be a point in L, let λ_i be a geodesic intersecting L at z_i, and let θ_i denote the (clockwise) angle between L and λ_i at z_i. Suppose λ_1, λ_2 are disjoint. Then*

$$|\theta_1 - \theta_2| \leq \mathrm{dist}(z_1, z_2).$$

Proof: Assume without loss of generality that $\theta_1 > \theta_2$. Then we can rotate λ_1 counterclockwise around z_1 until it is asymptotic to λ_2 on one side of L, and this only increases the left side of the inequality. Thus it suffices to bound this case. The hyperbolic triangle with one ideal vertex bounded by L, λ_1 and λ_2 has area $\pi - (\theta_2 + \pi - \theta_1) = \theta_1 - \theta_2$, by the Gauss-Bonnet formula. It has one finite edge of length $\mathrm{dist}(z_1, z_2)$, and an explicit computation shows that the area of a hyperbolic triangle with one side of length c is at most c. □

6. Nondivergence for horocycles

6.1. **Sets of disjoint saddle connections.** Let us first give an indication of our argument for proving non-divergence results, which is essentially an effective version of similar arguments in [KeMasSm] and [Ve]. As for earthquakes, we will use the sparse cover argument described in §3. The collection $\mathcal{F}$ of all length functions, with respect to the flat metric, is good (Proposition 4.5) but another condition of Proposition 3.2 is not satisfied: there is no uniform upper bound on the number of simultaneously short saddle connections. In the case of hyperbolic length functions, there is such an upper bound, as follows from the 'Collar Lemma' (see Proposition 4.3), which implies that if $\gamma \in \Gamma^S$ is sufficiently short, then any $\gamma' \in \Gamma^S$ which intersects γ must be long.

The replacement for this statement is a geometric lemma (Lemma 6.2) due to [KeMasSm] and [Ve], which implies that if δ is a saddle connection which is on the boundary of a subcomplex which is triangulated by short saddle connections, then δ has the following isolation property:

> *If δ is short then any saddle connection which intersects δ is long.*

We are able to apply the sparse cover argument by selecting a subset $\mathcal{F}_0 \subset \mathcal{F}$ which consists of saddle connections which have the isolation property.

We proceed to the details. First let us introduce some notation which will be used throughout this section. In the rest of the proof we will assume that S *is a closed surface*, and return in §6.4 to discuss the case with punctures.

Let M be as in Proposition 4.7, let $1 \leq r \leq M$, and let $\mathbf{q} \in \widetilde{\mathcal{Q}_1}$. Define

$$\mathcal{E}_r = \{E \subset \mathcal{L}_{\mathbf{q}} : E \text{ consists of } r \text{ disjoint segments}\}.$$

Define

$$l_{\mathbf{q},E}(t) = \max_{\delta \in E} l_{\mathbf{q},\delta}(t),$$

and

$$\alpha_r(t) = \alpha_{\mathbf{q},r}(t) = \min_{E \in \mathcal{E}_r} l_{\mathbf{q},E}(t).$$

Note that the minimum in this definition is attained because of Proposition 4.8. We denote $l_{\mathbf{q},\delta} \equiv l_{\mathbf{q},\delta}(0)$, $l_{\mathbf{q},E} \equiv l_{\mathbf{q},E}(0)$, and $\alpha_{\mathbf{q},r} = \alpha_{\mathbf{q},r}(0)$.

For $E \in \mathcal{E}_r$, define $S(E)$ as the closure of the union of the simply connected connected components of $S - \bigcup_{\delta \in E} \overline{\delta}$.

Proposition 6.1. *There exists $\rho_0 > 0$ (depending only on S) such that for every $\mathbf{q} \in \widetilde{\mathcal{Q}_1}$, every $r \in \{1, \dots, M\}$ and every $E \in \mathcal{E}_r$, if $S(E) = S$ then $l_{\mathbf{q},E} \geq \rho_0$. In particular $\alpha_{\mathbf{q},M} \geq \rho_0$.*

Proof: This follows from a standard area argument. See e.g. [KeMasSm, pg. 302, Lemma]. □

Lemma 6.2. *Let $\mathbf{q} \in \widetilde{\mathcal{Q}_1}$, and $r \in \{1, \dots, M-1\}$. Suppose that $E \in \mathcal{E}_r$ such that $l_{\mathbf{q},E} < \frac{\theta}{3\sqrt{2}}$ and $\alpha_{r+1} \geq \theta$ for some θ, and suppose that δ is a saddle connection on $\partial S(E)$.*

Then for any $\delta' \in \mathcal{L}_{\mathbf{q}}$ such that $\delta \neq \delta'$ and $\delta \cap \delta' \neq \emptyset$ we have $l_{\mathbf{q},\delta'} \geq \frac{\sqrt{2}\theta}{3}$.

This lemma follows from the arguments in [Ve, Corollary 4.19 and preceding discussion] or [KeMasSm, §3]. For the reader's convenience, a proof is included in §6.3.

6.2. **Proofs of the Theorems.** The following is the main result of this section:

Theorem 6.3. *There are positive constants C_1, C_2, α, ρ_0, depending only on S, such that if $\mathbf{q} \in \widetilde{\mathcal{Q}_1}$, an interval $I \subset \mathbb{R}$, and $0 < \rho' \leq \rho_0$ satisfy:*

$$\text{for any } \delta \in \mathcal{L}_{\mathbf{q}},\ \|l_{\mathbf{q},\delta}\|_I \geq \rho', \tag{29}$$

then for any $0 < \epsilon < C_1\rho'$ we have:

$$|\{t \in I : \alpha_1(t) < \epsilon\}| \leq C_2 \left(\frac{\epsilon}{\rho'}\right)^{\alpha} |I|. \tag{30}$$

Remark: The proof we shall give yields the following explicit bounds on the constants appearing in the statement of the theorem: one may take $\alpha = \frac{1}{M-1}$, $C_1 = (\sqrt{2}/9)^{\frac{1}{M-1}}$ and $C_2 = 9\sqrt{2}M(M-1)$.

First let us record a lemma, which follows from Lemma 4.4, and whose proof is deferred to §6.3:

Lemma 6.4. *Let f and $\widetilde{f}$ be two functions of the form $t \mapsto \max\{c, c|t - t_0|\}$. Suppose that for some $b > 0$ and $s \in \mathbb{R}$ we have $f(s) < b/3$ and $\widetilde{f}(s) < b/3$. Then, possibly after exchanging f and $\widetilde{f}$, $f(t) < b$ whenever $\widetilde{f}(t) < b/3$.*

Proof of Theorem 6.3: Let M and ρ_0 be as in Propositions 4.7 and 6.1, and let $\alpha = \frac{1}{M-1}$. Let $\mathbf{q}, I, \rho'$ satisfy (29), and let $0 < \epsilon < C_1\rho'$, where $C_1 \le 1$ is a constant to be chosen below. Let

$$V_\epsilon = \{t \in I : \alpha_1(t) < \epsilon\}.$$

For $k = 1, \ldots, M-1$, define

$$L_k = \epsilon \left(\frac{\rho'}{\epsilon}\right)^{\frac{k-1}{M-1}}.$$

Note that $L_1 = \epsilon$, $L_M = \rho'$, and the L_k's increase by a constant multiplicative factor:

$$\frac{L_k}{L_{k+1}} = \left(\frac{\epsilon}{\rho'}\right)^{\frac{1}{M-1}} < C_1^{\frac{1}{M-1}}. \tag{31}$$

For every $t \in V_\epsilon$ let

$$r(t) = \max\{k : \alpha_k(t) < L_k\}.$$

Since $\alpha_M(t) \ge \rho_0 \ge \rho' = L_M$ by Proposition 6.1, we have $r(t) \le M-1$ and

$$\alpha_{r(t)}(t) < L_{r(t)}, \quad \alpha_{r(t)+1}(t) \ge L_{r(t)+1}.$$

Let

$$V_k = \{t \in V_\epsilon : r(t) = k\}.$$

Then V_ϵ is the disjoint union of the measurable sets $V_1, \ldots, V_{M-1}$ and thus there is some r for which

$$|V_r| \ge \frac{|V_\epsilon|}{M-1}. \tag{32}$$

With this choice of r we can now define:

$$L = L_r, \qquad U = L_{r+1}$$

and

Definition 6.5. *For $\delta \in \mathcal{L}_{\mathbf{q}}$, let $H(\delta)$ be the set of $t \in I$ for which*

$$l_{\mathbf{q},\delta}(t) < L,$$

and whenever $\delta \cap \delta' \neq \emptyset$ for $\delta \neq \delta' \in \mathcal{L}_{\mathbf{q}}$, we have

$$l_{\mathbf{q},\delta'}(t) \ge \frac{U\sqrt{2}}{3}.$$

In other words, $H(\delta)$ is the set of times when δ is "isolating".

Define

$$\mathcal{F}_0 = \{\delta \in \mathcal{L}_{\mathbf{q}} : V_r \cap H(\delta) \neq \emptyset\},$$

the set of δ that are isolating at some point in V_r.

We will apply Proposition 3.2 (Sparse Cover), setting $\mathcal{F}, \epsilon, \rho, C, \alpha$ in the statement of the proposition equal respectively to $\mathcal{F}_0, L, U\sqrt{2}/9, 2$ and 1. Let us check that the hypotheses of the Proposition are satisfied. Hypotheses

1 and 2 follow from Proposition 4.5 and (29). By making C_1 small enough and using (31), we can ensure that $\epsilon < \rho$. Hypothesis 3 follows from:

Claim 6.6. *For all $t \in I$,*

$$\#\{\delta \in \mathcal{F}_0 : l_\delta(t) \leq \frac{U\sqrt{2}}{9}\} \leq M.$$

By making C_1 smaller if necessary and using (31), assume that $L \leq U\sqrt{2}/9$. Suppose $t \in I$ and $\delta, \delta' \in \mathcal{F}_0$ are such that $l_\delta(t), l_{\delta'}(t) \leq \frac{U\sqrt{2}}{9}$. By Lemma 4.4 and Proposition 6.4, we obtain (possibly after exchanging δ and δ') that if $l_\delta(s) < \frac{U\sqrt{2}}{9}$ then $l_{\delta'}(s) < \frac{U\sqrt{2}}{3}$. Since for $s \in V_r \cap H(\delta)$ (the last set is nonempty by the definition of $\mathcal{F}_0$) we must have $l_\delta(s) < L \leq \frac{U\sqrt{2}}{9}$, we get from the definition of $H(\delta)$ that $\delta \cap \delta' = \emptyset$. Since, by Proposition 4.7, the number of disjoint elements of $\mathcal{L}_{\mathbf{q}}$ is at most M, the claim follows.

From the Proposition and (31) we conclude:

$$\frac{|\bigcup_\delta H(\delta)|}{|I|} \leq \frac{2ML}{U\sqrt{2}/9} < C_2(\epsilon/\rho')^\alpha \tag{33}$$

(where $C_2 = C_2(M)$ is a constant).

The theorem now follows from (32), (33), and:

Claim 6.7.

$$V_r \subset \bigcup_{\delta \in \mathcal{L}_{\mathbf{q}}} H(\delta).$$

Indeed, let $t \in V_r$. Making C_1 small enough, in (31) we get that $L < U/3\sqrt{2}$. There is $E \in \mathcal{E}_r$ such that

$$l_{\mathbf{q},E}(t) = \alpha_r(t) < L < \frac{U}{3\sqrt{2}}.$$

By Proposition 6.1, $S(E) \neq S$, so let $\delta \in E$ be on the boundary of $S(E)$. Then for any $\delta' \in \mathcal{L}_{\mathbf{q}}$, if $\delta \neq \delta'$ and $\delta \cap \delta' \neq \emptyset$ then, by Lemma 6.2 (with $\theta = U$), we have $l_{\mathbf{q},\delta'}(t) \geq \frac{U\sqrt{2}}{3}$. Thus $t \in H(\delta)$, proving the claim, and the theorem. $\square$

For the remainder of this section, let C_1, C_2, ρ, α be as in Theorem 6.3.

Proof of Theorem H3: Given $\mathbf{q} \in \widetilde{\mathcal{Q}_1}$, let $\rho' \leq \rho_0$ be no larger than $\min\{l_\delta : \delta \in \mathcal{L}_{\mathbf{q}}\}$. The minimum exists because of Proposition 4.8.

Now let $\rho = C_1\rho'$ and let T and $0 < \epsilon < \rho$ be given. Applying Theorem 6.3 with $I = [0, T]$ we obtain:

$$|\{t \in I : \alpha_1(t) < \epsilon\}| \leq C_2 \left(\frac{\epsilon}{\rho'}\right)^\alpha |I| = C \left(\frac{\epsilon}{\rho}\right)^\alpha |I|.$$

From this the theorem follows. $\square$

Proof of Theorem H1: Given ϵ and η, let $\rho' = \min\{\eta, \rho_0\}$. Let $\eta' < \rho'$ be small enough so that $C_2 \left(\frac{\eta'}{\rho'}\right)^\alpha < \epsilon$. Then for any $T > 0$ and any $q \in K_\eta$, (29) is satisfied for $I = [0, T]$, $\mathbf{q} \in \pi^{-1}(q)$ and the theorem follows. $\square$

Proof of Theorem H2: Given ϵ let ϵ' be small enough so that

$$C_2 \left(\frac{\epsilon'}{\rho_0}\right)^\alpha < \epsilon \quad \text{and} \quad \epsilon' < C_1 \rho_0.$$

Let

$$K = \pi(\{\mathbf{q} \in \widetilde{\mathcal{Q}_1} : l_{\mathbf{q},\delta} \geq \epsilon', \ \forall \delta \in \mathcal{L}_{\mathbf{q}}\}).$$

K is compact by Proposition 4.6.

Let $\mathbf{q} \in \widetilde{\mathcal{Q}_1}$ and suppose the second alternative in the statement of the theorem does not hold. The set

$$\mathcal{L}_0 = \{\delta \in \mathcal{L}_{\mathbf{q}} : l_\delta < \rho_0\}$$

is finite by Proposition 4.8. Since we are assuming that none of the functions $t \mapsto l_\delta(t)$ are constant, they all diverge by Lemma 4.4. Thus there is some T_0 such that for all $\delta \in \mathcal{L}_0$, $l_\delta(T_0) \geq \rho_0$.

For any $T \geq T_0$ we can now apply Theorem 6.3 with $I = [0, T]$ and $\rho' = \rho_0$, and obtain that

$$\mathrm{Avg}_{T,\mathbf{q}}(\mathcal{Q}_1 - K) < \epsilon.$$

From this the theorem follows. $\square$

6.3. **Auxilliary Results.**

Proof of Lemma 6.2: The lemma is stated using lengths $l_{\mathbf{q}}$ defined as the maximum of horizontal and vertical components, but it is more convenient to prove using the Euclidean length, which we denote $L_{\mathbf{q}}$. Define $\widehat{\alpha}_r$ as the analogous quantity to α_r, with $L_{\mathbf{q}}$ replacing $l_{\mathbf{q}}$. Since the two lengths are related by:

$$\frac{1}{\sqrt{2}} L_{\mathbf{q}} \leq l_{\mathbf{q}} \leq L_{\mathbf{q}}$$

it suffices to prove

(*) *If $L_{\mathbf{q},E} < \theta/3$ and $\widehat{\alpha}_{r+1} \geq \theta$, and if δ is a saddle connection on $\partial S(E)$, then for any $\delta' \in \mathcal{L}_{\mathbf{q}}$ such that $\delta \neq \delta'$ and $\delta \cap \delta' \neq \emptyset$ we have $L_{\mathbf{q},\delta'} \geq \frac{2\theta}{3}$.*

Let

$$D = \{\sigma \in \mathcal{L}_{\mathbf{q}} : \delta \in E \Longrightarrow \sigma \cap \delta = \emptyset\}.$$

Recall that $E \in \mathcal{E}_r$, so $\widehat{\alpha}_{r+1} \geq \theta$ implies that the length of any element of D is at least θ.

For $A \subset \mathcal{L}_{\mathbf{q}}$ we let $\overline{A}$ denote the closure of the union of saddle connections in A (the closure just adds their endpoints). If Ω is a union of connected components of $S - \overline{E}$, we let $\overline{\Omega}$ denote the completion of Ω in the path metric – this adds a boundary $\partial\Omega$, which is mapped to $\overline{E}$ by the continuous extension of the inclusion map of Ω. In particular any segment of $\overline{E}$ which Ω meets on both sides is covered by two segments of $\partial\Omega$.

$\overline{\Omega}$ inherits the piecewise Euclidean metric of $\mathbf{q}$ from S, and any two adjacent segments of $\partial\Omega$ meet in a well-defined *internal angle.*

We first establish the following

Claim 6.8 ([Ve], Lemma 4.16)**.** *If Ω_0 is a component of $S - \overline{E}$, and at least one internal angle of $\partial\Omega_0$ is less than π, then $\Omega_0 \subset S(E)$.*

Suppose $\delta_1, \delta_2 \in \partial\Omega_0$ have endpoints x_1, z and z, x_2 respectively, such that the internal angle at z is less than π. Let η be the path which is the concatentation of δ_1 and δ_2 and let η' be the shortest representative of the homotopy class, rel x_1, x_2, in $\overline{\Omega_0}$, of η. Since η is not length minimizing, $\eta' \neq \eta$ and the homotopy between them covers a disk Ω_1 with nonempty interior. Then η' consists of saddle connections which are either on $\partial\Omega_0$ or contained in Ω_0, that is, the image η'' of η' in S consists of saddle connections in $D \cup E$. Since the length of η'' is less than $\frac{2\theta}{3}$, we have $\eta'' \subset E$, and hence $\eta' \subset \partial\Omega_0$. Thus $\partial\Omega_1 \subset \partial\Omega_0$, and hence $\Omega_1 = \Omega_0$. This makes Ω_0 a disk, so $\Omega_0 \subset S(E)$ and we have proved the claim.

Now recall we have a saddle connection δ in $\partial S(E)$ and $\delta' \neq \delta$ intersecting δ. Thus δ' cannot be contained in $S(E)$. Let $\omega = [x_1, x_2]$ be a segment on δ' whose interior is outside $S(E)$ and whose endpoints are either on $\partial S(E)$ or on saddle connections of $\mathbf{q}$. For each x_i that is not already a saddle, let δ_i be the saddle connection in $\partial S(E)$ containing it, and let σ_i be the shorter of the two components of $\delta_i - \{x_i\}$. If x_i is a saddle, let $\sigma_i = x_i$. Let η be the concatenation of σ_1, ω and σ_2.

If (*) is not true, then $L_{q,\delta'}(t) < \frac{2\theta}{3}$ and hence the length of η is at most θ. Let η' be the shortest path homotopic to η rel endpoints within $\overline{S - S(E)}$. Then its image η'' in S consists of saddle connections in $D \cup E$. Since its total length is less than θ, all of the saddle connections must be in E.

The homotopy between η and η' in $\overline{S - S(E)}$ must cover a disk Ω_1 with degree 1, and $\partial\Omega_1$ must map to $\overline{E}$ and ω. Thus, except possibly for the two angles corresponding to the endpoints of ω, all internal angles of $\partial\Omega_1$ are internal angles of $\overline{S - S(E)}$, and hence at least π, by Claim 6.8. However, a disk with a piecewise Euclidean metric, geodesic boundary, interior cone points with angle least 2π and only two internal boundary angles less than π violates the Gauss-Bonnet theorem. This contradiction establishes the lemma. $\square$

Proof of Lemma 6.4: Let c, t_0 (resp. $\widetilde{c}, \widetilde{t_0}$) be the constants in the definition of f (resp. $\widetilde{f}$). If either of $f, \widetilde{f}$ is a constant function there is nothing to prove. Exchanging f and $\widetilde{f}$ if necessary, assume that $0 < c \leq \widetilde{c}$. Since $\widetilde{f}(t) < b/3$ and $\widetilde{f}(s) < b/3$ we have that $\widetilde{c}|t - \widetilde{t_0}| < b/3$ and $\widetilde{c}|\widetilde{t_0} - s| < b/3$. Dividing these inequalities by $\widetilde{c}$ and adding yields

$$|t-s| \le |t-\widetilde{t}_0| + |\widetilde{t}_0 - s| < \frac{2b}{3\widetilde{c}}.$$

Since $f(s) < b/3$ we have $c < b/3$. We consider separately the two cases according as t_0 is or is not between s and t. If t_0 is between s and t then

$$c|t-t_0| \le \widetilde{c}|s-t| < \frac{2b}{3}.$$

If t_0 is not between s and t then

$$c|t-t_0| = c|s-t_0| + c|t-s| < f(s) + \frac{2b}{3} < b.$$

In either case we get that $f(t) < b$. □

6.4. **The case with punctures.** If S has punctures we may reduce to the case of a closed surface by using the branched cover given by Lemma 4.9. In particular the set of length functions of saddle connections is unaltered by passage to the branched cover, so that the proofs in the cover imply the same conclusions down in S.

6.5. **Veech's decay condition on measures.** In this section we show that the results of §6.2 can be generalized to averages with respect to measures which satisfy Veech's decay condition. The main observation we shall use is that if a suitable interval is chosen, length functions for saddle connections are good for any F-decaying measure.

For every function $f : \mathbb{R} \to \mathbb{R}_+$ and every $\theta > 0$, let

$$\mathbb{R}_{f,\theta} = \{t \in \mathbb{R} : f(t) < \theta\}.$$

Proposition 6.9. *Let*

$$\mathcal{F} = \{l_{\mathbf{q},\delta} : \mathbf{q} \in \widetilde{\mathcal{Q}_1},\ \delta \in \mathcal{L}_{\mathbf{q}}\}.$$

Let $I \subset \mathbb{R}$ be an interval and let $\rho > 0$ satisfy:

$$\mathbb{R}_{f,\rho} \cap I \neq \emptyset \Longrightarrow \mathbb{R}_{f,\rho} \subset I. \tag{34}$$

Then for any F-decaying measure μ, $\mathcal{F}$ is (μ, F, ρ)-good on I.

Proof: Given $0 < \epsilon < \rho$, if $I_{f,\epsilon} = \emptyset$ there is nothing to prove. Otherwise, we have from (34) that

$$I_{f,\epsilon} = \mathbb{R}_{f,\epsilon}, \quad I_{f,\rho} = \mathbb{R}_{f,\rho}.$$

From Proposition 4.4 there are $t_0 \in I$ and $r > 0$ such that

$$I_{f,\rho} = B(t_0, r), \quad I_{f,\epsilon} = B(t_0, \frac{\epsilon}{\rho} r).$$

Now using our assumption on μ we get that

$$\frac{\mu(I_{f,\epsilon})}{\mu(I_{f,\rho})} \le F(\epsilon/\rho).$$

□

Now repeating the proof of Theorem 6.3 (using Proposition 3.4 instead of Proposition 3.2), we obtain:

Theorem 6.10. *There are ρ_0, C_1, C_2, α (depending only on S) such that for every F-decaying measure μ the following holds.:*

Suppose $\mathbf{q} \in \widetilde{\mathcal{Q}_1}$, an interval $I \subset \mathbb{R}$, and $0 < \rho' \leq \rho_0$ satisfy:

$$\text{for any } \delta \in \mathcal{L}_{\mathbf{q}},\ \mathbb{R}_{l_\delta,\rho} \cap I \neq \emptyset \Longrightarrow \mathbb{R}_{l_\delta,\rho} \subset I. \tag{35}$$

Then for any $0 < \epsilon < C_1\rho'$ we have:

$$\frac{\mu(\{t \in I : \alpha_1(t) < \epsilon\})}{\mu(I)} \leq C_2 F\left(\frac{\epsilon}{\rho'}\right)^{\alpha}. \tag{36}$$

□

7. Finiteness, minimal sets and ergodicity

7.1. **Ergodic measures are finite.** In this section, unless otherwise indicated, the proofs for the earthquake flow and the Teichmüller horocycle flow are identical. For definiteness, all proofs are given for the Teichmüller horocycle flow. The constants C_1, C_2, ρ_0, α (resp. ρ, c) are as in Theorem 6.3 (resp. Proposition 5.1).

Recall the statement of the first corollary:

Corollary 2.6 *For both the earthquake flow and the Teichmüller horocycle flow, any locally finite ergodic invariant measure is finite.*

In the flow-space $\mathcal{Q}_1$, any locally finite invariant measure which is ergodic for the action of $\mathrm{SL}(2,\mathbb{R})$ is finite.

Proof of Corollary 2.6: The argument sketched below is due to Dani [Da1], and is included for the reader's convenience.

Let ν be an ergodic invariant measure, and let $f : \mathcal{Q}_1 \to \mathbb{R}$ be a continuous positive function such that $\int_{\mathcal{Q}_1} f d\nu = 1$ (one exists because the measure is locally finite). By the Birkhoff ergodic theorem (see [Kr]), for a.e. $q \in \mathcal{Q}_1$ the limit

$$\bar{f}(q) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(h_t q)dt \tag{37}$$

exists, is in $L^1(\mathcal{Q}_1, \nu)$, and is h_t-invariant.

Let us show that if q is any point for which the limit in (37) exists, then $\bar{f}(q) > 0$. By Theorem H3 let K be a compact subset of $\mathcal{Q}_1$ such that for all $T > 0$ we have $\mathrm{Avg}_{T,q}(K) \geq 1/2$. Let

$$\alpha = \min_{p\in K} f(p) > 0,$$

then it is clear that $\bar{f}(q) \geq \alpha/2$.

Now the sets $\mathcal{Q}_1^n = \{q \in \mathcal{Q}_1 : \bar{f}(q) \geq 1/n\}$ are measurable, h_t-invariant, and exhaust $\mathcal{Q}_1$, therefore there is n for which $\nu(\mathcal{Q}_1^n) > 0$. By ergodicity

$\nu(\mathcal{Q}_1) = \nu(\mathcal{Q}_1^n)$, and since $\bar{f} \in L^1$ we get that $\nu(\mathcal{Q}_1) < \infty$. This proves the first assertion.

Let a locally–finite measure μ on $\mathcal{Q}_1$ which is invariant and ergodic for the natural action of $G = \mathrm{SL}(2, \mathbb{R})$ be given. Let U denote the subgroup of upper–triangular matrices. An elementary calculation (see e.g. [Ma2, §1]) shows that the inclusion $U \subset G$ satisfies the *Mautner property*, i.e., for any unitary representation of G, any vector which is fixed by U is fixed by G. Considering the natural action of G on the Hilbert space $L^2(\mathcal{Q}_1, \mu)$, this implies that μ is already ergodic with respect to the Teichmüller horocycle flow. Thus the last assertion follows from the preceding one. $\square$

7.2. **Minimal invariant sets.** Recall the statement of the second corollary:

Corollary 2.7 *For both the earthquake flow and the Teichmüller horocycle flow, any closed invariant set contains a minimal closed invariant subset. Any minimal closed invariant set is compact.*

For the proof of Corollary 2.7 we will need the following:

Lemma 7.1. *Suppose X_0 is a closed invariant subset of $\mathcal{Q}_1$ such that*

$$\inf\{l_{\mathbf{q},\delta} : \mathbf{q} \in \pi^{-1}(X_0),\ \delta \in \mathcal{L}_{\mathbf{q}},\ l_{\mathbf{q},\delta}(t) \equiv \mathrm{const}\} > 0. \tag{38}$$

Then X_0 contains a minimal closed invariant set.

Proof: We claim that there is a compact $K \subset \mathcal{Q}_1$ which intersects the trajectory $\{h_t q\}$ for every $q \in X_0$. Indeed, let ρ' be a positive number smaller than the infimum in (38) and ρ_0. Let $\epsilon < C_1\rho'$ be small enough so that

$$C_2 \left(\frac{\epsilon}{\rho'}\right)^{\alpha} < 1,$$

and let $K = K_\epsilon$. Now, given $q \in X_0$ and arguing as in the proof of Theorem H2 we can find T large enough so that the hypotheses of Theorem 6.3 will be satisfied for $I = [0, T]$. We obtain from (30) that $\mathrm{Avg}_{q,T}(K) > 0$ and our claim follows.

We can now use Zorn's lemma. We order the closed invariant subsets of X_0 by inclusion. For any totally ordered family $\{X_\alpha\}$ of closed invariant subsets of X_0, the finite intersections $K \cap X_{\alpha_1} \cap \cdots \cap X_{\alpha_r}$ are nonempty since all trajectories meet K, hence by compactness $\bigcap_\alpha X_\alpha \neq \emptyset$. Then by Zorn's lemma there is a minimal closed invariant subset in X_0. $\square$

We will also use the following lemma. For a proof see [DaMa] and the references therein.

Lemma 7.2. *Let $\{T_t\}$ be an action of $\mathbb{R}$, by homeomorphisms, on a locally compact space Z. Suppose there is a compact $K \subset Z$ such that for every $z \in Z$ the subsets $\{t \geq 0 : T_t z \in K\}$ and $\{t \leq 0 : T_t z \in K\}$ are both unbounded. Then Z is compact.*

Proof of Corollary 2.7: First let us prove that if a minimal set exists, then it is compact. Let X be a minimal set. Using Lemma 7.2 it is enough to show that for every $q \in X$, the positive and negative semi–orbits $\{h_t q : t \geq 0\}$ and $\{h_t q : t \leq 0\}$ are both dense in X. Let $X^+(q)$ denote the accumulation points of the positive semi–orbit of q; then $X^+(q)$ is a closed invariant set, which is nonempty by Theorem H1. By minimality $X^+(q) = X$. The proof for the negative semi–orbit is similar.

Now to prove the existence of a minimal set, we must show that every closed invariant subset $X \subset \mathcal{Q}_1$ contains a closed invariant subset X_0 verifying condition (38) of Lemma 7.1. We will separate this part of the proof into two cases.

Verifying the condition for earthquakes. Let $p \in X$, let $\mathbf{p} \in \pi^{-1}(p)$, and let

$$L(p) = \min\{\ell_{\mathbf{p},\gamma} : \gamma \in \Gamma^S,\ \ell_{\mathbf{p},\gamma}(t) \equiv \text{const}\}.$$

By Proposition 4.3, $L(p) > 0$.

We will set $X_0 = \overline{\{E_t p : t \in \mathbb{R}\}}$, and argue that if p is suitably chosen then $L(p)$ is the infimum in (38) for X_0 as well.

In order to do this we introduce the notion of *extendability* $\varepsilon(\lambda)$ of a geodesic lamination λ: $\varepsilon(\lambda)$ is the largest number k such that there exists a sequence of distinct geodesic laminations $\lambda = \lambda_0 \subset \cdots \subset \lambda_k$.

The complement of λ is an open surface which we can endow with the path metric – in particular its inclusion in S is an injective immersion but typically not an embedding. Its completion R_λ in the path-metric is a surface of finite area with geodesic boundary. The inclusion $\iota : S - \lambda \to S$ extends to a map $\bar{\iota} : R_\lambda \to S$ which takes ∂R_λ onto the *boundary leaves* of λ – those that are isolated on at least one side. (See Figure 6 for an example). We will use the following facts, whose proof will appear at the end of the argument:

Claim 7.3.
(1) *$\varepsilon(\lambda)$ is bounded above by a number depending only on S.*
(2) *If $\lambda \subseteq \mu$ then $\varepsilon(\lambda) \geq \varepsilon(\mu)$, with equality only if $\lambda = \mu$.*
(3) *$\varepsilon(\lambda)$ depends only on the complementary surface R_λ.*

Now let p be chosen so that, for $\mathbf{p} \in \pi^{-1}(p)$, $\varepsilon(\lambda_{\mathbf{p}})$ is maximal in X (the choice of $\mathbf{p}$ does not change ε), and let X_0 be the closure of its orbit.

Fix any $p_\infty \in X_0$ and a sequence $t_i \in \mathbb{R}$ such that $p_i \equiv E_{t_i} p$ converges to p_∞. Choose lifts $\mathbf{p}_\infty \in \widetilde{\mathcal{P}_1}$ of p_∞ and $\mathbf{p}_i \in \widetilde{\mathcal{P}_1}$ of p_i so that $\mathbf{p}_i \to \mathbf{p}_\infty$.

By definition, the laminations $\lambda_i \equiv \lambda_{\mathbf{p}_i}$ converge to $\lambda_\infty \equiv \lambda_{\mathbf{p}_\infty}$, in $\mathcal{PML}(S)$. The metrics $\sigma_i \equiv \sigma_{\mathbf{p}_i}$ converge to $\sigma_\infty \equiv \sigma_{\mathbf{p}_\infty}$ in Teichmüller space, and after suitable reparametrization we may assume they converge pointwise. Possibly taking a subsequence, we may also assume that λ_i converge to a lamination λ_H in the Hausdorff topology on closed subsets of S, and λ_H contains the support of λ_∞. Our goal now is to prove that $\lambda_H = \lambda_\infty$. By part (2) of Claim 7.3 we already have $\varepsilon(\lambda_H) \leq \varepsilon(\lambda_\infty)$, and it suffices to show that $\varepsilon(\lambda_H) = \varepsilon(\lambda_\infty)$.

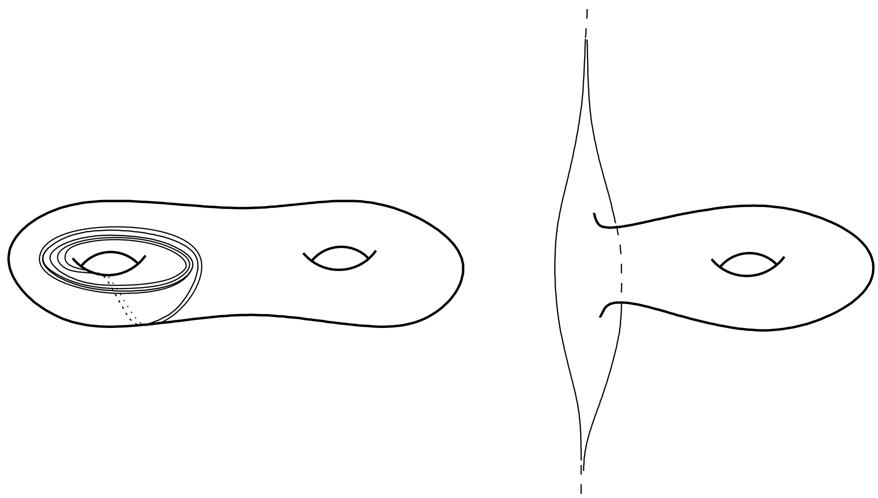

FIGURE 6. An example of a minimal lamination λ and its associated surface R_λ. Here $\chi(S) = -2$, $\chi(R_\lambda) = -1$ and $\beta = 2$.

Let $\lambda \equiv \lambda_{\mathbf{p}}$. Since all the p_i are on a single earthquake path, there is for each i an isometry $\varphi_i : R_\lambda \to R_{\lambda_i}$, where R_{λ_i} is considered with the σ_i metric. This map is obtained from the restriction of the earthquake map to the complement of λ. Since $\sigma_i \to \sigma_\infty$ in Teich(S) we may assume that the metrics are converging pointwise. Letting $\iota_i : S - \lambda_i \to S$ be the inclusion map, and using the definition of the Hausdorff topology, we find that the maps $\iota_i \circ \varphi_i$ converge to an isometric immersion of $int(R_\lambda)$ into S with the σ_∞ metric, whose image is the complement of λ_H. In other words R_{λ_H} is isometric to R_λ, and we conclude that $\varepsilon(\lambda_H) = \varepsilon(\lambda)$ (by part (3) of Claim 7.3). Since $\varepsilon(\lambda)$ is maximal by choice of p, we have $\varepsilon(\lambda_H) \geq \varepsilon(\lambda_\infty)$, and hence the two are equal.

The condition that $\ell_{\mathbf{p},\gamma}$ is a constant function is equivalent to the condition that γ does not intersect $\lambda_{\mathbf{p}}$. Now since the support of λ_∞ equals λ_H, any curve γ disjoint from λ_∞ is disjoint from λ_H, and hence eventually from λ_i. It follows that $\ell(\gamma)$ is already on our list of constant length functions, and so $L(p_\infty) = L(p)$, as desired.

This completes the proof for earthquakes, modulo the proof of the claim:

Proof of Claim 7.3: Part (2) of the claim is immediate from the definition.

Let us now consider the structure of R_λ (for details see [Bon1, Bon2], [CasBle]). The area of R_λ is the same as that of S, namely $-2\pi\chi(S)$ by the Gauss-Bonnet formula. The boundary components of R_λ are either closed geodesics, or infinite geodesics. At each end of an infinite geodesic it is asymptotic to the end of another (possibly the same) geodesic. The region between two such asymptotic ends is called a "spike". The Gauss-Bonnet formula applied to R_λ gives a second expression for its area, $-2\pi\chi(R_\lambda)+\pi\beta$, where $\beta = \beta(R_\lambda)$ is the number of spikes (or the number of infinite boundary components). Equating these two expressions tells us that β is even.

The laminations μ containing λ are in one-to-one correspondence with the laminations in R_λ containing the boundary, via the map $\bar\iota : R_\lambda \to S$. Thus part (3) of the claim follows immediately.

Finally to give the bound in part (1) we apply an argument suggested by F. Bonahon.

Let $\partial R_\lambda \subset \mu_1 \subset \cdots \subset \mu_n = \mu$ be a maximal-length sequence of laminations in R_λ. We may assume R_μ is a union of ideal triangles, since otherwise we may enlarge μ by adding isolated leaves in R_μ that are asymptotic to the boundary.

A basic structure theorem about geodesic laminations (see [CaEpGr, Thm 4.2.8]) says that any geodesic lamination is a finite union of minimal laminations and isolated leaves which accumulate onto them. A minimal lamination is either a simple closed curve, or an uncountable union of infinite leaves, each of which is dense. Applying this to the lamination μ, we see that each step of the sequence μ_i must involve the addition of either a minimal lamination or an isolated leaf, and so our task is to bound the number of such elements of μ.

Each isolated leaf of μ is the $\bar\iota$-image of two leaves of ∂R_μ – one from each side. Similarly a minimal sublamination ν that is not a closed leaf has leaves that are the images of at least two leaves of ∂R_μ. This is seen as follows: $\beta(R_\nu)$ is even, as observed before, so there are at least two infinite leaves in ∂R_ν (if there were none then ν would be a closed leaf). A leaf of μ cannot accumulate onto an infinite leaf l of ∂R_ν from within R_ν, since as soon as it is sufficiently close it is forced to continue into one of the spikes of l, and cannot return. Thus l is also a boundary leaf of R_μ.

This tells us that the number of isolated leaves and non-closed minimal components is bounded by $\frac{1}{2}\beta(R_\mu)$. R_μ is a union of N ideal triangles, each of area π so $N = -2\chi(S)$, and clearly $\beta(R_\mu) = 3N$. Thus the bound becomes $-3\chi(S)$. (In fact it is $-3\chi(S) - \beta(R_\lambda)$, since each infinite boundary leaf of R_λ is also a boundary leaf of R_μ.)

The number of simple closed curves in μ is bounded by $-(3\chi(R_\lambda) + \kappa)/2$ by an Euler characteristic argument, where κ is the number of punctures of R_λ. Putting these two bounds together (and noting that $-\chi(R_\lambda)$ is at most $-\chi(S)$) we have our bound on $\varepsilon(\lambda)$. $\square$

Verifying the condition for horocycles. We will prove the condition for the case where S has no punctures. The general case reduces to this one, again via Lemma 4.9.

For $\mathbf{q} \in \widetilde{\mathcal{Q}_1}$ and $\delta \in \mathcal{L}_{\mathbf{q}}$ we have that $l_{\mathbf{q},\delta} \equiv \text{const}$ if and only if δ is contained in a horizontal leaf for $\mathbf{q}$. Let $\Xi_{\mathbf{q}}$ be the union of all horizontal leaves of $\mathbf{q}$ that meet singularities, endowed with the path-metric restricted from $\mathbf{q}$. This is a graph with finitely many edges, some of which may be infinite rays. It is injectively immersed in S.

$\Xi_{\mathbf{q}}$ will play the same role that R_λ played in the case of earthquakes. Indeed if $\lambda_{\mathbf{q}}$ is the geodesic lamination associated to the horizontal foliation of $\mathbf{q}$ (as discussed in §4.4), then we can relate $R_{\lambda_{\mathbf{q}}}$ to $\Xi_{\mathbf{q}}$ as follows: Lift $\Xi_{\mathbf{q}}$ to $\widetilde{\Xi}_{\mathbf{q}}$ in the universal cover $\mathbb{H}^2$ of S, and let $\mathcal{N}$ denote an embedded regular neighborhood of it. Each component of $\mathcal{N}$ is simply connected, and each boundary component of $\mathcal{N}$ is an infinite path terminating at two distinct points in S^1, and hence is homotopic to a unique geodesic. Thus, the closure of any component of $\mathcal{N}$ is properly isotopic to a subset of $\mathbb{H}^2$ with geodesic boundary, and the union of these projects down to $R_{\lambda_{\mathbf{q}}}$ in S (see Levitt [Lev] for details).

Note in particular that $R_{\lambda_{\mathbf{q}}}$ is homotopy-equivalent to $\Xi_{\mathbf{q}}$, by a map that takes boundary components of $R_{\lambda_{\mathbf{q}}}$ to "boundary paths" of $\Xi_{\mathbf{q}}$. A boundary path is an immersed path in $\Xi_{\mathbf{q}}$ with these properties: whenever it traverses a singularity it enters and exits along two adjacent prongs. The other prongs are either always to the right of the path (fixing an orientation), or to the left.

Define $\varepsilon(\mathbf{q})$ to be $\varepsilon(\lambda_{\mathbf{q}})$. By Claim 7.3, $\varepsilon(\mathbf{q})$ depends only on $\Xi_{\mathbf{q}}$.

Now choose $q \in X$ so that $\varepsilon(\mathbf{q})$ is maximal (where $\mathbf{q} \in \pi^{-1}(q)$), and let X_0 be the closure of $\{h_t q\}$. Let q_∞ be a point of X_0 and $q_i = h_{t_i} q$ a sequence coverging to q_∞. Lift these to a convergent sequence $\mathbf{q}_i \to \mathbf{q}_\infty$.

We wish to prove that $\Xi_{\mathbf{q}}$ and $\Xi_{\mathbf{q}_\infty}$ are isometric. Indeed, because the Teichmüller horocyclic action leaves the horizontal direction invariant, $\Xi_{\mathbf{q}}$ is isometric to $\Xi_{\mathbf{q}_i}$ for each i. Let ψ_i be this isometry. Let $f_i : S \to S$ be the comparison maps described in §4.3 in the discussion of convergence in $\widetilde{\mathcal{Q}}$. It follows from that discussion that the maps $f_i \circ \psi_i$ converge, possibly after restriction to a subsequence, to a map $\psi_\infty : \Xi_{\mathbf{q}} \to \Xi_{\mathbf{q}_\infty}$, which is a local isometric embedding. In particular a point of $\Xi_{\mathbf{q}}$ with $k \geq 2$ prongs ($k = 2$ applies to a non-singular point) must map to a point of $\Xi_{\mathbf{q}_\infty}$ with at least k prongs, and the images of the prongs are distinct.

We claim that $\varepsilon(\mathbf{q}_\infty) \geq \varepsilon(\mathbf{q})$, with equality only if ψ_∞ is an isometry.

To see this, lift to the universal cover. The map ψ_∞ lifts to $\widetilde{\psi}_\infty : \widetilde{\Xi}_{\mathbf{q}} \to \widetilde{\Xi}_{\mathbf{q}_\infty}$, which is a local isometry (but not necessarily globally injective). It is surjective since every singular leaf of $\mathbf{q}_\infty$ is a limit of images of singular leaves in $\mathbf{q}_i$ (§4.3). Every boundary leaf of $\widetilde{\Xi}_{\mathbf{q}}$ maps to some path in $\widetilde{\Xi}_{\mathbf{q}_\infty}$ (not necessarily a boundary path), and after the straightening step we obtain, in the quotient surface, an embedding of $R_{\lambda_{\mathbf{q}}}$ in $R_{\lambda_{\mathbf{q}_\infty}}$. We conclude $\varepsilon(\mathbf{q}_\infty) \geq \varepsilon(\mathbf{q})$, by part (2) of Claim 7.3. By maximality of $\varepsilon(\mathbf{q})$, we have $\varepsilon(\mathbf{q}) = \varepsilon(\mathbf{q}_\infty)$. Therefore every boundary leaf of $\widetilde{\Xi}_{\mathbf{q}}$ must in fact map to a boundary leaf of $\widetilde{\Xi}_{\mathbf{q}_\infty}$. It follows that a k-pronged point must map to a k-pronged point (if it mapped to point with $k' > k$ prongs then there would have to be two adjacent prongs mapping to non-adjacent prongs, and the corresponding boundary leaf would not map to a boundary leaf), so that $\widetilde{\psi}_\infty$ is a covering map. Since each component of $\widetilde{\Xi}_{\mathbf{q}}$ is simply connected, i.e. a tree, $\widetilde{\psi}_\infty$ is a homeomorphism, hence an isometry, from each component to its image component.

It remains to verify that $\widetilde{\psi}_\infty$ cannot identify two different components of $\widetilde{\Xi}_{\mathbf{q}}$. Suppose otherwise, and let Ξ_1, Ξ_2 be two such components. Since each component contains at least one singularity with $k \geq 3$ prongs, let L_1 be a leaf of Ξ_1 passing through a singularity v and let P_1 be a half-leaf emanating from v to one side of L_1. Let L_2, P_2 be the leaves of Ξ_2 identified to L_1, P_1 by $\widetilde{\psi}_\infty$. Since $\widetilde{\psi}_\infty$ is a limit of embeddings $\widetilde{\psi}_i$, for each i we find that $\widetilde{\psi}_i(L_1)$ separates the disk into two components, and $\widetilde{\psi}_i(L_2)$ is eventually arbitrarily close to $\widetilde{\psi}_i(L_1)$ on one side. Since the images are disjoint, the half-leaf $\widetilde{\psi}_i(P_1)$ is on the other side. It follows that $\widetilde{\psi}_i(P_2)$ is separated from $\widetilde{\psi}_i(P_1)$ by $\widetilde{\psi}_i(L_1)$, and hence they cannot have the same limit.

We conclude, since $\widetilde{\psi}_\infty$ is surjective, it must in fact be a global isometry of $\widetilde{\Xi}_{\mathbf{q}}$ to $\widetilde{\Xi}_{\mathbf{q}_\infty}$, and downstairs ψ_∞ is an isometry.

Thus $\Xi_{\mathbf{q}_\infty}$ and $\Xi_{\mathbf{q}}$ are isometric, and in particular their horizontal saddle connections have the same lengths. This verifies condition (38). □

7.3. **Unique Ergodicity.** Recall that $q \in \mathcal{Q}_1$ is called *minimal* if there are no proper closed subsets of $\mathcal{Q}_1$ which are union of leaves for the vertical foliation of q. Let $\{g_t : t \in \mathbb{R}\}$ denote the Teichmüller geodesic flow. An orbit $\{g_t q : t \geq 0\}$ is called *divergent* if for any compact $K \subset \mathcal{Q}_1$ there is t_0 such that

$$t \geq t_0 \Longrightarrow g_t q \notin K.$$

To prove Corollaries 2.8 and 2.9 we will follow Masur's strategy of using non–divergence results in conjunction with the following:

Proposition 7.4 (Masur [Mas1], [Mas2])**.** *Suppose $q \in \mathcal{Q}_1$ is minimal and not uniquely ergodic. Then $\{g_t q : t \geq 0\}$ is divergent.*

Recall the statement of the corollary:

Corollary 2.8 *Let $q \in \mathcal{Q}_1$. For almost every $s \in \mathbb{R}$, $h_s q$ is uniquely ergodic.*

Proof of Corollary 2.8: Arguing by contradiction, we obtain the existence of $q \in \mathcal{Q}_1$ and a finite interval $I \subset \mathbb{R}$ such that

$$\eta = \frac{|\{s \in I : h_s q \text{ is not uniquely ergodic }\}|}{|I|} > 0. \tag{39}$$

Pick some $s_0 \in I$ and let

$$\rho'' = \min\{l_{\delta,q}(s_0) : \delta \in \mathcal{L}_{\mathbf{q}}\},$$

and

$$\rho' = \min\{\rho'', \frac{|I|\rho''}{2}, \rho_0\}.$$

Claim 7.5. *For any $\delta \in \mathcal{L}_{\mathbf{q}}$ and any $t \geq 0$ there is $s \in I$ such that the length of the horizontal component of δ in the flat metric corresponding to $g_t h_s q$ is at least ρ'.*

Suppose the coordinates of δ with respect to the flat metric corresponding to q are (x, y). It follows from the discussion in §2.5 that the coordinates of δ with respect to the flat metric corresponding to $g_t h_s q$ are:

$$\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{t/2}(x+sy) \\ e^{-t/2}y \end{pmatrix}.$$

Since $e^{t/2} \geq 1$ it suffices to find $s \in I$ such that

$$|x + sy| \geq \rho'.$$

If $y = 0$ then $l_{\mathbf{q},\delta}(s_0) = x$, and by the definitions of ρ' and ρ'' we have $x \geq \rho'$. If $y \neq 0$ and $x + s_0 y < \rho'$ then by the definition of ρ'' we must have $|y| \geq \rho''$. In this case the function $s \mapsto |x + sy|$ has slope $\pm y$ and thus for at least one of the endpoints s of I we will have:

$$|x + sy| \geq \frac{|I|}{2}|y| \geq \frac{\rho''|I|}{2} \geq \rho'.$$

This proves the claim.

Now let $\epsilon > 0$ be small enough so that

$$C_2 \left(\frac{\epsilon}{\rho'}\right)^{\alpha} < \frac{\eta}{2}.$$

Let $K = K_\epsilon$. For any $t_0 > 0$, let $I' = \{e^{t_0}s : s \in I\}$ and let $q' = g_{t_0} q$. Using Theorem 6.3 and Claim 7.5 we obtain that

$$\frac{|\{s \in I : g_{t_0} h_s q \notin K\}|}{|I|} = \frac{|\{s \in I' : h_s q' \notin K\}|}{|I'|} \leq \frac{\eta}{2}. \tag{40}$$

Let

$$S_0 = \{s \in I : h_s q \text{ is not minimal}\}.$$

Claim 7.6. *S_0 is countable.*

Using (16), for any $\delta \in \mathcal{L}_{\mathbf{q}}$ which is not horizontal, there is at most one $s = s(\delta) \in I$ such that δ is vertical with respect to $h_s q$. Now let $s_0 \in S_0$, and let $\sigma \subset S$ be a segment which is in a horizontal leaf for q. Since the vertical foliation of $h_{s_0} q$ is not minimal, neither is the interval exchange $\sigma \to \sigma$ given by the first return map along the vertical foliation of $h_{s_0} q$. It then follows from a result of Keane [Kea] that there is a saddle connection δ which is contained in the vertical foliation for $h_{s_0} q$, i.e., $s_0 = s(\delta)$. Thus the countability of S_0 follows from that of $\mathcal{L}_{\mathbf{q}}$.

From the claim and Proposition 7.4 it follows that the measure of the set of points in I whose forward trajectory under $\{g_t\}$ is divergent has measure at least $\eta|I|$. Since all divergent trajectories eventually leave K, there is t_0 such that

$$|\{s \in I : g_{t_0} h_s q \notin K\}| \geq \frac{2\eta}{3} \cdot |I|.$$

This contradicts (40). $\square$

We now turn to

Corollary 2.9 *If μ is a measure on $\mathbb{R}$ which satisfies Veech's decay condition, then for μ-almost every s, $h_s q$ is uniquely ergodic.*

Proof of Corollary 2.9: Again by contradiction, there is $q \in \mathcal{Q}_1$ and an interval $I = [b, c]$ for which (39) (with μ instead of $|\cdot|$) holds. Construct an interval $J = [a, d]$ containing I by adjoining two intervals $I_1 = [a, b]$ and $I_2 = [c, d]$ to I on both sides. Making η smaller if necessary, we obtain that

$$\eta = \frac{\mu(\{t \in I : h_t q \text{ is not uniquely ergodic}\})}{\mu(J)} > 0.$$

This implies that for any interval $I' = [s_1, s_2]$, with $s_i \in I_i$, we have

$$\frac{\mu(\{t \in I' : h_t q \text{ is not uniquely ergodic}\})}{\mu(I')} \geq \eta.$$

Picking some $x_i \in I_i,\ i = 1, 2$ we let

$$\rho'' = \min\{l_{\delta,q}(x_i) : \delta \in \mathcal{L}_{\mathbf{q}},\ i = 1, 2\},$$

and

$$\rho' = \min\{\rho'', \frac{|I_i|\rho''}{2}, \rho_0\}.$$

Repeating the proof of Claim 7.5 for both I_1 and I_2 we obtain:

For $i = 1, 2$, any $\delta \in \mathcal{L}_{\mathbf{q}}$ and any $t \geq 0$ there is $s \in I_i$ such that the length of the horizontal component of δ in the flat metric corresponding to $g_t h_s q$ is at least ρ'.

Now choosing $\widetilde{\rho}$ small enough so that

$$C_2 \left(\frac{\widetilde{\rho}}{\rho'}\right)^{\alpha} < 1,$$

we obtain from Theorem 6.3 that for each $t \geq 0$ and each i, there is $s_i = s_i(t) \in I_i$ such that for all $\delta \in \mathcal{L}_{\mathbf{q}}$,

$$l_{g_t h_{s_i} q, \delta} \geq \widetilde{\rho}.$$

This implies that (35) is satisfied for $I' = [s_1, s_2]$ instead of I and $\widetilde{\rho}$ instead of ρ.

Now suppose that μ is F-decaying for some function F with $\lim_{x \to 0} F(x) = 0$. Choosing ϵ small enough and using Theorem 6.10 we obtain

$$\frac{\mu(\{s \in I : g_t h_s \notin K_\epsilon\})}{\mu(I)} \leq C_2 F\left(\frac{\epsilon}{\widetilde{\rho}}\right)^{\alpha} < \frac{\eta}{2},$$

and arguing as before, arrive at a contradiction. $\square$

8. Comparing horocycles and earthquakes

In this section we compare, in the special case of one complex dimensional Teichmüller spaces, the earthquake flow to the Teichmüller horocyclic flow.

Teich(S) has one complex dimension exactly when S is a torus with at most one puncture, or a 4-punctured sphere. In this case Teich(S) can be identified with the upper half-plane $\mathbb{H}^2$. The Teichmüller horocycle flow is just the regular horocycle flow for $\mathbb{H}^2$. The Teichmüller metric is exactly the hyperbolic metric.

An unpunctured torus admits no hyperbolic metric, so the earthquake flow is not defined. The naturally corresponding definition using the Euclidean metric works out to be identical to the horocyclic flow.

From now on we will assume that S is a once-punctured torus or 4-punctured sphere. The identification of both $\widetilde{\mathcal{P}_1}$ and $\widetilde{\mathcal{Q}_1}$ with Teich$(S) \times \mathcal{PML}(S)$ preserves asymptotic behavior with respect to Thurston's compactification $\overline{\text{Teich}}(S) = \text{Teich}(S) \cup \mathcal{PML}(S)$. That is, let $\pi : \text{Teich}(S) \times \mathcal{PML}(S) \to \text{Teich}(S)$ denote projection to the first factor and let $\mathbf{p} \in \widetilde{\mathcal{P}_1}$ and $\mathbf{q} \in \widetilde{\mathcal{Q}_1}$ be given by pairs (x, λ) and (y, λ), respectively. Then the projected orbits $\{\pi \circ E_t \mathbf{p}\}$ and $\{\pi \circ h_t \mathbf{q}\}$ limit to λ in the compactification, as $t \to \pm\infty$.

In spite of this, it turns out that the flows are quite different. We shall prove:

Proposition 8.1. *Let S be a once-punctured torus or a four-times punctured sphere. For every point $x \in Teich(S)$ and every irrational lamination $\lambda \in \mathcal{PML}(S)$, the projected flow lines $\{\pi \circ h_t(x, \lambda)\}$ and $\{\pi \circ E_t(x, \lambda)\}$ are an infinite Hausdorff distance apart in $\mathbb{H}^2$.*

We recall that the Hausdorff distance between two sets in a metric space is the infimum of all δ for which each set is in a δ-neighborhood of the other, or ∞ if there is no such δ.

A lamination λ is rational if and only if its support is a simple closed curve. In the rational case both earthquake and horocyclic orbits project in moduli space to homotopic closed curves, and hence are a finite Hausdorff distance apart.

In this setting, both projected orbits are level sets of a length function. The projection of the earthquake path $\pi \circ E_t(x, \lambda)$ is exactly the set of points

$$\{z : \ell_\lambda(z) = 1\}. \tag{41}$$

The horocycle $\pi \circ h_t(x, \lambda)$ is the set

$$\{z : \mathbf{e}_\lambda(z) = 1\} \tag{42}$$

where $\mathbf{e}_\lambda(z)$ denotes *extremal length* of the lamination λ in the conformal structure of z. For a definition of Ahlfors-Beurling's notion of extremal length of curves see [Ah]. Kerckhoff [Ker1] generalized this to a continuous function on measured laminations (or foliations), analogous to, but quantitatively quite different from, the hyperbolic length ℓ. In particular if λ is equivalent to the horizontal foliation of a holomorphic quadratic differential q then $\mathbf{e}_\lambda$ is equal to the area of q, and (42) follows.

A K-quasiconformal map distorts all extremal lengths by a factor of at most K, and the definition of Teichmüller distance then implies:

$$\mathrm{dist}_{\mathrm{Teich}(S)}(w,z) \geq \sup_{\lambda\in\mathcal{ML}(S)} \frac{1}{2}\log\frac{\mathbf{e}_\lambda(z)}{\mathbf{e}_\lambda(w)}. \tag{43}$$

(In fact Kerckhoff shows in [Ker1] that this inequality is an equality, but we will not need this).

Our strategy will be based on the following estimate, showing that extremal length of λ grows without bound if the earthquake path is not confined to a compact part of the moduli space.

Lemma 8.2. *There exist constants c, ϵ_0 such that, if γ is a simple closed curve and the length function $\ell_\gamma(E_t(x,\lambda))$ reaches a minimum of $\epsilon < \epsilon_0$ at $t = t_\gamma$, then*

$$\mathbf{e}_\lambda(t_\gamma) \geq \frac{3}{\epsilon(c+2\log(1/\epsilon))^2}.$$

We then argue as follows. By a theorem of Hedlund [Hed] when λ is irrational the horocycle $h(x,\lambda)$ is dense in the moduli space, and in particular unbounded. If $E(x,\lambda)$ stays in a compact part of the moduli space then automatically the Hausdorff distance between the two paths is infinite, and we are done. Hence assume that $E(x,\lambda)$ also leaves every compact part of the moduli space, and so there are curves γ with $\ell_\gamma(E_{t_\gamma}(x,\lambda)) = \epsilon$ for arbitrarily small ϵ. Lemma 8.2 then implies that $\mathbf{e}_\lambda$ is unbounded along the earthquake path, whereas on the horocycle path it is constant, and hence by (43) the Hausdorff distance is again infinite. $\square$

It remains to establish Lemma 8.2:

Let γ be a simple closed curve on S, let t_γ be the minimum point for $\ell_\gamma(t) \equiv \ell_\gamma(E_t(x,\lambda))$, and let $\epsilon = \epsilon_\gamma$ be the minimum value of $\ell_\gamma(t)$.

Consider the hyperbolic metric corresponding to $E_{t_\gamma}(x,\lambda)$. For intersection points y of γ with leaves of λ, the angles of intersection θ_y vary in an interval of the form $[\theta-\epsilon/2, \theta+\epsilon/2]$, by Lemma 5.7. Since t_γ is a minimum point for ℓ_γ, Kerckhoff's equation for the derivative of ℓ_γ gives us $\int_\gamma \cos(\theta_y)d\mu(y) = 0$ where $d\mu$ is the transverse measure on γ induced by λ. Integrating, it follows that $\cos(\theta+\epsilon/2) < 0 < \cos(\theta-\epsilon/2)$ and hence $|\theta - \pi/2| < \epsilon/2$. Thus each θ_y satisfies

$$\left|\theta_y - \frac{\pi}{2}\right| < \epsilon. \tag{44}$$

We can use this to estimate $\ell_\lambda(t_\gamma)$, as follows. There is a maximal collar C_γ around the geodesic γ, of radius r satisfying

$$r \leq \log\frac{1}{\epsilon} + c_1 \tag{45}$$

with a constant c_1 independent of any of the data (see Buser [Bu, Chap 4] for a more precise version). Every component b of $\lambda \cap C_\gamma$ is a geodesic segment connecting the boundaries of the collar and crossing its core γ at one of the angles θ_y. Its length has a bound

$$l(b) \leq 2r + c_2 \tag{46}$$

following from the bound (44) on θ_y, and the hyperbolic sine law. (That is, lifting to the universal cover, the segment of $\widetilde{b}$ from its midpoint on $\widetilde{\gamma}$ to a boundary component of $\widetilde{C}_\gamma$, together with the perpendicular dropped down to $\widetilde{\gamma}$ and a segment on $\widetilde{\gamma}$, form a right triangle for which we have $\sinh(l(b)/2)/\sin(\pi/2) = \sinh(r)/\sin(\theta_y)$.)

Each segment a of $\lambda \cap (S - C_\gamma)$ has length bounded by a uniform c_3, for ϵ sufficiently small. This is because γ cuts S into one or two hyperbolic thrice-punctured spheres, with ends that are cusps or of length ϵ, and a is a properly embedded arc in the complement of the collars and cusp neighborhoods. This is the only place where the fact that S is a once-punctured torus or four-times-punctured sphere plays an essential role.

Putting these together we find that the length of λ is bounded by

$$\ell_\lambda \leq i(\lambda, \gamma) \left(2 \log \frac{1}{\epsilon} + c_4 \right) \tag{47}$$

for a uniform c_4. On the other hand since we are on the earthquake path $\ell_\lambda = 1$ and we conclude

$$i(\lambda, \gamma) \geq \frac{1}{2 \log \frac{1}{\epsilon} + c_4}. \tag{48}$$

A result of Maskit [Mskt, Corollary 2] implies that there exists $\epsilon' > 0$ such that, for any curve γ with length $\ell_\gamma < \epsilon'$ and any hyperbolic metric,

$$\mathbf{e}_\gamma < \ell_\gamma / 3. \tag{49}$$

Let us now assume $\epsilon < \epsilon'$ so that $\mathbf{e}_\gamma < \epsilon/3$ in our setting. It is also a standard consequence of the definition of extremal length (see e.g. [Min]) that for any two measured laminations α, β and any surface,

$$\mathbf{e}_\alpha \mathbf{e}_\beta \geq i(\alpha, \beta)^2. \tag{50}$$

Thus we have

$$\begin{aligned} \mathbf{e}_\lambda &\geq \frac{i(\lambda, \gamma)^2}{\mathbf{e}_\gamma} \\ &\geq \frac{3}{\epsilon (2 \log \frac{1}{\epsilon} + c_4)^2} \end{aligned}$$

which establishes lemma 8.2. $\square$

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