Problem:

There are three colleges in a town. Each college has n students. Any student of any college knows n+1 students of the other two colleges. Prove that it is possible to choose a student from each of the three colleges so that all three students would know each other.

Solution:

First note that we can reformulate the problem as follows; consider a tri-partite graph of 3-n nodes \( G_n := K_{n,n,n} \) where each node has degree n+1. We want to show that this graph contains a \( K_3 \) subgraph (3-cycle). Next we note that each node \( v \in G_n \) has edges connecting it to one of two partite regions of \( G_n \); we define a function \( f(v) \) that maps each node \( v \in G_n \) to the lesser of the two numbers of edges connecting \( v \) to a partite region (e.g. if \( v \) has i edges connected to nodes in region A and j edges connected to nodes in region B then \( f(v) = \min\{i,j\} \)). We also define the function \( \sigma(G_n) \) which maps the graph \( G_n \) to the integer \( \min\{f(v) \mid v \in G_n\} \).

Notice that \( \sigma(G_n) > 0 \) since each node has degree n+1 and each region contains n nodes; therefore each node must contain at least one node in each partite region (e.g. \( f(v) > 0 \forall v \in G_n \)). Now suppose that \( \sigma(G_n) = k \) and choose a node \( v1 \) in the partite region A such that \( f(v1) = k \). Let \( v2 \) be one of the k nodes in the partite region B that share an edge with \( v1 \). We know that \( v1 \) must share an edge with n-k+1 nodes in region C and that \( v2 \) must share an edge with at least k nodes in region C. But \( (n-k+1)+k = n+1 \) and there are only n nodes in region C, therefore there must be some node \( v3 \) in region C that shares an edge with both \( v1 \) and \( v2 \) so that \( \{v1,v2,v3\} \) form a 3-cycle. \( \Box \)