Problem of the Month
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1 Problems

- Is there a polyhedron and a point outside of it, from which no vertex of the polyhedron is visible?
- Is there a convex polyhedron and a point outside of it, from which no vertex of the polyhedron is visible?

2 Solutions

2.1 Solution to the first problem

Let ϵ be a some small number greater than 0. Consider a point at the origin and six bars of boxes surrounding the point at the origin in a following way: each bar has width and height of length $1 - 2\epsilon$ and depth of arbitrary length greater than 3. Let each center of a box is placed at $(1, 0, 0)$, $(-1, 0, 0)$, $(0, 1, 0)$, $(0, -1, 0)$, $(0, 0, 1)$, $(0, 0, -1)$ respectively. Also let boxes with centers having nonzero x-coordinate values be aligned parallel to the y-axis, boxes with centers having nonzero y-coordinate values be aligned parallel to the z-axis, and boxes with centers having nonzero z-coordinate values be aligned parallel to the x-axis.

Then the point at the origin cannot see any vertex of any of the boxes. The point is surrounded by the boxes and has very little visible area in space, mostly obscured by the boxes. Now, we can "connect" the boxes to make a single polyhedron without having any vertex being visible to the point by adding more boxes on the "outer side" (this is more or so trivial if one can visualize how six boxes are oriented in the space). Therefore there exists a polyhedron and a point outside of it, from which no vertex of the polyhedron is visible. ■

2.2 Solution to the second problem

Let $\mathbb{S}$ be a convex polyhedron in $\mathbb{R}^3$. Let P be a point lying outside of $\mathbb{S}$. Now, find a closest point in $\mathbb{S}$ to P and call it Q. Then the following are the possible cases:

- Q is a vertex of $\mathbb{S}$.
- Q is a point on an edge of $\mathbb{S}$.
- Q is a point on a face of $\mathbb{S}$.

The point Q must be clearly visible to P, otherwise there must be a point in $\mathbb{S}$ in between P and Q and this is not possible since Q is the closest point to P.

If Q is a vertex of $\mathbb{S}$, then no furthur work is required to show a vertex of $\mathbb{S}$ is visible to P.

If Q is a point on an edge of $\mathbb{S}$, then the edge must be tangent to the circle centered at P with radius $|\overline{PQ}|$, otherwise a part of the edge will lie inside the circle and this means there is a point in $\mathbb{S}$ that is closer to P than Q, which is not possible. Then we see that both ends of the edge must be visible to P. If an end of the edge, say A, weren't visible to P, then that means there is a point in $\mathbb{S}$ on $\overline{AP}$ other than A. Pick any point on $\overline{AP}$ that is in $\mathbb{S}$ other than A and call it B. By the definition of a convex polyhedron, $\overline{BQ}$ must be in $\mathbb{S}$. Since some part of $\overline{BQ}$ must lie inside the circle, this raises a contradiction. So we see that A must be visible to P. Knowing that an end of an edge in a polyhedron is a vertex, we know a vertex of $\mathbb{S}$ is visible to P.

If Q is a point on a face of $\mathbb{S}$, for similar reasons above, we see that all the vertices of the face must be visible to P. To outline the reasons, the face must be tangent to the sphere centered at P with radius $|\overline{PQ}|$, and any vertex of the face, say A, must be visible to P, because if A weren't visible to P, then there must be a closer point in $\mathbb{S}$ to P than Q is to P.

So for any case, given a convex polyhedron and a point outside of it, a vertex of a polyhedron must be visible to the point. Therefore there doesn't exist a convex polyhedron and a point outside of it, from which no vertex of the polyhedron is visible. ■