OCTOBER 2005 PROBLEM OF THE MONTH

JERRY MCMAHAN JR.

1. Introduction

This is a solution for the October 2005 problem of the month by Jerry McMahan Jr. (jerry@ensomnya.net).

The problem defines a sequence, $a_0, a_1, \dots$ as

$$a_0 = 1, a_1 = 1, a_n = 3a_{n-1} - a_{n-2} \forall n \geq 2$$

The problem then asks to show that $\frac{a_n}{a_{n-1}}$ has a limit as n approaches infinity, and to find the limit. It also asks to show that the above sequence can be obtained from every other term in the Fibonacci sequence, defined as

$$f_0 = 1, f_1 = 1, f_n = f_{n-1} + f_{n-2} \forall n \geq 2$$

2. Convergence

To follow the algebra below, it is useful to keep in mind that a_n can be represented three different ways using the definition of the sequence above:

$$a_n = 3a_{n-1} - a_{n-2} \tag{1}$$

$$a_n = \frac{a_{n+1} + a_{n-1}}{3} \tag{2}$$

$$a_n = 3a_{n+1} - a_{n+2} \tag{3}$$

Take $\frac{a_n}{a_{n-1}}$ and perform the following algebraic manipulations:

Date: October 21, 2005.

$$\frac{a_n}{a_{n-1}} = \frac{(a_{n-1}+a_{n+1})}{3a_{n-1}} = \frac{1}{3} + \frac{a_{n+1}}{3a_{n-1}} \tag{4}$$

$$= \frac{1}{3} + \frac{a_{n+1}}{3a_{n-1}} + (\frac{5}{3} - \frac{5}{3}) = 2 + \frac{a_{n+1} - 5a_{n-1}}{3a_{n-1}} \tag{5}$$

$$= 2 + \frac{(3a_n - a_{n-1}) - 5a_{n-1}}{3a_{n-1}} = 2 + \frac{3a_n - 6a_{n-1}}{3a_{n-1}} \tag{6}$$

$$= 2 + \frac{a_n - 2a_{n-1}}{a_{n-1}} = 2 + \cfrac{1}{\cfrac{a_{n-1}}{a_n - 2a_{n-1}}} \tag{7}$$

$$= 2 + \cfrac{1}{(1-1) + \cfrac{a_{n-1}}{a_n - 2a_{n-1}}} = 2 + \cfrac{1}{1 + \cfrac{a_{n-1} - a_n + 2a_{n-1}}{a_n - 2a_{n-1}}} \tag{8}$$

$$= 2 + \cfrac{1}{1 + \cfrac{3a_{n-1} - a_n}{a_n - 2a_{n-1}}} = 2 + \cfrac{1}{1 + \cfrac{a_{n-2}}{a_n - 2a_{n-1}}} \tag{9}$$

$$= 2 + \cfrac{1}{1 + \cfrac{1}{\cfrac{a_n - 2a_{n-1}}{a_{n-2}}}} = 2 + \cfrac{1}{1 + \cfrac{1}{\cfrac{a_n - 2a_{n-1} + (a_{n-1} - a_{n-1})}{a_{n-2}}}} \tag{10}$$

$$= 2 + \cfrac{1}{1 + \cfrac{1}{\cfrac{(a_n - 3a_{n-1}) + a_{n-1}}{a_{n-2}}}} = 2 + \cfrac{1}{1 + \cfrac{1}{\cfrac{(-a_{n-2}) + a_{n-1}}{a_{n-2}}}} \tag{11}$$

$$= 2 + \cfrac{1}{1 + \cfrac{1}{-1 + \cfrac{a_{n-1}}{a_{n-2}}}} = 2 + \cfrac{1}{1 + \cfrac{1}{(2-2) - 1 + \cfrac{a_{n-1}}{a_{n-2}}}} \tag{12}$$

$$= 2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{a_{n-1} - 2a_{n-2}}{a_{n-2}}}} \tag{13}$$

Compare the term $\frac{a_{n-1}-2a_{n-2}}{a_{n-2}}$ in equation (13) with the term $\frac{a_n-2a_{n-1}}{a_{n-1}}$ in the left part of equation (7). These two terms are of the same form. Therefore, by repeating the same algebraic steps in equations (7) through (13) for the term $\frac{a_{n-1}-2a_{n-2}}{a_{n-2}}$ will expand the fraction in the same way. For any $n \geq 2$, repeating this process until the term becomes $\frac{a_2-2a_1}{a_1}=0$ yields the simple continued fraction for the term a_n. It is clear that as n approaches infinity, this becomes the infinite simple continued fraction:

$$2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\dots}}} \tag{14}$$

Since any infinite simple continued fraction is equal to some irrational number, then the sequence of ratios of consecutive terms in the sequence $a_0, a_1, ...$ defined above converges to this number (to see the proof that an infinite simple continued fraction converges to a value, reference a number theory text such as [1]).

3. Value of the Limit

Since we know the limit exists, we can do the following:

$$\lim_{n\to\infty}\frac{a_n}{a_{n-1}} = \lim_{n\to\infty}\frac{3a_{n-1}-a_{n-2}}{a_{n-1}} \tag{15}$$

$$= \lim_{n\to\infty} 3-\frac{a_{n-2}}{a_{n-1}} \tag{16}$$

Notice that $\frac{a_{n-2}}{a_{n-1}}$ is the inverse of the ratio whose limit we are looking for. So as n approaches infinity, this should approach the inverse of the desired limit. We can use this to set up an equation which will let us solve for the limit's value. Let $x=\lim_{n\to\infty}\frac{a_n}{a_{n-1}}$. Then

$$x = 3-\frac{1}{x} \tag{17}$$

Multiplying through by x and arranging all terms on one side, we get

$$x^2-3x+1=0 \tag{18}$$

Using the quadratic equation to solve for x yields:

$$x=\frac{3\pm\sqrt{5}}{2} \tag{19}$$

The value $\frac{3-\sqrt{5}}{2}$ can be eliminated by observing that $a_2 = 2 > a_1 = 1$, and that if $a_k > a_{k-1}$ then

$$a_k > a_{k-1}$$

$$a_k > 3a_k - a_{k+1}$$

so

$$a_{k+1} > 2a_k > a_k \tag{20}$$

That $2a_k > a_k$ follows once we know that $a_k > 0$. If this were not true, since we know the sequence starts out positive, then there would be some $a_n, n < k$ that was the first negative element. This means

$$a_n = 3a_{n-1} - a_{n-2} < 0$$

which gives us

$$3a_{n-1} < a_{n-2} \tag{21}$$

But since a_n is the first negative element, a_{n-1} and a_{n-2} are both positive. Since we've assumed $a_k > a_{k-1}$ for all a_n up to a_k, then $a_{n-1} > a_{n-2}$. This contradicts equation (21). By a similar argument, a_k cannot be 0. Thus the sequence is always positive, justifying the inequality in equation (20), which in turn allows us to determine by induction that the sequence is always increasing for $a_n, n \geq 2$. Since $a_n > a_{n-1}, \forall n \geq 2$, we have

$$\frac{a_n}{a_{n-1}} > 1, \forall n \geq 2 \tag{22}$$

However

$$\sqrt{5} > \sqrt{4} = 2$$

$$-\sqrt{5} < -2$$

$$3 - \sqrt{5} < 3 - 2$$

$$\frac{3-\sqrt{5}}{2} < \frac{1}{2} < 1$$

Since the ratio in equation (22) is always greater than 1 and it is always increasing, and the value in question is less than 1, it is impossible for the sequence to converge to this value. So the sequence converges to

$$\frac{3+\sqrt{5}}{2} = 1 + \frac{1+\sqrt{5}}{2}$$

4. Relationship to the Fibonacci Sequence

As mentioned in the problem statement the sequence we defined can be obtained from the Fibonacci sequence. Note that equations analogous to (1), (2), and (3) are used in the following manipulations.

$$f_n = f_{n-1} + f_{n-2} = f_{n-1} + (f_{n-1} - f_{n-3}) \tag{23}$$
$$= 2f_{n-1} - f_{n-3} = 2(f_{n-2} + f_{n-3}) - f_{n-3} \tag{24}$$
$$= 2f_{n-2} + f_{n-3} = 2f_{n-2} + (f_{n-2} - f_{n-4}) \tag{25}$$
$$= 3f_{n-2} - f_{n-4} \tag{26}$$

Notice now that equation (26) has the same recursive form as the sequence we worked with earlier, only every other element is being used rather than every element. The original sequence and the one defined in (26) will equal one another only for f_n with n even which you can verify through substituting the first values in the equation. For n odd, the sequences will not be the same. However, as n approaches infinity, the sequence of f_n for n odd will converge to the same value as the sequence with n even. The reason for this is that the ratio defined by $\frac{f_n}{f_{n-1}}$ converges to the value $\frac{1+\sqrt{5}}{2}$, which can be proved using similar methods as the proof for the above sequence. In fact, given this information, you could prove the first sequence converges like so:

$$\lim_{n\to\infty} \frac{f_{2n}}{f_{2n-2)}} = \lim_{n\to\infty} \frac{f_{2n-1} + f_{2n-2}}{f_{2n-2}} = \lim_{n\to\infty} 1 + \frac{f_{2n-1}}{f_{2n-2}}$$

This would equal the same value that the first sequence converged to, or equivalently, the limit of the ratio of adjacent Fibonacci numbers plus 1, which is

$$1 + \frac{1+\sqrt{5}}{2}$$

This number has some significance. $\frac{1+\sqrt{5}}{2}$ is known as the golden ratio. The ratio of two numbers, a and b, is the same as the golden ratio if $\frac{a+b}{b} = \frac{a}{b}$. The ratio of every Fibonacci number to the Fibonacci number previous to it converges to this value, and the first sequence given converges to one plus this value. The number occurs frequently in nature, an often-cited example being the proportions of the spiral on a conch shell. The number was known to the Greeks and possibly the Egyptians. It has been used at times by modern artists to arrange elements in their work in what was considered to be an aesthetically pleasing manner. [2]

References

[1] Niven, Ivan and Zuckerman, Herbert, "An Introduction To The Theory Of Numbers," John Wiley and Sons, USA, 1962.

[2] Wikipedia, "Golden Ratio," http://en.wikipedia.org/wiki/Golden_Ratio