Every Triangle is Isosceles

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July 19th 2023

During the talk we saw the definition of the p-adic norm.

For a fraction m/n, it is given by

$$\left|\left|\frac{m}{n}\right|\right|_p = \frac{1}{p^{\mathrm{ord}_p(m)-\mathrm{ord}_p(n)}}.$$

Since this is a metric, for any three rational numbers a, b, and c, we have the *triangle inequality* given by

$$||a-b||_p \leq ||a-c||_p + ||c-b||_p.$$

However, it can be further shown that we have something stronger for the p-adic norm. The following condition defines an *ultra metric*, and it is what the p-adic numbers have.

$$||a-b||_p \leq \max\left(||a-c||_p, ||c-b||_p\right).$$

We can now state and prove a small theorem that shows how and why the geometry of the p-adic numbers is vastly different to that of Euclidean numbers.

Theorem. Every p-adic triangle is isosceles.

Proof. Let x and y be two p-adic numbers. The three points of our triangle will then be 0, x, and y. To obtain the *side lengths*, we have to take the difference between the points, and then apply the p-adic metric. The three side lengths of our triangle are thus

$$||x||_p, \qquad ||y||_p, \qquad \text{and} \qquad ||x-y||_p.$$

To show the triangle is isosceles, we must prove that two of the side lengths are equal.

Suppose that $||x||_p$ and $||y||_p$ are not already equal. We may further assume that $||x||_p > ||y||_p$.

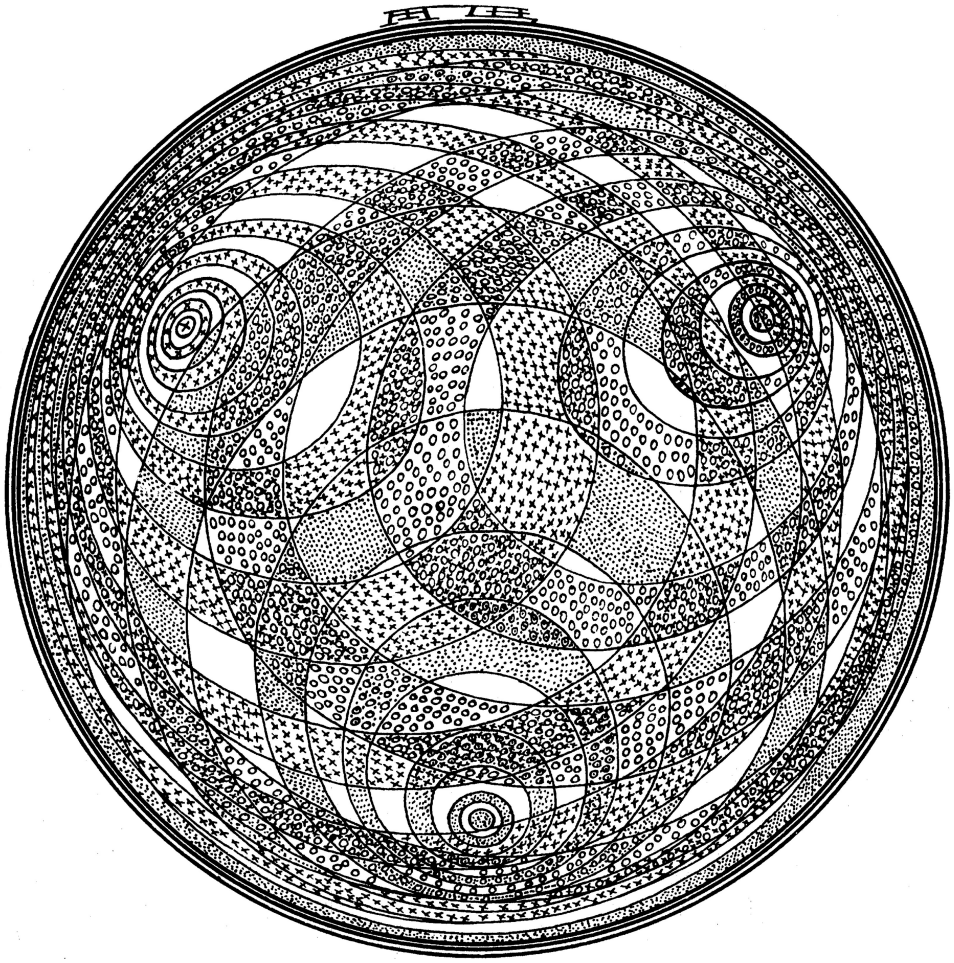
By the *ultra triangle inequality* (we make $a = x$, $b = y$, $c = 0$ in the above definition), we observe that

$$||x-y||_p \leq \max(||x||_p, ||y||_p) = ||x||_p.$$

We note that both $||y||_p$ and $||x-y||_p$ are smaller than $||x||_p$. Finally,

$$||x||_p = ||(x-y)+y||_p \leq \max\left(||x-y||_p, ||y||_p\right) \leq ||x||_p.$$

Whether $\max\left(||x-y||_p, ||y||_p\right) = ||x-y||_p$ or $\max\left(||x-y||_p, ||y||_p\right) = ||y||_p$ holds, the above inequality asserts that one of the side lengths must be equal to $||x||_p$. Hence, the triangle is isosceles.



Artist's conception of the 3-adic unit disk.

Drawing by A.T. Fomenko of Moscow State University, Moscow, U.S.S.R.