Game Theory

Day 2
Game Theory Day 2

Last time, we discussed two person zero sum games

Recall: These games were specified by a “payoff matrix”

Ex

<table>
<thead>
<tr>
<th></th>
<th>H</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>P1</strong></td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td><strong>P2 (Me)</strong></td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Recall: For two person zero sum games, one player's gain is another player's loss

Main Takeaway from last time: For two person zero sum games, the safety strategies for each player are optimal

More precisely: There is a number $V$, called the value of the game, such that

$$\text{Maximum worst case} = V = \text{Minimum worst case}$$

Rmk: Safety strategies are optimal since neither player gains anything by changing their strategy
Recall: Last time, we argued that in our example each player had a "dominant strategy."

\[
\begin{array}{c|cc}
& H & T \\
\hline
P1 & 2 & 3 \\
(You) & 0 & 1
\end{array}
\]

i.e. you should always pick (left)
I should always pick heads

The dominant strategies in this case are also the safety strategies

Computation for P1:
Suppose you choose L with probability \( x \),
R with probability \( 1-x \),

\[
\begin{array}{c|cc}
& H & T \\
\hline
P1 & x, L & 2 & 3 \\
(You) & 1-x, R & 0 & 1
\end{array}
\]

Expected gain if I pick H: \( 2x + 0 \cdot (1-x) = 2x \),
Expected gain if I pick T: \( 3x + 1 \cdot (1-x) = 1+2x \),

worst case expected gain: \( \min(2x, 1+2x) = 2x \).
Max worst case expected gain occurs at: \( x = 1 \)

so safety strategy is to pick L with probability (}
Today, we will discuss a different class of games called two person general sum games.

Remark For two person general sum games

* One person's gain is not necessarily the other person's loss. Each player can have different payoffs.

* Safety strategies still exist, but they may no longer be optimal, i.e. von Neumann may fail.

* Instead of finding strategies for one player that consider all other strategies for the other, we will be interested in pairs of strategies, one for each player, such that each is the best response to the other.

Example: Prisoner's Dilemma

Two suspects (me and you) are imprisoned by the police for a crime and they want us to confess.

The charge is serious, but the police don't have enough evidence to convict.
The police offer you the following deal:

- If you confess and I stay silent, you go free
- If you stay silent and I confess, you get 10 years free
- If we both confess, you get 8 years in jail
- If we both remain silent, you get (year in jail)

Task: Summarize this situation in a "payoff matrix" where the numbers in the matrix represent time in prison for you (use negative numbers).

Payoff matrix:

<table>
<thead>
<tr>
<th></th>
<th>S</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>You</td>
<td>-1</td>
<td>-10</td>
</tr>
<tr>
<td>Me</td>
<td>0</td>
<td>-8</td>
</tr>
</tbody>
</table>

- If we both stay silent, you would get less time in jail if you confessed instead.
If you stay silent and I confess, you would get less time in jail if you confessed instead.

It is in your interest to always confess.

i.e. confessing is the dominant strategy.

Why is this a dilemma?

Suppose that separately, I was offered the same deal.

We can then summarize the situation in the following payoff matrix:

<table>
<thead>
<tr>
<th></th>
<th>S</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>You</td>
<td>(-1, -1)</td>
<td>(-10, 0)</td>
</tr>
<tr>
<td>C</td>
<td>(0, -10)</td>
<td>(-8, -8)</td>
</tr>
</tbody>
</table>

Note: (-10, 0) means 10 years in jail for you and 0 for me.

This is a general sum game since the payoffs to each player are not the same, one person's gain is not the others' loss.

In this example, the best outcome for both of us is to both stay silent.

However, as we discussed, each player has an incentive to confess, in particular confessing is the dominant strategy.
Me

<table>
<thead>
<tr>
<th>S</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>S</td>
<td>(-1,-1)</td>
</tr>
<tr>
<td>C</td>
<td>(0,-10)</td>
</tr>
</tbody>
</table>

(C,C) is coming from

You S (S,S) (S,C, C) (C,S) is stable dominant strat for both players

In this case: the strategy (C,C) is an equilibrium solution

- The pair (C,C) is stable, neither player has incentive to change their strategy

- No other pair of strategies (S,S), (S,C), (C,S) is stable
  Given any of the pairs, one player has incentive to deviate

John Nash
A Beautiful Mind

Def A pair of mixed strategies (x,y) are in Nash Equilibrium if neither player gains by switching their strategy while the other leaves theirs unchanged

Rmk Equilibrium solutions are stable since neither player has incentive to deviate
Ed Stag Hunt

Two Hunters (Me and you) are following a stag when a hare runs by.

We each have to make the following split second decision:

Do we continue tracking the stag or do we chase the hare?

We must cooperate to catch the stag since neither of us can catch it on our own.

If we both go for the hare, we must share it.

A stag is worth 4 times as much as a hare.

We can summarize this situation in the following payoff matrix:

<table>
<thead>
<tr>
<th></th>
<th>S</th>
<th>H</th>
</tr>
</thead>
<tbody>
<tr>
<td>You</td>
<td>(4,4)</td>
<td>(0,2)</td>
</tr>
<tr>
<td></td>
<td>(2,0)</td>
<td>(1,1)</td>
</tr>
<tr>
<td>Payoff to You</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Me</td>
<td>S</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>2</td>
</tr>
</tbody>
</table>

Q Does either player have a dominant strategy?

A No, no dominant strategy for you or me.

If you pick H and I pick S, you gain more by switching to S.

If you pick S and I pick H, you gain more by switching to H.
Q: What is your safety strategy, i.e. the one that maximizes your worst case expected gain?

<table>
<thead>
<tr>
<th></th>
<th>S</th>
<th>H</th>
</tr>
</thead>
<tbody>
<tr>
<td>You S</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>1-x, H</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Me payoff to you

Expected gain if I pick S: \(4x + 2(1-x) = 2x + 2\)

Expected gain if I pick H: \(0x + 1(1-x) = 1-x\)

Worst case expected gain: \(\min(2x + 2, 1 - x) = 1 - x\)

Max worst case expected gain occurs at: \(x = 0\)

Picking S 0% of the time \(\rightarrow\) safety strategy is to always pick H

At 100% of the time

Q: What are the equilibrium solutions?

- \((H, H)\) is an equilibrium solution
- \((S, S)\) is also an equilibrium solution

Rmk: These different equilibrium solutions have different payoffs!

Q: Which equilibrium solution is best?

A: It's not so clear
There is actually a third equilibrium in which each of us has a purely mixed strategy!

Suppose you pick $S$ with probability $X$,
$H$ with probability $1-X$.

Suppose I pick $S$ with probability $Y$,
$H$ with probability $1-Y$.

Suppose the pair $(X, 1-X)$ and $(Y, 1-Y)$ are a Nash equilibrium pair.

Then, both of your possible actions must have the same pay off!

If not, you’d have incentive to deviate

<table>
<thead>
<tr>
<th></th>
<th>$S$</th>
<th>$H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>You</td>
<td>$(4,4)$</td>
<td>$(0,2)$</td>
</tr>
<tr>
<td></td>
<td>$(2,0)$</td>
<td>$(1,1)$</td>
</tr>
</tbody>
</table>

Expected gain for you if you go $S$: $4X - 0(1-X) = 4X$.
Expected gain for you if you go $H$: $2Y + 1(1-Y) = Y + 1$

Payoffs must be the same $\Rightarrow 4X = Y + 1$
$\Rightarrow 3X = 1$
$\Rightarrow X = \frac{1}{3}$
Similarly, both of my possible actions must have the same payoffs:

<table>
<thead>
<tr>
<th></th>
<th>S</th>
<th>H</th>
</tr>
</thead>
<tbody>
<tr>
<td>You</td>
<td>(4,4)</td>
<td>(0,2)</td>
</tr>
<tr>
<td>1-x, H</td>
<td>(2,0)</td>
<td>(1,1)</td>
</tr>
</tbody>
</table>

Expected gain for you if I go S: \(4x, \text{to } (1-x,1) = 4x\).

Expected gain for you if I go H: \(2x, +1(1-x) = x, +1\).

Payoffs must be the same \(\Rightarrow 4x = x, +1\)

\(\Rightarrow 3x = 1\)

\(\Rightarrow x = \frac{1}{3}\)

\(\Rightarrow\) The following pair of mixed strategies are in equilibrium:

Me: Stag \(\frac{1}{3}\) of the time
Hare \(\frac{2}{3}\) of the time

You: Stag \(\frac{1}{3}\) of the time
Hare \(\frac{2}{3}\) of the time

Why are Nash Equilibria important in Game Theory?

Thm {(Nash)} Every two person general sum game has a Nash equilibrium

Remark one pair of eq. solutions for two person zero sum games are the safety strategies
Main Takeaway: For two-person general sum games, there may not be an optimal strategy for either player, but there is always a pair of strategies which are the best response to each other.

**Remarks:**
- Concept of Nash Equilibria extends to games with more than two players.
- Game Theory/Nash equilibria is a useful tool in decision making.
- Nash Equilibria may be hard to find explicitly in general.
- Hard to decide which equilibrium strategy to pick when there are multiple Nash equilibria.
Exercise Chicken

Two players (you and me) run towards each other unless one chicken out and swerves

If we both swerve, both get a payoff of 1

If you run and I swerve, you get 2, I get -1

If you swerve and I run, you get -1, I get 2

If we both run, we both get -10

\[
\begin{array}{c|cc}
\text{Me} & R & S \\
\hline
\text{You} & (-10,-10) & (2,-1) \\
& (-1,2) & (1,1) \\
\end{array}
\]

Q: Are there any dominant strategies for either player?
A: No

Q: What is the safety strategy for each player?
A: What is the strategy that maximizes worst case expected gain

Q: Swerve
What are the pure Nash Equilibria?

\[(S, R), (R, S)\]