2 kinds
continuous (flows)
- flow of water in pipe
- billiards

* discrete, iterative dynamics.
- $x_{n+1} = f(x_n)$
- $x_n = f^n(x_0) = \underbrace{f \circ \cdots \circ f}_{n-\text{times}}(x_0)$

orbit of $x_0$, $(x_0, x_1, x_2, x_3, \ldots)$
\[ f(x) = x^2 \quad f : \mathbb{R} \to \mathbb{R} \]

**Population of Rabbits**

Logistic

\[ x_{n+1} = r \cdot x_n \cdot (1 - x_n) \]

\[ y = r \cdot x \cdot (1 - x) \]

\[ x = \% \text{ of carrying cap.} \]

\[ g_r(x) = r \cdot x \cdot (1 - x) \]

\[ = r \cdot x - r \cdot x^2 \]

\[ g_r \text{ is conjugate to a quadratic of the form } f_c(x) = x + c \]

by a linear function.

There is a linear function

\[ l_{a,b}(x) = ax + b \]

\[ l_{0.5} \circ g_r \circ l_{0.5}^{-1} = f_c(r) \]

\[ f^n_c = (l_{0.5} \circ g_r \circ l_{0.5}) \circ \ldots \circ (l_{0.5} \circ g_r \circ l_{0.5}) \]

\[ = l_{0.5} \circ g_r^n \circ l_{0.5}^{-1} \]

**Challenge:** Try to conj. \( r \cdot x \cdot (1 - x) \)

to \( x^2 + c \), where \( c \) depends on \( r \).
Complex Dynamics

\[ f_c : \mathbb{C} \to \mathbb{C}, \quad f_c(z) = z^2 + c. \]

\[ f_0(z) = z^2 \]

\[ (6i)^2 = 6^2 i^2 = -36 \]

\[ (6i, -36, 6^2, 6^4, \ldots) \]

\[ n \to \infty \]

\[ f_b^n(6i) \to \infty \]

\[ 3 + 2i = r e^{i	heta} \]

\[ (3 + 2i)^2 = r^2 e^{2i	heta} \]

if \( |z| > 1 \) then \( f_0^n(z) \to \infty \)

if \( |z| < 1 \), say \( z = \frac{1}{2} \) \( \left( \frac{1}{2}, \frac{1}{4}, \frac{1}{16}, \ldots \right) \)

\[ f_0^n \left( \frac{1}{2} \right) = \frac{1}{2}^{2^n} \to 0 \]

\[ z^2 = 1 \]

\[ f(1) = 1^2 = 1. \]

\[ f(0) = 0^2 = 0 \]

0 is an attracting

\[ \ldots \]
To be an attracting fixed point, nearby points must converge to the fixed point. $z = 1$ is a repelling fixed point.

- $z \mapsto z^2$

2 branches of the inverse are

- $z \mapsto \sqrt{z}$
- $z \mapsto -\sqrt{z}$

E.g. $\sqrt{(-1)^2} = \sqrt{1} = 1 \neq -1$
- $-\sqrt{(-1)^2} = -1$

Fatou set (or stable set)
- Small perturbations don’t matter
- Nearby points, stay close by

Julia set (or chaotic set)
- Complement of Fatou set
\[ F(f) \cup J(f) = \mathbb{C} \]
- the smallest of perturbations can cause huge changes in orbit

\underline{Filled Julia set}

\[ K(f) = \{ z \in \mathbb{C} \mid f^n(z) \not\to \infty \} \]

\[ J(f) = \partial K(f) \]

What is \( M \)?

\[ M = \{ c \in \mathbb{C} \mid f_c^n(0) \not\to \infty \} \]

equiv. = \{ c \in \mathbb{C} \mid K(f) \text{ is connected} \}

\underline{Thm For } \text{deg} \geq 2, \text{ } f(z) \text{ a polynomial,}

\[ J(f) \neq \emptyset \]

\[ \begin{align*}
0^2 - 1 &= -1 \\
(-1)^2 - 1 &= 1 - 1 = 0
\end{align*} \]
Thm: $M$ is connected.

 Conj. $M$ is locally connected.

Ex. not loc. conn.

[Diagram with points labeled 0, 1/4, 1/2, 1]