Polynomial skew products with wandering Fatou-disks

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Goal

Describe the Fatou set for rational maps in two complex variables.

1. Classify and describe the periodic Fatou components.
2. Investigate the existence of wandering Fatou components.

Special interest: Hénon maps.

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Fatou components for Hénon maps

There is always the escaping region.

Bedford-Smillie, '91

The only other Fatou components of **hyperbolic** Hénon maps are *basins of attracting cycles*.

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Dissipative Hénon maps

Bedford-Smillie '91, Lyubich-P. '14

Let Ω be an invariant Fatou component with bounded forward orbits, and assume that $|Jac(f)| < \frac{1}{d^2}$. Then either

1. Ω is the basin of an attracting fixed point.
2. Ω is the basin of an invariant closed Riemann surface $\Sigma \subset \Omega$. Σ is an embedded disk or annulus (??), and f acts on Σ as an irrational rotation.
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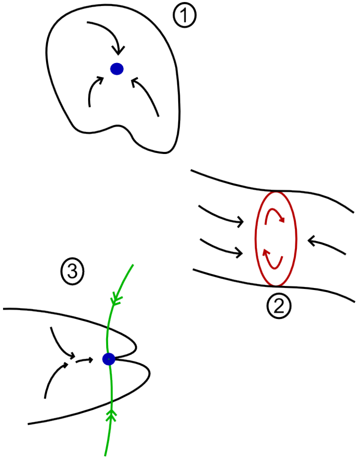
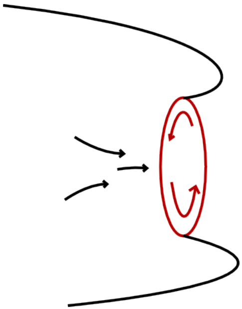


Dissipative Hénon maps

Note that a crossbreed between parabolic and Siegel



cannot exist. (for Hénon maps)

Main open question

Can non-hyperbolic dissipative Hénon maps have wandering Fatou components?

Skew-products

We will consider polynomial maps of the form

$$F : (t, z) \mapsto (p(t), q(t, z)).$$

Question

Can such maps have wandering Fatou components?

Note: if (t, z) lies in the Fatou set of F, then t lies in the Fatou set of p.
The "dissipative case": t in the basin of an attracting periodic point.

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with $p(0) = 0$ and $|p'(0)| < 1$.

Question

Can F have wandering Fatou components near $\{t = 0\}$?

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The super-attracting case

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Lilov, '04

There are no wandering Fatou components near $\{t = 0\}$.

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Stronger results of Lilov

Bulging Fatou components

Let F a polynomial skew product with an attracting invariant fiber $\{t = 0\}$, and let $U \subset \{t = 0\}$ be a Fatou component of $q(0, \cdot)$. Then there exist a Fatou component V of F with

$$V \cap \{t = 0\} = U.$$

Near super-attracting fibers

The forward orbit of every vertical disk will intersect a bulging Fatou component.

The non-existence of wandering Fatou components follows.

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A lemma

Two ingredients: a geometric description of the bulging Fatou components, and this:

Shrinking rate of vertical disks

Let D be a vertical disk of radius r, lying sufficiently close to the invariant fiber $\{t = 0\}$. Then $F(D)$ contains a disk of radius $C \cdot r^d$.

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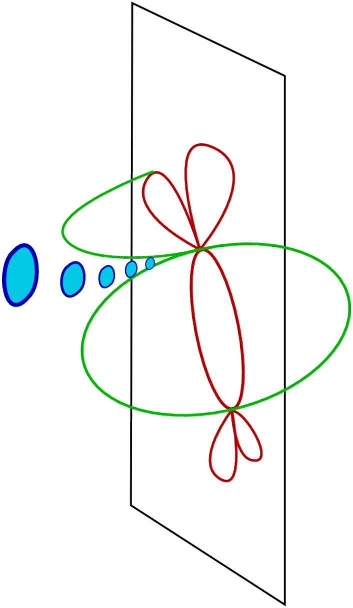
The super-attracting picture



Geometrically attracting skew products

Can we generalize Lilov's stronger results to the attracting case? No.

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There exist polynomial skew products with attracting invariant fibers and nearby vertical disks whose forward orbits avoid the bulging Fatou components.

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Why call them wandering?

Remark: If U is a (pre-)periodic Fatou component and $f^{n_j} \to h$ on U, then $h(U)$ is periodic.

For the disks we construct, the ω-limit sets contain non-periodic points. Hence if such a disk intersects the Fatou set, we must have found a *wandering* Fatou component.

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Fatou disks in the Julia set

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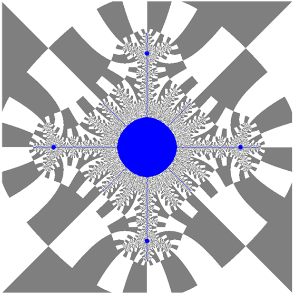
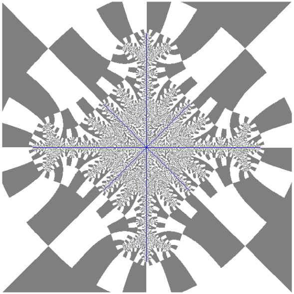
Proof by picture

$$F : (t, z) \mapsto (\frac{t}{8}, 2(z+1)^4 - 2 + t - \frac{641}{4165}t^2).$$

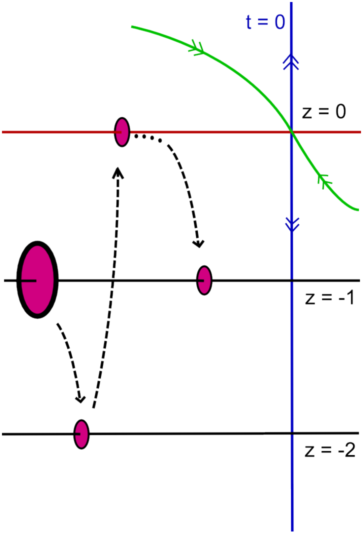


Wandering Fatou disks



Typical nearby fiber

Sketch of the proof



Linearization in $\mathbb{C}$

Let $g : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, with $g'(0) = \lambda$ and $|\lambda| > 1$.

Koenigs, 1884

The functions $\varphi_n(w) := g^n(\lambda^{-n}(w))$ converge to a holomorphic function Φ satisfying

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For a polynomial we obtain a global linearization:

$$\Phi : \mathbb{C} \to \mathbb{C} \setminus \mathcal{E}$$

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Hyperbolic Linearization

Let $F : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$, with

$$DF(0) = \begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix},$$

where $|\mu| < 1$ and $|\lambda| > 1$. Locally $W^u_F(0)$ is a graph over $\{z = 0\}$. Denote the projection

$$\pi : \{z = 0\} \to W^u_F(0),$$

and define

$$\varphi_n(w) = F^n \circ \pi(\lambda^{-n} w).$$

The maps φ_n converge to a (local) linearization Φ of the unstable manifold, satisfying the same functional equation.

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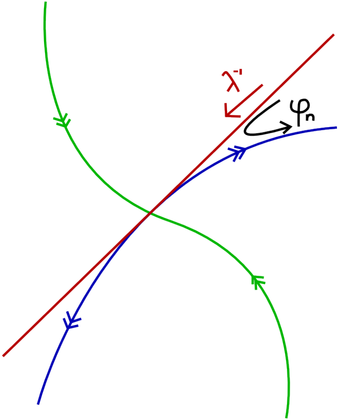

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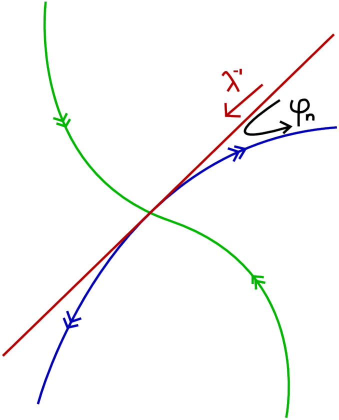
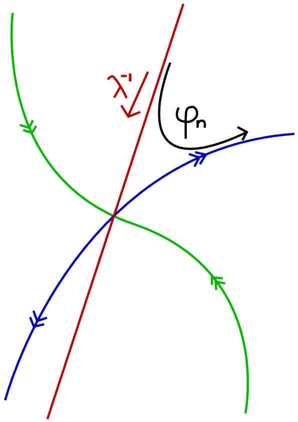


Hyperbolic Linearization

In fact, we can work with any other complex line transverse to the attracting direction and obtain (a multiple of) the same linearization map.



Rate of convergence

Now consider *resonant* germs,

$$DF(0) = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}, \quad |\lambda| > 1$$

Typically one has

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Applying linearization to skew products

$$F(t, z) = (\mu t, \lambda z + at + h.o.t.),$$

with $|\mu| < 1$, $|\lambda| > 1$ and $a \neq 0$.

Corollary

The functions $\phi_n(w) := F^n(\lambda^{-n} w, 0)$ converge to a map Φ satisfying

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Note that $a \neq 0$ guarantees that the horizontal axis is transverse to the attracting direction.

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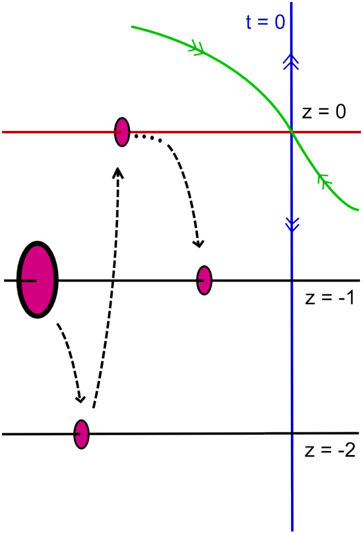

Setting up the construction

Let $F(t,z) = (p(t), g(z) + h(t))$, with $(0,0)$ a hyperbolic fixed point.

1. Let -1 be a critical point of g, with $g^2(-1) = 0$.
2. Define $p(t) = \lambda^{-1} t$. (resonance!)
3. Let w be such that $\Phi(\lambda^{-2} w) = -1$. (We can do this because $-1 \notin \mathcal{E}$.)

Conclusion:

$$F^n(\lambda^{-n} w, -1) \sim (\lambda^{-2n} w, -1).$$

By choosing $h(t)$ "degenerate" we can achieve

$$\left| F^n(\lambda^{-n} w, -1) - (0, -1) \right| \sim \lambda^{2n}.$$

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2. Define $p(t) = \lambda^{-1} t$. (resonance!)
3. Let w be such that $\Phi(\lambda^{-2} w) = -1$. (We can do this because $-1 \notin \mathcal{E}$.)

Conclusion:

$$F^n(\lambda^{-n} w, -1) \sim (\lambda^{-2n} w, -1).$$

By choosing $h(t)$ “degenerate” we can achieve

$$\left| F^n(\lambda^{-n} w, -1) - (0, -1) \right| \sim \lambda^{2n}.$$

$$F : (t, z) \mapsto \left(\frac{t}{8}, 2(z+1)^4 - 2 + t - \frac{641}{4165}t^2\right)$$

Definition

$$D_n = \left\{(8^{-n}w, z) : |z - (-1)| < 8^{-\frac{3}{4}n}\right\}$$

Then $F(D_n)$ has radius $\sim 8^{-3n}$, so $F^n(D_n)$ has radius $\sim 8^{-2n} < 8^{-\frac{3}{4}2n}$.

Hence for large n:

$$F^n(D_n) \subset D_{2n}.$$

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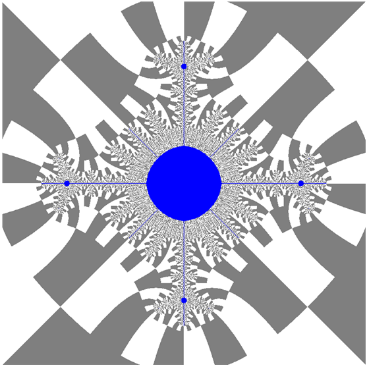
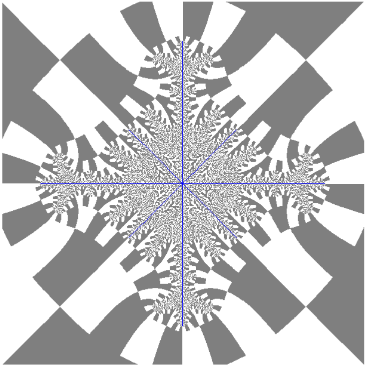


A fiber $t = 8^{-n} w$



A nearby fiber

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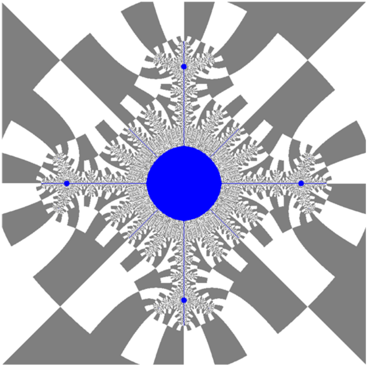

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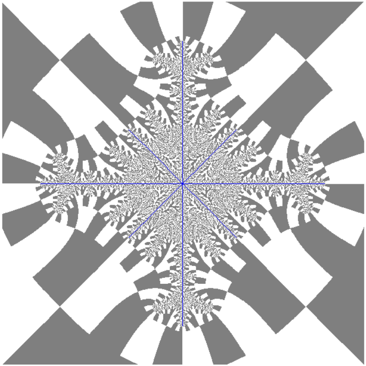

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Nearby fibers

Recall $g : z \mapsto 2(z-1)^4 - 2$, and recall the linearization map $\Phi : \mathbb{C} \to \mathbb{C} \setminus \mathcal{E}$.

The filled Julia set of g has no interior, hence for w generic, $\Phi(w)$ lies in the basin of infinity.

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Conclusion

There exist (resonant and degenerate) polynomial skew products with *wandering vertical Fatou disks*. However, for the maps *we constructed*, there exist no wandering Fatou components.

Question

As we move further and further away from one-dimensional rational functions, we can expect wandering Fatou components to arise at some point. Will they already arise for skew-products? Will they arive for Hénon maps?

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